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Determinant and rank functions in semisimple pivotal Ab-categories

Khalid Draoui*, Hanan Choulli, and Hakima Mouanis

Abstract. We investigate and generalize quantum determinants to semisimple spherical and pivotal categories. It is well known that traces are preserved by strong tensor functors; we show on one hand that in fact, weaker conditions on a functor are sufficient to continue preserving traces. On the other hand, we prove that these determinants are well-behaved under strong tensor functors. Further, we introduce a notion of domination rank for objects of a semisimple pivotal category and prove similar properties of the ordinary case. Furthermore, we expand the determinantal and McCoy ranks to introduce a morphism quantum rank function on a semisimple pivotal category.

1 Introduction

Given an m by n matrix M over a field, its (column) rank, defined as the dimension of the subspace generated by its columns, coincides with the (row) rank defined identically; which is exactly the size of the largest submatrix

* Corresponding author

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of M of non zero determinant (determinantal rank) or equivalently, equals the smallest k for which $M = BC$ with B is an m by k matrix and C is a k by n matrix (Schein rank). These rank scalars do not all coincide no longer in case of a commutative ring. Under reasons of searching rank conditions for solvability of linear systems of equations, the McCoy rank (as the best possible generalization for this purpose, of the classical above recalled ranks) is introduced in [1], see also [8], and it is defined as

$$rk(M) = max\{t, Ann_R(I_t(M)) = (0)\}$$

where, $I_t(M)$ is the ideal in R generated by all the $(t \times t)$ minors of M and

$$Ann_R(I_t(M)) = \{r \in R,\ ra = 0,\ \forall a \in I_t(M)\}.$$

In [3], the authors defined rank functions in the context of a triangulated category. In this paper, we begin with the introduction of a *domination rank* concept for an object V of a *semisimple* (in the sense of [13]) *pivotal* category C, using the fact that for any two objects U and V of C, $Hom_C(U, V)$ is a finitely generated and projective module [12, Lemma 4.2.1, page 100] over the commutative ground ring $End_C(I)$ of C. Inspired by the work [2] on a categorification of the classical determinants to semisimple ribbon categories, the first aim of this work is to extend the above notions of determinantal and McCoy ranks into the context of a semisimple pivotal category C, profiting from the fact that to each endomorphism f of an object V of C, there is associated a square matrix over the commutative ground ring $End_C(I)$ of C, with entries expressed by means of the quantum trace and dimension defined in this context.

The second interest of this paper is the *determinant* concept, as an essential element in linear algebra and one of most meaningful invariants related to square matrices, investigated by Cramer in order to solve systems of linear equations. The authors in [2] gave a positive answer to the following question:

Can we define a notion of determinant (giving a formula) of an endomorphism of an object of a semisimple ribbon Ab−category [12], in such a way that

(a) We meet the classical determinant when we consider the category $vect_{\mathtt{K}}$ of finite dimensional vector spaces over a field $\mathtt{K}$.

(b) We keep similar properties as the classical case.

Hence, a categorical generalization of the classical *determinants*. For this end, they exploited the fact that traces are being categorified and widely studied in general contexts; namely, *quantum traces* of endomorphisms of dualizable objects in different monoidal categories [5–7, 11] thanks to the tensor products therein, these traces yield the notion of the *quantum dimension* of an object, as an element of $End_C(I)$, as the trace of the identity map, which find their interesting applications for example in quantum topology, by constructing quantum invariants [12]. They also profited from the fact that these two ingredients (traces and determinants) are intimately related. Pivotal categories allow the definition of left and right traces. Hence, we generalize the above categorification to the context of a semisimple pivotal/spherical category. We define the left *quantum determinant* of an endomorphism $f : V \to V$ as the left trace of the endomorphism $f^n l\Lambda_V^n$, namely:

$$ldet_V^n(f) := Tr_l(f^n l\Lambda_V^n);$$

where $l\Lambda_V^n$ is the endomorphism of $V^n(= V^{\otimes^n})$ given by:

$$l\Lambda_V^n = \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_l^{n_1}(id_{W_1}))^{-1}D_\sigma^1 \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_l^{n_m}(id_{W_m}))^{-1}D_\sigma^m;$$

where $(W_j)_{1\leq j\leq m}$ are representative objects of their classes of isomorphic objects in the considered dominating family $(V_{i(r)}; \varepsilon_r; \mu_r)_{1\leq r\leq n}$ of V, of simple objects, and D_σ^j is the endomorphism of $V^{n_j}(= V^{\otimes^{n_j}})$ defined by (the details are presented in section 4):

$$D_\sigma^j = \mu_{s_1^j}\varepsilon_{\sigma(s_1^j)} \otimes .... \otimes \mu_{s_{n_j}^j}\varepsilon_{\sigma(s_{n_j}^j)};$$

where $I_j = [\![s_1^j, s_{n_j}^j]\!]$, such that $[\![1, n]\!] = \dot{\bigcup}_{1\leq j\leq m} I_j$, with $n_j = |I_j|$.

Note here that these are not to confuse with the "quantum determinants" introduced in the case of $q-$matrices [9, page 79].

It is well known that strong tensor functors preserve traces [14, page 49]. We provide a sufficient condition for a Frobenius tensor (monoidal) functor [4] to continue preserving traces. Furthermore, we show that strong tensor functors preserve also quantum determinants.

2 Preliminaries

We refer to [9, 10, 13, 14] for more details of the following reminder of some notions from the theory of monoidal categories.

A monoidal category $(C;\otimes;I;\alpha;l;r)$ is given by a category C, a bifunctor $\otimes : C \times C \longrightarrow C$, a unit object I and natural isomorphisms $\alpha : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$, $l : I \otimes U \longrightarrow U$ and $r : U \otimes I \longrightarrow U$, for all $U, V, W \in Ob(C)$, called associativity constraint, left and right unitality constraints respectively, such that the pentagon and triangle axioms hold. If moreover, α, l and r are identities, then C is called strict.

A tensor (monoidal) functor $(F;F_0;F_2)$ between strict monoidal categories $(C;\otimes;I)$ and $(D;\otimes';I')$ is a functor $F : C \longrightarrow D$ together with maps $F_0 : I' \longrightarrow F(I)$ and $F_{2\,U,V} : F(U) \otimes F(V) \longrightarrow F(U \otimes V)$, for all $U, V \in Ob(C)$, such that the associativity, left and right unitality diagrams commute [14, page 15].

$(F;F_0;F_2)$ is called strong if F_0 and $F_{2\,U,V}$ are isomorphisms for all $U, V \in Ob(C)$.

A category C is an $Ab-$category (or preadditive), provided that the hom sets $Hom_C(U, V)$ are additive abelian groups, and the composition and tensor product of morphisms are bilinear.

A left (resp. right) duality for a strict monoidal category C, consists of a left (resp. right) duality for every object V of C; namely an object V^* of C and morphisms $d_V : V^* \otimes V \longrightarrow I$; (resp. $d'_V : V \otimes V^* \longrightarrow I$) and $b_V : I \longrightarrow V \otimes V^*$ (resp. $b'_V : I \longrightarrow V^* \otimes V$) such that

$$(id_V \otimes d_V)(b_V \otimes id_V) = id_V \;\; and \;\; (d_V \otimes id_{V^*})(id_{V^*} \otimes b_V) = id_{V^*};$$

respectively

$$(d'_V \otimes id_V)(id_V \otimes b'_V) = id_V \;\; and \;\; (id_{V^*} \otimes d'_V)(b'_V \otimes id_{V^*}) = id_{V^*}.$$

For any morphism $f : U \longrightarrow V$ between left dualizable objects, its left dual morphism $f^* : V^* \longrightarrow U^*$ is defined by

$$f^* = (d_V \otimes id_{U^*})(id_{V^*} \otimes f \otimes id_{U^*})(id_{V^*} \otimes b_U).$$

Similarly one defines its right dual morphism using this time the right duality structures $(V^*; d'_V; b'_V)$, $V \in Ob(C)$.

A pivotal category is a monoidal category C, endowed with left duality $(V^*; d_V; b_V)$ and right duality $(V^*; d'_V; b'_V)$ structures, for every object V of C, such that the induced left and right dual functors coincide as monoidal functors, and the induced isomorphisms

$$V^* \otimes U^* \xrightarrow{\lambda_{U,V}} (U \otimes V)^*$$

are the same (see [14, 1.7.1, page 26], also [11]).

Left and right traces of an endomorphism $f : V \longrightarrow V$ in a pivotal category C are defined as

$$Tr_l(f) = d_V(1 \otimes f)b'_V \ ; \quad Tr_r(f) = d'_V(f \otimes 1)b_V.$$

Furthermore, left and right partial traces of an endomorphism $f \in End_C(U \otimes V)$ in a pivotal category C are defined as

$$pTr_r(f) = (id_U \otimes d'_V)(f \otimes id_{V^*})(id_U \otimes b_V) \in End_C(U);$$

$$pTr_l(f) = (d_U \otimes id_V)(id_{U^*} \otimes f)(b'_U \otimes id_V) \in End_C(V).$$

The left and right *dimensions* of an object V of a *pivotal category* C are defined by

$$dim_l(V) = Tr_l(id_V) \ ; \quad dim_r(V) = Tr_r(id_V).$$

Consequently, we have $dim_l(V^*) = dim_r(V)$; $dim_r(V^*) = dim_l(V)$ and any two isomorphic objects have equal left (resp. right) dimensions.

A spherical category is a pivotal category where left and right traces coincide (denoted simply $Tr(f)$, for all $f \in End_C(V)$).

The following definitions follow [12] (without restriction to ribbon $Ab-$categories).

Let C be a pivotal $Ab-$category. An object V of C is called simple, if the map $End_C(I) \longrightarrow End_C(V)$, $k \mapsto k \otimes id_V$ is a bijection, and V is said to be dominated by simple objects if there exist a finite set of simple objects $\{V_i\}_i$ of C (the same simple object may be repeated) and morphisms $\varepsilon_i : V \to V_i$; $\mu_i : V_i \to V$, for all i, such that $\sum_i \mu_i \varepsilon_i = id_V$.

