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Action graph of a semigroup act & its functorial connection

P. Mukherjee, R. Mukherjee, and S.K. Sardar*

Abstract. In this paper we define *C-induced action graph* $G(S, a, C; A)$ corresponding to a semigroup act (S, a, A) and a subset C of S. This generalizes many interesting graphs including *Cayley Graph* of groups and semigroups, *Transformation Graphs (TRAG)*, *Group Action Graphs (GAG)*, *Derangement Action Graphs*, *Directed Power Graphs of Semigroups* etc. We focus on the case when $C = S$ and name the digraph, so obtained, as *Action Graph of a Semigroup Act* (S, a, A). Some basic structural properties of this graph follow from algebraic properties of the underlying semigroup and its action on the set. Action graph of a strongly faithful act is also studied and graph theoretic characterization of a strongly faithful semigroup act as well as that of idempotents in a semigroup are obtained. We introduce the notion of *strongly transitive digraphs* and based on this we characterize action graphs of semigroup acts in the class of simple digraphs. The simple fact that morphism between semigroup acts leads to digraph homomorphism between corresponding action graphs, motivates us to represent action graph construction as a functor from the category of semigroup acts to the category of certain digraphs. We capture its functorial properties, some of which

* Corresponding author

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signify previous results in terms of Category Theory.

1 Introduction

Graphs are very useful objects in both pure and applied Mathematics as they have a natural geometric essence and can be easily used to model several physical problems related to interactions between objects. In 1736 Graphs first came into being when famous mathematician Leonhard Euler gave a logical solution for the Königsberg bridge problem [27]. From then centuries have passed in the development of Graph theory. In modern times *Algebraic Graph Theory* has opened up a new direction of studying graph properties. It is unique in its own treatment where algebraic methods are applied to study properties and problems about graphs. The literature of algebraic graph theory has grown enormously since 1974 when the original version of the book *Algebraic Graph Theory* [4] by Norman L. Biggs was published. It can be considered as one of the two pioneer books on the subject. The other one is *Algebraic Graph Theory* [15] by Chris Godsil and Gordon Royle, first published in 2001.

In 1967, J. Dénes [7] focused on the interconnection between semigroups and digraphs. In [8], Dénes considered graphical representation of a transformation of degree n (which can be treated as an element of the transformation semigroup of a set of n elements) and obtained some combinatorial results. This is undoubtedly a beginning in the study of action graphs. Action (di)graphs have been considered by many authors by different names like *TRAG* [2], *GAG* [2], *Schreier Graph* [14]. In 2002, A. Malnič [25] gave a beautiful description of an action (di)graph as a combinatorial representation of a group acting on a set. In 1878 Arthur Cayley first considered Cayley graphs of groups which is induced by the action of some suitably chosen connection set known as *Cayley set*. The construction is nicely described in [4], [15], [22]. Cayley graphs of groups have been extensively studied and beautiful results have been obtained (see [4]). Cayley graphs of semigroups was first considered by Zelinka [28] in 1981. Studying properties of Cayley graphs of different classes of semigroups and characterizing them have opened a separate branch of research among mathematicians. A lot of works have been done in this area. For this, one is referred to [13], [18], [20], [23], [26] and their references. Though several graphs have been constructed from semigroup acts (for example see [9], [10], [12]), but so far possibly except in [2], [7], [17], [25] no attention has been paid to the study of action graphs of semigroup acts (in [2] it is known as TRAG). The reason may be that this study seems to be almost similar to the study of graphs of group actions since as an algebraic structure semigroup generalizes group. But it is well known that lack of inverses and possible existence of idempotents other than identity in semigroups make their study interesting. Moreover it is worth motivating how action graphs of semigroup acts unify so many important classes of graphs including Group action graphs ([2], [25]), Derangement action digraphs ([17]) and Cayley graphs of groups and semigroups. Also Power graph $Pow(S)$ of a semigroup S ([5], [19]) can be obtained as a loop free copy of action graph of the particular action of $\mathbb{N}$ on S (*cf.* Remark 3.24). Thus investigation of properties and characterization of action graphs of semigroup acts urges serious attention. This motivates us to renew the study of action graphs initiated in different ways by Dénes [7], Annexstein et al. [2] and Malnič [25].

In Section 2 we have stated the preliminary concepts. Section 3 has been started by defining a left semigroup act (S, a, A) and morphism between acts. After establishing that semigroup acts form a category, we have defined C-induced action graph of a semigroup act (S, a, A) with respect to a subset C of the semigroup S. Cayley graphs of groups and semigroups have been exhibited as particular instances of this graph (*cf.* Example 3.7, 3.8). We have mainly focused on the graph obtained by taking $C = S$, named it as *Action Graph of the Semigroup Act* (S, a, A) and obtained some of its structural properties (*cf.* Proposition 3.12). Action graph construction has been presented as a functor (*cf.* Proposition 3.14) from the category of semigroup acts to the category of particular digraphs. Section 4 has been devoted to the study of strongly faithful semigroup acts and corresponding action graphs. In this section along with other results we have given a graph theoretic characterization (*cf.* Proposition 4.10) of strongly faithful (finite) semigroup acts. In Section 5 we have asked the question, 'When a given digraph will be the action graph of some semigroup act?' and in the search of an answer, we have shown that every simple, transitive digraph is isomorphic to the action graph of some semigroup act ignoring loops (*cf.* Theorem 5.4). Further we have characterized action graphs of semigroup acts in the class of simple digraphs (*cf.* Theorem 5.10, Corollary

5.11) and identified the image of the previously defined functor. Some more properties (*cf.* Theorem 6.7, Theorem 6.8) of this functor and its restriction to the subcategory of strongly faithful acts have been investigated in Section 6. In this paper some results have been obtained (*cf.* Proposition 4.10, Proposition 6.2) which can be seen as generalization of analogous results studied for Cayley graphs of semigroups and Derangement action graphs while some results have been derived (*cf.* Proposition 4.21, Theorem 5.4) which have no satisfactory analogue in group action graphs.

