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Composition series on (Rees) congruences of S-acts

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Abstract. In this paper, we study composition series of subacts or congruences of S-acts. It is shown that composition series of subacts are exactly those that are both Rees artinian and Rees noetherian, i.e. those satisfying both ascending and descending chain conditions on subacts. But this is not valid for the case of composition series of congruences in general. We prove that the properties of having composition series of subacts or congruences are inherited in Rees short exact sequences. Also, we discuss whenever two composition series of subacts or congruences have the same length and they are equivalent.

1 Introduction and Preliminaries

Chain conditions are classical finiteness conditions in algebra, see for example [1, 14]. Related to the notion of chain conditions is that of composition series and the Jordan-Holder Theorem. The importance of a composition

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series in abstract algebra is to provide a way to break up an algebraic structure into simple pieces. Motivated by this fact, we study the concept of composition series in the category of S-acts.

Right noetherian semigroups were introduced in [6], and then further studied in [8, 9, 13]. For S-acts, in [3] the descending and ascending chain conditions on commutative monoids with zero as S-acts are investigated. Then, as the dual concept of finitely generated S-acts, in [11] the concepts of finitely (Rees) cogenerated S-acts are investigated. In [12], several basic properties of (Rees) noetherian and artinian S-acts are studied. Using these results, in Section 2, we study composition series of subacts as a strictly decreasing sequence of subacts in which no further subact can be inserted. Then we prove that an S-act has a composition series of subacts if and only if it is both Rees artinian and Rees noetherian. It is well known that congruences play a fundamental role in the study of quotient structures of S-acts. Moreover, unlike the case for module theory, not every congruence on an S-act is associated with a subact. Subacts only determine Rees congruences and Rees factor S-acts. So in Section 3, we introduce composition series of congruences. Then we establish some connections between composition series of congruences with the notions related to chain condition on congruences. The lattice of congruences is not generally modular. When the lattice of congruences on an S-act is a modular, some of the main results in the study of composition series of congruences are proved.

Throughout this paper, S will stand for a monoid and A is a right S-act. Recall that a *congruence* ρ on an S-act A is an equivalence relation on A such that $a\ \rho\ a'$ implies that $as\ \rho\ a's$ for each $a, a' \in A$ and $s \in S$. The lattice of all congruences on A is denoted by $(Con(A), \subseteq, \cap, \vee)$ which is a complete lattice with greatest element $\nabla_A = A \times A$ and smallest element $\Delta_A = 1_A$. An S-act is called *congruence-free* if it contains no congruences other than Δ_A and ∇_A. Moreover, a *simple* S-act is an S-act with no subacts other than itself, and a *θ-simple* S-act is an S-act which contains no subacts other than itself and the one element subact Θ. For general background on S-acts the reader can consult [7].

In the reminder of this section, we recall some results of [12] without proof which will be needed in the sequel. As we mentioned earlier, an S-act A is called *artinian* (*noetherian*) in case $Con(A)$ satisfies the descending (ascending) chain condition, equivalently, the minimal (maximal) condition. By [12, Theorems 5 and 6], artinian S-acts are those which all their factor acts are finitely cogenerated, and noetherian S-acts are those which all their congruences are finitely generated.

An S-act A is called *Rees artinian* (*Rees noetherian*) if it satisfies the descending (ascending) chain condition on its Rees congruences, equivalently, on its subacts (or, equivalently, the minimal (maximal) condition on its subacts). By [12, Proposition 7], Rees artinian (Rees noetherian) S-acts are those which all their factor acts (subacts) are finitely Rees cogenerated (generated). The notions *right* (*Rees*) *artinian* (*noetherian*) *monoids* apply for a monoid S with this property as a right S-act.

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be S-morphisms. Recall from [16] that the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a *Rees short exact sequence* if f is one-to-one, g is onto, and $\ker g = \mathcal{K}_{\mathrm{Imf}}$ where $\mathcal{K}_{\mathrm{Imf}} = (f(A) \times f(A)) \cup \Delta_B$. The following two results will serve as useful tools for our study of composition series.

Theorem 1.1. ([12]) *Let $A \longrightarrow B \longrightarrow C$ be a Rees short exact sequence of S-acts. Then, B is (Rees) artinian (noetherian) if and only if both A and C are (Rees) artinian (noetherian).*

Lemma 1.2. ([12]) *Let A be an S-act, and $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A$. Then A is (Rees) artinian (noetherian) if and only if A_1 and factor S-acts A_{i+1}/A_i are (Rees) artinian (noetherian) for all $1 \leq i \leq n-1$.*

2 Composition series of subacts

In this section we introduce composition series of subacts for S-acts, and study the relation between such series and Rees artinian and noetherian S-acts. Let us first define a composition series of subacts.