C is called dominated by a family $\{V_i\}_{i\in J}$ of simple objects, where J is an index set, if every object V of C is dominated by a finite sub-family of $\{V_i\}_{i\in J}$. This family will be denoted by $(V_{i(r)}; \varepsilon_r; \mu_r)_{1\leq r\leq n}$ and referred to as a dominating family of V, where $i(r) \in J$ and $\varepsilon_r : V \to V_{i(r)}$; $\mu_r : V_{i(r)} \to V$ as above, for all $1 \leq r \leq n$. Seen as a function, $i : \mathbb{N} \longrightarrow J$ is one to one.

A monoidal category C is called pure [14, page 14] if for all $k \in End_C(I)$, and for all $f \in Mor(C)$, one has $k.f = f.k$, where $k.f := k \otimes f$.

In the sequel; without loss of generality, C will always mean a strict monoidal category $(C; \otimes; I)$ (by Mac-Lane's coherence Theorem [10], asserting that every monoidal category is equivalent to a strict one). $\mathtt{K}_C$ denotes the commutative ground ring $End_C(I)$ of C and $\mathtt{K}$ denotes a base field with unit.

3 Domination rank in semisimple pivotal categories

A semisimple category is usually defined in the literature to be an abelian category, where each of its objects splits as a direct sum of simple ones. Here we follow a different terminology, Turaev [12], that does not involve direct sums . Let's start with a slightly different (more general) definition of a semisimple category than the one given in [12], where the author restricts the study to ribbon categories, but here we shall let the category C be pivotal or particularly spherical and keep the same definition, as long as the definition of semisimplicity on C is independant and does not require the ribbon structure on it as introduced by the same author in [13, page 416].

Definition 3.1. [12, page 99] A semisimple category is a couple $(C, \{V_i\}_{i\in J})$, consisting of a pivotal $Ab-$category C and a family $\{V_i\}_{i\in J}$ of simple objects of C, such that the following hold.

(a) There exists $0 \in J$, such that $V_0 = I$ (Normalization axiom).

(b) For all $i \in J$, there exists $i^* \in J$, such that $V_{i^*} \simeq V_i^*$ (Duality axiom).

(c) C is dominated by the family $\{V_i\}_{i\in J}$ (Domination axiom).

(d) For any non isomorphic simple objects V_i and V_j; $i, j \in J$, we have $Hom_C(V_i, V_j) = \{0\}$ (Schur's axiom).

Sometimes we will simply write C for $(C, \{V_i\}_{i\in J})$.

Remark 3.2. In a semisimple category, every object is dominated by a finite family of simple objects, but the cardinal of such family is not unique in the sense that one object can be dominated by families of different cardinals. To see this, consider the following situation: Let V be a simple object of a semisimple category C, then V is obviousely dominated by the family $(V; id_V; id_V)$. Let now W be another simple object of C, isomorphic to V, say via $\alpha : W \longrightarrow V$. Consider the family $(W; \varepsilon_i; \mu_i)_{1\le i\le 3}$, where $\mu_i = \alpha$ for $i = 1, 2$, $\mu_3 = -\alpha$, and $\varepsilon_i = \alpha^{-1}$ for $i = 1, 2, 3$. Thus defined, it is not difficult to see that this is a dominating family of V, consisting of one simple object W.

The following remark will serve for making clear a definition and some results of a *domination rank*.

Remark 3.3. (a) $\text{rank}(Hom_C(V_i, V_j)) = \delta_{i,j}$ (the Kronecker symbol), for all $i, j \in J$. In fact $Hom_C(V_i, V_j)$ is a free $\mathtt{K}_C-$module, of rank 1 if $V_i \simeq V_j$, and 0 otherwise by Shur's axiom.

(b) $Hom_C(V_i, V_j) \simeq Hom_C(V_j, V_i)$, for all $i, j \in J$, and hence we also have that $Hom_C(V_i^*, V_j^*) \simeq Hom_C(V_j^*, V_i^*)$ by the duality axiom.

(c) $\text{rank}(Hom_C(V_i, V_j)) = 1$, if and only if, $\text{rank}(Hom_C(V_i^*, V_j^*)) = 1$.

Definition 3.4. Let $(C; \{V_i\}_{i\in J})$ be a semisimple category and U an object of C, dominated by a family of simple objects of cardinal n. We call domination rank of U and denote by $rank_d(U)$, the positive integer defined by

$$rank_d(U) := \max_{1\le r_0\le n} \left(\sum_{r=1}^{n} rank(Hom_C(V_{i(r_0)}, V_{i(r)})) \right) \tag{3.1}$$

where, $(V_{i(r)}; \varepsilon_r; \mu_r)_{1\le r\le n}$ runs over all families of simple objects and of cardinal n, that are dominating U. In other words, $rank_d(U)$ is nothing but the maximum number of isomorphic dominating simple objects among all possible dominating families of U, of cardinal n.

Remark 3.5. The above defined quantity $rank_d(U)$ is well defined as $1 \leq ran_d(U) \leq n$ for any family of cardinal n, and it depends on the chosen dominating family, hence whenever unmentioned for the rest of the paper, the domination rank of an object (in a semisimple category) is defined on the picked dominating family. In case the dominating family was unique, up to isomorphisms between the simple objects (in the sense that if there exists another dominating family of U, of simple objects $(V'_{i(s)})_{1\leq s\leq n}$, there is some permutation $\pi \in \mathfrak{S}_n$, such that $V_{i(r)} \simeq V'_{\pi(i(r))}$ for all $1 \leq r \leq n$, where $\mathfrak{S}_n$ is the set of all permutations of the elements of the set $\{1, ..., n\}$), the domination rank in this case is the maximal number of isomorphic simple objects therein.

Theorem 3.6. *Let $(C; \{V_i\}_{i\in J})$ be a semisimple category and U and V be isomorphic objects of C. Then*

(a) $rank_d(V_i) = 1$, *for all* $i \in J$.

(b) $rank_d(V) = rank_d(U)$.

(c) $rank_d(U^*) = rank_d(U)$.

Proof. (a): For all $i \in J$, V_i is dominated by itself and $Hom_C(V_i, V_i)$ is a free module of rank 1.

(b): U is dominated by n simple objects $(V_i)_i$, if and only if, V is dominated by the same simple objects $(V_i)_i$ (even though with different domination structure maps).

(c): U is dominated by n simple objects $(V_i)_i$, if and only if, U^* is dominated by $(V_i^*)_i$. □

Proposition 3.7. *Let $(C; \{V_i\}_{i\in J})$ be a semisimple category and U an object of C dominated by a family of simple objects $(V_{i(r)})_{1\leq r\leq n}$, unique up to isomorphisms between the simple objects in the dominating families. Then $V_{i(r)} \simeq V_{i(s)}$, for all $1 \leq r, s \leq n$, if and only if, $rank_d(U) = n$.*

Proof. Immediate. □

Example 3.8. Let V be an object of the category $vect_{\mathtt{K}}$ of finite dimensional vector spaces over a filed $\mathtt{K}$. Then, $rank_d(V)$ as defined above, coincides with its dimension.

Example 3.9. Let V be an object of the category Mod_R of finitely generated projective modules over a local ring R (R is the unique simple object of Mod_R). Then, $rank_d(V) = rank(V)$.

In fact, V being projective and R local, V is free by the Kaplansky's Theorem. But V is finitely generated, then it has a finite free basis, hence a finite rank n. Consequently, $rank_d(V) = n$.

Example 3.10. Let R be a commutative ring with unit, G a multiplicative abelian group with neutral element e, and $c : G \otimes G \to R^\times$ a bilinear map, where $R^\times$ is the set of invertible elements of R. Consider the category $\mathcal{V}$ whose objects are elements of G and morphisms are defined as: $Hom_{\mathcal{V}}(g, g) = R$ and $Hom_{\mathcal{V}}(g, h) = \{0\}$ if $g \neq h$, for all $g, h \in G$. The identity morphism is given by the unit of R, the tensor product of objects is given by the product in G and the composition and tensor product of morphisms is given by the product in R. $\mathcal{V}$ can be equipped with a braiding $gh \to gh$ by means of c as $c(g, h) \in R$, and a twist: $g \to g$ as $c(g, g) \in R$ for all $g, h \in G$; making $\mathcal{V}$ into a ribbon category with e as unit object, see [12, Sect. I.1.7.2]. In particular, $\mathcal{V}$ is semisimple with $\{V_i\}_{i\in J}$ consists of all objects of $\mathcal{V}$ as they are all simple. Thus, every object V of $\mathcal{V}$ is dominated by itself and so $rank_d(V) = 1$. This construction can be generalized by fixing a group homomorphism $\alpha : G \to R^\times$ satisfying $\alpha(g^2) = 1$, for all $g \in G$; with the same structures as above, except the twist which is defined now to be $g \to g$ as $\alpha(g)c(g, g) \in R$, for all $g \in G$.

4 Quantum determinants in semisimple pivotal categories

Definition 4.1. Let $(C; \{V_i\}_{i\in J})$ be a semisimple category, V an object of C and let $\mathcal{F} = (V_{i(r)}; \varepsilon_r; \mu_r)_{1\leq r\leq n}$ and $\mathcal{F}' = (V'_{i(s)}; \varepsilon'_s; \mu'_s)_{1\leq s\leq m}$ be dominating families of V, of simple objects.

(1) We say that $\mathcal{F}$ and $\mathcal{F}'$ are equivalent and we write $\mathcal{F} \sim \mathcal{F}'$ if

   (a) $m = n$.

   (b) There exists some permutation $\pi \in \mathfrak{S}_n$, such that $V_{i(r)} \simeq V'_{\pi(i(r))}$ for all $1 \leq r \leq n$.

(2) $\mathcal{F}$ is called standard if

(a) $\varepsilon_r \mu_s = \delta_{r,s}\, id_{V_{i(r)}}$, for all $1 \leq r, s \leq n$.

(b) $\mathcal{F}$ is minimal in the sense that if $\mathcal{F}'$ is another dominating family satisfying (a), then the cardinal of $\mathcal{F}$ is less than that of $\mathcal{F}'$.

