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Green's relations on ordered n-ary semihypergroups

Jukkrit Daengsaen and Sorasak Leeratanavalee*

Abstract. In this paper, we introduce the concept of weak i-hyperfilters of ordered n-ary semihypergroups, where a positive integer $1 \leq i \leq n$ and $n \geq 2$, and discuss their related properties. We define Green's relations $\mathcal{M}_i$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{K}$ on ordered n-ary semihypergroups and investigate the relationships between Green's relations and the equivalence relation $\mathcal{W}_i$, which is generated by the weak i-hyperfilters. Also, we give the characterizations of intra-regular ordered n-ary semihypergroups via the properties of weak i-hyperfilters. Finally, we introduce the concepts of (i-, Λ-)duo ordered n-ary semihypergroups and establish some interesting properties.

1 Introduction

The theory of algebraic hyperstructures was initiated in 1934 when Marty [13] introduced the concept of hypergroups based on the notion of hyperoperations in order to study problems in non-commutative algebras. After-

* Corresponding author
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wards, this topic have been investigated and applied to several branches of pure and applied mathematics. Also, different aspects of algebraic hyperstructures have been introduced and studied by many mathematicians. In 2011, Heidari and Davvaz [10] applied the notion of ordered semigroups to algebraic hyperstructures. They introduced the concept of ordered semihypergroups as a natural generalization of ordered semigroups and discussed their related properties. The relationships between ordered semihypergroups and ordered semigroups were investigated in [5, 9]. The notion of (left, right) hyperideals of ordered semihypergroups, which is a nice generalization of (left, right) ideals of ordered semigroups, was considered by Changphas and Davvaz [1]. Tang et al. [17] introduced the notion of (left, right) hyperfilters of ordered semihypergroups and characterized them in terms of completely prime hyperideals. Such notion can be considered as an extension of (left, right) filters of ordered semigroups. As we have known from [11, 12] that the equivalence relation $\mathcal{N}$, which is generated by the filters of ordered semigroups, plays a significant role in studying the structural properties of intra-regular and duo ordered semigroups. In 2016, Omidi and Davvaz [14] defined and investigated the equivalence relation $\mathcal{N}$ on ordered semihypergroups by using the hyperfilters. Tang and Davvaz [16] defined the hyper version of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ on ordered semihypergroups and explained the relationships between the Green's relations and the equivalence relation $\mathcal{N}$. Tang et al. [18] introduced the concept of weak hyperfilters of ordered semihypergroups and established the connections of Green's relations, the equivalence relation $\mathcal{N}$ and the equivalence relation $\mathcal{W}$, which is defined by weak hyperfilters. Also, they gave some characterizations of intra-regular, left duo, right duo and duo ordered semihypergroups by the properties of weak hyperfilters and the relation $\mathcal{W}$. We notice here that every hyperfilter is always a weak hyperfilter and the relation $\mathcal{W} \neq \mathcal{N}$ in general.

Recently, a new class of algebraic hyperstructures called an ordered n-ary semihypergroup, where $n \geq 2$, was introduced and studied by Daengsaen and Leeratanavalee [2]. Such new class represents a natural generalization of ordered semigroups, ordered semihypergroups and ordered n-ary semigroups. Some interesting results concerning hyperideals, hyperfilters and regularities in ordered n-ary semihypergroups have been investigated by the same authors in [2, 3]. In this paper, we introduce the concept of weak

i-hyperfilters of ordered n-ary semihypergroups, where a positive integer $1 \leq i \leq n$ and $n \geq 2$. Also, we define the Green's relations $\mathcal{M}_i$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{K}$ on ordered n-ary semihypergroups and establish connections between Green's relations and the equivalence relation $\mathcal{W}_i$, which is generated by the weak i-hyperfilters. Furthermore, the characterizations of intra-regular ordered n-ary semihypergroups by the properties of weak i-hyperfilters are given. Finally, we introduce the concepts of i-duo , Λ-duo and duo ordered n-ary semihypergroups and discuss their related properties. As an application of the results, the corresponding results on n-ary semihypergroups are also obtained.

2 Preliminaries

In this section, we recall some basic definitions and some results of n-ary semihypergroups and ordered n-ary semihypergroups, that will be used throughout the paper. For more detail, we refer the readers to see [2–4, 6–8].

Let $\mathbb{N}$ be the set of all positive integers and $i, j, k, m, n \in \mathbb{N}$. Recall that an *$n$-ary hypergroupoid* [8] (S, f) is a nonempty set S endowed with an n-ary hyperoperation f, i.e., a mapping $f : S \times \cdots \times S \to \mathcal{P}^*(S)$ where S appears $n \geq 2$ times and $\mathcal{P}^*(S)$ denotes the set of all nonempty subsets of S. According to the abbreviated symbols in the theory of n-ary systems, the sequence $x_i, x_{i+1}, ..., x_j$ of S is denoted by x_i^j. In the case $j < i$, this symbol is empty. If $x_{i+1} = x_{i+2} = ... = x_j = x$, then we write x^{j-i} instead of x_{i+1}^j. In this convention, we write $f(x_1, x_2, ..., x_n) := f(x_1^n)$ and $f(x_1, ..., x_i, y, ..., y, z_{j+1}, ..., z_n) := f(x_1^i, y^{j-i}, z_{j+1}^n)$. For the abbreviated symbol of a sequence of subsets of S, we define analogously. For nonempty subsets $X_1, ..., X_n$ of S, we use the following notation:

$$f(X_1^n) = f(X_1, ..., X_n) := \bigcup_{a_i \in X_i, i=1,...,n} f(a_1^n).$$

If $X_i = \{a\}$ for some $1 \leq i \leq n$, then we write $f(X_1^{i-1}, a, X_{i+1}^n)$ instead of $f(X_1^{i-1}, \{a\}, X_{i+1}^n)$. If $X_1 = ... = X_i = Y$ and $X_{i+1} = ... = X_n = Z$, then we write $f(Y^i, Z^{n-i})$ instead of $f(X_1^n)$.

An n-ary hypergroupoid (S, f) is called an *n-ary semihypergroup* [7] if the equation

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

holds for all $x_1^{2n-1} \in S$ and all $1 \leq i \leq j \leq n$. If $m = k(n-1)+1$ where $k \geq 2$, then the m-ary hyperoperation f_k on S is defined by

$$f_k(x_1^{k(n-1)+1}) = \underbrace{f(f(...f(f(x_1^n}_{f \text{ appears } k \text{ times}}), x_{n+1}^{2n-1}), ...), x_{(k-1)(n-1)+2}^{k(n-1)+1}).$$

In a particular case $k = 2$, $f_2(x_1^{2n-1}) = f(f(x_1^n), x_{n+1}^{2n-1})$.

An *ordered n-ary semihypergroup* [2] $(S, f, \leq)$ is an n-ary semihypergroup (S, f) together with a partial order $\leq$ that is compatible with f, i.e., for any $x, y \in S$,

$$x \leq y \text{ implies } f(z_1^{i-1}, x, z_{i+1}^n) \preceq f(z_1^{i-1}, y, z_{i+1}^n)$$

for all $z_1^n \in S$ and all $1 \leq i \leq n$. Note that, for any $X, Y \in \mathcal{P}^*(S)$, $X \preceq Y$ means for every $a \in X$ there exists $b \in Y$ such that $a \leq b$. A nonempty subset A of an ordered n-ary semihypergroup $(S, f, \leq)$ is called an n-ary subsemihypergroup of S if $f(A^n) \subseteq A$. In this case, $(A, f, \leq)$ is also an ordered n-ary semihypergroup.

Throughout this paper, S stands for an ordered n-ary semihypergroup $(S, f, \leq)$, where $n \geq 2$, unless specified otherwise. For a nonempty subset X of S, we denote

$$(X] = \{a \in S \mid a \leq b \text{ for some } b \in X\}.$$

Lemma 2.1. [2] *Let $X, Y, X_1, ..., X_n$ be nonempty subsets of S. Then the following statements hold.*

(i) $X \subseteq (X]$.

(ii) $((X]] = (X]$.

(iii) $f((X_1], (X_2], ..., (X_n]) \subseteq (f(X_1^n)]$.

(iv) $(X \cup Y] = (X] \cup (Y]$.

(v) $X \subseteq Y$ *implies* $(X] \subseteq (Y]$.

Definition 2.2. [2] Let A be a nonempty subset of S. For any $1 \leq i \leq n$ and $n \geq 2$, A is called an *i-hyperideal* of S if the following assertions are satisfied.

(1) $f(x_1^{i-1}, y, x_{i+1}^n) \subseteq A$ for all $y \in A$ and all $x_1^{i-1}, x_{i+1}^n \in S$.

(2) For every $x \in A$ and $y \in S$, if $y \leq x$ then $y \in A$. Equivalently, $(A] \subseteq A$.