Definition 2.1. Let A be an S-act. Then,

(i) A finite chain of $n+1$ subacts of A

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n,$$

is called a *composition series of subacts* of length n for A provided that A_i/A_{i+1} is θ-simple, and A_n is a simple S-act, i.e., provided each term in the chain is maximal in its predecessor. In addition, the θ-simple S-acts A_i/A_{i+1}, $0 \leq i \leq n$, where A_n/A_{n+1} denotes $A_n \cup \{\theta\}$, are called the *composition factors* of the series. We denote the minimum length of such a series for A by $l_s(A)$.

(ii) Two composition series of subacts

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n$$

and

$$A = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_m$$

of A are said to be *equivalent* in case $n = m$ and there is a permutation ι on$\{1, 2, \ldots, n\}$ such that

$$A_i/A_{i+1} \cong B_{\iota(i)}/B_{\iota(i+1)}$$

for each $i = 1, 2, \ldots, n$.

First we prove that the right S-acts that are both Rees artinian and Rees noetherian are precisely those with a composition series of subacts.

Theorem 2.2. *Let S be a monoid. An S-act A has a composition series of subacts if and only if it is both Rees noetherian and Rees artinian.*

Proof. Necessity. Suppose that A has a composition series of subacts

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n.$$

Since each A_i/A_{i+1} is θ-simple, then all factor acts A_i/A_{i+1} and A_n are both Rees artinian and Rees noetherian. By Lemma 1.2, A is both Rees artinian and Rees noetherian.

Sufficiency. Suppose that A is Rees noetherian and Rees artinian. Since A is Rees noetherian, A has a maximal proper subact A_1. Now, A_1 is Rees noetherian, so if A_1 is not simple, then A_1 has a maximal proper subact A_2. Similarly, either A_2 is simple or it contains a maximal proper subact A_3. Continuing in this way, we must eventually obtain a simple subact A_n, for otherwise we would have a strictly descending chain

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots ,$$

contradicting the fact that A is Rees artinian. Moreover, each A_i/A_{i+1} is θ-simple, we have a composition series of subacts for A. $\square$

The following result is a consequence of Theorems 1.1 and 2.2.

Corollary 2.3. *Let $A \longrightarrow B \longrightarrow C$ be a Rees short exact sequence of S-acts. Then, B has a composition series of subacts if and only if both A and C have composition series of subacts.*

Clearly, the above corollary implies that the property of having a composition series of subacts is closed under subacts and factor acts.

The following theorem is the analogue for S-acts of the Jordon-Holder Theorem for modules.

Theorem 2.4. (*Jordan-Holder*) *If an S-act A has a composition series of subacts, then every pair of composition series of subacts for A are equivalent.*

Proof. Suppose that A has a composition series of subacts. We use induction on $l_s(A)$. If $l_s(A) = 0$, then A is simple, so obviously A has only one composition series. Now suppose that $l_s(A) = n \geq 1$, and assume that for any S-act with a composition series of subacts of smaller length, all of its composition series are equivalent. Let

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \tag{2.1}$$

be a composition series of subacts for A with the minimal length $l_s(A) = n$. Moreover, let

$$A = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_m, \tag{2.2}$$

be another composition series of subacts for A. If $A_1 = B_1$, then by the induction hypothesis, the pair of composition series

$$A_1 \supset A_2 \supset \cdots \supset A_n \quad \text{and} \quad A_1 = B_1 \supset B_2 \supset \cdots \supset B_m$$

are equivalent, and hence the composition series (1) and (2) are equivalent. So we may assume that $A_1 \neq B_1$. First suppose that $n = 1$. Since we have

$$A \supseteq A_1 \cup B_1 \supseteq A_1 \supseteq A_1 \cap B_1,$$

and A_1 and B_1 are simple with $A_1 \neq B_1$, it follows that $A_1 \cap B_1 = \emptyset$ and $A = A_1 \cup B_1$. Therefore, we have $A/A_1 \cong B_1 \cup \{\theta\}$ and $A/B_1 \cong A_1 \cup \{\theta\}$, so that the composition series (1) and (2) are equivalent. Now suppose that $n > 1$. Then since A_1 is a maximal subact of A, we have $A_1 \cup B_1 = A_1 \cup B_2 = A$ which implies that $A_1 \cap B_1 \neq \emptyset$. In addition, $A/A_1 = (A_1 \cup B_1)/A_1 \cong B_1/(A_1 \cap B_1)$, and $A/B_1 = (A_1 \cup B_1)/B_1 \cong A_1/(A_1 \cap B_1)$. Since both A/B_1 and A/A_1 are θ-simple, $A_1 \cap B_1$ is maximal in both A_1 and B_1. By Corollary 2.3, $C = A_1 \cap B_1$ has a composition series

$$C = C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_k.$$

So

$$A_1 \supset C \supset C_1 \supset C_2 \supset \cdots \supset C_k$$

and

$$B_1 \supset C \supset C_1 \supset C_2 \supset \cdots \supset C_k$$

are composition series of subacts for A_1 and B_1, respectively. By the induction hypothesis, since $l_s(A_1) < n$, every two composition series of subacts for A_1 are equivalent. Thus, the composition series of subacts