Denote by $Dom(V)$ the set of all dominating families of simple objects of V, and by $Dom^n(V)$ the sub-set consisting of those families of same cardinal $n \in \mathbb{N}^*$ of the form:

$$Dom^n(V) = \Big\{ (V_{i(r)}; \varepsilon_r; \mu_r)_{1\leq r\leq n} \in Dom(V),\ \textit{such that} \\ \varepsilon_r \mu_s = \delta_{r,s}\, id_{V_{i(r)}},\ \ \textit{for all}\ \ 1 \leq r, s \leq n \Big\}.$$

Proposition 4.2. *The relation " $\sim$ " defined in Definition* 4.1 between dominating families on V, is an equivalence relation. Moreover, let $(V_{i(r)}; \varepsilon_r; \mu_r)_{1\leq r\leq n}$ be a fixed dominating family in $Dom^n(V)$. Then, all elements of $Dom^n(V)$, equivalent to this family, are of the form:

$$\Big(V'_{\pi(i(r))}; \alpha_r \varepsilon_r h^{-1}; h\mu_r \alpha_r^{-1}\Big)_{1\leq r\leq n}$$

for some automorphism $h \in Aut_C(V)$, and a family of isomorphisms $\alpha_{i(r)} : V_{i(r)} \to V'_{\pi(i(r))}$, for all $1 \leq r \leq n$.

Proof. " $\sim$ " is clearly an equivalence relation. For any $h \in Aut_C(V)$, and any isomorphism $\alpha_r : V_{i(r)} \longrightarrow V'_{\pi(i(r))}$, for all $1 \leq r \leq n$, the element $\Big(V'_{\pi(i(r))}; \alpha_r \varepsilon_r h^{-1}; h\mu_r \alpha_r^{-1}\Big)_{1\leq r\leq n}$ is obviously a dominating family of V and belongs to $Dom^n(V)$.

Conversely, let $(V'_{i(s)}; \varepsilon'_s; \mu'_s)_{1\leq s\leq n}$ be a family in $Dom^n(V)$, equivalent to the fixed one. The fact that they are equivalent implies that there exists a permutation $\pi \in \mathfrak{S}_n$ and a family of isomorphisms $\{\alpha_r : V_{i(r)} \longrightarrow V'_{\pi(i(r))}\}_{1\leq r\leq n}$. In this way, the family $(V'_{i(s)}; \varepsilon'_s; \mu'_s)_{1\leq s\leq n}$ can be written as $(V'_{\pi(i(r))}; \varepsilon'_{\pi(r)}; \mu'_{\pi(r)})_{1\leq r\leq n}$. Define $h : V \longrightarrow V$ by $h := \sum_r \mu'_{\pi(r)} \alpha_r \varepsilon_r$, then $h \in Aut_C(V)$ and it is invertible with inverse given by $h^{-1} := \sum_s \mu_s \alpha_s^{-1} \varepsilon'_{\pi(s)}$. In fact:

$$hh^{-1} = \sum_{r,s} \mu'_{\pi(r)} \alpha_r \varepsilon_r \mu_s \alpha_s^{-1} \varepsilon'_{\pi(s)} = \sum_r \mu'_{\pi(r)} \varepsilon'_{\pi(r)} = id_V.$$

Similarly, $h^{-1}h = id_V$. Furthermore, for all $1 \leq r \leq n$, we clearly have

$$\varepsilon'_{\pi(r)} = \alpha_r \varepsilon_r h^{-1} \ \ and \ \ \mu'_{\pi(r)} = h\mu_r\alpha_r^{-1}.$$

□

Remark 4.3. Let $\mathcal{F}$ be a standard dominating family of V. Then, for any $\mathcal{F}' \sim \mathcal{F}$, $\mathcal{F}'$ is standard as well.

Proposition 4.4. *Let* $\mathtt{K}$ *be a field and* $(C; \{V_i\}_{i\in J})$ *a semisimple category, where* $\{V_i\}_{i\in J}$ *consists of all the simple objects of* C*. Assume that* C *is enriched over finite dimensional* $\mathtt{K}-$*vector spaces, such that* $\mathtt{K}_C \simeq \mathtt{K}$*. Then, every object of* C *admits a standard dominating family, and if the simple objects of* C *are all isomorphic, then all standard dominating families on any object are equivalent.*

Proof. Let V be an object of C and $\{V_j : j \in J\}$ a set of simple objects of C, then for all $j \in J$, $Hom_C(V, V_j)$ is a finite dimensional $\mathtt{K}_C-$vector space and its dual is $Hom_C(V_j, V)$. Picking a basis $(\mu_{j(r)})_{1\leq r\leq n}$ of $Hom_C(V, V_j)$ and its dual basis $(\varepsilon_{j(r)})_{1\leq r\leq n}$. Then V is dominated by $(V_j; \varepsilon_{j(r)}; \mu_{j(r)})_{1\leq r\leq n}$, and $\varepsilon_{j(r)}\mu_{j(s)} = \delta_{r,s}\ id_{V_j}$, for all $1 \leq r, s \leq n$. The minimal of such resulting families is standard, and they are equivalent if C owns only one class of isomorphic simple objects. □

Assume for this section and the forthcoming ones that all standard families on the objects of the considered semisimple category C are equivalent.

Let C be a semisimple category and V an object of C dominated by a family $(V_{i(r)}; \varepsilon_r; \mu_r)_{1\leq r\leq n}$ of simple objects. Note that all simple objects of C have invertible left and right dimensions in $\mathtt{K}_C$ ([12, Lemma 4.2.4, page 103]). Let $[\![1, n]\!] = I_1 \cup I_2 \cup ... \cup I_m$, where $m \leq n$, be a partition of $[\![1, n]\!] := [1, n] \cap \mathbb{N}$ into classes of isomorphic simple objects. Without loss of generality, we can assume, in the sequel, that all the simple objects in the class I_j, for all $1 \leq j \leq m$, are equal to a simple object W_j which will represent its class. This is by constructing a new dominating family. This resulting family, with simple objects the W_j's, is equivalent to the initial one. To make it clear, we shall illustrate the situation via an example. If

I_1 contains $V_{i(1)}, ..., V_{i(5)}$, choose any candidate, e.g: $V_{i(1)} = W_1$, and let $\varepsilon'_k = \alpha_k^{-1}\varepsilon_k$, and $\mu'_k = \mu_k\alpha_k$, for all $1 \leq k \leq 5$, where $\alpha_k : W_1 \to V_{i(k)}$ is an isomorphism, as the objects in the same class are isomorphic. Now replace $V_{i(1)}$, ε_k, μ_k by W_1, ε'_k, μ'_k respectively, and repeat the same procedure for the rest of classes.

Denote by $l\Lambda^n_V$, the endomorphism of $V^n(= V^{\otimes^n})$ defined by

$$l\Lambda^n_V = \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_l^{n_1}(id_{W_1}))^{-1}D^1_\sigma \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_l^{n_m}(id_{W_m}))^{-1}D^m_\sigma \tag{4.1}$$

where, for every $1 \leq j \leq m$ and every permutation $\sigma \in \mathfrak{S}_{n_j}$, W_j is a representative object of its class I_j as in the above discussion, and D^j_σ is the endomorphism of $V^{n_j}(= V^{\otimes^{n_j}})$ defined by:

$$D^j_\sigma = \mu_{s^j_1}\varepsilon_{\sigma(s^j_1)} \otimes .... \otimes \mu_{s^j_{n_j}}\varepsilon_{\sigma(s^j_{n_j})}; \tag{4.2}$$

where $I_j = [\![s^j_1, s^j_{n_j}]\!]$, with $s^j_1, ..., s^j_{n_j}$ are elements of the set $\{1, ..., n\}$, distinct two by two, and $n_j = |I_j|$ (its cardinal); such that $[\![1, n]\!] = \dot{\bigcup}_{1\leq j\leq m} I_j$ a disjoint union.

Similarly, denote by $r\Lambda^n_V$, the endomorphism of V^n defined by:

$$r\Lambda^n_V = \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_r^{n_1}(id_{W_1}))^{-1}D^1_\sigma \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_r^{n_m}(id_{W_m}))^{-1}D^m_\sigma. \tag{4.3}$$

Definition 4.5. Let $(C; \{V_i\}_{i\in J})$ be a semisimple category; V an object of C dominated by a standard family of cardinal n and $f \in End_C(V)$. We call left quantum determinant of f, and denote it by $ldet^n_V(f)$, the element of the ring $\mathtt{K}_C$ defined by

$$ldet^n_V(f) = Tr_l(f^n l\Lambda^n_V).$$

Similarly, we call right quantum determinant of f, and denote it by $rdet^n_V(f)$, the element of $\mathtt{K}_C$ defined by

$$rdet^n_V(f) = Tr_r(f^n r\Lambda^n_V).$$

Remark 4.6. Note that for all $1 \leq r, s \leq n$, if $V_{i(r)}$ is not isomorphic to $V_{i(s)}$, then $\varepsilon_r f \mu_s = 0$ by Schur's axiom, and in this case, $Tr_l(\varepsilon_r f \mu_s) = Tr_r(\varepsilon_r f \mu_s) = 0$. On the other hand, if $V_{i(r)} \simeq V_{i(s)}$, then $Tr_l(\varepsilon_r f \mu_s)$ and $Tr_r(\varepsilon_r f \mu_s)$ appearing in the explicit expressions of $Tr_l(f^n l\Lambda_V^n)$ and $Tr_r(f^n r\Lambda_V^n)$ are well defined by the above discussion. Hence, $ldet_V^n(f)$ and $rdet_V^n(f)$ are well defined and do not depend on the choice of the repartition of the simple objects in the dominating family. In fact, in Lemma 4.11, we prove that these two elements coincide with the usual determinant of some corresponding matrices, and the change of the order of two isomorphic simple objects in the partition yields a change of the corresponding lines (resp. columns) in the associated matrix, followed by a change of the corresponding columns (resp. lines), hence resulting equal determinants. Similarly, the change now of the order of two classes I_i and I_j yields a change of lines and columns by an even number, hence the determinant is still invariant. We prove in Proposition 4.13 that $ldet_V^n(f)$ and $rdet_V^n(f)$ are invariant under the choice of the standard dominating family until they are assumed to be equivalent (Definition 4.1).

Theorem 4.7. *Let $(C; \{V_i\}_{i\in J})$ be a semisimple pure category; V an object of C dominated by a standard family $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ of cardinal n and $f \in End_C(V)$. Then*

(a) $ldet_V^n(id_V) = rdet_V^n(id_V) = id_I$.