Among other motivations as mentioned above, the relevant treatment of Cayley graphs and other topics in the monograph [22] by Knauer et al. has greatly inspired us in this work.

2 Preliminaries

Before going into the details, we mention some basic graph and semigroup theoretic definitions which will be used throughout the paper. Some particular definitions are given later along with the context when necessary.

Definition 2.1. A digraph G is defined as a triple $G = (V(G), h, E(G))$ where $V(G)$ and $E(G)$ are sets and $h : E(G) \longrightarrow V(G) \times V(G)$ is a map.

The elements of $V(G)$ and $E(G)$ are respectively called vertices and arcs of G. For any arc $e \in E(G)$, the first vertex of the pair $h(e)$ is called the *tail* (or, *source*) and the second is the *head* (or, *range*) of the arc e. If h is injective then G is called *simple*. In this case any arc $e \in E(G)$ can be represented by its image under h, that is by the ordered pair $h(e)$ and consequently $E(G)$ can be identified by its image $h(E(G))$ (a subset of $V(G) \times V(G)$) and we represent e simply as an ordered pair of vertices ((tail,head)). Digraphs having finite sets of vertices and arcs are called *finite digraphs*. A vertex is called *isolated* if it is not the head or tail of any arc. Digraph which consists only of isolated vertices is called a *null graph*. A *sub-digraph* of a digraph $G = (V(G), h, E(G))$ is a digraph $H = (V(H), h', E(H))$ such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and $h|_{E(H)} = h'$. If $V(H) = V(G)$ then H is called a *spanning sub-digraph* of G. An $x - y$ *path* (precisely, *directed path*) in a digraph G is an alternating sequence of distinct vertices and arcs $\{v_1 = x, e_1, v_2, e_2, ..., v_{n-1}, e_{n-1}, v_n = y\}$ such that $head(e_i) = v_{i+1}$, $tail(e_i) = v_i$ for all $i = 1, 2, ..., n - 1$. A closed directed path (where $v_1 = v_n$) is called a *directed cycle*. The *length* of a directed path or a cycle is the number of distinct arcs on the path or cycle. A digraph G is called *strongly connected* if for any two distinct vertices $x, y \in G$ there exists a directed $x - y$ path. It is called *weakly connected* if the underlying undirected graph is connected. A maximal strongly (respectively, weakly) connected sub-digraph is called a *strong* (respectively, *weak*) *component* of G. The length of the shortest cycle (if exists) of G is called its *girth* and it is denoted by $gr(G)$. A simple digraph G is called *complete* if for any two distinct vertices x, y in G there is an arc (x, y) in G. A subset W of $V(G)$ is called a *clique* if for any two distinct vertices $x, y \in W$, $(x, y) \in E(G)$. A subset I of $V(G)$ is called an *independent set* if there is no arc in G between any two vertices of I. G is called *bipartite* if the set of vertices $V(G)$ can be partitioned into two disjoint sets A, B such that every arc connects a vertex in A to a vertex in B or vice versa.

Definition 2.2. Suppose $G_1 = (V(G_1), h_1, E(G_1)), G_2 = (V(G_2), h_2, E(G_2))$ are two digraphs. A *digraph homomorphism* is defined by a pair $(f, g) : G_1 \longrightarrow G_2$ such that $f : V(G_1) \longrightarrow V(G_2)$, $g : E(G_1) \longrightarrow E(G_2)$ are morphisms in $\mathbb{S}\text{et}$ for which the following diagram commutes.

$$\begin{array}{ccc} E(G_1) & \xrightarrow{\quad g \quad} & E(G_2) \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ V(G_1) \times V(G_1) & \xrightarrow[f \times f]{} & V(G_2) \times V(G_2) \end{array}$$

For simple digraphs $G_1 = (V(G_1), h_1, E(G_1))$ and $G_2 = (V(G_2), h_2, E(G_2))$, any digraph homomorphism $(f, g) : G_1 \longrightarrow G_2$ is completely determined by the map $f : V(G_1) \longrightarrow V(G_2)$ in the sense that if $(x, y) \in E(G_1)$ then $(f(x), f(y)) \in E(G_2)$. Digraphs together with digraph homomorphisms as defined in Definition 2.2 form a category $\mathbb{DG}\text{raph}$.

For any simple digraph G, the elements of the set $N_G^+(x) := \{y \in V(G) \mid (x, y) \in E(G)\}$ (respectively, $N_G^-(x) := \{y \in V(G) \mid (y, x) \in E(G)\}$) are called the *out-neighbours* (respectively, *in-neighbours*) of x and the number of distinct out-neighbours (respectively, in-neighbours) is called the *out-degree* (respectively, *indegree*) of x. A graph G is called *k-out regular* (respectively, *k-in regular*) if $outdegree(x) = k$ (respectively, $indegree(x) = k$) for all $x \in V(G)$.