$$A_1 \supset A_2 \supset \cdots \supset A_n$$

and

$$A_1 \supset C \supset C_1 \supset C_2 \supset \cdots \supset C_k$$

are equivalent. So $k = n - 1$ implies that $l_s(B_1) < n$. Again, by induction hypothesis, every two composition series of subacts for B_1 are equivalent. Then the two composition series

$$B_1 \supset B_2 \supset \cdots \supset B_m$$

and

$$B_1 \supset C \supset C_1 \supset C_2 \supset \cdots \supset C_k$$

are equivalent. It follows that $n = m$. Moreover, we mentioned that $A/A_1 \cong B_1/C_0$, and $A/B_1 \cong A_1/C_0$. Now, we have $A_1/C_0 \cong A_j/A_{j+1}$ for some $j \in \{1, \ldots, n\}$, and there is a bijection

$$\sigma : \{0, \ldots, n-2\} \to \{1, \ldots, n\} \backslash \{j\}$$

such that $C_i/C_{i+1} \cong A_{\sigma(i)}/A_{\sigma(i+1)}$. Similarly, $B_1/C_0 \cong B_l/B_{l+1}$ for some $l \in \{1, \ldots, n\}$, and there is a bijection

$$\tau : \{0, \ldots, n-2\} \to \{1, \ldots, n\} \backslash \{l\}$$

such that $C_i/C_{i+1} \cong B_{\tau(i)}/B_{\tau(i+1)}$. Now define a permutation ι on $\{0, \ldots, n\}$ by $\iota(0) = l$, $\iota(j) = 0$ and $\iota(\tau(i)) = \sigma(i)$. Then $A_i/A_{i+1} \cong B_{\iota(i)}/B_{\iota(i+1)}$ for each $i \in \{0, \ldots, n\}$. Therefore, the two composition series (2.3) and (2.4) are equivalent, as desired. □

The following result is an immediate consequence of the Jordan-Holder Theorem which states that for any S-act having a composition series of subacts, all composition series for that S-act have the same length. So such S-acts are said to be *of finite length of subacts.*

Proposition 2.5. *Suppose that A has a composition series of subacts of length n. Then every composition series of subacts of A has length n, and every chain in A can be extended to a composition series.*

Corollary 2.6. *Every finitely generated S-act A over Rees artinian commutative monoid S is of finite length of subacts.*

Proof. By [12, Theorem 20], S is Rees noetherian, hence [12, Proposition 11] implies that every finitely generated S-act A is both Rees artinian and Rees noetherian. Consequently, A has finite length of subacts. □

In the category S-Act$_0$, the coproducts of θ-simple S-acts are called *semisimple.* If an S-act A is a coproduct of finitely many θ-simple S-acts, then A is of finite length of subacts. Conversely, if A is semisimple and of finite length of subacts, then A is a finite coproduct of θ-simple subacts.

As a direct consequence of [12, Proposition 11], we obtain the following corollary.

Corollary 2.7. *For a monoid S, the following statements are true.*

(i) *A is of finite length of subacts if and only if $A\coprod\Theta$ is of finite length of subacts.*

(ii) *If A is of finite length of subacts, then $\coprod_{i=1}^{i=n} A$ is of finite length of subacts for each $n \in \mathbb{N}$.*

(iii) *If S contains a zero and $A_1, \ldots, A_n$ are S-acts, then $A = \coprod_{i=1}^{i=n} A_i$ is of finite length of subacts if and only if each A_i, $1 \leq i \leq n$, is of finite length of subacts.*

3 Composition series of congruences

Given the importance of the role of congruences in the study of S-acts, in this section we consider composition series of congruences. Then we study the connections of composition series of congruences with artinian and noetherian properties. First, we extend the definition of composition series of subacts to composition series of congruences.

Definition 3.1. Let A be an S-act. Then,

(i) A finite chain of $n+1$ congruences of A

$$\nabla_A = \rho_0 \supset \rho_1 \supset \rho_2 \supset \cdots \supset \rho_n = \Delta_A,$$

is called a *composition series of congruences* of length n for A provided that ρ_{i+1} is maximal congruence in ρ_i, in other words, ρ_i covers ρ_{i+1}, which is denoted by $\rho_{i+1} \leq_m \rho_i$. We denote the minimum length of a composition series of congruences on A by $l_c(A)$.

(ii) Two composition series of congruences

$$\nabla_A = \rho_0 \supset \rho_1 \supset \rho_2 \supset \cdots \supset \rho_n = \Delta_A,$$

and

$$\nabla_A = \sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_m = \Delta_A,$$

of A are said to be *equivalent* in case $n = m$, and there is a permutation ι on $\{1, 2, \ldots, n\}$ such that $\rho_i/\rho_{i+1} \cong \sigma_{\iota(i)}/\sigma_{\iota(i+1)}$ for each $i = 1, 2, \ldots, n$.