(b) $ldet_V^n(q \otimes f) = q^n ldet_V^n(f)$ *and* $rdet_V^n(q \otimes f) = q^n rdet_V^n(f)$*; for all* $q \in \mathtt{K}_C^\times$ *($\mathtt{K}_C^\times$ denotes the set of invertible morphisms of $\mathtt{K}_C$).*

(c) $ldet_V^n(f^*) = rdet_V^n(f)$ *and* $rdet_V^n(f^*) = ldet_V^n(f)$.

Proof. C being pure, the left and right traces are $\otimes-$multiplicative, i.e, $Tr_l(f \otimes g) = Tr_l(f) \otimes Tr_l(g)$, for any endomorphisms f and g of C. Similarly for the right trace. Then we have

(a): $ldet_V^n(id_V) = Tr_l(l\Lambda_V^n)$

$$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_l^{n_1}(id_{W_1}))^{-1} Tr_l(D_\sigma^1) \ldots$$

$$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_l^{n_m}(id_{W_m}))^{-1} Tr_l(D_\sigma^m)$$

$$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_l^{n_1}(id_{W_1}))^{-1} Tr_l(\varepsilon_{\sigma(s_1^1)}\mu_{s_1^1}) \ldots Tr_l(\varepsilon_{\sigma(s_{n_1}^1)})\mu_{s_{n_1}^1}) \ldots$$

$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_l^{n_m}(id_{W_m}))^{-1}Tr_l(\varepsilon_{\sigma(s_1^m)}\mu_{s_1^m}) \ldots Tr_l(\varepsilon_{\sigma(s_{n_m}^m)}\mu_{s_{n_m}^m})$
$= (Tr_l^{n_1}(id_{W_1}))^{-1}Tr_l^{n_1}(id_{W_1}) \ldots (Tr_l^{n_m}(id_{W_m}))^{-1}Tr_l^{n_m}(id_{W_m})$
$= id_I.$

(b): C being pure, the tensor product is bilinear, then $(q\otimes f)^n = q^n.f^n$, hence $ldet_V^n(q\otimes f) = Tr_l((q\otimes f)^n l\Lambda_V^n) = q^n Tr_l(f^n l\Lambda_V^n) = q^n ldet_V^n(f)$; and similarly for the second claim.

(c): V^* is dominated by $(V_{i(r)}^*;\mu_r^*;\varepsilon_r^*)_r$, and for all $1\leq j\leq m$ and all $1\leq k\leq n_j$, we have

$$(Tr_l^{n_j}(id_{W_j^*}))^{-1} = (Tr_r^{n_j}(id_{W_j}))^{-1} \tag{4.4}$$

and

$$Tr_l(f^*\varepsilon_{s_k^j}^*\mu_{\sigma(s_k^j)}^*) = Tr_r(\mu_{\sigma(s_k^j)}\varepsilon_{s_k^j}f) = Tr_r(f\mu_{\sigma(s_k^j)}\varepsilon_{s_k^j}). \tag{4.5}$$

Hence, we obtain that
$ldet_V^n(f^*) = Tr_l((f^n)^* l\Lambda_{V^*}^n)$
$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_l^{n_1}(id_{W_1^*}))^{-1}Tr_l(f^*\varepsilon_{s_1^1}^*\mu_{\sigma(s_1^1)}^*) \ldots Tr_l(f^*\varepsilon_{s_{n_1}^1}^*\mu_{\sigma(s_{n_1}^1)}^*) \ldots$
$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_l^{n_m}(id_{W_m^*}))^{-1}Tr_l(f^*\varepsilon_{s_1^m}^*\mu_{\sigma(s_1^m)}^*) \ldots Tr_l(f^*\varepsilon_{s_{n_m}^m}^*\mu_{\sigma(s_{n_m}^m)}^*)$
$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_r^{n_1}(id_{W_1}))^{-1}Tr_r(f\mu_{\sigma(s_1^1)}\varepsilon_{s_1^1}) \ldots Tr_r(\mu_{\sigma(s_{n_1}^1)}\varepsilon_{s_{n_1}^1}) \ldots$
$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_r^{n_m}(id_{W_m}))^{-1}Tr_r(f\mu_{\sigma(s_1^m)}\varepsilon_{s_1^m}) \ldots Tr_r(f\mu_{\sigma(s_{n_m}^m)}\varepsilon_{s_{n_m}^m})$
$= Tr_r(f^n r\Lambda_V^n)$ (replacing the permutation σ by σ^{-1})
$= rdet_V^n(f).$

Similarly, $rdet_V^n(f^*) = ldet_V^n(f)$. □

Remark 4.8. We immediately deduce from the previous Theorem 4.7, (c), that

$$ldet_V^n((f^*)^*) = ldet_V^n(f) \text{ and } rdet_V^n((f^*)^*) = rdet_V^n(f).$$

The following Proposition relates the quantum determinant of the partial trace of an endomorphism, with this last's trace.

Proposition 4.9. *Let $(C;\{V_i\}_{i\in J})$ be a semisimple category, U and V objects of C and $f\in End_C(U\otimes V)$. Then*

(a) *If* $V \in \{V_i\}_{i\in J}$, *then* $dim_l(V)\ ldet^1_V(pTr_l(f)) = Tr_l(f)$.

(b) *If* $U \in \{V_i\}_{i\in J}$, *then* $dim_r(U)\ rdet^1_U(pTr_r(f)) = Tr_r(f)$.

Proof. (a): we have:

$$ldet^1_V(pTr_l(f)) = Tr_l(pTr^n_l(f) l\Lambda^1_V) = dim_l^{-1}(V) Tr^1_l(f).$$

(b): Similar to (a). □

Proposition 4.10. *Let* $(C; \{V_i\}_{i\in J})$ *be a semisimple category such that* C *is spherical;* V *an object of* C *dominated by a standard family of cardinal* n *and* $f \in End_C(V)$. *Then*

$$ldet^n_V(f) = rdet^n_V(f)$$

which we simply denote by $det^n_V(f)$.

Proof. Straightforward. □

The following Lemmas serve in the coming Theorem for proving multiplicativity of quantum determinants and later on for *rank* introduction.

Lemma 4.11. *Let* $(C; \{V_i\}_{i\in J})$ *be a spherical semisimple category;* V *an object of* C *dominated by a family* $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ *of cardinal* n *and* $f \in End_C(V)$. *Then*

$$det^n_V(f) = det(M^f) \tag{4.6}$$

where $M^f = (a^f_{r,s})_{1\le r,s\le n} := \Big(Tr(\varepsilon_r f \mu_s) dim^{-1}(W_{r,s})\Big)_{1\le r,s\le n}$, *where we denote by* $W_{r,s}$ *the chosen representative simple object of the class that contains* r *and* s.

Proof. Note that the entries $a^f_{r,s}$ are well defined as if $V_{i(r)}$ and $V_{i(s)}$ are not isomorphic, we have that $a^f_{r,s} = 0$ by Schur's axiom. Now, M^f is a block diagonal matrix: $M^f = diag(M^f_1, ..., M^f_m)$ where, for all $1 \le j \le m$:

$$M^f_j = \Big(Tr(\varepsilon_l f \mu_k) dim(W_j)^{-1}\Big)_{s^j_1 \le l,k \le s^j_{n_j}}.$$

Recall that W_j is a representative object of its class I_j. Then, we have
$det(M^f) = det(M_1^f) \; ... \; det(M_m^f)$
$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(dim^{n_1}(W_1))^{-1}Tr(\varepsilon_{s_1^1} f\mu_{\sigma(s_1^1)})...Tr(\varepsilon_{s_{n_1}^1} f\mu_{\sigma(s_{n_1}^1)})...$
$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(dim^{n_m}(W_m))^{-1}Tr(\varepsilon_{s_1^m} f\mu_{\sigma(s_1^m)})...Tr(\varepsilon_{s_{n_m}^1} f\mu_{\sigma(s_{n_m}^1)})$
$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr^{n_1}(id_{W_1}))^{-1}Tr(f^{n_1}D_\sigma^1) \; ...$
$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr^{n_m}(id_{W_m}))^{-1}Tr(f^{n_m}D_\sigma^m)$
$= \; Tr(f^{n_1}(\sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr^{n_1}(id_{W_1}))^{-1}D_\sigma^1) \; ...$
$Tr(f^{n_m}(\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr^{n_m}(id_{W_m}))^{-1}D_\sigma^m))$
$= Tr(f^n(\sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr^{n_1}(id_{W_1}))^{-1}D_\sigma^1 \otimes \; ...$
$\otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr^{n_m}(id_{W_m}))^{-1}D_\sigma^m))$
$= det_V^n(f).$ □

Lemma 4.12. *Let C be a semisimple spherical category and V an object of C dominated by a family $(V_{i(r)};\varepsilon_r;\mu_r)_r$ of cardinal n and $f, g \in End_C(V)$. Then $M^{fg} = M^f M^g$.*

Proof. If $V_{i(r)}$ is not isomorphic to $V_{i(s)}$, then $a_{r,s}^f = a_{r,s}^g = 0 = a_{r,s}^{fg}$.
If $V_{i(r)} \simeq V_{i(s)}$, let $W_{r,s}$ be as in the previous Lemma 4.11. Then we have

$$\begin{aligned}
a_{r,s}^{fg} &= Tr(\varepsilon_r \; fg \; \mu_s)dim^{-1}(W_{r,s}) \\
&= Tr(\varepsilon_r \; f(id_V)g \; \mu_s)dim^{-1}(W_{r,s}) \\
&= Tr(\varepsilon_r f(\sum_{l=1}^{n} \mu_l\varepsilon_l)g\mu_s)dim^{-1}(W_{r,s}) \\
&= \sum_{l=1}^{n} Tr(\varepsilon_r f\mu_l \; \varepsilon_l g\mu_s)dim^{-1}(W_{r,s}) \\
&= \sum_{l=1}^{n} Tr(\; (k_{r,l} \otimes id_{W_{l,s}}) \; \varepsilon_l g\mu_s)dim^{-1}(W_{r,s})
\end{aligned}$$

$$= \sum_{l=1}^{n} Tr(k_{r,l} \otimes \varepsilon_l g \mu_s) dim^{-1}(W_{r,s})$$
$$= \sum_{l=1}^{n} Tr(k_{r,l}) Tr(\varepsilon_l g \mu_s) dim^{-1}(W_{r,s})$$
$$= \sum_{l=1}^{n} Tr(k_{r,l} \otimes id_{W_{r,l}}) dim^{-1}(W_{r,l}) Tr(\varepsilon_l g \mu_s) dim^{-1}(W_{r,s})$$
$$= \sum_{l=1}^{n} a^f_{r,l} a^g_{l,s}$$

for some unique $k_{r,l}$ in $\mathtt{K}_C$, as $\varepsilon_r f \mu_l$ is an endomorphism of a simple object $W_{r,l}$. Finally, $M^{fg} = M^f M^g$. □