A is called a *hyperideal* of S if it is an i-hyperideal of S for all $i = 1, ..., n$. From now on, we denote by $M^i(A)$ (respectively, $J(A)$) the i-hyperideal (respectively, hyperideal) of S generated by A. In the case $A = \{x\}$, we write $M^i(x)$ instead of $M^i(\{x\})$.

Lemma 2.3. [2] *Let A be a nonempty subset of S. Then the following statements hold.*

(i) $M^1(A) = \big(f(A, S^{n-1}) \cup A\big]$.

(ii) $M^n(A) = \big(f(S^{n-1}, A) \cup A\big]$.

(iii) *For any $1 \leq i \leq n$ and $n \geq 3$,*

$$M^i(A) = \left(\bigcup_{k \geq 1} f_k(S^{k(i-1)}, A, S^{k(n-i)}) \cup A\right].$$

Let A be a nonempty subset of S. A is called *prime* if, for any $x_1^n \in S$, $f(x_1^n) \subseteq A$ implies $x_k \in A$ for some $k = 1, 2, ..., n$. A is called *semiprime* if, for any $x \in S$, $f(x^n) \subseteq A$ implies $x \in A$. A is called *completely prime* if, for any $x_1^n \in S$, $f(x_1^n) \cap A \neq \emptyset$ implies $x_k \in A$ for some $k = 1, 2, ..., n$. A is called *completely semiprime* if, for any $x \in S$, $f(x^n) \cap A \neq \emptyset$ implies $x \in A$.

Definition 2.4. [3] Let A be an n-ary subsemihypergroup of S. For any $1 \leq i \leq n$ and $n \geq 2$, A is called an *i-hyperfilter* of S if the following assertions are satisfied.

(1) If $f(x_1^n) \cap A \neq \emptyset$, for all $x_1^n \in S$, then $x_i \in A$.

(2) For every $x \in A$ and $y \in S$, if $x \leq y$, then $y \in A$.

If A is an i-hyperfilter of S, for all $1 \leq i \leq n$, then A is called a *hyperfilter* of S.

In what follows, we denote by $N^i(x)$ (respectively, $N(x)$) the i-hyperfilter (respectively, hyperfilter) of S generated by x. The equivalence relations $\mathcal{N}_i$ and $\mathcal{N}$ [3] on S are defined as follows:

$$\mathcal{N}_i = \left\{(x, y) \in S \times S \mid N^i(x) = N^i(y)\right\},$$

$$\mathcal{N} = \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

For every nonempty subset A of S, we define

$$\delta_A := \{(x, y) \in S \times S \mid x, y \in A \text{ or } x, y \notin A\}.$$

Clearly, δ_A is an equivalence relation on S. The following theorems are already proved in [3].

Theorem 2.5. *Let A be a nonempty subset of S. For any $1 \leq i \leq n$ and $n \geq 2$, A is a (i-)hyperfilter of S if and only if $S \setminus A = \emptyset$ or $S \setminus A$ is a completely prime (i-)hyperideal of S.*

Theorem 2.6. *Let $\mathcal{CP}(S)$ and $\mathcal{CP}_i(S)$ be the set of all completely prime hyperideals and the set of all completely prime i-hyperideals of S, respectively. Then*

$$\mathcal{N} = \bigcap \{\delta_A \mid A \in \mathcal{CP}(S)\} \text{ and } \mathcal{N}_i = \bigcap \{\delta_A \mid A \in \mathcal{CP}_i(S)\}.$$

3 Green's relations on n-ary semihypergroups

In this section, we introduce the concept of weak i-hyperfilters of ordered n-ary semihypergroups, where $1 \leq i \leq n$ and $n \geq 2$, and investigate their related properties.

Definition 3.1. Let S be an ordered n-ary semihypergroup with $n \geq 2$. For any $1 \leq i \leq n$, a nonempty subset A of S is called a *weak i-hyperfilter* of S if it satisfies the following conditions.

(i) If $x_1^n \in A$ then $f(x_1^n) \cap A \neq \emptyset$.

(ii) If $f(x_1^n) \cap A \neq \emptyset$ for all $x_1^n \in S$ then $x_i \in A$.

(iii) For any $x \in A$ and $y \in S$, if $x \leq y$ then $y \in A$.

A is said to be a *weak hyperfilter* of S if it is a weak i-hyperfilter of S for all $1 \leq i \leq n$.

Clearly, every (i-)hyperfilter of S is always a weak (i-)hyperfilter of S. Moreover, Definition 3.1 coincides with Definition 3.1 given in [18] if $n = 2$. The following theorems describe characterizations of weak (i-)hyperfilter of ordered n-ary semihypergroups in terms of their prime (i-)hyperideal.

Theorem 3.2. *Let S be an ordered n-ary semihypergroup with $n \geq 2$ and let A be a nonempty subset of S. For any $1 \leq i \leq n$, A is a weak i-hyperfilter of S if and only if $S \setminus A = \emptyset$ or $S \setminus A$ is a prime i-hyperideal of S.*

Proof. Let i be a fixed positive integer satisfying $1 \leq i \leq n$. Let A be a weak i-hyperfilter of S and $S \setminus A \neq \emptyset$. To show that $S \setminus A$ is a prime i-hyperideal of S, let $x \in S \setminus A$ and $y_1^{i-1}, y_{i+1}^n \in S$. If $f(y_1^{i-1}, x, y_{i+1}^n) \cap A \neq \emptyset$, then, since A is an i-hyperfilter of S, we have $x \in A$. It is impossible. So $f(y_1^{i-1}, x, y_{i+1}^n) \subseteq S \setminus A$. Let $y \in S \setminus A$ and $x \in S$ be such that $x \leq y$. Then $x \in S \setminus A$. In fact, if $x \in A$, then, since A is a weak i-hyperfilter of S and $x \leq y$, we have $y \in A$. It is impossible. Consequently, $S \setminus A$ is an i-hyperideal of S. Next, let $x_1^n \in S$ be such that $f(x_1^n) \subseteq S \setminus A$. Then $x_j \in S \setminus A$ for some $j = 1, 2, ..., n$. In fact, if $x_j \in A$ for all $j = 1, 2, ..., n$, then, since A is an i-hyperideal of S, we have $f(x_1^n) \subseteq A$. It is impossible. Thus $S \setminus A$ is a prime i-hyperideal of S. Conversely, if $S \setminus A = \emptyset$, then we are done. Suppose that $S \setminus A$ is a prime i-hyperideal of S. To show that A is a weak i-hyperfilter of S, let $x_1^n \in A$. Then $f(x_1^n) \cap A \neq \emptyset$. Indeed, if $f(x_1^n) \cap A = \emptyset$, then $f(x_1^n) \subseteq S \setminus A$. Since $S \setminus A$ is a prime i-hyperideal of S, we have $x_k \in S \setminus A$ for some $k = 1, 2, .., n$. It is impossible. Next, let $y_1^n \in S$ be such that $f(y_1^n) \cap A \neq \emptyset$. Then $y_i \in A$. In fact, if $y_i \in S \setminus A$, then, since $S \setminus A$ is an i-hyperideal of S, we have $f(y_1^n) \subseteq S \setminus A$. It is impossible. Next, let $x \in A$ and $y \in S$ be such that $x \leq y$. Then $y \in A$. Indeed, if $y \in S \setminus A$, then, since $S \setminus A$ is an i-hyperideal of S, we have $x \in S \setminus A$, which leads to contradict with $x \in A$. Therefore A is a weak i-hyperfilter of S. □

Theorem 3.3. *Let S be an ordered n-ary semihypergroup with $n \geq 2$ and let A be a nonempty subset of S. Then, A is a weak hyperfilter of S if and only if $S \setminus A = \emptyset$ or $S \setminus A$ is a prime hyperideal of S.*

Proof. The proof is similar to Theorem 3.2. □

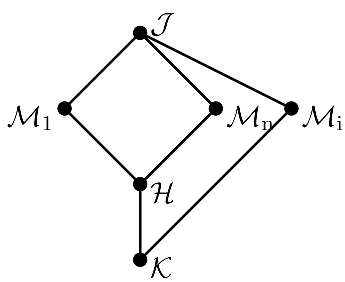
Let S be an ordered n-ary semihypergroup with $n \geq 3$. For any $1 \leq i \leq n$ and $x \in S$, we denote by $W^i(x)$ (respectively, $W(x)$) the weak i-hyperfilter (respectively, weak hyperfilter) of S generated by x and called the *principal weak i-hyperfilter (respectively, hyperfilter) generated by x*. Firstly, we introduce the Green's relations $\mathcal{M}_i$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{K}$ on S as follows:

$$\mathcal{M}_i := \left\{(x,y) \in S \times S \mid M^i(x) = M^i(y)\right\},$$

$$\mathcal{J} := \{(x,y) \in S \times S \mid J(x) = J(y)\},$$

$$\mathcal{H} := \mathcal{M}_1 \cap \mathcal{M}_n \text{ and } \mathcal{K} := \bigcap_{i=1}^{n} \mathcal{M}_i.$$

The connections among Green's relations on S can be expressed by the following diagram.