Recall that a lattice $(Con(A), \subseteq, \cap, \vee)$ is *modular* if for each $\rho, \sigma, \delta \in Con(A)$, we have $\rho \subseteq \delta$ implies that $\rho \vee (\delta \cap \sigma) = \delta \cap (\rho \vee \sigma)$. Before proceeding to the explicit study of composition series of congruences, we shall prove two following lemmas which will be needed subsequently.

Lemma 3.2. *Let A be an S-act which the lattice $Con(A)$ is modular. If $\rho_1, \rho_2, \sigma \in Con(A)$ and $\rho_2 \leq_m \rho_1$, then*

(i) $\rho_1 \vee \sigma = \rho_2 \vee \sigma$ *or* $\rho_2 \vee \sigma \leq_m \rho_1 \vee \sigma$*;*

(ii) $\rho_1 \cap \sigma = \rho_2 \cap \sigma$ *or* $\rho_2 \cap \sigma \leq_m \rho_1 \cap \sigma$*.*

Proof. (i) Let $\rho_1, \rho_2, \sigma \in Con(A)$ and $\rho_2 \leq_m \rho_1$. Clearly,

$$\rho_2 \subseteq \rho_1 \cap (\rho_2 \vee \sigma) \subseteq \rho_1,$$

and maximality of ρ_2 in ρ_1 implies that $\rho_2 = \rho_1 \cap (\rho_2 \vee \sigma)$ or $\rho_1 = \rho_1 \cap (\rho_2 \vee \sigma)$. If $\rho_1 = \rho_1 \cap (\rho_2 \vee \sigma)$, then, using the fact that $Con(A)$ is modular, we have

$$\rho_1 \vee \sigma = [\rho_1 \cap (\rho_2 \vee \sigma)] \vee \sigma = [\rho_2 \vee (\rho_1 \cap \sigma)] \vee \sigma = \rho_2 \vee \sigma.$$

Now, suppose that $\rho_2 = \rho_1 \cap (\rho_2 \vee \sigma)$, and consider δ such that $\rho_2 \vee \sigma \subseteq \delta \subseteq \rho_1 \vee \sigma$. Then $\rho_2 \subseteq \rho_1 \cap \delta \subseteq \rho_1$, which implies that $\rho_2 = \rho_1 \cap \delta$ or $\rho_1 = \rho_1 \cap \delta$. If $\rho_2 = \rho_1 \cap \delta$, then

$$\delta = \delta \cap (\rho_1 \vee \sigma) = (\rho_1 \cap \delta) \vee \sigma = \rho_2 \vee \sigma.$$

If $\rho_1 = \rho_1 \cap \delta$, then $\rho_1 \subseteq \delta$, and so $\rho_1 \vee \sigma = \delta$ since $\sigma \subseteq \delta$. Therefore, $\rho_2 \vee \sigma \leq_m \rho_1 \vee \sigma$.

Part (ii) can be proved in an analogous way. □

Lemma 3.3. *Let A, B be S-acts which the lattices $Con(A)$ and $Con(B)$ are modular, and $f : A \to B$ be an S-morphism. The following hold.*

(i) *Let $\sigma_1, \sigma_2 \in Con(A)$ with $\sigma_2 \leq_m \sigma_1$. Letting*

$$\rho_i = \{(f(a), f(b)) |\ (a, b) \in \rho_i \vee \ker f\} \cup \Delta_B,$$

we have $\rho_i \in Con(B)$ and ($\rho_2 = \rho_1$ or $\rho_2 \leq_m \rho_1$). Moreover, if f is a monomorphism, $\rho_2 \leq_m \rho_1$.

(ii) *Let* $\gamma_1, \gamma_2 \in Con(B)$ *with* $\gamma_2 \leq_m \gamma_1$*, Letting*

$$\delta_i = \{(a, b)|\ (f(a), f(b)) \in \gamma_i\},$$

we have $\delta_i \in Con(A)$ *and* ($\delta_2 = \delta_1$ *or* $\delta_2 \leq_m \delta_1$)*. Moreover, if* f *is an epimorphism,* $\delta_2 \leq_m \delta_1$*.*

Proof. (i) Let $\sigma_1, \sigma_2 \in Con(A)$ and $\sigma_2 \leq_m \sigma_1$. Let

$$\rho_i = \{(f(a), f(b))|\ (a, b) \in \sigma_i \vee \ker f\} \cup \Delta_B.$$

If $(c, d) \in \rho_i$, then $c = d$ or $c = f(a),\ d = f(b)$ such that $(a, b) \in \sigma_i \vee \ker f$. Clearly, for each $s \in S$, $(as, bs) \in \sigma_i \vee \ker f$ and so $(cs, ds) = (f(as), f(bs)) \in \rho_i$ which implies that $\rho_i \in Con(B)$. By Lemma 3.2, $\sigma_1 \vee \ker f = \sigma_2 \vee \ker f$, or $\sigma_2 \vee \ker f \leq_m \sigma_1 \vee \ker f$. In the first case, $\rho_1 = \rho_2$. Otherwise, suppose that $\rho_2 \subseteq \delta \subseteq \rho_1$. Take

$$\xi = \{(a, b)|\ (f(a), f(b)) \in \delta\}.$$

Clearly, $\xi \in Con(A)$ and $\sigma_2 \vee \ker f \subseteq \xi \subseteq \sigma_1 \vee \ker f$, and since $\sigma_2 \vee \ker f \leq_m \sigma_1 \vee \ker f$, we have $\sigma_2 \vee \ker f = \xi$ or $\sigma_1 \vee \ker f = \xi$. Hence, $\delta = \rho_2$ or $\delta = \rho_1$, and the result follows. The second part is clear.