Proposition 4.13. *Let C be a semisimple spherical category and V an object of C dominated by a standard family of cardinal n and let $f \in End_C(V)$. Then, $det^n_V(f)$ is invariant under the choice of the standard dominating family.*

Proof. Let $\mathcal{F} = (V_{i(r)}; \varepsilon_r; \mu_r)_r$ and $\mathcal{F}' = (V'_{i(s)}; \varepsilon'_s; \mu'_s)_s$ be two standard families on V. Then $\mathcal{F} \sim \mathcal{F}'$, by assumption. There exists then a permutation $\pi \in \mathfrak{S}_n$, such that $\mathcal{F}' = (V'_{\pi(i(r))}; \varepsilon'_{\pi(r)}; \mu'_{\pi(r)})_r$. By Proposition 4.2, we have

$$\begin{aligned} M^f_{\mathcal{F}'} &= \Big(Tr(\varepsilon'_{\pi(k)} f \mu'_{\pi(l)}) dim^{-1}(W'_{\pi(k),\pi(l)})\Big)_{k,l} \\ &= \Big(Tr(\alpha_k \varepsilon_k h^{-1} f h \mu_l \alpha_l^{-1}) dim^{-1}(W_{k,l})\Big)_{k,l} \\ &= \Big(Tr(\alpha_l^{-1} \alpha_k \varepsilon_k h^{-1} f h \mu_l) dim^{-1}(W_{k,l})\Big)_{k,l} \\ &= \Big(Tr(\varepsilon_k h^{-1} f h \mu_l) dim^{-1}(W_{k,l})\Big)_{k,l} \\ &= M^{h^{-1}fh}_{\mathcal{F}}, \end{aligned}$$

where $W_{k,l}$ and $W'_{\pi(k),\pi(l)}$ are the chosen representatives of their classes and $\alpha_k = \alpha_l : W_{k,l} \longrightarrow W'_{\pi(k),\pi(l)}$ is an isomorphism between them. Using now Lemma 4.12, we obtain that

$$det_{\mathcal{F}'}(M^f) = det_{\mathcal{F}}(M^{h^{-1}fh})$$

$$= det_{\mathcal{F}}(M^{h^{-1}} M^f M^h)$$
$$= det_{\mathcal{F}}(M^f),$$

where $h = \sum_r \mu'_{\pi(r)}\varepsilon_r$ and its inverse $h^{-1} = \sum_s \mu_s \varepsilon'_{\pi(s)}$, which justifies the last passage. In fact, $M^{h^{-1}} = (M^h)^{-1}$ by the fact that $\varepsilon_r \mu_s = \delta_{r,s}\ id_{V_{i(r)}}$, for all $1 \leq r, s \leq n$. Hence, as required. □

Theorem 4.14. *Let $(C; \{V_i\}_{i\in J})$ be a spherical semisimple category; V an object of C dominated by a standard family of cardinal n and $f \in End_C(V)$. Then, $det^n_V(f)$ verifies the following*

(a) $det^n_V(id_V) = id_I$.

(b) $det^n_V(q \otimes f) = q^n det^n_V(f)$; *for all* $q \in \mathtt{K}^{\times}_C$.

(c) $det^n_V(f^*) = det^n_V(f)$.

(d) $det^n_V(fg) = det^n_V(f)det^n_V(g)$, *for all* $g \in End_C(V)$.

Proof. C is spherical (not assumed to be pure), then trace is $\otimes-$multiplicative [14, page 50]. Hence:

(a), (c): Immediate from Theorem 4.7.

(b): Using the fact that $Tr(q.f) = qTr(f)$ and note that $(q.f)^n := (q \otimes f)^n$ need not be equal to $q^n.f^n$ (which holds if C is pure).

(d): This holds by Lemma 4.12. □

The quantum determinant is not additive for arbitrary objects, but for simple objects, this holds. Moreover we have

Corollary 4.15. *Let V be a simple object of a spherical semisimple category C. Then, the map*

$$det : End_C(V) \longrightarrow \mathtt{K}_C,\ f \mapsto det(f) := det^1_V(f)$$

is an isomorphism of $\mathtt{K}_C-$algebras.

Proof. Let $k \in \mathtt{K}_C$ and $f, g \in End_C(V)$. We have $det(k.f) = k.det(f)$ by Theorem 4.14, (b). On the other hand, we have $det(f + g) = det^1_V(f +$

$g) = Tr((f+g)dim^{-1}(V).id_V) = dim^{-1}(V)Tr(f+g) = dim^{-1}(V)(Tr(f)+Tr(g)) = det(f) + det(g)$. This proves linearity.

Let $f \in End_C(V)$, such that $det(f) = 0$. Then $dim^{-1}(V)Tr(f) = 0$, but f is an endomorphism of a simple object, hence there exists a unique $k \in \mathtt{K}_C$, such that $f = k \otimes id_V$, then $k = 0$ and so $f = 0$. This proves injectivity.

Let $k \in \mathtt{K}_C$, and put $f = k.id_V$. Then we have $det(f) = k$, hence surjectivity holds.
Finally, by Theorem 4.14, (a), (d), det is an isomorphism of $\mathtt{K}_C-$algebras. □

Definition 4.16. Let $F : C \longrightarrow D$ be a tensor $(F; r; r_0)$ and cotensor $(F; i; i_0)$ functor between monoidal categories. We say that F is conatural if for any morphisms $f : U \longrightarrow V$ and $g : X \longrightarrow Y$ in C, where $U, V, X, Y \in Ob(C)$, the following diagrams commute:

$$\begin{array}{ccc} F(U)\otimes F(X) & \xleftarrow[i]{} & F(U\otimes X) \\ {\scriptstyle F(f)\otimes F(g)}\downarrow & & \downarrow{\scriptstyle F(f\otimes g)} \\ F(V)\otimes F(Y) & \xrightarrow{r} & F(V\otimes Y) \end{array} \qquad \begin{array}{ccc} F(U)\otimes F(X) & \xrightarrow{r} & F(U\otimes X) \\ {\scriptstyle F(f)\otimes F(g)}\downarrow & & \downarrow{\scriptstyle F(f\otimes g)} \\ F(V)\otimes F(Y) & \xleftarrow[i]{} & F(V\otimes Y) \end{array}$$

Example 4.17. Any strong tensor functor is conatural.

Proof. Immediate from naturality of both F_2 and F_2^{-1}. □

We know (e.g, [14, page 49]) that traces are preserved under pivotal functors (see [14, page 29], for definition of the latter). Here we present weaker conditions on a functor to continue preserving traces.

Proposition 4.18. *Any conatural Frobenius tensor functor* ([4])*, between pivotal categories, preserves left and right traces.*

Proof. Let $F : C \longrightarrow D$ be a conatural *Frobenius* tensor functor between pivotal categories with tensor structure $(F; r; r_0)$ and cotensor structure $(F; i; i_0)$, V an object of C with left duality structures denoted by $(V^*; d_V; b_V)$ and $f \in End_C(V)$. We have $Tr_l(F(f)) = i_0 F(Tr_l(f)) r_0$. In fact, being a Frobenius tensor functor, F preserves duals and then

$(F(V^*); i_0 F(d)r; iF(b)r_0)$ are left duality structures on $F(V)$. On another hand, we have

$$\begin{aligned} Tr_l(F(f)) &= i_0 F(d) r (id \otimes F(f)) i F(b) r_0 \\ &= i_0 F(d) F(id \otimes f) F(b) r_0 \\ &= i_0 F(Tr_l(f)) r_0. \end{aligned}$$

The first and third equalities hold by definition, whereas the second is due to the conaturality assumption. The same thing holds for right traces. □

Lemma 4.19. *Let $(C; \{V_i\}_{i \in J})$ be a semisimple category and*

$$F = (F; F_0; F_2) : C \longrightarrow D$$

an additive strong tensor equivalence from C to a pivotal $Ab-$category D. Let V be an object of C, dominated by a family $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ of cardinal n. Then, $F(V)$ is dominated by the family $(F(V_{i(r)}); F(\varepsilon_r); F(\mu_r))_r$.

Proof. We have to show that for all $1 \leq r \leq n$, the map $\psi_r : \mathtt{K}_D \longrightarrow End_D(F(V_{i(r)}))$, $k \mapsto k \otimes id_{F(V_{i(r)})}$ is a bijection. Let $f_r \in End_D(F(V_{i(r)}))$. F being a tensor equivalence, it is in particular fully faithful, hence there exists a unique $g_r \in End_C(V_{i(r)})$ such that $F(g_r) = f_r$. But $V_{i(r)}$ is simple, thus, there exists a unique $h_r \in \mathtt{K}_C$ such that $g_r = h_r \otimes id_{V_{i(r)}}$. Finally, for any $1 \leq r \leq n$, we have

$$\begin{aligned} f_r &= F(g_r) \\ &= F(h_r \otimes id_{V_{i(r)}}) \\ &= F_2 F_2^{-1} F(h_r \otimes id_{V_{i(r)}}) F_2 F_2^{-1} \\ &= F_2 (F(h_r) \otimes F(id_{V_{i(r)}})) F_2 \\ &= F_2 (F_0 F_0^{-1} F(h_r) F_0 F_0^{-1} \otimes id_{F(V_{i(r)})}) F_2^{-1} \\ &= F_2 (F_0 \otimes id_{F(V_{i(r)})})(F_0^{-1} F(h_r) F_0 \otimes id_{F(V_{i(r)})})(F_0^{-1} \otimes id_{F(V_{i(r)})}) F_2^{-1} \\ &= F_0^{-1} F(h_r) F_0 \otimes id_{F(V_{i(r)})}. \end{aligned}$$

Where, the fourth equality holds by naturality of F_2, whereas the last one holds by the unitality constraint of F_2 and F_2^{-1}.
h_r being unique, $F_0^{-1}F(h_r)F_0$ is unique in $\mathtt{K}_D$ and we have

$$\psi(F_0^{-1}F(h_r)F_0) = f_r.$$

Hence, $(F(V_{i(r)}))_r$ is a family of simple objects. Moreover, we have

$$\sum_i F(\mu_r)F(\varepsilon_r) = F(\sum_r \mu_r\varepsilon_r) = F(id_V) = id_{F(V)}.$$

□

The following Theorem shows that the quantum determinants are well-behaved under strong tensor functors.