Next, we define the equivalence relation $\mathcal{W}_i$ (respectively, $\mathcal{W}$) on S, which is generated by the same principal weak i-hyperfilter (respectively, hyperfilter) of S, as follows:

$$\mathcal{W}_i := \left\{(x,y) \in S \times S \mid W^i(x) = W^i(y)\right\},$$

$$\mathcal{W} := \{(x,y) \in S \times S \mid W(x) = W(y)\}.$$

Note that $(x)_{\mathcal{M}_i}$ (respectively, $(x)_{\mathcal{J}}$, $(x)_{\mathcal{W}_i}$ and $(x)_{\mathcal{W}}$) denotes the $\mathcal{M}_i$-class (respectively, $\mathcal{J}$-class, $\mathcal{W}_i$-class and $\mathcal{W}$-class) containing x. The following theorem provides the relationship between the Green's relations and the equivalence relations $\mathcal{N}$, $\mathcal{W}$.

Theorem 3.4. *Let S be an ordered n-ary semihypergroup with $n \geq 3$. For any $1 \leq i \leq n$, the following statements hold.*

(i) *If $\mathcal{A}_i(S)$ is the set of all i-hyperideals of S, $\mathcal{A}(S)$ is the set of all hyperideals of S, $\mathcal{P}_i(S)$ is the set of all prime i-hyperideals of S and $\mathcal{P}(S)$ is the set of all prime hyperideals of S, then*

$$\mathcal{M}_i = \bigcap \{\delta_A \mid A \in \mathcal{A}_i(S)\}, \quad \mathcal{J} = \bigcap \{\delta_B \mid B \in \mathcal{A}(S)\},$$

$$\mathcal{W}_i = \bigcap \{\delta_C \mid C \in \mathcal{P}_i(S)\} \text{ and } \mathcal{W} = \bigcap \{\delta_D \mid D \in \mathcal{P}(S)\}.$$

(ii) $\mathcal{K} \subseteq \mathcal{M}_i \subseteq \mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{N}$, $\mathcal{K} \subseteq \mathcal{M}_i \subseteq \mathcal{W}_i \subseteq \mathcal{W} \subseteq \mathcal{N}$ *and* $\mathcal{K} \subseteq \mathcal{M}_i \subseteq \mathcal{W}_i \subseteq \mathcal{N}_i \subseteq \mathcal{N}$.

(iii) *If A is an i-hyperideal of S, B is a hyperideal of S, C is a prime i-hyperideal of S and D is a prime hyperideal of S, then*

$$A = \bigcup \{(x)_{\mathcal{M}_i} \mid x \in A\}, \quad B = \bigcup \{(x)_{\mathcal{J}} \mid x \in B\},$$

$$C = \bigcup \{(x)_{\mathcal{W}_i} \mid x \in C\} \text{ and } D = \bigcup \{(x)_{\mathcal{W}} \mid x \in D\}.$$

Proof. (i) We first prove that $\mathcal{M}_i = \bigcap \{\delta_A \mid A \in \mathcal{A}_i(S)\}$. Let $(x, y) \in \mathcal{M}_i$. Then $M^i(x) = M^i(y)$. To show that $(x, y) \in \delta_A$ for all $A \in \mathcal{A}_i(S)$, assume that $(x, y) \notin \delta_A$ for some $A \in \mathcal{A}_i(S)$. Then, we obtain two cases as follows.
Case 1.1: $x \in A$ and $y \notin A$. Since A is an i-hyperideal of S, we have $y \in M^i(y) = M^i(x) = \left(\bigcup_{k\geq 1} f_k(S^{k(i-1)}, x, S^{k(n-i)}) \cup \{x\}\right] \subseteq \left(\bigcup_{k\geq 1} f_k(S^{k(i-1)}, A, S^{k(n-i)}) \cup A\right] \subseteq (A] = A$. It is impossible.
Case 1.2: $x \notin A$ and $y \in A$. Using the similar processes as in Case 1.1, we have $x \in A$. It is impossible. So $\mathcal{M}_i \subseteq \bigcap\{\delta_A \mid A \in \mathcal{A}_i(S)\}$. Conversely, suppose that $(x, y) \in \delta_A$ for all $A \in \mathcal{A}_i(S)$. Since $(x, y) \in \delta_{M^i(y)}$ and $y \in$

$M^i(y)$, we get $x \in M^i(y)$. It follows that $M^i(x) \subseteq M^i(y)$. Similarly, since $(x,y) \in \delta_{M^i(x)}$ and $x \in M^i(x)$, we obtain $M^i(y) \subseteq M^i(x)$. Consequently, $M^i(x) = M^i(y)$ and so $(x,y) \in \mathcal{M}_i$. It follows that $\bigcap\{\delta_A \mid A \in \mathcal{A}_i(S)\} \subseteq \mathcal{M}_i$. Thus $\mathcal{M}_i = \bigcap\{\delta_A \mid A \in \mathcal{A}_i(S)\}$.

Next, we show that $\mathcal{W}_i = \bigcap\{\delta_C \mid C \in \mathcal{P}_i(S)\}$. Let $(x,y) \in \mathcal{W}_i$. Then $W^i(x) = W^i(y)$. Assume that $(x,y) \notin \delta_C$ for some $C \in \mathcal{P}_i(S)$. Then, there are two cases as follows.
Case 2.1: $x \in C$ and $y \notin C$. Then $y \in S \setminus C$. Since $S \setminus (S \setminus C) = C$ is a prime i-hyperideal of S, by Theorem 3.2, $S \setminus C$ is a weak i-hyperfilter of S. Since $y \in S \setminus C$, we obtain $x \in W^i(x) = W^i(y) \subseteq S \setminus C$, which is a contradiction.
Case 2.2: $x \notin C$ and $y \in C$. Using the similar proof as in Case 2.1, we also get a contradiction. Thus $\mathcal{W}_i \subseteq \bigcap\{\delta_C \mid C \in \mathcal{P}_i(S)\}$. To show the reverse subset, let $(x,y) \in \delta_C$ for all $C \in \mathcal{P}_i(S)$. Assume that $(x,y) \notin \mathcal{W}_i$. Then, we have two cases to be considered as follows.
Case 3.1: $x \notin W^i(y)$. Then $x \in S \setminus W^i(y)$. Since $W^i(y)$ is a weak i-hyperfilter of S and $S \setminus W^i(y) \neq \emptyset$, by Theorem 3.2, $S \setminus W^i(y)$ is a prime i-hyperideal of S. By hypothesis, $(x,y) \in \delta_{S \setminus W^i(y)}$. Since $y \notin S \setminus W^i(y)$, we have $x \notin S \setminus W^i(y)$. This is a contradiction.
Case 3.2: $y \notin W^i(x)$. Using the similar processes as in Case 3.1, we also get a contradiction. So $(x,y) \in \mathcal{W}_i$. It follows that $\bigcap\{\delta_C \mid C \in \mathcal{P}_i(S)\} \subseteq \mathcal{W}_i$. Therefore $\mathcal{W}_i = \bigcap\{\delta_C \mid C \in \mathcal{P}_i(S)\}$. For the rest, their proofs are analogous.

(ii) Clearly, $\mathcal{K} \subseteq \mathcal{M}_i$. Since $\mathcal{CP}(S) \subseteq \mathcal{P}(S) \subseteq \mathcal{A}(S) \subseteq \mathcal{A}_i(S)$, by Theorem 2.6 and (i), we have $\mathcal{K} \subseteq \mathcal{M}_i \subseteq \mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{N}$. Similarly, since $\mathcal{CP}(S) \subseteq \mathcal{P}(S) \subseteq \mathcal{P}_i(S) \subseteq \mathcal{A}_i(S)$ and $\mathcal{CP}(S) \subseteq \mathcal{CP}_i(S) \subseteq \mathcal{P}_i(S) \subseteq \mathcal{A}_i(S)$, we obtain $\mathcal{K} \subseteq \mathcal{M}_i \subseteq \mathcal{W}_i \subseteq \mathcal{W} \subseteq \mathcal{N}$ and $\mathcal{K} \subseteq \mathcal{M}_i \subseteq \mathcal{W}_i \subseteq \mathcal{N}_i \subseteq \mathcal{N}$.