A nonempty set S together with an associative binary operation is called a *semigroup*. For the definitions of *identity in a semigroup*, *subsemigroup*, *left ideal* (respectively, *right ideal*), *semigroup homomorphism* and other terminologies we refer to [16]. A semigroup can also be considered as a *semicategory* (category without the requirement of identity morphisms on each object) with only one object. Semigroups and their morphisms form a category $\mathbb{S}$emi$\mathbb{G}$rp which is simultaneously complete and cocomplete (since it is locally finitely presentable) which means that there is a small category $\mathbb{S}_{\mathbb{S}emi\mathbb{G}rp}$ such that the category $\mathbb{M}$od$(\mathbb{S}_{\mathbb{S}emi\mathbb{G}rp})$ consisting of functors from $\mathbb{S}_{\mathbb{S}emi\mathbb{G}rp}$ to $\mathbb{S}$et which preserve small limits and the natural transformations as morphisms is equivalent to the category $\mathbb{S}$emi$\mathbb{G}$rp. It is also interesting to note that semigroups are naturally algebras for a monad on the category $\mathbb{S}$et since the following forgetful functor U is monadic

$$\begin{array}{c} \mathbb{S}emi\mathbb{G}rp \\ {\scriptstyle U}\downarrow \\ \mathbb{S}et \end{array}$$

Let S be a semigroup. S is called *left* (respectively, *right*) *simple* if S has no proper left (respectively, right) ideal. An element $x \in S$ is called an *idempotent* if $xx = x^2 = x$. In this paper the set of all idempotents in S is denoted by $E(S)$. An element $e \in S$ is said to be a *right* (respectively, *left*) *identity* of S if $se = s$ (respectively, $es = s$) for all $s \in S$. Suppose $s \in S$. If the set $\{x \in \mathbb{N} \mid (\exists\ y \in \mathbb{N})\ s^x = s^y; x \neq y\}$ is nonempty then its least element exists by well ordering principle and is called the *index* of s. If m is the index of s, then the set $\{x \in \mathbb{N} \mid s^{m+x} = s^m\}$ is also nonempty and its least element denoted by t, is called the *period* of s. In this case, we say that s is a periodic element with index m and period t. Let A be a nonempty set and S_A be the set of all finite nonempty words $a_1a_2...a_m$ in the 'alphabet' A. Define a binary operation on S_A by juxtaposition of words: $(a_1a_2...a_m)(b_1b_2...b_n) := a_1a_2...a_mb_1b_2...b_n$. With respect to this operation, S_A becomes a semigroup and it is called the *free semigroup* on A. A is called the *generating set* for S_A. In terms of Category theory, F is called a free semigroup on A if

(i) there exists a map $\alpha : A \longrightarrow F$

(ii) (*universal property*) for every semigroup S and every map $\phi : A \longrightarrow S$, there exists a unique semigroup homomorphism $\psi : F \longrightarrow S$ making the following diagram (Fig-2.1) commutative, that is $\psi \circ \alpha = \phi$.

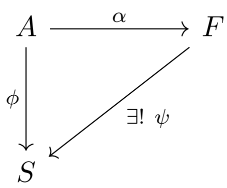


Fig-2.1

3 Action graphs of semigroup acts

We start by defining a left semigroup act which is going to play a crucial role in this work.

Definition 3.1. Let S be a semigroup and A be a nonempty set. S is said to *act on A from left* if there exists a structural map $a : S \times A \longrightarrow A$ such that

(i) $a(s, a(t, x)) = a(st, x)$ for all $s, t \in S$ and $x \in A$;

(ii) if S contains identity 1 then $a(1, x) = x$ for all $x \in A$.

We denote this by (S, a, A) and call it a *left semigroup act.*

Throughout the paper 'semigroup act' unless otherwise mentioned denotes a 'left semigroup act'. When the structural map is clear from the context (or there is no other action mentioned) we simply denote the image of $(s, x) \in S \times A$ by sx instead of $a(s, x)$. Now we describe morphisms between semigroup acts.

Definition 3.2. Let $(S, a, A), (T, b, B)$ be left semigroup acts. Suppose $\mu : S \longrightarrow T$ is a semigroup homomorphism and $f : A \longrightarrow B$ is a set mapping, then a *morphism of left semigroup* acts is defined by the pair of maps $(\mu, f) : (S, a, A) \longrightarrow (T, b, B)$ such that the following diagram commutes in the category $\mathbb{S}\text{et}$ of sets.

$$\begin{array}{ccc} S \times A & \xrightarrow{\mu\times f} & T \times B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

Lemma 3.3. *Left semigroup acts together with morphisms of acts form a category.*

Proof. Suppose $(S,a,A),(T,b,B),(U,c,C)$ are semigroup acts and $(\mu,f):(S,a,A)\longrightarrow(T,b,B)$, $(\eta,g):(T,b,B)\longrightarrow(U,c,C)$ are morphisms of acts. We define $(\eta,g)\circ(\mu,f):=(\eta\circ\mu,g\circ f)$. Clearly $\eta\circ\mu:S\longrightarrow U$ is a semigroup homomorphism and $g\circ f:A\longrightarrow B$ is a mapping such that $(g\circ f)(sx)=g(f(sx))=g(\mu(s)f(x))=\eta(\mu(s))g(f(x))=(\eta\circ\mu)(s)(g\circ f)(x)$ for all $s\in S$ and for all $x\in A$. Thus $(\eta,g)\circ(\mu,f)$ is a morphism of acts from (S,a,A) to (U,c,C). Clearly (I_S,I_A) plays the role of identity morphism on (S,a,A), that is $I_{(S,a,A)}=(I_S,I_A)$. To check that the law of composition of morphisms is associative, consider $(\mu,f):(S,a,A)\longrightarrow(T,b,B)$, $(\eta,g):(T,b,B)\longrightarrow(U,c,C)$ and $(\gamma,h):(U,c,C)\longrightarrow(V,d,D)$. Then $(\gamma,h)\circ((\eta,g)\circ(\mu,f))=(\gamma,h)\circ(\eta\circ\mu,g\circ f)=(\gamma\circ(\eta\circ\mu),h\circ(g\circ f))=((\gamma\circ\eta)\circ\mu,(h\circ g)\circ f)=(\gamma\circ\eta,h\circ g)\circ(\mu,f)=((\gamma,h)\circ(\eta,g))\circ(\mu,f)$. Hence the lemma follows. $\square$

We denote the category of left semigroup acts by $\mathbb{S}\text{gr}\mathbb{A}\text{ct}_l$. The following remark is in order.