(ii) Let $\gamma_1, \gamma_2 \in Con(B)$ and $\gamma_2 \leq_m \gamma_1$. Let

$$\delta_i = \{(a, b)|\ (f(a), f(b)) \in \gamma_i\}.$$

Clearly, $\delta_i \in Con(A)$. By Lemma 3.2, $\sigma_1 \cap \mathcal{K}_{\mathrm{Imf}} = \sigma_2 \cap \mathcal{K}_{\mathrm{Imf}}$, or $\sigma_2 \vee \mathcal{K}_{\mathrm{Imf}} \leq_m \sigma_1 \vee \mathcal{K}_{\mathrm{Imf}}$. Now, suppose that $\delta_2 \subseteq \varphi \subseteq \delta_1$. Take

$$\xi = \{(f(a), f(b))|\ (a, b) \in \varphi \vee \ker f\} \cup \Delta_B.$$

Clearly, $\xi \in Con(B)$ and $\gamma_2 \cap \mathcal{K}_{\mathrm{Imf}} \subseteq \xi \subseteq \gamma_1 \cap \mathcal{K}_{\mathrm{Imf}}$. Since $\gamma_2 \cap \mathcal{K}_{\mathrm{Imf}} \leq_m \gamma_1 \cap \mathcal{K}_{\mathrm{Imf}}$, we have $\gamma_2 \cap \mathcal{K}_{\mathrm{Imf}} = \xi$ or $\gamma_1 \cap \mathcal{K}_{\mathrm{Imf}} = \xi$. Hence, $\varphi = \delta_2$ or $\varphi = \delta_1$, as desired. Now, suppose that f in epimorphism. Take $(c, d) \in \gamma_1 \setminus \gamma_2$. Since f is an epimorphism, there exist $a, b \in A$ such that $c = f(a)$ and $d = f(b)$. Then $(a, b) \in \delta_1 \setminus \delta_2$, and so $\delta_2 \neq \delta_1$. □

We now prove that for an S-act whose congruence lattice is modular, any two compositions series have the same length.

Proposition 3.4. *Suppose that A has a composition series of congruences of length n, and the lattice $Con(A)$ is modular. Then every composition series of A has length n, and every chain of congruences in A can be extended to a composition series of congruences.*

Proof. Suppose that lattice $Con(A)$ is modular, and

$$\nabla_A = \rho_0 \supset \rho_1 \supset \rho_2 \supset \cdots \supset \rho_n = \Delta_A$$

is a composition series of congruences of A with the least length $l_c(A) = n$. First we show that $l_c(A/\sigma) < l_c(A)$ for each $\sigma \in Con(A)$, $\sigma \neq \Delta_A$. We have a series

$$\nabla_{A/\sigma} = (\rho_0 \vee \sigma)/\sigma \supseteq (\rho_1 \vee \sigma)/\sigma \supseteq (\rho_2 \vee \sigma)/\sigma \supseteq \cdots \supseteq (\rho_n \vee \sigma)/\sigma = \Delta_{A/\sigma}.$$

Using Lemma 3.2, we deduce that $(\rho_i \vee \sigma)/\sigma = (\rho_{i+1} \vee \sigma)/\sigma$ or $(\rho_{i+1} \vee \sigma)/\sigma \leq_m (\rho_i \vee \sigma)/\sigma$. Hence, removing repeated terms, we obtain a composition series of congruences for A/σ, and so $l_c(A/\sigma) \leq l_c(A)$. If $l_c(A/\sigma) = l_c(A) = n$, then $\rho_{i+1} \vee \sigma \leq_m (\rho_i \vee \sigma)$ for each $i = 1, 2, \ldots, n$. From the proof of Lemma 3.2, in this case $\rho_{i+1} = \rho_i \cap (\rho_{i+1} \vee \sigma)$ for each $i = 1, 2, \ldots, n$. This implies that

$$\rho_{i+1} \cap \sigma = [\rho_i \cap (\rho_{i+1} \vee \sigma)] \cap \sigma = (\rho_i \cap \sigma) \vee (\rho_{i+1} \cap \sigma) = \rho_i \cap \sigma,$$

for each $i = 1, 2, \ldots, n$. But then,

$$\sigma = \rho_0 \cap \sigma = \rho_1 \cap \sigma = \cdots = \rho_n \cap \sigma = \Delta_A,$$

a contradiction. Hence $l_c(A/\sigma) < l_c(A)$.