Theorem 4.20. *Let $(C; \{V_i\}_{i\in J})$ be a semisimple pure category; D a pivotal pure Ab−category; $F = (F; F_0; F_2) : C \longrightarrow D$ an additive strong tensor equivalence and let V be an object of C dominated by a family $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ of cardinal n. Then*

$$ldet_V^n(F(f)) = F_0^{-1}F(ldet_V^n(f))F_0;$$

and

$$rdet_V^n(F(f)) = F_0^{-1}F(rdet_V^n(f))F_0.$$

Proof. We have that left and right traces verify

$$Tr_l(F(f)) = F_0^{-1}F(Tr_l(f))F_0,$$

for all $f \in End_C(V)$. Consider the same partition of $[\![1, n]\!]$ for the family $(F(V_{i(r)}))_r$ as taken for $(V_{i(r)})_r$, and $F(W_1)$, ..., $F(W_m)$ as representatives of their classes. On the other hand, using Lemma 4.19, we have

$$ldet_V^n(F(f)) = Tr_l(F^n(f)\ l\Lambda^n_{F(V)})$$

$$= Tr_l(\sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(Tr_l^{n_1}(id_{F(W_1)}))^{-1}F^{n_1}(f)D^1_\sigma \otimes ...$$
$$\otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(Tr_l^{n_m}(id_{F(W_m)}))^{-1}F^{n_m}(f)D^m_\sigma)$$

$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)F_0^{-1}F((Tr_l^{n_1}(id_{W_1}))^{-1})F_0Tr_l(F^{n_1}(f)D_\sigma^1) \otimes ...$
$\otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)F_0^{-1}(Tr_l^{n_m}(id_{W_m})^{-1})F_0Tr_l(F^{n_m}(f)D_\sigma^m)$

$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)F_0^{-1}F((Tr_l^{n_1}(id_{W_1}))^{-1})F_0Tr_l(F(f)F(\mu_{s_1^1}\varepsilon_{\sigma(s_1^1)})\otimes...\otimes F(f)F(\mu_{s_{n_1}^1}\varepsilon_{\sigma(s_{n_1}^1)}))$ ...
$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)F_0^{-1}F((Tr_l^{n_m}(id_{W_m}))^{-1})F_0Tr_l(F(f)F(\mu_{s_1^m}\varepsilon_{\sigma(s_1^m)}) \otimes ... \otimes F(f)F(\mu_{s_{n_m}^m}\varepsilon_{\sigma(s_{n_m}^m)})$
$= F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)F((Tr_l^{n_1}(id_{W_1}))^{-1})F_0Tr_l(F(f\mu_{s_1^1}\varepsilon_{\sigma(s_1^1)}))...Tr_l(F(f\mu_{s_{n_1}^1}\varepsilon_{\sigma(s_{n_1}^1)}))$ ...
$F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)F((Tr_l^{n_m}(id_{W_m}))^{-1})F_0Tr_l(F(f\mu_{s_1^m}\varepsilon_{\sigma(s_1^m)}))...Tr_l(F(f\mu_{s_{n_m}^m}\varepsilon_{\sigma(s_{n_m}^m)})$
$= F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)F((Tr_l^{n_1}(id_{W_1}))^{-1})F_0Tr_l(F(f\mu_{s_1^1}\varepsilon_{\sigma(s_1^1)}))...Tr_l(F(f\mu_{s_{n_1}^1}\varepsilon_{\sigma(s_{n_1}^1)}))$ ...
$F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)F((Tr_l^{n_m}(id_{W_m}))^{-1})F_0Tr_l(F(f\mu_{s_1^m}\varepsilon_{\sigma(s_1^m)}))...Tr_l(F(f\mu_{s_{n_m}^m}\varepsilon_{\sigma(s_{n_m}^m)})$
$= F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)F((Tr_l^{n_1}(id_{W_1}))^{-1})F(Tr_l(f\mu_{s_1^1}\varepsilon_{\sigma(s_1^1)}))F_0...F_0^{-1}F(Tr_l(f\mu_{s_{n_1}^1}\varepsilon_{\sigma(s_{n_1}^1)}))F_0$ ...
$F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)F((Tr_l^{n_m}(id_{W_m}))^{-1})F(Tr_l(f\mu_{s_1^m}\varepsilon_{\sigma(s_1^m)}))F_0...F_0^{-1}F(Tr_l(f\mu_{s_{n_m}^m}\varepsilon_{\sigma(s_{n_m}^m)}))F_0$
$= F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)F((Tr_l^{n_1}(id_{W_1}))^{-1})F(Tr_l(f\mu_{s_1^1}\varepsilon_{\sigma(s_1^1)}))...F(Tr_l(f\mu_{s_{n_1}^1}\varepsilon_{\sigma(s_{n_1}^1)}))F_0$ ...
$F_0^{-1} \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)F((Tr_l^{n_m}(id_{W_m}))^{-1})F(Tr_l(f\mu_{s_1^m}\varepsilon_{\sigma(s_1^m)}))...F(Tr_l(f\mu_{s_{n_m}^m}\varepsilon_{\sigma(s_{n_m}^m)}))F_0$
$= F_0^{-1}F(ldet_V^n(f))F_0.$

The second claim for right quantum determinant holds with a similar procedure. □

5 Quantum rank of endomorphisms in semisimple category

We discuss some facts which will be of our interest, about dominating families of simple objects.

Proposition 5.1. *Let C be a semisimple category, V and W isomorphic objects of C and $\mathcal{F} = (V_{i(r)}; \varepsilon_r; \mu_r)_r$ a standard dominating family of V, of cardinal n. Then*

(a) *Let $\mathcal{F}' = (V'_{i(s)}; \varepsilon'_s; \mu'_s)_s$ be another dominating family of V, such that $\mathcal{F}' \sim \mathcal{F}$. Then, the following are equivalent*

  (i) $\varepsilon'_k\mu'_l = \delta_{k,l}\, id_{V'_{i(k)}}$, *for all $1 \le k, l \le n$.*

(ii) $\mathcal{F}'$ *is standard.*

(b) $\mathcal{F}^* := (V^*_{i(r)}; \mu^*_r; \varepsilon^*_r)_r$ *is a standard dominating family of* V^*.

(c) $\mathcal{F}' := (V_{i(r)}; \varepsilon_r f^{-1}; f\mu_r)_r$ *is a standard dominating family of* $W \in Ob(C)$, *for any isomorphism* $f : V \to W$ *in* C.

(d) $\mathcal{F}^{**}$ *is a standard dominating family of* V, *and we have* $\mathcal{F}^{**} \sim \mathcal{F}$, *where* $\mathcal{F}^{**} = (\mathcal{F}^*)^*$.

(e) *Let* $\mathcal{F}'$ *be a dominating family of* V. *Then* $\mathcal{F} \sim \mathcal{F}' \Leftrightarrow \mathcal{F}^* \sim \mathcal{F}'^*$.

Proof. (a): In fact, $\mathcal{F}' \sim \mathcal{F}$ ensures that $\mathcal{F}'$ is minimal. The second condition: $\varepsilon'_k \mu'_l = \delta_{k,l}\, id_{V'_{i(k)}}$, for all $1 \leq k, l \leq n$ is present in both assertions.

(b): $(V^*_{i(r)})_r$ is a family of simple objects, and we have $\sum\limits_r \varepsilon^*_r \mu^*_r = (\sum\limits_r \mu_r \varepsilon_r)^* = id_{V^*}$. On the other hand, we have $\mu^*_k \varepsilon^*_l = (\varepsilon_l \mu_k)^* = \delta_{l,k}\, id_{V^*_{i(k)}}$, for all $1 \leq k, l \leq n$.

(c): Straightforward.

(d): This holds from Proposition 5.1, (b) and the fact that $V^{**}_{i(r)} \cong V_{i(r)}$, for all $1 \leq r \leq n$.

(e): In fact, duals of two isomorphic objects are also isomorphic. □

Proposition 5.2. *Let* C *be a semisimple spherical category and* V *an object of* C *dominated by a family* $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ *of cardinal* n. *Then, the map*

$$\varphi : End_C(V) \longrightarrow M_n(\mathtt{K}_C), f \mapsto M^f$$

is a morphism of $\mathtt{K}_C-$*algebras. If moreover,* $\varepsilon_r \mu_s = \delta_{r,s}\, id_{V_{i(r)}}$, *for all* $1 \leq r, s \leq n$. *Then,* φ *is an isomorphism of* $\mathtt{K}_C-$*algebras.*

Proof. Let $f,\ g \in End_C(V)$ and $k \in \mathtt{K}_C$. Then
$\varphi(f + k.g) = M^{f+k.g} = M^f + kM^g = \varphi(f) + k\varphi(g)$. So, φ is a morphism of $\mathtt{K}_C-$modules. Further, we have

$$\varphi(fg) := M^{fg} = M^f M^g := \varphi(f)\varphi(g),$$

where the second equality is justified by Lemma 4.12. Hence, φ is a morphism of $\mathtt{K}_C-$algebras.