Remark 3.4. If we fix a semigroup S then all left semigroup acts (S,a,A) (cf. Definition 3.1) form a category whose morphisms are given by the following commutative diagram in $\mathbb{S}\text{et}$.

$$\begin{array}{ccc} S \times A & \xrightarrow{id_S\times f} & S \times A' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

Drawing analogy with R-$\mathbb{M}$od, the category of all left R-modules for a given ring R, we can denote this category by S-$\mathbb{M}$od. Clearly $\mathbb{S}$gr$\mathbb{A}$ct$_l$ and S-$\mathbb{M}$od are not the same.

Now we are going to define a particular type of digraphs.

Definition 3.5. The *(uncolored) C-induced action (di)graph* corresponding to a semigroup act (S, a, A) is a simple digraph

$$G(S, a, C; A) = (V(G(S, a, C; A)), E(G(S, a, C; A)))$$

such that the set of vertices $V(G(S, a, C; A)) := A$ and the set of arcs $E(G(S, a, C; A)) := \{(x, y) \in A \times A \ : \exists\ s \in C,\ y = a(s, x) = sx\}$.

In what follows, the term 'action graph' unless otherwise stated represents a digraph. The term 'uncolored' signifies that for more than one $s \in C$ with $sx = y$, the choice of s is immaterial for assigning the arc (x, y). Later we will explore the case when there is only one $s \in C$ for which $sx = y$; accordingly we can color the arcs and obtain the colored C-induced action graph.

Remark 3.6. By definition $G(S, a, C; A)$ is a simple digraph. It may have loops at some vertices. In fact loops in the C-induced action graph of a semigroup act (S, a, A) correspond to the fixed points ($cf.$ Definition 4.5) of the elements of C and vice-versa.

Interestingly some special digraphs come as particular C-induced action graphs.

Example 3.7. Consider the semigroup act $(X, *, X)$ where $(X, *)$ is a finite group and the action is given by the multiplication $*$ of X. Then for any nonempty subset C of X, the C-induced action digraph $G(X, *, C; X)$ is the well known Cayley digraph $Cay(X, C)$ for the connection set C. Undirected Cayley graph (also known as König graph) (see [4], [22]) can be obtained just by imposing two extra conditions on the connection set C namely (i) $e_X \notin C$ where e_X is the identity of the group X and (ii) whenever $x \in C$ it implies $x^{-1} \in C$.

Example 3.8. Let $(S, *)$ be any semigroup and $C \subseteq S$. Then S acts on itself via its multiplication $*$ and $G(S, *, C; S) = Cay(S, C)$, the Cayley graph of S relative to C given by left action of the elements of C (see [18]).

In this paper we mainly focus on the particular C-induced action graph of a semigroup act (S, a, A) namely when $C = S$, call it the *action graph of a semigroup act* (S, a, A) and denote it by $G(S, a, A)$, that is $G(S, a, S; A) = G(S, a, A)$. Note that for any $C \subseteq S$, $G(S, a, C; A)$ is a spanning sub-digraph of $G(S, a, A)$. Before giving concrete examples, we here capture some fundamental properties of action graph of a semigroup act. In order to do this, we first need to define the following.

Definition 3.9. Let $D = (V(D), h, E(D))$ be a simple, uncolored digraph. D is called *strongly transitive* if for any three vertices a, b, c (may not be distinct), $(a, b), (b, c) \in E(D)$ implies that $(a, c) \in E(D)$.

Remark 3.10. For an arbitrary digraph (not necessarily simple and uncolored) the above definition can be reformulated in the following way: A digraph is called a 1-*magma* if

$$x \xrightarrow{g} y \xrightarrow{f} z$$

are two adjacent arcs, then the following arc

$$x \xrightarrow{f \circ g} z$$

exists. If the operation $f \circ g$ is associative then the 1-magma is called a *semicategory*. Hence we can say that a digraph $D = (V(D), h, E(D))$ is strongly transitive if and only if it is equipped with a structure of a semicategory.

Definition 3.11. A semigroup act (S, a, A) is said to be *transitive* if for any two elements $x, y \in A$, there exists $s \in S$ such that $a(s, x) = sx = y$. In other words the action of S on A is said to be transitive if for any $x \in A$, the map $a_x : S \longrightarrow A$ defined by $a_x(s) := a(s, x)$ is an epic (surjective map) in the category $\mathbb{S}$et of sets.

Proposition 3.12. *Let (S, a, A) be a semigroup act where S is a semigroup and A is a nonempty set. Then the following properties follow for the digraph $G(S, a, A)$.*

(1) *$G(S, a, A)$ is a strongly transitive digraph and each of its strong components is a complete sub-digraph.*

(2) *The following are equivalent for the action graph $G(S, a, A)$ where S is a semigroup with* 1*.*

(i) $G(S, a, A)$ *is a complete digraph with loops.*

(ii) $G(S, a, A)$ *is strongly connected.*

(iii) (S, a, A) *is a transitive act.*

(3) *Every vertex of $G(S, a, A)$ has positive out-degree. If A is finite then $G(S, a, A)$ must have a loop.*

(4) *If S is left simple then for all $x \in A$, $N^{+}_{G(S,a,A)}(x)$ forms a clique in $G(S, a, A)$.*

Proof. (1) The first part follows from the relevant definitions. To prove the second part, take any two distinct vertices x, y from any strong component of $G(S, a, A)$. Then there exists a directed path $v_0 = x \to v_1 \to v_2 \to ... \to v_n = y$ in $G(S, a, A)$. Since $G(S, a, A)$ is strongly transitive this implies $(x, y) \in E(G(S, a, A))$. So any strong component is a complete sub-digraph.

(2) (i) $\Rightarrow$ (ii) This is trivial.