(iii) We only show the first equality. Let A be an i-hyperideal of S. If $y \in A$, then $y \in (y)_{\mathcal{M}_i} \subseteq \bigcup\{(x)_{\mathcal{M}_i} \mid x \in A\}$. So $A \subseteq \bigcup\{(x)_{\mathcal{M}_i} \mid x \in A\}$. Conversely, let $y \in (x)_{\mathcal{M}_i}$ for some $x \in A$. By (i), we have $(y,x) \in \mathcal{M}_i = \bigcap\{\delta_A \mid A \in \mathcal{A}_i(S)\}$. So $(y,x) \in \delta_A$. Since $x \in A$, we obtain $y \in A$. Therefore $\bigcup\{(x)_{\mathcal{M}_i} \mid x \in A\} \subseteq A$ and this completes the proof. □

4 Intra-regular ordered n-ary semihypergroups

In this section, we give some characterizations of intra-regular ordered n-ary semihypergroups by means of Green's relations $\mathcal{M}_i$ and $\mathcal{W}$.

Lemma 4.1. [2] *Let S be an ordered n-ary semihypergroup with $n \geq 3$. For any $1 < j < n$, the following statements are equivalent.*

(i) *For each $a \in S$ there exist $x_1^{j-1}, x_{j+1}^n \in S$ such that*

$$a \in \left(f(x_1^{j-1}, f(a^n), x_{j+1}^n)\right].$$

(ii) *For each $a \in S$ there exists $y_1^{2n-2} \in S$ such that*

$$a \in \left(f(y_1^{n-1}, f(f(a^n), y_n^{2n-2}))\right].$$

Definition 4.2. [2] Let S be an ordered n-ary semihypergroup with $n \geq 3$. An element $a \in S$ is called *intra-regular* if it satisfies one of the equivalent conditions in Lemma 4.1. S is called *intra-regular* if every element of S is intra-regular.

Remark 4.3. Let S be an ordered n-ary semihypergroup with $n \geq 3$. Then, the following statements are equivalent.

(i) S is intra-regular.

(ii) For any $1 < j < n$, $a \in \left(f(S^{j-1}, f(a^n), S^{n-j})\right]$ for all $a \in S$.

(iii) $a \in \left(f(S^{n-1}, f(f(a^n), S^{n-1}))\right]$ for all $a \in S$.

Applying previous results, the following lemmas are obtained.

Lemma 4.4. *Let S be an ordered n-ary semihypergroup with $n \geq 3$. For any $1 < i < n$, the following statements are equivalent.*

(i) *S is intra-regular.*

(ii) $A \subseteq \left(f(S^{j-1}, f(A^n), S^{n-j})\right]$ $\left(A \subseteq \left(f(S^{n-1}, f(f(A^n), S^{n-1}))\right]\right)$ *for all $\emptyset \neq A \subseteq S$.*

(iii) $a \in \left(f(S^{j-1}, f(a^n), S^{n-j})\right]$ $\left(a \in \left(f(S^{n-1}, f(f(a^n), S^{n-1}))\right]\right)$ *for all* $a \in S$.

Proof. The proof is straightforward. □

Firstly, we characterize intra-regular ordered n-ary semihypergroups in terms of the weak hyperfilters.

Theorem 4.5. *Let* S *be an ordered* n*-ary semihypergroup with* $n \geq 3$*. Then,* S *is intra-regular if and only if* $W(x) = \left\{y \in S \mid x \in \left(f(f(S, y^{n-1}), S^{n-1})\right]\right\}$ *for all* $x \in S$*.*

Proof. ($\Longrightarrow$) Let S be an intra-regular ordered n-ary semihypergroup and $x \in S$. Let

$$A := \left\{y \in S \mid x \in \left(f(f(S, y^{n-1}), S^{n-1})\right]\right\}.$$

Obviously, $x \in A$. In fact, since S is intra-regular, by Lemma 4.4, we have $x \in \left(f(S, f(x^n), S^{n-2})\right] = \left(f(f(S, x^{n-1}), x, S^{n-2})\right] \subseteq \left(f(f(S, x^{n-1}), S^{n-1})\right]$. Hence $x \in A$. To show that A is a weak hyperfilter of S generated by x, let $a_1^n \in A$. Then $f(a_1^n) \cap A \neq \emptyset$. Indeed, since $a_1^n \in A$, we have $x \in \left(f(f(S, \{a_i\}^{n-1}), S^{n-1})\right]$ for all $i = 1, 2, ..., n$. Since

$$x \in \left(f(f(S, \{a_n\}^{n-1}), S^{n-1})\right], \ x \in \left(f(f(S, \{a_{n-1}\}^{n-1}), S^{n-1})\right]$$

we have

$$\begin{aligned}
x &\in (f(S, f(x^n), S^{n-2})] \\
&\subseteq (f_2(S, (f(f(S, \{a_n\}^{n-1}), S^{n-1})], (f(f(S, \{a_{n-1}\}^{n-1}), S^{n-1})], x^{n-2}, S^{n-2})] \\
&\subseteq (f_2(S, f(f(S, \{a_n\}^{n-1}), S^{n-1}), f(f(S, \{a_{n-1}\}^{n-1}), S^{n-1}), x^{n-2}, S^{n-2})] \\
&\subseteq (f_4(S^2, a_n, S^{n-3}, f(S, f(S^{n-1}, S), S^{n-2}), a_{n-1}, S^{n-1}, x^{n-2}, S^{n-2})] \\
&\subseteq (f_3(S^2, f(a_n, S^{n-2}, a_{n-1}), S^{n-1}, x^{n-2}, S^{n-2})], \text{by Lemma 4.4,} \\
&\subseteq (f_3(S^2, (f(S, f(f(a_n, S^{n-2}, a_{n-1})^n), S^{n-2})], S^{n-1}, x^{n-2}, S^{n-2})] \\
&\subseteq (f_3(S^2, f(S, f(f(a_n, S^{n-2}, a_{n-1})^n), S^{n-2}), S^{n-1}, x^{n-2}, S^{n-2})] \\
&\subseteq (f_5(S^2, S, S^{n-2}, f(a_n, S^{n-2}, a_{n-1}), f(a_n, S^{n-2}, a_{n-1}), S^{n-2}, S^{n-1}, x^{n-2}, S^{n-2})] \\
&= \Big(f_3(S, f(S, f(S, S^{n-2}, a_n), S^{n-2}), a_{n-1}, a_n, S^{n-3}, f(f(S, a_{n-1}, S^{n-2}), S^{n-1}), \\
&\qquad x^{n-2}, S^{n-2}) \Big]
\end{aligned}$$

$$\begin{aligned}
&\subseteq \left(f_3(S^2, a_{n-1}, a_n, S^{n-2}, x^{n-2}, S^{n-2})\right] \\
&\subseteq \ ... \\
&\subseteq \left(f_3(S^n, a_1, ..., a_n, S^{n-2})\right] \\
&= \left(f(f(S^n), f(a_1^n), S^{n-2})\right] \\
&\subseteq \left(f(S, f(a_1^n), S^{n-2})\right].
\end{aligned}$$

Then, there exists $y \in f(a_1^n)$ such that $x \in \left(f(S, y, S^{n-2})\right]$. Since S is intra-regular, we obtain $y \in \left(f(S, f(y^n), S^{n-2})\right]$. Then

$$\begin{aligned}
x &\in \left(f(S, y, S^{n-2})\right] \\
&\subseteq \left(f(S, \left(f(S, f(y^n), S^{n-2})\right], S^{n-2})\right] \\
&\subseteq \left(f(S, f(S, y, f(y^{n-1}, S), S^{n-3}), S^{n-2})\right] \\
&\subseteq \left(f(S, f(S, y, S^{n-2}), S^{n-2})\right] \\
&\subseteq \ ... \\
&\subseteq \left(\underbrace{f(S,\ f(\ ..., f(S}_{f \text{ appears } n-1 \text{ times}}\ , y, S^{n-2}), ...), S^{n-2})\right] \\
&\subseteq \left(\underbrace{f(S, f(..., f(S, f(S, f}_{f \text{ appears } n+1 \text{ times}}(y^n), S^{n-2}), S^{n-2}), ...), S^{n-2})\right] \\
&= \left(f(f(f(S^n), y^{n-1}), y, \underbrace{f(S^n), f(S^n), ..., f(S^n)}_{f \text{ appears } n-2 \text{ times}})\right] \\
&\subseteq \left(f(f(S, y^{n-1}), S^{n-1})\right].
\end{aligned}$$

Consequently, $y \in A$. It implies $A \cap f(a_1^n) \neq \emptyset$.