Now suppose that

$$\nabla_A = \sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_m = \Delta_A$$

is another composition series of congruences for A of length m. So we have a chain of epimorphisms

$$A \xrightarrow{\pi_m} A/\sigma_{m-1} \xrightarrow{\pi_{m-1}} A/\sigma_{m-2} \xrightarrow{\pi_{m-2}} \cdots \xrightarrow{\pi_1} A/\sigma_0$$

where $\pi_i([a]_{\sigma_i}) = [a]_{\sigma_{i-1}}$. Then, we have

$$l_c(A) > l_c(A/\sigma_{m-1}) > \cdots > l_c(A/\sigma_m) = l_c(\Theta) = 0,$$

and hence $l_c(A) \geq m$. So minimality of $l_c(A)$ implies that $m = l_c(A)$. Thus all composition series have the same length $l_c(A)$. Finally, consider any chain such that its length is less than $l_c(A)$. Since such a chain is not a composition series for A, we can insert new terms until it can be refined to a composition series. As all composition series have the same length $l_c(A)$, new terms can be inserted until the length is $l_c(A)$. $\square$

The following example demonstrates that the condition of modularity in Proposition 3.4 is necessary.

Example 3.5. Let $S = \{1, x\}$ be the cyclic group of order 2, and let A be the S-act $A = \{a, b, c, d\}$ with action given by

$$ax = b,\ bx = a,\ cx = d,\ dx = c.$$

Let ρ be the congruence on A with classes $\{a, b\}$ and $\{c, d\}$; let ρ' be the congruence on A with classes $\{a, b\}$, $\{c\}$ and $\{d\}$; and let σ be the congruence on A with classes $\{a, c\}$ and $\{b, d\}$. Then A has the following composition series of congruences:

$$\nabla_A \supset \rho \supset \rho' \supset \Delta_A \quad \text{and} \quad \nabla_A \supset \sigma \supset \Delta_A.$$

To see that $Con(A)$ is not modular, observe that

$$\sigma \vee (\rho \cap \rho') = \sigma \vee (\rho') = \nabla_A,$$

while

$$\rho \cap (\sigma \vee \rho') = \rho \cap \nabla_A = \rho.$$

If an S-act has a composition series of congruences, in addition, all composition series of congruence for that S-act have the same length, then the length of composition series denoted by $l_c(A)$. So such S-acts are said to be *of finite length of congruences*. Clearly, every congruence-free S-act is of finite length of congruences, artinian and noetherian.

The following is an example of an S-act of finite length of congruences having a non-modular lattice of congruences. Indeed, the converse of Proposition 3.4 is not valid. For $X \subseteq A \times A$, we denote the congruence on A generated by X by $\rho(X)$.

Example 3.6. Suppose that $S = \{1, a, b, c, d\}$ where a, b, c, d are right zeros. Consider the S-act $A = \{a, b, c, d\}$. To show that $Con(A)$ is not modular, let $\rho_1 = \rho(a, b)$, $\rho_2 = \rho((a, b), (a, d))$, and $\sigma = \rho(a, c)$. It is easily checked that $\rho_1 \leq \rho_2$, and $\rho_1 = \rho_1 \vee (\rho_2 \cap \sigma) \neq \rho_2 \cap (\rho_1 \vee \sigma) = \rho_2$. On the other hand,

$$Con(A) = \{\Delta_A, \rho(a,b), \rho(a,c), \rho(a,d), \rho(b,c), \rho(b,d), \rho(c,d),$$
$$\rho((a,b),(a,d)), \rho((a,b),(a,c)), \rho((a,c),(a,d)), \rho((b,c),(b,d)),$$
$$\rho_{\{a,b,c\}}, \rho_{\{a,b,d\}}, \rho_{\{b,c,d\}}, \nabla_A\}.$$

A has the composition series $\nabla_A = \rho_0 \supset \rho_1 \supset \rho_2 \supset \rho_3 = \Delta_A$. One can see that all composition series of congruences in $Con(A)$ have the length of 3. Therefore, A is of finite length of congruences.

Theorem 3.7. *Let S be a monoid and A an S-act. If A is both noetherian and artinian, then A has a composition series of congruences. The converse is valid when A is of finite length of congruences, in particular, when the lattice $Con(A)$ is modular.*

Proof. Suppose that A is noetherian and artinian. Since A is artinian, if A_S is not congruence-free then it has a proper minimal congruence ρ_1. Now, A/ρ_1 is artinian, so if A/ρ_1 is not congruence-free, it has a proper minimal congruence, σ_2. Take

$$\rho_2 = \{(a, b) \in A \times A |\ ([a]_{\rho_1}, [b]_{\rho_1}) \in \sigma_2\},$$

then $\rho_1 \subseteq \rho_2$, and by Lemma 3.3, $\rho_1 \leq_m \rho_2$. Continuing in this way, we must eventually obtain a congruence $\rho_n = \nabla_A$, for otherwise we would have a strictly descending chain

$$\Delta_A = \rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \cdots,$$

contradicting the fact that A is noetherian. Moreover, since each ρ_i is maximal in ρ_{i+1}, we have a composition series of congruences for A.