(ii) $\Rightarrow$ (iii) Let x, y be any two distinct elements in A. There exists a directed path $v_0 = x \to v_1 \to v_2 \to ... \to v_n = y$ in $G(S, a, A)$, that is there exist $s_i \in S$; $i = 1, 2, ..., n$ such that $v_i = s_i v_{i-1}$. Then $y = v_n = s_n v_{n-1} = (s_n s_{n-1}) v_{n-2} = ... = (s_n s_{n-1} ... s_1)x$. Again for any $x \in A$, $x = 1x$. So (S, a, A) is a transitive act.

(iii) $\Rightarrow$ (i) This holds from the definition of transitive act.

(3) For any $s \in S$ and $x \in A$, $(x, sx) \in E(G(S, a, A))$. So $outdegree(x) \geqslant 1$ for all $x \in A$. From the finiteness of A, it is evident that $G(S, a, A)$ contains a directed cycle of length $\geqslant 1$ e.g., $x \to sx \to ... \to s^{k-1}x \to s^k x = x$ where $k \geqslant 1$. Then $(x, s^k x)$ is a loop at x.

(4) Let $x \in A$. Take $y, z \in N^{+}_{G(S,a,A)}(x)$. Then there exist $s, t \in S$ such that $y = sx$ and $z = tx$. Now by left simplicity of S, there exists $u \in S$ such that $us = t$ and so $uy = u(sx) = (us)x = tx = z$, that is $(y, z) \in E(G(S, a, A))$. Hence $N^{+}_{G(S,a,A)}(x)$ forms a clique. □

Proposition 3.13. *If $(\mu, f) : (S, a, A) \longrightarrow (T, b, B)$ is a morphism of acts then $G(\mu, f) : V(G(S, a, A)) \longrightarrow V(G(T, b, B))$ defined by $G(\mu, f)(x) := f(x)$ for all $x \in V(G(S, a, A)) = A$ is a digraph homomorphism from $G(S, a, A)$ to $G(T, b, B)$.*

Proof. Let $(x, y) \in E(G(S, a, A))$. Then $y = sx$ for some $s \in S$. Therefore $f(y) = f(sx) \Rightarrow f(y) = \mu(s)f(x) \Rightarrow G(\mu, f)(y) = \mu(s)G(\mu, f)(x) \Rightarrow (G(\mu, f)(x), G(\mu, f)(y)) \in E(G(T, B))$. Hence $G(\mu, f)$ is a digraph homomorphism. $\square$

Let us denote by $\mathbb{S}$imp-$\mathbb{TDG}$raph, the full subcategory of $\mathbb{DG}$raph consisting of strongly transitive simple digraphs in which every vertex has positive out-degree. Interestingly we can represent action graph construction as a functor from $\mathbb{S}$gr$\mathbb{A}$ct$_l$ to $\mathbb{S}$imp-$\mathbb{TDG}$raph. Such a functorial connection can also be found in [23].

Proposition 3.14. *Let $(S, a, A), (T, b, B)$ be any two semigroup acts and $(\mu, f) : (S, a, A) \longrightarrow (T, b, B)$ be any morphism of acts. Then*

$$\mathcal{F} : \ \mathbb{S}gr\mathbb{A}ct_l \longrightarrow \mathbb{S}imp - \mathbb{TDG}raph$$

defined by $\mathcal{F}(S, a, A) := G(S, a, A)$, $\mathcal{F}(T, b, B) := G(T, b, B)$, and $\mathcal{F}(\mu, f) := G(\mu, f)$, is a functor.

$$\begin{array}{ccc} \mathbb{S}\text{gr}\mathbb{A}\text{ct}_l & \xrightarrow{\mathcal{F}} & \mathbb{S}\text{imp-}\mathbb{TD}\text{graph} \\ & & \\ (S, a, A) & \longmapsto & G(S, a, A) \\ \Big\downarrow{\scriptstyle (\mu, f)} & & \Big\downarrow{\scriptstyle G(\mu, f)} \\ (T, b, B) & \longmapsto & G(T, b, B) \end{array}$$

Proof. It is evident from (1) and (3) of Proposition 3.12 that $G(S, a, A) \in Obj(\mathbb{S}\text{imp-}\mathbb{TDG}\text{raph})$ for all semigroup act (S, a, A).

(i) By Proposition 3.13, it follows that $\mathcal{F}(\mu, f)$ is a digraph homomorphism of $\mathcal{F}(S, a, A)$ to $\mathcal{F}(T, b, B)$.

(ii) $\mathcal{F}(I_{(S,a,A)}) = \mathcal{F}(I_S, I_A) = G(I_S, I_A)$. Now for any $x \in V(G(S, a, A)) = A$, $G(I_S, I_A)(x) = I_A(x) = x = I_{G(S,a,A)}(x)$ and so $\mathcal{F}(I_{(S,a,A)}) = I_{\mathcal{F}(S,a,A)}$.

(iii) Let $(\mu, f) : (S, a, A) \longrightarrow (T, b, B)$ and $(\eta, g) : (T, b, B) \longrightarrow (U, c, C)$ be morphisms of acts. Then we can easily show that $\mathcal{F}((\eta, g) \circ (\mu, f)) = \mathcal{F}(\eta \circ \mu, g \circ f) = G(\eta \circ \mu, g \circ f) = G(\eta, g) \circ G(\mu, f) = \mathcal{F}(\eta, g) \circ F(\mu, f)$.

Hence $\mathcal{F}$ is a covariant functor. □

The above functorial construction leads to the natural questions:

(i) Is $\mathcal{F}$ faithful?

(ii) Is $\mathcal{F}$ full?

(iii) Can we characterize the image of $\mathcal{F}$?

In the subsequent sections we will try to answer these questions.

Example 3.15. Suppose S is any semigroup and $A = \{a, b\}$. There is a trivial action of S on A defined by $sa, sb := a$ for all $s \in S$. Corresponding action graph is shown below (Fig-3.1).