Next, let $b_1^n \in S$ be such that $f(b_1^n) \cap A \neq \emptyset$. Then, there exist $y \in f(b_1^n)$ and $y \in A$. Then $x \in \left(f(f(S, y^{n-1}), S^{n-1})\right] \subseteq \left(f(f(S, f(b_1^n)^{n-1}), S^{n-1})\right]$. We consider the following three cases.
Case 1: $j = 1$. Then $x \in \left(f(f(S, f(b_1^n)^{n-1}), S^{n-1})\right] \subseteq \left(f(f(S, f(b_1^n), S^{n-2}), S^{n-1})\right] = \left(f(S, b_1, f(f(b_2^n, S), S^{n-2}), S^{n-3})\right] \subseteq \left(f(S, b_1, S^{n-2})\right]$.
Case 2: $1 \leq j \leq n$. Then

$$\begin{aligned}
x &\in \left(f(f(S, f(b_1^n)^{n-1}), S^{n-1})\right] \\
&\subseteq \left(f(f(S, f(b_1^n)^{n-j}, f(b_1^{j-1}, b_j, b_{j+1}^n), f(b_1^n)^{j-2}), S^{n-1})\right] \\
&\subseteq \left(f(f(S, S^{n-j}, f(S^{j-1}, b_j, S^{n-j}), S^{j-2}), S^{n-1})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(f(f(S^n), b_j, S^{n-3}, f(S^n))\right] \\
&\subseteq \left(f(S, b_j, S^{n-2})\right].
\end{aligned}
$$

Case 3: $j = n$. Using the similar proof as in Case 1, we also get $x \in \left(f(S, b_n, S^{n-2})\right]$. From Case 1-3, we have $x \in \left(f(S, b_j, S^{n-2})\right]$ for all $j = 1, 2, ..., n$. Since S is intra-regular, we obtain $b_j \in \left(f(S^{n-1}, f(f(\{b_j\}^n), S^{n-1}))\right]$. Then

$$
\begin{aligned}
x &\in \left(f(S, b_j, S^{n-2})\right] \\
&\subseteq \left(f(S, f(S^{n-1}, f(f(\{b_j\}^n), S^{n-1})), S^{n-2})\right] \\
&= \left(f(f(f(S^n), \{b_j\}^{n-1}), f(b_j, S^{n-1}), S^{n-2})\right] \\
&\subseteq \left(f(f(S, \{b_j\}^{n-1}), S^{n-1})\right].
\end{aligned}
$$

So $b_j \in A$. Next, let $y \in A$ and $z \in S$ be such that $y \leq z$. Then $y \in (z]$. Since $y \in A$, we have $x \in \left(f(f(S, y^{n-1}), S^{n-1})\right] \subseteq \Big(f(f(S, (z]^{n-1}), S^{n-1})\Big]$ $\subseteq \left(f(f(S, z^{n-1}), S^{n-1})\right]$. Hence $z \in A$. Thus A is a weak hyperfilter of S containing x. It follows that $W(x) \subseteq A$.

Finally, let B be a weak hyperfilter of S containing x. To show that $A \subseteq B$, let $y \in A$. Then $x \in \left(f(f(S, y^{n-1}), S^{n-1})\right] = \left(f(S, y^{n-2}, f(y, S^{n-1}))\right]$. Hence, there exists $a \in f(S, y^{n-2}, f(y, S^{n-1}))$ such that $x \leq a$. Since B is a weak hyperfilter of S containing x, we have $a \in B$. Since $a \in f(S, y^{n-2}, f(y, S^{n-1}))$, there exist $u \in S$ and $v \in f(y, S^{n-1})$ such that $a \in f(u, y^{n-2}, v)$. It follows that $B \cap f(u, y^{n-2}, v) \neq \emptyset$. Since B is a weak hyperfilter of S, we obtain $y \in B$. Consequently, $A \subseteq B$. Therefore $W(x) = A$.

($\Longleftarrow$) Let $x \in S$. Since $x \in W(x)$, by Definition 3.1(i), we have $f(x^n) \cap W(x) \neq \emptyset$. Then, there exist $y \in f(x^n)$ and $y \in W(x)$. By hypothesis, we get $x \in \left(f(f(S, y^{n-1}), S^{n-1})\right] \subseteq \left(f(f(S, y^{n-2}, f(x^n)), S^{n-1})\right]$ $\subseteq \left(f(S^{n-1}, f(f(x^n), S^{n-1}))\right]$. Therefore S is intra-regular. □

In the following theorem, the connections of intra-regular ordered n-ary semihypergroups and Green's relations are provided.

Theorem 4.6. *Let S be an ordered n-ary semihypergroup with $n \geq 3$. Then, the following statements hold.*

(i) *If S is intra-regular, then $\mathcal{W} = \mathcal{W}_i = \mathcal{M}_i = \mathcal{J}$ for all $1 < i < n$.*

(ii) *If $\mathcal{W} = \mathcal{M}_i$ for some $1 \leq i \leq n$, then S is intra-regular.*

Proof. (i) Let S be an intra-regular ordered n-ary semihypergroup with $n \geq 3$ and let i be a fixed positive integer satisfying $1 < i < n$. Let $(x, y) \in \mathcal{W}$. Then $x \in W(x) = W(y)$. By Theorem 4.5, we obtain $y \in \big(f(f(S, x^{n-1}), S^{n-1})\big] = \big(f(S, x^{n-2}, f(x, S^{n-1}))\big] \subseteq \big(f(S, x^{n-2}, S)\big] = \big(f(S, x^{i-2}, x, x^{n-i-1}, S)\big] \subseteq \big(f(S^{i-1}, x, S^{n-i})\big] \subseteq M^i(x)$. So $M^i(y) \subseteq M^i(x)$. Similarly, since $y \in W(y) = W(x)$, using the analogous process, we have $M^i(x) \subseteq M^i(y)$ and then $M^i(x) = M^i(y)$. Consequently, $(x, y) \in \mathcal{M}_i$ and so $\mathcal{W} \subseteq \mathcal{M}_i$. By Theorem 3.4(ii), we conclude that $\mathcal{W} = \mathcal{M}_i = \mathcal{W}_i = \mathcal{J}$.

(ii) Let $\mathcal{W} = \mathcal{M}_i$ for some $1 \leq i \leq n$. Let $x \in S$. Since $x \in W(x)$, by Definition 3.1(i), we have $f(x^n) \cap W(x) \neq \emptyset$. Then, there exist $y \in f(x^n)$ and $y \in W(x)$. It follows that $W(y) \subseteq W(x)$. Since $W(y) \cap f(x^n) \neq \emptyset$ and $W(y)$ is a weak hyperfilter of S, we get $x \in W(y)$. Consequently, $W(x) \subseteq W(y)$ and so $W(x) = W(y)$. Using the similar processes, we have $W(y) = W(z)$ for some $z \in f(f(x^n)^n)$. So $W(x) = W(z)$. It implies that $(x, z) \in \mathcal{W} = \mathcal{M}_i$. Since $x \in M^i(x) = M^i(z)$, we have

$$
\begin{aligned}
x &\in M^i(z) \\
&\subseteq M^i(f(f(x^n)^n)) \\
&= \left(\bigcup_{k \geq 1} f_k(S^{k(i-1)}, f(f(x^n)^n), S^{k(n-i)}) \bigcup f(f(x^n)^n)\right] \\
&\subseteq \left(\bigcup_{j=2}^{n-1} f(S^{j-1}, f(f(x^n)^n), S^{n-j}) \bigcup f(S^{n-1}, f(f(f(x^n)^n), S^{n-1})) \bigcup f(f(x^n)^n)\right] \\
&\subseteq \Bigg(\bigcup_{j=2}^{n-1} f(S^{j-1}, f(f(x^n)^{n-j}, f(x^n), f(x^n)^{j-1}), S^{n-j}) \bigcup \\
&\quad f(S^{n-1}, f(f(x^n), f(f(x^n)^{n-1}, S), S^{n-2})) \bigcup \\
&\quad f(x^{n-2}, f(x^2, f(x^n)^{n-3}, x), f(f(x^n), x^{n-1}))\Bigg] \\
&\subseteq \big(f(S^{n-1}, f(f(x^n), S^{n-1}))\big].
\end{aligned}
$$

Therefore S is an intra-regular ordered n-ary semihypergroup. □

5 Duo ordered n-ary semihypergroups

In this section, we introduce the concept of i-duo and Λ-duo ordered n-ary semihypergroups and characterize them in terms of Green's relations $\mathcal{M}_i$ and the relation $\mathcal{W}_i$. We notice here that the notion of i-duo ordered n-ary semihypergroup is an extension of a left and a right duo ordered semihypergroup and the notion of Λ-duo ordered n-ary semihypergroups is a generalization of a duo ordered semihypergroup, see [18].

Definition 5.1. Let S be an ordered n-ary semihypergroup with $n \geq 2$. For any $1 \leq i \leq n$ and $i \neq \frac{n+1}{2}$, S is called *i-duo* if every i-hyperideal of S is an $(n-i+1)$-hyperideal of S.

Here, every 1-duo ordered n-ary semihypergroup is said to be a *right duo* ordered n-ary semihypergroup and every n-duo ordered n-ary semihypergroup is called a *left duo* ordered n-ary semihypergroup.