For the converse, suppose that A has a composition series of congruences of length n and all composition series of congruences have the length n. (By Proposition 3.4, this holds in the case that $Con(A)$ is modular.) So all ascending and descending chains of congruences eventually stabilise. Therefore, A is both noetherian and artinian. □

The following theorem considers the behavior of composition series of congruences on Rees short exact sequences.

Theorem 3.8. *Let $A \longrightarrow B \longrightarrow C$ be a Rees short exact sequence of S-acts where $Con(A)$, $Con(B)$ and $Con(C)$ are modular. Then, B has a composition series of congruences if and only if both A and C have composition series of congruences. Moreover, if A and C are of finite length of congruences, then $l_c(B) = l_c(A) + l_c(C)$.*

Proof. Necessity. Let

$$\nabla_B = \rho_0 \supset \rho_1 \supset \rho_2 \supset \cdots \supset \rho_n = \Delta_B$$

be a composition series of congruences on B. Then,

$$\sigma_i = \{(a, b) |\ (f(a), f(b)) \in \rho_i\}$$

is a congruence on A for each $i = 1, 2, \ldots, n$. Then

$$\nabla_A = \sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_n = \Delta_A.$$

Using Lemma 3.3, we imply that $\sigma_{i+1} = \sigma_i$ or $\sigma_{i+1} \leq_m \sigma_i$. Hence, removing repeated terms, we obtain a composition series of congruences on A. On the other hand, take $\varepsilon_i = \rho_i \vee \ker g$ and

$$\gamma_i = \{(g(a), g(b)) |\ (a, b) \in \varepsilon_i\}$$

for each $i = 1, 2, \ldots, n$. It is routine to check that $\gamma_i \in Con(C)$. Moreover,

$$\nabla_C = \gamma_0 \supset \gamma_1 \supset \gamma_2 \supset \cdots \supset \gamma_k = \Delta_C.$$

Using Lemma 3.3, $\gamma_{i+1} = \gamma_i$ or $\gamma_{i+1} \leq_m \gamma_i$. Again, removing repeated terms, we obtain a composition series of congruences on C.

Sufficiency. Let

$$\nabla_A = \sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_m = \Delta_A$$

be a composition series of congruences on A. Then,

$$\rho_i = \{(f(a), f(b)) |\ (a, b) \in \sigma_i\} \cup \Delta_B$$

is a congruence on B for each $i = 1, 2, \ldots, m$. Then

$$\mathcal{K}_{\mathrm{Imf}} = \rho_0 \supset \rho_1 \supset \rho_2 \supset \cdots \supset \rho_m = \Delta_B$$

is a chain of congruences on B such that, by Lemma 3.3, $\rho_{i+1} \leq_m \rho_i$. On the other hand, suppose that

$$\nabla_C = \gamma_0 \supset \gamma_1 \supset \gamma_2 \supset \cdots \supset \gamma_k = \Delta_C$$

is a composition series of congruences on C. Let

$$\xi_i = \{(a, b) |\ (g(a), g(b)) \in \gamma_i\}$$

for each $i = 1, 2, \ldots, k$. Thus

$$\nabla_B = \xi_0 \supset \xi_1 \supset \xi_2 \supset \cdots \supset \xi_k = \ker g$$

is a chain of congruences on B such that, by Lemma 3.3, $\xi_{i+1} \leq_m \xi_i$. Therefore,

$$\nabla_B = \xi_0 \supset \xi_1 \supset \xi_2 \supset \cdots \supset \xi_k = \ker g$$

$$= \mathcal{K}_{\mathrm{Imf}} = \rho_0 \supset \rho_1 \supset \rho_2 \supset \cdots \supset \rho_m = \Delta_B$$

is a composition series of congruences on A. Consequently, we see that if all composition series of congruences have the same length, then $l_c(B) = m + k = l_c(A) + l_c(C)$. □

We remark that, without the assumption of modularity, the above statement no longer holds. Indeed, consider the above example where $S = \mathbb{Z}_2$ and $A = \{a, b, c, d\}$. Let B denote the subact $\{a, b\}$. It was shown above that $l_c(A) = 3$. On the other hand, since $|B| = |A/B| = 2$, it is clear that $l_c(B) = l_c(A/B) = 1$.

From the previous theorem we deduce easily the following result.