Now, assume that $\varepsilon_r\mu_s = \delta_{r,s}\ id_{V_{i(r)}}$, for all $1 \le r,s \le n$ and let $M = (a_{r,s})_{1\le r,s\le n} \in M_n(\mathtt{K}_C)$. Put $f = \sum\limits_{r,s} a_{r,s}.\mu_r\varepsilon_s$. Then

$$\begin{aligned}
\varphi(f) &= \Big(Tr(\varepsilon_l(\sum_{r,s} a_{r,s}.\mu_r\varepsilon_s)\mu_k)dim^{-1}(W_{l,k})\Big)_{1\le l,k\le n}\\
&= \Big(\sum_{r,s} a_{r,s}\ Tr(\varepsilon_l\mu_r\varepsilon_s\mu_k)dim^{-1}(W_{l,k})\Big)_{1\le l,k\le n}\\
&= \Big(Tr(id_{W_{l,k}})a_{l,k}\ dim^{-1}(W_{l,k})\Big)_{1\le l,k\le n}\\
&= (a_{l,k})_{1\le l,k\le n}\\
&= M.
\end{aligned}$$

Thus, φ is surjective. For the injectivity, let $f, g \in End_C(V)$ such that $\varphi(f) = \varphi(g)$. Then

$$\begin{aligned}
\varphi(f) = \varphi(g) &\Rightarrow Tr(\varepsilon_r f\mu_s) = Tr(\varepsilon_r g\mu_s), \quad \textit{for all}\ 1\le r,s\le n\\
&\Rightarrow Tr(\varepsilon_r(f-g)\mu_s) = 0, \quad \textit{for all}\ 1\le r,s\le n\\
&\Rightarrow k_{r,s}\ dim(W_{r,s}) = 0, \quad \textit{for all}\ 1\le i,j\le n\\
&(\textit{for some unique}\ k_{r,s}\in \mathtt{K}_C, \textit{such that}\ \varepsilon_r(f-g)\mu_s = k_{r,s}\otimes id_{W_{r,s}})\\
&\Rightarrow k_{r,s} = 0, \quad \textit{for all}\ 1\le r,s\le n\\
&\Rightarrow \varepsilon_r(f-g)\mu_s = 0, \quad \textit{for all}\ 1\le r,s\le n\\
&\Rightarrow f = g,
\end{aligned}$$

where, the last step is obtained by composing by μ_r on the left, then summing over r and composing by ε_s on the right, then summing over s. $\square$

Let C be a semisimple category and V an object of C dominated by a family $(V_{i(r)};\varepsilon_r;\mu_r)_r$ of simple objects, of cardinal n. Let $[\![1,n]\!] = I_1 \cup I_2 \cup ... \cup I_m$ be a partition of $[\![1,n]\!]$ into classes of isomorphic simple objects. For any $f \in End_C(V)$, we associate two matrices $M_l^f = (la^f_{r,s})_{1\le r,s\le n}$ and $M_r^f = (ra^f_{r,s})_{1\le r,s\le n}$, where

$$la^f_{r,s} = Tr_l(\varepsilon_r f\mu_s)dim_l^{-1}(V_{i(r)});$$

and

$$ra^f_{r,s} = Tr_r(\varepsilon_r f\mu_s)dim_r^{-1}(V_{i(r)}).$$

Note that it is not to confuse the notations r, s that mean "left" and "right", with the indices meaning.

Definition 5.3. Let C be a semisimple category such that its commutative ground ring $\mathtt{K}_C$ is a field, and let V be an object of C dominated by a standard family, and $f \in End_C(V)$. The left quantum rank of f, denote by $ran_l(f)$, is defined as

$$ran_l(f) = ran(M_l^f).$$

Similarly, the right quantum rank of f, denote by $ran_r(f)$, is defined as

$$ran_r(f) = ran(M_r^f),$$

where, M_l^f, M_r^f are the square matrices of $M_n(\mathtt{K}_C)$ associated to f as defined above and $ran(M_l^f)$, $ran(M_r^f)$ are the ordinary (determinantal) ranks.

Remark 5.4. Note that some authors as in [13], impose $\mathtt{K}_C$ in definition of a semisimple category to be a field, by the axiom that a morphism between simple objects is either zero or an isomorphism.

Remark 5.5. $ran_l(f)$ and $ran_r(f)$ are well defined and do not depend on the choice of the repartition of the simple objects in the dominating family by similar discussion as in Remark 4.6. We show in Proposition 5.7 that $ran_l(f)$ and $ran_r(f)$ are invariant under the choice of the standard dominating family, until they are assumed to be equivalent (Definition 4.1).

Remark 5.6. For the rest of the paper, all the results for left quantum rank hold in the same way for right quantum rank, it suffices to replace the word "left" by "right" except when there is an interaction between the two notions and this manifests mainly in presence of duality (for objects and morphisms of C). For this reason, in such case, the involved rank is made precise (Lemma 5.9, Theorem 5.12, (a) and Proposition 5.23). Otherwise, we simply write "$ran(-)$" to designate "$ran_l(-)$", "$ran_r(-)$" and M^f to designate M_l^f, M_r^f. The two notions of rank coincide if C is spherical, and the associated matrices to f are equal.

Proposition 5.7. *Let C be a semisimple category and V an object of C dominated by a standard family and let $f \in End_C(V)$. Then, $ran(f)$ is invariant under the choice of the equivalent dominating families.*

Proof. Let $\mathcal{F} = (V_{i(r)}; \varepsilon_r; \mu_r)_r$ be a standard dominating family of V and $\mathcal{F}' = (V'_{i(s)}; \varepsilon'_s; \mu'_s)_s$, such that $\mathcal{F}' \sim \mathcal{F}$. By Proposition 4.13, we have

$$M^f_{\mathcal{F}'} = M^{h^{-1}fh}_{\mathcal{F}}$$

and with the same arguments therein, we obtain that

$$\begin{aligned} ran_{\mathcal{F}'}(f) &:= ran_{\mathcal{F}'}(M^f) \\ &= ran_{\mathcal{F}}(M^{h^{-1}fh}) \\ &= ran_{\mathcal{F}}(M^{h^{-1}} M^f M^h) \\ &= ran_{\mathcal{F}}(M^f) \\ &:= ran_{\mathcal{F}}(f). \end{aligned}$$

The fourth passage holds by the fact that the rank of a matrix multiplied by invertible matrices does not change. □

Proposition 5.8. *Let V be an object of C, dominated by a standard family $(V_{i(r)}; \varepsilon_r; \mu_r)_{1\leq r\leq n}$. For all $1 \leq k \leq n$, let $f_k = \mu_k\varepsilon_k$ and $g_k = \sum_{l=1}^{n} \mu_l\varepsilon_l$. Then, $ran(f_k) = 1$ and $ran(g_k) = k$.*

Proof. Straightforward. □

The following Lemmas will serve for proof of some properties of the very defined rank, similar to those of a square matrix rank, in the ordinary case.

Lemma 5.9. *Let C be a semisimple category; V an object of C dominated by a family $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ of cardinal n and $f \in End_C(V)$. Then*

$$M_l^{f^*} = (M_r^f)^T \quad and \quad M_r^{f^*} = (M_l^f)^T$$

where, the right hand sides are the transpose matrices.

Proof. $V_{i(r)} \simeq V_{i(s)} \Leftrightarrow V^*_{i(r)} \simeq V^*_{i(s)}$, and using Proposition 5.1, (2), we have: If $V^*_{i(r)} \simeq V^*_{i(s)}$, then

$$la^{f^*}_{r,s} = Tr_l(\mu^*_r f^* \varepsilon^*_s) dim_l^{-1}(V^*_{i(r)})$$

$$\begin{aligned} &= Tr_l((\varepsilon_s f \mu_r)^*) dim_l^{-1}(V_{i(s)}^*) \\ &= Tr_r(\varepsilon_s f \mu_r) dim_r^{-1}(V_{i(s)}) \\ &= ra_{s,r}^f. \end{aligned}$$

Otherwise, $la_{r,s}^{f^*} = 0 = ra_{s,r}^f$ (by Schur's axiom). The second claim is proved similarly. □

Remark 5.10. (1) The following are equivalent

(a) $la_{s,r}^f = 0$ (resp. $ra_{s,r}^f = 0$).
(b) $ra_{r,s}^{f^*} = 0$ (resp. $al_{r,s}^{f^*} = 0$).

(2) M_l^f, M_r^f, $M_l^{f^*}$ and $M_r^{f^*}$ are block diagonal matrices.

(3) The domination rank of V is just the size of the largest sub-matrix in the diagonal blocks.

Lemma 5.11. *Let C be a semisimple category and V an object of C dominated by a standard family. Then, M^f is invertible and $(M^f)^{-1} = M^{f^{-1}}$ for every $f \in Aut_C(V)$.*

Proof. Immediate from Proposition 5.2. □

Theorem 5.12. *Let C be a semisimple category; V an object of C dominated by a standard family $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ and $f \in End_C(V)$. Then*

(a) $ran_l(f^*) = ran_r(f)$ *and* $ran_r(f^*) = ran_l(f)$.

(b) $ran(id_V) = n$.

(c) $ran(f^{-1}) = ran(f)$, *when f is an isomorphism.*

Proof. (a): Proposition 5.1, (b), (d) and (e), ensures that all standard dominating families on V^* are equivalent whenever those on V are assumed to be; i.e, whitout taking the assumption on V^*. Hence, $ran_l(f^*)$ and $ran_r(f^*)$ are well defined, and using Lemma 5.9, we have

$$ran_l(f^*) := ran_l(M_l^{f^*}) = ran_r((M_r^f)^T) = ran_r(M_r^f) := ran_r(f).$$

Similar proof holds for the second claim.

(b): By Proposition 5.2, $ran(id_V) := ran(M^{id_V}) = n$.

(c): This holds by Lemma 3.3. □

Remark 5.13. By Proposition 5.1, (d) and (e), we immediately see that

$$ran_l(f^{**}) = ran_l(f) \ \ and \ \ ran_r(f^{**}) = ran_r(f).$$

Theorem 5.14. *Under the same hypotheses of Theorem* 5.12, for all $f, g \in End_C(V)$, we have

(a) $ran(f + g) \leq ran(f) + ran(g)$.

(b) $ran(fg) \leq ran(f)$ and $ran(fg) \leq ran(g)$.

(c) $ran(fg) = ran(gf) = ran(f)$, for every $g \in Aut_C(V)$.

Proof. (a): $ran(f + g) := ran(M^{f+g}) = ran(M^f + M^g) \leq ran(M^f) + ran(M^g)$ $(:= ran(f) + ran(g))$.

(b): $ran(fg) := ran(M^{fg}) = ran(M^f M^g) \leq ran(f)$.
Similarly, $ran(fg) \leq ran(g)$.