Example 5.2. Let $S = \{a, b, c, d, e\}$. Define a ternary hyperoperation $f : S \times S \times S \to \mathcal{P}^*(S)$ by $f(x_1^3) = (x_1 \circ x_2) \circ x_3$, for all $x_1^3 \in S$, where $\circ$ is defined by the following table.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$
e	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a, d, e\}$

Define a partial order $\leq$ on S as follows:

$$\leq := \{(a,a), (a,e), (b,b), (c,c), (d,d), (d,e), (e,e)\}.$$

Then $(S, f, \leq)$ is an ordered ternary semihypergroup. Clearly, the sets $\{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e\}$ and S are all 1-hyperideals of S. Also, they are 3-hyperideals of S. Therefore S is a right duo ordered ternary semihypergroup. However, S is not a left duo ordered ternary semihypergroup since $\{a, c, d\}$ is a 3-hyperideal of S but $\{a, c, d\}$ is not a 1-hyperideal of S.

Example 5.3. Let $S = \{a, b, c, d, e\}$. Define a hyperoperation $f : S \times S \times S \times S \to \mathcal{P}^*(S)$ by $f(x_1^4) = ((x_1 \circ x_2) \circ x_3) \circ x_4$, for all $x_1^4 \in S$, where $\circ$ is defined by the following table. Define a partial order $\leq$ on S as follows:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{b\}$	$\{c\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a, d\}$
e	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a, e\}$

$$\leq := \{(a, a), (a, d), (a, e), (b, b), (c, c), (d, d), (e, e)\}.$$

Then $(S, f, \leq)$ is an ordered 4-ary semihypergroup. It is not difficult to show that S is not a 1-duo ordered 4-ary semihypergroup because the set $\{a, b, e\}$ is a 1-hyperideal of S but it is not a 4-hyperideal of S. Similarly, since the set $\{a, d, e\}$ is a 4-hyperideal of S but it is not a 1-hyperideal of S, S is not a 4-duo ordered 4-ary semihypergroup. On the other hand, S is a 2-duo ordered 4-ary semihypergroup because every 2-hyperideal of S, i.e., the sets $\{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}$, $\{a, b, d, e\}$ and S, is a 3-hyperideal of S.

Theorem 5.4. *Let S be an ordered n-ary semihypergroup with $n \geq 3$. Then, the following statements hold.*

(i) *If S is intra-regular, then S is i-duo and every i-hyperideal of S is semiprime for all $1 < i < n$.*

(ii) *For any $1 < i < n$, every i-hyperideal of S is semiprime if and only if $\mathcal{W} = \mathcal{M}_i$.*

Proof. (i) Suppose that S is an intra-regular ordered n-ary semihypergroup. Let i be a fixed positive integer satisfying $1 < i < n$. To show that S is an i-duo ordered n-ary semihypergroup, i.e. every i-hyperideal of S is a $(n-i+1)$-hyperideal of S, let A be an i-hyperideal of S. Let $x \in A$ and $y_1^{n-i}, y_{n-i+2}^n \in S$. Since S is intra-regular, we have $x \in \big(f(S^{n-1}, f(f(x^n), S^{n-1}))\big] = \big(f(f(S^{n-1}, x), x^{i-2}, x, x^{n-i-1}, f(x, S^{n-1}))\big] \subseteq \big(f(S^{i-1}, x, S^{n-i})\big]$. By associativity, we have

$$f(y_1^{n-i}, x, y_{n-i+2}^n) \quad \subseteq \quad f(S^{n-i}, x, S^{i-1})$$

$$\begin{aligned}
&\subseteq f(S^{n-i}, \big(f(S^{i-1}, x, S^{n-i})\big], S^{i-1}) \\
&\subseteq \big(f(S^{n-i}, f(S^{i-1}, x, S^{n-i}), S^{i-1})\big] \\
&= \big(f(S^{n-1}, f(x, S^{n-1}))\big] \\
&\subseteq \big(f(S^{n-1}, f(f(S^{i-1}, x, S^{n-i}), S^{n-1}))\big] \\
&= \big(f(f(S^n), S^{i-2}, x, S^{n-i-1}, f(S^n))\big] \\
&\subseteq \big(f(S^{i-1}, x, S^{n-i})\big].
\end{aligned}$$

Since A is an i-hyperideal of S, we have $f(S^{i-1}, x, S^{n-i}) \subseteq A$ and then $\big(f(S^{i-1}, x, S^{n-i})\big] \subseteq (A] = A$. Consequently, A is a $(n-i+1)$-hyperideal of S and so S is an i-duo ordered n-ary semihypergroup. Next, let J be an i-hyperideal of S. Let $x \in S$ be such that $f(x^n) \subseteq J$. Since S is intra-regular, we have $x \in \big(f(S^{n-1}, f(f(x^n), S^{n-1}))\big] \subseteq \big(f(S^{n-1}, f(J, S^{n-1}))\big] = \big(f(S^{n-i}, f(S^{i-1}, J, S^{n-i}), S^{i-1})\big]$. Since J is an i-hyperideal of S, we have $x \in \big(f(S^{n-i}, f(S^{i-1}, J, S^{n-i}), S^{i-1})\big] \subseteq \big(f(S^{n-i}, J, S^{i-1})\big]$. Since S is i-duo, J is also a $(n-i+1)$-hyperideal of S. It follows that $x \in \big(f(S^{n-i}, J, S^{i-1})\big] \subseteq (J] = J$. Therefore, J is a semiprime i-hyperideal of S.

(ii) Suppose that every i-hyperideal of S is semiprime. In this case, we will apply the proof of Theorem 4.6(ii). First of all, we will show that $x \in M^i(f(f(x^n)^n))$ for all $x \in S$. Let $x \in S$ and let $y \in f(x^n)$. Since $f(y^n) \subseteq f(f(x^n)^n) \subseteq M^i(f(f(x^n)^n))$ and $M^i(f(f(x^n)^n))$ is a semiprime i-hyperideal of S, we have $y \in M^i(f(f(x^n)^n))$. So $f(x^n) \subseteq M^i(f(f(x^n)^n))$. Since $M^i(f(f(x^n)^n))$ is a semiprime i-hyper-ideal of S, we obtain $x \in M^i(f(f(x^n)^n))$. Using the similar poof as in Theorem 4.6(ii), we obtain that S is an intra-regular ordered n-ary semihypergroup. By Theorem 4.6(i), we have $\mathcal{W} = \mathcal{M}_i$. Conversely, it is obvious by (i) and Theorem 4.6(ii). □

Next, the relationships between the weak hyperfilters and the i-duo ordered n-ary semihypergroups, where $i = 1 (i = n)$ and $n \geq 3$, are discussed.

Theorem 5.5. *Let S be an ordered n-ary semihypergroup with $n \geq 3$. Then, every n-hyperideal of S is semiprime and S is left duo if and only if $W(x) = \{y \in S \mid x \in \big(f(S^{n-1}, y)\big]\}$ for all $x \in S$.*

Proof. ($\Longrightarrow$) Let S be a left duo ordered n-ary semihypergroup and suppose that every n-hyperideal of S is semiprime. Let $x \in S$. Put $A :=$

$\{y \in S \mid x \in (f(S^{n-1}, y)]\}$. Obviously, $x \in A$. Indeed, since $f(x^n) \subseteq f(S^{n-1}, x) \subseteq (f(S^{n-1}, x)]$ and $(f(S^{n-1}, x)]$ is a semiprime n-hyperideal of S, we have $x \in (f(S^{n-1}, x)]$. Hence $x \in A$. In order to show that A is a weak hyperfilter of S generated by x, we now consider the following assertions.

(1) Let $y_1^n \in A$. Then $f(y_1^n) \cap A \neq \emptyset$. In fact, since $y_i \in A$, we have $x \in (f(S^{n-1}, y_i)]$ for all $i = 1, 2, .., n$. By associativity, we have

$$\begin{aligned} f(x^n) &\subseteq f((f(S^{n-1}, y_1)], (f(S^{n-1}, y_2)], (f(S^{n-1}, y_3)], ..., (f(S^{n-1}, y_n)]) \\ &\subseteq (f(f(S^{n-1}, y_1), f(S^{n-1}, y_2), f(S^{n-1}, y_3), ..., f(S^{n-1}, y_n))] \\ &= (f(f(f(S^{n-1}, y_1), S^{n-1}), y_2, f(S^{n-1}, y_3), ..., f(S^{n-1}, y_n))]. \end{aligned}$$