Corollary 3.9. (*The Dimension Theorem.*) *Let A be an S-act of finite length of congruences, and let B and C be subacts of A such that $B \cap C \neq \emptyset$. Then*

$$l_c(B \cup C) + l_c(B \cap C) = l_c(B) + l_c(C).$$

Proof. We apply Theorem 3.8 to the two Rees exact sequences
$B \longrightarrow B\cup C \longrightarrow (B\cup C)/B$ and $B\cap C \longrightarrow C \longrightarrow C/(B\cap C)$,
to get $l_c(B\cup C) = l_c(B)+l_c((B\cup C)/B)$ and $l_c(C) = l_c(B\cap C)+l_c(C/(B\cap C))$. Then, the fact that $C/(B\cap C) \cong (B\cup C)/B$ implies the result. □

Corollary 3.10. *Suppose that A_i are S-acts, $1 \le i \le n$, of finite length of congruences and each A_i contains a zero. Then $A = \coprod_{i=1}^{i=n} A_i$ ($A = \prod_{i=1}^{i=n} A_i$) is of finite length of congruences and $l_c(A) = \sum_{i=1}^{i=n} l_c(A_i)$.*

Proof. Let $A_1, \dots, A_n$ be S-acts with have zeros $0_1, \dots, 0_n$, respectively. We have a surjective homomorphism $g_k : \coprod_{i=1}^{k} A_i \to A_k$ given by $g_k(a) = a$ if $a \in A_k$ and $g_k(a) = 0_k$ otherwise. Using the following Rees short exact sequences for $k = 2, \dots, n$:

$$\coprod_{i=1}^{i=k-1} A_i \to \coprod_{i=1}^{i=k} A_i \to A_k,$$

we apply Theorem 3.8 to get the result.

We use a similar proof for the direct product case, by defining an injective homomorphism

$$f_k : \prod_{i=1}^{k-1} A_k \to \prod_{i=1}^{k} A_i, (a_1, \dots, a_{k-1}) \mapsto (a_1, \dots, a_{k-1}, 0_k),$$

and a surjective homomorphism

$$g_k : \prod_{i=1}^{k} A_i \to A_k, (a_1, \dots, a_k) \mapsto a_k.$$

□

The above results are also valid for the length of composition series of subacts. In fact, by similar arguments, if $B \cap C \neq \emptyset$ then

$$l_s(B\cup C) + l_s(B\cap C) = l_s(B) + l_s(C).$$

Moreover, for a Rees short exact sequence of S-acts $A \longrightarrow B \longrightarrow C$,

$$l_s(B) = l_s(A) + l_s(C).$$

In addition, if A_i contains a zero for each $i = 1, \ldots, n$, then

$$l_s(\coprod_{i=1}^{i=n} A_i) = \sum_{i=1}^{i=n} l_s(A_i) \quad \text{and} \quad l_s(\prod_{i=1}^{i=n} A_i) = \sum_{i=1}^{i=n} l_s(A_i).$$

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References

[1] Anderson, A. and Fuller, K., "Rings and Categories of Modules", Springer-Verlag, 1992.

[2] Dandan, Y., Gould, V., Hartmann M., Rusuk N., and Zenab R.E., *Coherency and construction for monoids*, Quart. J. Math. 71 (2020), 1461-1488.

[3] Davvaz, B. and Nazemian, Z., *Chain conditions on commutative monoids*, Semigroup Forum 100 (2020), 732-742.

[4] Estaji, A.A. and Shabani, M., *A note on multiplication S-acts*, Far East J. Math. Sci. 36(2) (2010), 133-150.

[5] Gould, V., *Coherent monoids*, J. Australian Math. Soc. 53 (1992), 166-182.

[6] Hotzel, E., *On semigroups with maximal conditions*, Semigroup Forum 11 (1975), 337-362.

[7] Kilp, M., Knauer, U., and Mikhalev, A., "Monoids, Acts and Categories", Walter de Gruyter, 2000.

[8] Kozhukhov, I., *On semigroups with minimal or maximal condition on left congruences*, Semigroup Forum 21 (1980), 337-350.

[9] Kozkukhov, I., *Semigroups with certain conditions on congruences*. J. Math. Sci. 114 (2003), 1119-1126.

[10] Khosravi, R. and Roueentan, M., *Co-uniform and hollow S-acts over monoids*, Commun. Korean Math. Soc. 37(2) (2022), 347-358.

[11] Khosravi, R., Liang, X., and Roueentan, M., *On Cogenerating and finitely cogenerated S-acts*, `https://arxiv.org/abs/2109.00344`.

[12] Khosravi, R. and Roueentan, M., *Chain conditions on (Rees) congruences of S-acts*, J. Algebra Appl., `https://doi.org/10.1142/S0219498824500853`.

[13] Miller, C. and Rusuk, N., *Right Noetherian semigroups*, Int. J. Algebra Comput. 30 (2020), 13-48.

[14] Wisbauer, R., "Foundations of module and ring theory", Gordon and Breach Science Publishers, 1991.

[15] Barzegar, H., *Essentiality in the category of S-acts*, Eur. J. Pure App. Math. 9(1)(2016), 19-26.

[16] Chen, Y. and Shum, K.P., *Rees short exact sequences of S-systems*, Semigroup Forum 65 (2002), 141-148.

[17] Roueentan, M. and Sedaghatjoo, M., *On uniform acts over semigroups*, Semigroup Forum 97 (2018), 229-243.

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