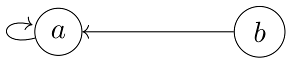


Fig-3.1

Example 3.16. Let $S = \mathbb{N}$, the semigroup of all natural numbers under usual multiplication and A be the cyclic group $\mathbb{Z}_6$. Corresponding to the action ν of S on A, defined by $\nu(n, a) = na$ for all $n \in S$ and for all $a \in A$, the action graph is shown in Fig-3.2.

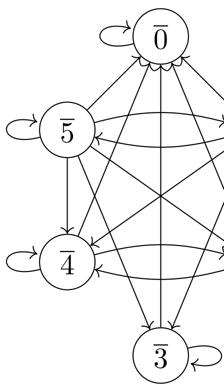


Fig-3.2

Observation 3.17. Suppose $(S, a, A), (T, b, B)$ are semigroup acts. Then $(S, \overline{a}, A \times B), (T, \overline{b}, A \times B)$ and $(S \times T, a \times b, A \times B)$ are all semigroup acts defined respectively as follows: $\overline{a}(s, (x, y)) := (sx, y)$, $\overline{b}(t, (x, y)) := (x, ty)$ and $(a \times b)((s, t), (x, y)) := (sx, ty)$ for all $s \in S$, $t \in T$ and for all $(x, y) \in A \times B$.

The definition of the action of $S \times T$ on $A \times B$ hints towards the monoidality of the category $\mathbb{S}\text{gr}\mathbb{A}\text{ct}_l$. In Section 6 we concentrate on this (*cf.* Lemma 6.5). For our next proposition we recall definitions of some binary graph operations from [22, Chapter 4].

Definition 3.18. Let $D_1 = (V(D_1), h_1, E(D_1))$, $D_2 = (V(D_2), h_2, E(D_2))$ be two simple digraphs.

(i) If $V(D_1) = V(D_2) = V$, then the *edge sum* is denoted by $D_1 \oplus D_2$ whose set of vertices is V and the set of arcs is $E(D_1) \cup E(D_2)$.

(ii) The *cross product* is denoted by $D_1 \times D_2$ whose set of vertices is $V(D_1) \times V(D_2)$ and $((x, y), (x', y')) \in E(D_1 \times D_2)$ if $(x, x') \in E(D_1)$ and $(y, y') \in E(D_2)$.

(iii) The *box product* is denoted by $D_1 \square D_2$ whose set of vertices is $V(D_1) \times V(D_2)$ and $((x, y), (x', y')) \in E(D_1 \square D_2)$ if either $x = x'$ and $(y, y') \in E(D_2)$ or $y = y'$ and $(x, x') \in E(D_1)$.

(iv) The *boxcross product* is denoted by $D_1 \boxtimes D_2$ and defined to be $D_1 \boxtimes D_2 := (D_1 \times D_2) \oplus (D_1 \square D_2)$.

The next proposition can be easily proved by using Observation 3.17 and Definition 3.18.

Proposition 3.19. *Let $(S, a, A), (T, b, B)$ be two semigroup acts. Then*

(i) $G(S, a, A) \times G(T, b, B) = G(S \times T, a \times b, A \times B)$

(ii) $G(S, a, A) \square G(T, b, B) = G(S, \overline{a}, A \times B) \oplus G(T, \overline{b}, A \times B)$

(ii) $G(S, a, A) \boxtimes G(T, b, B) = G(S, \overline{a}, A \times B) \oplus G(T, \overline{b}, A \times B) \oplus G(S \times T, a \times b, A \times B)$

Remark 3.20. The categorical interpretation of Proposition 3.19 (i) is given in Section 6 (*cf.* Theorem 6.7) via the functor $\mathcal{F}$ obtained in Proposition 3.14.

Proposition 3.12 (2) characterizes strongly connectedness of $G(S, a, A)$ in terms of transitivity of the action of S on A. Now we are keen to know about the weakly connectedness of $G(S, a, A)$, that is the connectedness of the underlying graph of $G(S, a, A)$.

Definition 3.21. Let (S, a, A) be a semigroup act. Define an undirected graph $\Gamma(S, a, A)$ whose set of vertices is A and two vertices x, y are adjacent if $y = sx$ or $x = ty$ for some $s, t \in S$. We call $\Gamma(S, a, A)$, the *undirected action graph* of (S, a, A).

Remark 3.22. Actually $\Gamma(S, a, A)$ is obtained from $G(S, a, A)$ by replacing each arc by an undirected edge and removing any resulting multiple edges (keeping one copy). In this way $\Gamma(S, a, A)$ is the underlying graph of $G(S, a, A)$.

Theorem 3.23. *Let (S, a, A) be a semigroup act where S is a commutative semigroup. Then the equivalence classes under the equivalence relation ϱ, defined on A by $x\varrho y$ if and only if there exist $s, t \in S$ such that $sx = ty$, are precisely the connected components of* $\Gamma(S, a, A)$.

Proof. It will be sufficient if we can prove that for $x, y \in A$, $x\varrho y$ if and only if x, y are connected by a path in $\Gamma(S, a, A)$. If $x\varrho y$ then $sx = ty$ for some $s, t \in S$. Let $sx = ty = z$. Then x–z–y is a path connecting x and y. Conversely assume that $x(= z_0)$–z_1–z_2–...–z_{k-1}–$y(= z_k)$ is a path in $\Gamma(S, a, A)$ connecting x and y. Then $z_{i-1}\varrho z_i$ for all $i = 1, 2, ..., k$ and since ϱ is transitive this implies $z_0\varrho z_k$, that is $x\varrho y$. □

Remark 3.24. A part of Theorem 2.3 for undirected power graphs of semigroups in [5], follows from the above theorem when $S = \mathbb{N}$, the semigroup of natural numbers under usual multiplication, A is any finite semigroup and the action is defined by $na := a^n$ for all $n \in \mathbb{N}$ and for all $a \in A$.