Since $(f(S^{n-1}, y_1)]$ is an n-hyperideal of S and S is left duo, $(f(S^{n-1}, y_1)]$ is a 1-hyperideal of S. Since $f(S^{n-1}, y_1) \subseteq (f(S^{n-1}, y_1)]$, we have

$$f(f(S^{n-1}, y_1), S^{n-1}) \subseteq (f(S^{n-1}, y_1)].$$

Then we have

$$\begin{aligned} f(x^n) &\subseteq (f((f(S^{n-1}, y_1)], y_2, f(S^{n-1}, y_3), ..., f(S^{n-1}, y_n))] \\ &\subseteq (f(f(S^{n-1}, y_1), y_2, f(S^{n-1}, y_3), ..., f(S^{n-1}, y_n))] \\ &= (f(S, f(S^{n-2}, y_1, y_2), f(S^{n-1}, y_3), ..., f(S^{n-1}, y_n))] \\ &= (f(S, f(f(S^{n-2}, y_1^2), S^{n-1}), y_3, f(S^{n-1}, y_4), ..., f(S^{n-1}, y_n))], \\ &\quad \text{since } (f(S^{n-2}, y_1^2)] \text{ is an } n\text{-hyperideal of } S \text{ and } S \text{ is left duo,} \\ &\subseteq (f(S, f(S^{n-2}, y_1^2), y_3, f(S^{n-1}, y_4), ..., f(S^{n-1}, y_n))] \\ &= (f(S^2, f(S^{n-3}, y_1^3), f(S^{n-1}, y_4), ..., f(S^{n-1}, y_n))] \\ &\subseteq ... \\ &\subseteq (f(S^{n-2}, f(S, y_1^{n-1}), f(S^{n-1}, y_n))] \\ &= (f(S^{n-2}, f(f(S, y_1^{n-1}), S^{n-1}), y_n)], \\ &\quad \text{since } (f(S, y_1^{n-1})] \text{ is an } n\text{-hyperideal of } S \text{ and } S \text{ is left duo,} \\ &\subseteq (f(S^{n-2}, f(S, y_1^{n-1}), y_n)] \\ &= (f(S^{n-1}, f(y_1^n))]. \end{aligned}$$

Since $(f(S^{n-1}, f(y_1^n))]$ is a semiprime n-hyperideal of S, we have $x \in (f(S^{n-1}, f(y_1^n))]$. Then, there exist $v \in f(y_1^n)$ and $u \in f(S^{n-1}, v)$ such that $x \leq u$. Consequently, $x \in (f(S^{n-1}, v)]$ and so $v \in A$. It implies that $f(y_1^n) \cap A \neq \emptyset$.

(2) Let $y_1^n \in S$ be such that $f(y_1^n) \cap A \neq \emptyset$. Then there exist $w \in f(y_1^n)$ and $w \in A$. It follows that $x \in \big(f(S^{n-1}, w)\big] \subseteq \big(f(S^{n-1}, f(y_1^n))\big]$. For each $1 \leq j \leq n$, we have

$$\begin{aligned} f(x^n) &\subseteq \big(f(f(S^{n-1}, f(y_1^n))^n)\big] \\ &= \big(f(f(S^{n-1}, f(y_1^n))^{n-j}, f(S^{n-1}, f(y_1^n)), f(S^{n-1}, f(y_1^n))^{j-1})\big] \\ &\subseteq \big(f(S^{n-j}, f(S^{n-1}, f(y_1^n)), S^{j-1})\big] \\ &= \Big(f(S^{n-j}, f(f(S^{n-1}, y_1), y_2^{j-1}, y_j, y_{j+1}^n), S^{j-1})\Big] \\ &\subseteq \big(f(S^{n-j}, f(S^{j-1}, y_j, S^{n-j}), S^{j-1})\big] \\ &= \big(f(f(S^{n-1}, y_j), S^{n-1})\big]. \end{aligned}$$

Since $\big(f(S^{n-1}, y_j)\big]$ is an n-hyperideal of S and S is left duo, $\big(f(S^{n-1}, y_j)\big]$ is a 1-hyperideal of S. Hence $f(f(S^{n-1}, y_j), S^{n-1}) \subseteq \big(f(S^{n-1}, y_j)\big]$ and then $f(x^n) \subseteq \big(f(S^{n-1}, y_j)\big]$. Since $\big(f(S^{n-1}, y_j)\big]$ is a semiprime n-hyperideal of S, we have $x \in \big(f(S^{n-1}, y_j)\big]$. Hence $y_j \in A$.

(3) Let $y \in A$ and $z \in S$ be such that $y \leq z$. Then $y \in (z]$. Since $y \in A$, we get $x \in \big(f(S^{n-1}, y)\big] \subseteq \big(f(S^{n-1}, (z])\big] \subseteq \big(f(S^{n-1}, z)\big]$. So $z \in A$.

From (1)–(3), we conclude that A is a weak j-hyperfilter of S containing x. For arbitrary $1 \leq j \leq n$, A is a weak hyperfilter of S containing x. It follows that $W(x) \subseteq A$.

Finally, let B be a weak hyperfilter of S containing x. To show that $A \subseteq B$, let $y \in A$. Then $x \in \big(f(S^{n-1}, y)\big]$. Hence, there exists $u \in f(S^{n-1}, y)$ such that $x \leq u$. Also, there are $z_1^{n-1} \in S$ such that $u \in f(z_1^{n-1}, y)$. Since $B \ni x \leq u$ and B is a weak hyperfilter of S, we get $u \in B$. It follows that $f(z_1^{n-1}, y) \cap B \neq \emptyset$. Again, since B is a weak hyperfilter of S, we have $y \in B$. So $A \subseteq B$. It implies that $A \subseteq W(x)$. Therefore $W(x) = A$.

($\Longleftarrow$) Let A be an n-hyperideal of S. Let $x \in S$ be such that $f(x^n) \subseteq A$. Since $x \in W(x)$, we have $f(x^n) \cap W(x) \neq \emptyset$. Then there exist $y \in f(x^n)$ and $y \in W(x)$. By hypothesis, we have $x \in \big(f(S^{n-1}, y)\big] \subseteq \big(f(S^{n-1}, f(x^n))\big] \subseteq \big(f(S^{n-1}, A)\big] \subseteq A$. So A is a semiprime n-hyperideal of S. To show that S is left duo, let J be an n-hyperideal of S. Let $a \in J$ and $z_1^{n-1} \in S$. Let $y \in f(a, z_1^{n-1})$. Since $y \in W(y) \cap f(a, z_1^{n-1})$ and $W(y)$ is a weak hyperfilter of S, we have $a, z_1^{n-1} \in W(y)$. By Definition 3.1(i), we obtain $f(z_1^{n-1}, a) \cap W(y) \neq \emptyset$. Then, there exist $b \in f(z_1^{n-1}, a)$ and $b \in W(y)$. By hypothesis, we have $y \in \big(f(S^{n-1}, b)\big] \subseteq \big(f(S^{n-1}, f(z_1^{n-1}, a))\big] =$

$\big(f(S^{n-2}, f(S, z_1^{n-1}), a)\big] \subseteq \big(f(S^{n-1}, a)\big] \subseteq \big(f(S^{n-1}, A)\big] \subseteq (A] = A$. So $f(a, z_1^{n-1}) \subseteq A$, i.e., A is a 1-hyperideal of S. Therefore S is a left duo ordered n-ary semihypergroup and this completes the proof. □

Theorem 5.6. *Let S be a left duo ordered n-ary semihypergroup with $n \geq 3$. Then, every n-hyperideal of S is semiprime if and only if $\mathcal{W} = \mathcal{M}_n$.*

Proof. Suppose that every n-hyperideal of S is semiprime. Let $(x, y) \in \mathcal{W}$. Then $x \in W(x) = W(y)$. By Theorem 5.5, we have $y \in \big(f(S^{n-1}, x)\big] \subseteq M^n(x)$. It follows that $M^n(y) \subseteq M^n(x)$. Similarly, since $y \in W(y) = W(x)$, we obtain $M^n(x) \subseteq M^n(y)$. Consequently, $M^n(x) = M^n(y)$ and then $(x, y) \in \mathcal{M}_n$. It follows that $\mathcal{W} \subseteq \mathcal{M}_n$. By Theorem 3.4(ii), we conclude that $\mathcal{W} = \mathcal{M}_n$. Conversely, let J be an n-hyperideal of S. Let $x \in S$ be such that $f(x^n) \subseteq J$. Using the similar proof as in Theorem 4.6(ii), there exists $y \in f(x^n)$ such that $W(x) = W(y)$. It implies that $(x, y) \in \mathcal{W} = \mathcal{M}_n$. Then $x \in M^n(y) \subseteq M^n(f(x^n)) = \big(f(x^n) \cup f(S^{n-1}, f(x^n))\big] \subseteq \big(J \cup f(S^{n-1}, J)\big] \subseteq (J] = J$. Therefore J is a semiprime n-hyperideal of S. □

Theorem 5.7. *Let S be a right duo ordered n-ary semihypergroup with $n \geq 3$. Then, the following statements are equivalent.*

(i) *Every 1-hyperideal of S is semiprime.*

(ii) $\mathcal{W} = \mathcal{M}_1$.

(iii) $W(x) = \big\{y \in S \mid x \in \big(f(y, S^{n-1})\big]\big\}$ *for all $x \in S$.*

Proof. The proof is similar to Theorems 5.5 and 5.6. □

Finally, we introduce the concept of Λ-duo ordered n-ary semihypergroups and investigate some related properties.