(c): Let $g \in Aut_C(V)$, by Proposition 5.2, M^g is invertible (with inverse M^f where f is the inverse of g). Thus

$$ran(gf) := ran(M^g M^f) = ran(f);$$

$$ran(fg) := ran(M^f M^g) = ran(f).$$

□

Remark 5.15. (a) $ran(fg) \neq ran(gf)$ in general.

(b) For all $f \in Aut_C(V)$, $ran(f^{-1}) = ran(f) = ran(id_V) = n$ by Theorem 5.12, (b), (c) and Theorem 5.14, (b).

Corollary 5.16. *Under the same hypotheses of Theorem* 5.12, f is an isomorphism, if and only if, $ran(f) = n$.

Proof. By Theorem 5.14, $ran(f) = ran(fid_V) = ran(id_V) = n$.

Conversely, if f is an endomorphism of V such that $ran(f) = n$, then M^f is invertible. Hence, f is an isomorphism by Proposition 5.2. □

Corollary 5.17. *The set* $K_V := \{f \in End_C(V),\ ran(f) = 0\}$ *is an ideal of the ring* $(End_C(V), +, \circ)$.

Proof. Let $f \in K_V$, $g \in End_C(V)$. Then, $ran(fg) \leq ran(f) = 0$, by Theorem 5.14, (b). □

Lemma 5.18. $ran(k.f) = ran(f)$, *for all* $k \in \mathtt{K}_C^{\times}$, *and* $f \in End_C(V)$.

Proof. $ran(k.f) := ran(M^{k.f}) = ran(kM^f) = ran(f)$. □

Corollary 5.19. *Let* $f, g \in End_C(V)$, *such that* $ran(g) = 0$. *Then*

$$ran(f + g) = ran(f).$$

Proof. By Theorem 5.14, (a), we have $ran(f + g) \leq ran(f)$. On the other hand, by Lemma 5.18, we have $ran(f) = ran(f + g - g) \leq ran(f + g) + ran(g)$. □

Remark 5.20. Let $mf := \underbrace{f + ... + f}_{m-times}$. Then, by Theorem 5.14, (a), we have

$$ran(mf) \leq m\, ran(f).$$

Remark 5.21. Let V be an object of C, dominated by a standard family $(V_{i(r)}; \varepsilon_r; \mu_r)_r$ of cardinal n, of non isomorphic simple objects. Then, $ran(f) = |r|$, for which $Tr(\varepsilon_r f \mu_r) \neq 0$. In fact, by Schur's axiom, $Tr(\varepsilon_r f \mu_s) = 0$ for any $r \neq s$.

Now, without assuming $\mathtt{K}_C$ to be a field, we define a rank of $f \in End_C(V)$ using the McCoy rank of a matrix over a commutative ring [1] as follows.

Definition 5.22. Let C be a semisimple category, V an object of C dominated by a standard family, and $f \in End_C(V)$. The rank of f, denoted by $rk(f)$, is the integer

$$rk(f) = rk(M^f)$$

where, $rk(M^f)$ is the McCoy rank of the associated matrix M^f.

Theorem 5.23. *Let C be a spherical semisimple category; V an object of C dominated by a standard family of cardinal n and $f \in End_C(V)$. Then, $rk(f)$ verifies*

(a) $rk(f^*) = rk(f)$.

(b) $rk(fg) \leq min\{rk(f), rk(g)\}$.

(c) $rk(f) = rk(fg) = rk(hf)$, *for all* $g, h \in Aut_C(V)$.

(d) *If* $\mathtt{K}_C$ *is an integral domain, then* $rk(f) = ran(f)$, (*the latter is computed over the quotient field of* $\mathtt{K}_C$).

(e) $rk(f) < n$ *if and only if* $det^n_V(f)$ *is a zero divisor.*

(f) *If* $det^n_V(f)$ *is invertible, then* $rk(f) = n$.

Proof. This holds from [1, pages 31-32] and with similar arguments as in Theorems 5.12 and 5.14. □

6 Quantum rank of morphisms in semisimple category

For this section, C is a spherical semisimple category, and $\mathtt{K}_C$ is assumed to be a field.

Let U and V be two objects of C, dominated by standard families $\mathcal{F} = (V_{i(s)}; \varepsilon_s; \mu_s)_s$ and $\mathcal{F}' = (V'_{i(r)}; \varepsilon'_r; \mu'_r)_r$, of cardinals m and n respectively. We want to define a *quantum rank* for a morphism $f : U \longrightarrow V$ in the same way as we did in the previous section (3) for endomorphisms. Let then $ran(f) := ran(M^f)$, where $M^f = (a_{r,s})_{1\leq r\leq n,\, 1\leq s\leq m}$ with

$$a_{r,s} = Tr(\varepsilon'_r f \mu_s) dim^{-1}(V_{i(s)}).$$

By Proposition 4.2, if $\overline{\mathcal{F}} = (\overline{V}_{i(s)}; \overline{\varepsilon_s}; \overline{\mu_s})_s$ and $\overline{\mathcal{F}}' = (\overline{V}'_{i(r)}; \overline{\varepsilon_r}'; \overline{\mu_r}')_r$ are other equivalent dominating families of U and V respectively, then there exist $h \in Aut_C(U)$, $t \in Aut_C(V)$, suth that $\overline{\varepsilon_s}' = \varepsilon'_s h^{-1}$ and $\overline{\mu_r} = t\mu_r$, hence

$$\overline{a}_{r,s} = Tr(\overline{\varepsilon_r}' f \overline{\mu_s}) dim^{-1}(\overline{V}_{i(s)}) = Tr(\varepsilon'_r h^{-1} f t \mu_s) dim^{-1}(V_{i(s)}),$$

where, the $\overline{a}_{r,s}$'s are the entries of M^f, relatively to $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}'$. Then

$$\begin{aligned} ran_{\overline{\mathcal{F}},\overline{\mathcal{F}}'}(f) :&= ran_{\overline{\mathcal{F}},\overline{\mathcal{F}}'}(M^f) \\ &= ran_{\mathcal{F},\mathcal{F}'}(M^{h^{-1}ft}) \\ &= ran_{\mathcal{F},\mathcal{F}'}(M^{h^{-1}} M^f M^t) \\ &= ran_{\mathcal{F},\mathcal{F}'}(M^f) \\ :&= ran_{\mathcal{F},\mathcal{F}'}(f). \end{aligned}$$

The fourth passage is justified by the fact that $M^{h^{-1}}$ and M^t are invertible matrices. Hence, $ran(f)$ is well defined.

Remark 6.1. (a) If $V_{i(s)}$ is not isomorphic to any $V'_{i(r)}$, for all $1 \leq s \leq m$ and $1 \leq r \leq n$. Then $ran(f) = 0$, for all $f \in Hom_C(U, V)$.
(b) For all $f \in Hom_C(U, V)$, $ran(f)$ does not exceed the minimum of n and m.

Proposition 6.2. *Let C be a semisimple spherical category and U and V be objects of C dominated by standard families of one same simple object, and let $f : U \longrightarrow V$ be a split monomorphism (resp. a split epimorphism). Then*

$$ran(f) = n, \ (resp.\ ran(f) = m).$$

Proof. $f : U \longrightarrow V$ is a split monomorphism implies that there exists $g : V \longrightarrow U$ such that, $gf = id_U$, hence, $n = ran(id_U) = ran(gf) \leq ran(f)$ by Theorem 5.14, (b). Similarly for the second claim. □

Remark 6.3. One can immediately see that for any isomorphism $f : V \longrightarrow W$ in C, we have

$$ran(f) = ran(f^{-1}).$$

Remark 6.4. The above defines a morphism rank function on C as:

$$r : f \in Hom_C(U, V) \longmapsto r(f) := ran(f),$$

for all $U, V \in Ob(C)$, satisfying:

(a) $r(k) = 1$, for every non zero endomorphism k of a simple object X of C.

(b) $r(f + g) \leq r(f) + r(g)$, for all $f, g \in End_C(V)$.

(c) $r(fg) \leq min\{r(f);\ r(g)\}$, for all $f, g \in End_C(V)$.

Remark 6.5. For every object V of C, let $r_V = r_{|End_C(V)}$ and define

$$ker(r_V) := \{f \in End_C(V),\ r_V(f) = 0\}.$$

Then, $ker(r_V)$ is just the ideal K_V of Corollary 5.17.

Example 6.6. The category $vect_{\mathtt{K}}$ of finite dimensional vector spaces over a field $\mathtt{K}$ is a semisimple category and the quantum rank of an endomorphism f of a vector space is nothing but the rank of a representative matrix of f.

Example 6.7. Let C be a semisimple category such that all the simple objects of C are isomorphic to the unit object I. Assume that $\mathtt{K}_C$ is a local ring and that there exists an object V of C dominated by a family of cardinal n, where $n = dim(Hom_C(V, I))$. Then, V admits a standard dominating family of cardinal n and $rank_d(V) = n$. In fact, $\mathtt{K}_C$ being local and $Hom_C(V, I)$ a projective module over $\mathtt{K}_C$, then, $Hom_C(V, I)$ is free. Hence, $rank(Hom_C(V, I))$ is well defined. Picking a basis $(\mu_i)_{1 \leq i \leq n}$ of $Hom_C(V, I)$ and its dual basis $(\varepsilon_i)_{1 \leq i \leq n}$, we obtain that V is dominated by $(I; \varepsilon_i; \mu_i)_i$, and $\varepsilon_i \mu_j = \delta_{i,j}\ id_I$, for all $1 \leq i, j \leq n$.

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Khalid Draoui *Mathematical Sciences and Applications Laboratory, Department of Mathematics, Faculty of Sciences Dhar Al Mahraz, P. O. Box 1796, University Sidi Mohamed Ben Abdellah Fez, Morocco.*
Email: khalid.draoui@usmba.ac.ma

Hanan Choulli *Mathematical Sciences and Applications Laboratory, Department of Mathematics, Faculty of Sciences Dhar Al Mahraz, P. O. Box 1796, University Sidi Mohamed Ben Abdellah Fez, Morocco.*
Email: hanan.choulli@usmba.ac.ma

Hakima Mouanis *Mathematical Sciences and Applications Laboratory, Department of Mathematics, Faculty of Sciences Dhar Al Mahraz, P. O. Box 1796, University Sidi Mohamed Ben Abdellah Fez, Morocco.*
Email: hakima.mouanis@usmba.ac.ma