Theorem 3.23 reflects the following.

Corollary 3.25. *For a commutative semigroup S and a semigroup act (S, a, A), $G(S, a, A)$ is weakly connected if and only if any two distinct vertices have a common out-neighbour.*

The following example makes it clear that if we remove the commutativity of S in Theorem 3.23, the conclusion may not follow.

Example 3.26. Let $S = \{s, t\}$ be a two element right zero semigroup and $A = \{a, b, c, x, y\}$. Define an action of S on A by $sa := b$, $sb := b$, $sc := c$, $sx := b$, $sy := c$ and $ta := c$, $tb := b$ and $tc := c$, $tx := b$, $tx := c$. It is a matter of routine verification to show that it is a semigroup act. The corresponding action graph is shown below (Fig-3.3). It can be easily observed that x, y are not ϱ related although they are in the same connected component of $\Gamma(S, a, A)$.

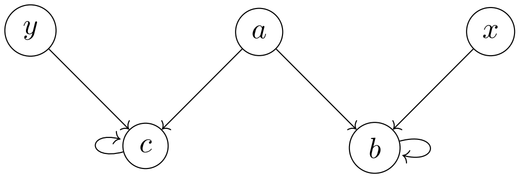


Fig-3.3

4 Strongly faithful acts and action graphs

A group action (G, a, A) is called free (or fixed point free) if for $g \in G$ and $x \in A$, $gx = x$ implies that $g = e_G$ where e_G is the identity of the group G. Equivalently we can say that the action is free if $gx = hx$ for $g, h \in G$ and $x \in A$ implies that $g = h$. The semigroup analogue of this is given in the following definition.

Definition 4.1. [21] Let (S, a, A) be a semigroup act. We say that the action of S on A is *strongly faithful* if for $s, t \in S$ and $x \in A$, $sx = tx$ implies that $s = t$ or in other words if $s \neq t$ then $sx \neq tx$ for all $x \in A$. Categorically the action is strongly faithful if for all $x \in A$, the map $a_x : S \longrightarrow A$ defined by $a_x(s) := a(s, x) = sx$ is monic (injective) in the category $\mathbb{S}\text{et}$.

In case of strongly faithful semigroup act, we can color the arcs of the corresponding action graph. This is described in the following definition.

Definition 4.2. Suppose (S, a, A) is a strongly faithful act. Then the color of an arc $(x, y) \in E(G(S, a, A))$ is denoted by $col(x, y)$ and it is defined to be the unique $s \in S$ such that $sx = y$.

Example 4.3. Let R be an integral domain. Let S be any multiplicatively closed subset of R not containing 0 and $A = I^* = I \setminus \{0\}$ for any ideal I of R. Then S is a semigroup which acts on A form left by the multiplication of R and we get a strongly faithful act.

Example 4.4. Any fixed point free group action can be seen as a strongly faithful semigroup action by considering the group as a semigroup.

Definition 4.5. Let (S, a, A) be a left semigroup act. An element $x \in A$ is called a *fixed point* of $s \in S$ if $sx = x$.

We know that an action of a group G on a set is free if and only if the identity of G is the only element having a fixed point. Similarly strongly faithful (left) action of a semigroup has the following connection with right identities of the semigroup.

Lemma 4.6. *Let (S, a, A) be a strongly faithful semigroup act. Then $s \in S$ has a fixed point if and only if s is a right identity of S.*

Proof. Suppose $x \in A$ is a fixed point of s. Therefore for all $t \in S$, $tsx = tx$ which implies $ts = t$ since (S, a, A) is a strongly faithful act. Hence s is a right identity of S. Conversely if s is a right identity of S then for any $x \in A$, $s(sx) = (ss)x = sx$ which shows that sx is a fixed point of s. □

Lemma 4.6 together with Proposition 3.12 (3) and Remark 3.6 gives the following.

Proposition 4.7. *If S is a semigroup which has no right identity then S can not act strongly faithfully on a finite nonempty set A.*

Remark 4.8. The above result appears to be a digression, but it is actually a consequence of Proposition 3.12 (3), which is a graph theoretic result of the present study.

Corollary 4.9. *A right zero semigroup can not act strongly faithfully on a finite set unless it is a one element trivial semigroup.*

In the following result, strongly faithful (finite) semigroup action is characterized by a graph theoretic property of the corresponding action graph.

Proposition 4.10. *Let (S, a, A) be a semigroup act where S is a finite semigroup. Then (S, a, A) is a strongly faithful act if and only if $G(S, a, A)$ is a $|S|$-out regular digraph.*

Proof. Choose any vertex x in $G(S, a, A)$. Define $\varphi_x : S \longrightarrow N^+_{G(S,a,A)}(x)$ by $\varphi_x(s) := sx$ for all $s \in S$. Since (S, a, A) is a strongly faithful act, so φ_x is injective. By definition φ_x is surjective. Therefore φ_x is a bijection and $|N^+_{G(S,a,A)}(x)| = |S|$. Since this holds for all $x \in V(G(S, a, A))$, so $G(S, a, A)$ is $|S|$-out regular digraph. Conversely assume that $G(S, a, A)$ is a $|S|$-out regular digraph. The mapping φ_x defined above, is always surjective and since $|S| = |N^+_{G(S,a,A)}(x)|$ for all $x \in A$, so φ_x is also injective. Therefore (S, a, A) is a strongly faithful act. □

The following is a passing remark.