Definition 5.8. Let S be an ordered n-ary semihypergroup with $n \geq 2$ and let $\emptyset \neq \Lambda \subseteq \{1, 2, ..., n\}$. A nonempty subset A of S is called a Λ-*hyperideal* [15] of S if $f(S^{i-1}, A, S^{n-i}) \subseteq A$ for all $i \in \Lambda$ and $(A] \subseteq A$. S is called Λ-*duo* if, for any $i \in \Lambda$, it is i-duo and every i-hyperideal of S is a Λ-hyperideal of S. S is called *duo* if it is Λ-duo where $\Lambda = \{1, 2, ..., n\}$. In other words, S is duo if and only if every i-hyperideal of S is a hyperideal of S for all $i = 1, 2, .., n$.

Example 5.9. Let $S = \{a, b, c, d, e\}$. Define a ternary hyperoperation $f : S \times S \times S \to \mathcal{P}^*(S)$ by the following tables. Define a partial order $\leq$ on

f	a	b	c	d
aa	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ab	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ac	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
ad	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
f	a	b	c	d
ba	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
bb	$\{a\}$	$\{b\}$	$\{b\}$	$\{a, d\}$
bc	$\{a\}$	$\{b\}$	$\{b\}$	$\{a, d\}$
bd	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$
f	a	b	c	d
ca	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
cb	$\{a\}$	$\{b\}$	$\{b\}$	$\{a, d\}$
cc	$\{a\}$	$\{b\}$	$\{b\}$	$\{a, d\}$
cd	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$
f	a	b	c	d
da	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
db	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$
dc	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$
dd	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$

S as follows:

$$\leq := \{(a, a), (a, d), (b, b), (c, c), (d, d), (e, e)\}.$$

Then $(S, f, \leq)$ is an ordered ternary semihypergroup. Clearly, the sets $\{a, d\}, \{a, b, d\}$ and S are all (1-, 2-, 3-)hyperideals of S. Therefore S is a duo ordered ternary semihypergroup.

Example 5.10. Let $S = \{a, b, c, d, e, 0\}$. Define a hyperoperation $f : S \times$

$S \times S \times S \to \mathcal{P}^*(S)$ by

$$f(x_1^4) = \begin{cases} \{d, e\} & \text{if } x_1 = a, x_2 = x_3 = b, x_4 = c, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x_1^4 \in S$. Define a partial order $\leq$ on S as follows:

$$\begin{aligned} \leq \ := \ & \{(0,0),(0,a),(0,b),(0,c),(0,d),(0,e),(a,a),(b,b),(c,c), \\ & (d,a),(d,d),(e,e)\}. \end{aligned}$$

Then $(S, f, \leq)$ is an ordered 4-ary semihypergroup. Let $\Lambda = \{2,3\}$. Clearly, the sets $\{0\}, \{0,c\}, \{0,d\}, \{0,e\}, \{0,a,d\}, \{0,a,e\}, \{0,c,d\}, \{0,c,e\}$, $\{0,d,e\}, \{0,a,c,d\}, \{0,a,d,e\}, \{0,c,d,e\}, \{0,a,b,d,e\}, \{0,b,c,d,e\}$ and S are all Λ-hyperideals of S. This follows that S is a Λ-duo ordered 4-ary semihypergroup. On the other hand, S is not a Λ-duo ordered 4-ary semihypergroup, where $\Lambda = \{1,2,4\}$, since $H = \{0,a,c,d\}$ is a 2-hyperideal of S but H is not a 1-hyperideal and a 4-hyperideal of S. In fact, since $\{d,e\} = f(a,b,b,c) \subseteq f(H,S,S,S)$ and $\{d,e\} = f(a,b,b,c) \subseteq f(S,S,S,H)$, we obtain $f(H,S,S,S) \nsubseteq H$ and $f(S,S,S,H) \nsubseteq H$.

Theorem 5.11. *Let S be a Λ-duo ordered n-ary semihypergroup with $\emptyset \neq \Lambda \subseteq \{1,2,...,n\}$ and $n \geq 3$. Then, every Λ-hyperideal of S is semiprime if and only if $\mathcal{W} = \bigcap_{i \in \Lambda} \mathcal{M}_i$.*

Proof. Suppose that every Λ-hyperideal of S is semiprime. Let i be a fixed element in Λ and let J be an i-hyperideal of S. Since S is Λ-duo, J is a Λ-hyperideal of S. By hypothesis, J is semiprime. By Theorems 5.4, 5.6 and 5.7, we have $\mathcal{W} = \mathcal{M}_i$. So $\mathcal{W} = \bigcap_{i \in \Lambda} \mathcal{M}_i$. Conversely, let J be a Λ-hyperideal of S. Then J is an i-hyperideal of S for all $i \in \Lambda$. Since $\mathcal{W} = \bigcap_{i \in \Lambda} \mathcal{M}_i \subseteq \mathcal{M}_i$, using the similar processes as in Theorems 5.4, 5.6 and 5.7, we conclude that J is a semiprime Λ-hyperideal of S. This completes the proof. □

Corollary 5.12. *Let S be a Λ-duo ordered n-ary semihypergroup where $\Lambda = \{1,n\}$ and $n \geq 3$. Then, every Λ-hyperideal of S is semiprime if and only if $\mathcal{W} = \mathcal{H}$.*

Corollary 5.13. *Let S be a duo ordered n-ary semihypergroup with $n \geq 3$. Then, every hyperideal of S is semiprime if and only if $\mathcal{W} = \mathcal{K}$.*

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References

[1] Changphas, T. and Davvaz, B., *Hyperidel theory in ordered semihypergroups*, International Congress on Algebraic Hyperstructures and its Applications, Xanthi, Greece (2014), 51-54.

[2] Daengsaen, J. and Leeratanavalee, S., *Regularities in ordered n-ary semihypergroups*, Mathematics 9(16) (2021), 1857.

[3] Daengsaen, J. and Leeratanavalee, S., *Semilattice strongly regular on ordered n-ary semihypergroups*, AIMS Mathematics 7(1) (2022), 478-498.

[4] Daengsaen, J., Leeratanavalee, S., and Davvaz, B., *On minimal and maximal hyperideals in n-ary semihypergroups*, Mathematics 8(10) (2020), 1656.

[5] Davvaz, B., Corsini, P., and Changphas, T., *Relationship between ordered semihypergroups and ordered semigroups by using pseudoorders*, Eur. J. Combin. 44 (2015), 208-217.

[6] Davvaz, B., Dudek, W., and Mirvakili, S., *Neutral elements, fundamental relations and n-ary hypersemigroups*, Int. J. Algebra Comput. 19 (2009), 567-583.

[7] Davvaz, B., Dudek, W., and Vougiouklis, T., *A generalization of n-ary algebraic systems*, Comm. Algebra. 37 (2009), 1248-1263.

[8] Davvaz, B. and Vougiouklis, T., *n-ary hypergroups*, Iran. J. Sci. Technol. 30 (2006), 165-174.

[9] Gu, Z. and Tang, X.L., *Ordered regular equivalence relations on ordered semihypergroups*, J. Algebra 450 (2016), 384-397.

[10] Heidari, D. and Davvaz, B., *On ordered hyperstructures*, U.P.B. Sci. Bull. Series A. 73 (2011), 85-96.

[11] Kehayopulu, N., Lepouras, G., and Tsingelis, M., *On right regular and right duo ordered semigroups*, Math. Japonica. 46 (1997), 311-315.

[12] Kehayopulu, N. and Tsingelis, M., *On intra-regular ordered semigroups*, Semigroup Forum 57 (1998), 138-141.

[13] Marty, F., *Sur une generalization de la notion de group*, In: 8th Congres. Math. Stocholm, Scandinaves (1934), 45-49.

[14] Omidi, S. and Davvaz, B., *A short note on the relation $\mathcal{N}$ in ordered semihypergroups*, Gazi Univ. J. Sci. 29(3) (2016), 659-662.

[15] Pornsurat, P., Ayutthaya, P.P., and Pibaljommee, B., *Prime i-ideals in ordered n-ary semigroups*, Mathematics 9 (2021), 491.

[16] Tang, J. and Davvaz, B., *Study on Green's relations in ordered semihypergroups*, Soft Comput. 24 (2020), 11189-11197.

[17] Tang, J., Davvaz, B., and Luo, Y., *Hyperfilters and fuzzy hyperfilters of ordered semihypergroups*, J. Intell. Fuzzy Syst. 29 (2015), 75-84.

[18] Tang, J., Xie, X.Y., and Gu, Z., *A study on weak hyperfilters of ordered semihypergroups*, AIMS Mathematics 6 (2021), 4319-4330.

Jukkrit Daengsaen *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.*
Email: daengsaen.j@gmail.com

Sorasak Leeratanavalee *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.*
Email: sorasak.l@cmu.ac.th