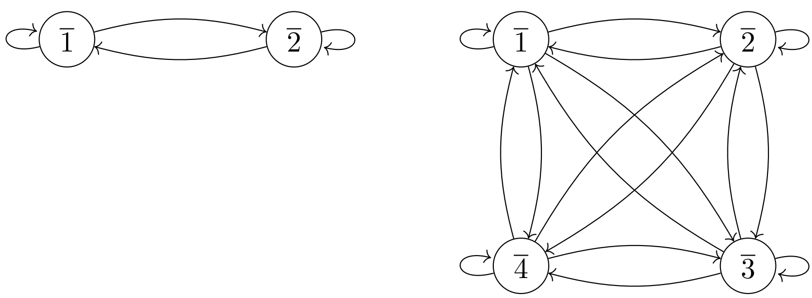
Remark 4.11. In case of strongly faithful action, for any $x \in A$, $N^+_{G(S,a,A)}(x)$ enjoys a semigroup structure induced by the bijection φ_x, defined in the proof of Proposition 4.10. A multiplication is defined on $N^+_{G(S,a,A)}(x)$ by $(sx)(tx) := (st)x$ which can be easily seen to be associative. Consequently φ_x becomes a semigroup isomorphism and so for strongly faithful semigroup act, $N^+_{G(S,a,A)}(x)$ is a semigroup isomorphic to S for all $x \in A$.

Suppose (S, a, A) and (T, b, B) are two strongly faithful semigroup acts. In Proposition 3.13, we have seen that a morphism of actions induces a digraph homomorphism. Now we are in a position to provide a counter example to illustrate that the converse of Proposition 3.13 is not true.

Example 4.12. We consider the strongly faithful semigroup acts

$$(\mathbb{Z}_3^*, \times_3, \mathbb{Z}_3^*), (\mathbb{Z}_5^*, \times_5, \mathbb{Z}_5^*)$$

with the actions being usual multiplications modulo 3 and modulo 5 of $\mathbb{Z}_3$ and $\mathbb{Z}_5$ respectively. The corresponding action graphs are shown below (Fig-4.1).



$G(\mathbb{Z}_3^*, \times_3, \mathbb{Z}_3^*)$ $G(\mathbb{Z}_5^*, \times_5, \mathbb{Z}_5^*)$

Fig-4.1

We do not mention the color of the arcs in the figures as they will make the figures clumsy. Note that $col(\overline{1}, \overline{2}) = col(\overline{2}, \overline{1}) = \overline{2}$ in $G(\mathbb{Z}_3^*, \times_3, \mathbb{Z}_3^*)$ and $col(\overline{1}, \overline{2}) = \overline{2}$, $col(\overline{2}, \overline{1}) = \overline{3}$ in $G(\mathbb{Z}_5^*, \times_5, \mathbb{Z}_5^*)$. Define a map

$$f : V(G(\mathbb{Z}_3^*, \times_3, \mathbb{Z}_3^*)) \longrightarrow V(G(\mathbb{Z}_5^*, \times_5, \mathbb{Z}_5^*))$$

by $f(\overline{1}) := \overline{1}$ and $f(\overline{2}) := \overline{2}$. Clearly f is a digraph homomorphism. If possible suppose $\mu : \mathbb{Z}_3^* \longrightarrow \mathbb{Z}_5^*$ is a semigroup homomorphism and $g : \mathbb{Z}_3^* \longrightarrow \mathbb{Z}_5^*$ is a mapping such that (μ, g) is a morphism of acts and $G(\mu, g) = f$. Then by definition $g = f$. Now observe that, in $(\mathbb{Z}_3^*, \times_3, \mathbb{Z}_3^*)$, $\overline{2} = \overline{2} \times_3 \overline{1}$ which implies that $f(\overline{2}) = g(\overline{2}) = \mu(\overline{2}) \times_5 g(\overline{1}) = \mu(\overline{2}) \times_5 f(\overline{1})$ and $\overline{1} = \overline{2} \times_3 \overline{2}$ which implies that $f(\overline{1}) = \mu(\overline{2}) \times_5 f(\overline{2})$. Therefore in $G(\mathbb{Z}_5^*, \times_5, \mathbb{Z}_5^*)$ it should happen that $col(f(\overline{1}), f(\overline{2})) = col(f(\overline{2}), f(\overline{1})) = \mu(\overline{2})$, but this is not the case as $col(f(\overline{1}), f(\overline{2})) = col(\overline{1}, \overline{2}) = \overline{2}$ and $col(f(\overline{2}), f(\overline{1})) = col(\overline{2}, \overline{1}) = \overline{3}$ in $G(\mathbb{Z}_5^*, \times_5, \mathbb{Z}_5^*)$. Thus f can't be induced by any morphism of acts.

In order to obtain a restricted converse of Proposition 3.13, we introduce the following notion.

Definition 4.13. Let $(S, a, A), (T, b, B)$ be strongly faithful semigroup acts. Let $f : G(S, a, A) \longrightarrow G(T, b, B)$ be a digraph homomorphism. f is called *color sensitive* if for any two arcs $(x_1, y_1), (x_2, y_2)$ in $E(G(S, A))$, $col(x_1, y_1) = col(x_2, y_2)$ implies that $col(f(x_1), f(y_1)) = col(f(x_2), f(y_2))$

First we observe that in case of strongly faithful semigroup action, morphisms of acts induce color sensitive homomorphisms.

Proposition 4.14. *Let $(\mu, f) : (S, a, A) \longrightarrow (T, b, B)$ be a morphism of acts where (S, a, A), (T, b, B) are strongly faithful semigroup acts. Then $G(\mu, f) : G(S, a, A) \longrightarrow G(T, b, B)$ is a color sensitive homomorphism.*

Proof. From Proposition 3.13, it is clear that $G(\mu, f)$ is a digraph homomorphism. To show that it is color sensitive, assume $col(x_1, y_1) = col(x_2, y_2) = s$. Then $G(\mu, f)(y_1) = f(y_1) = \mu(s)f(x_1) = \mu(s)G(\mu, f)(x_1)$ and similarly $G(\mu, f)(y_2) = \mu(s)G(\mu, f)(x_2)$. So $col(G(\mu, f)(x_1), G(\mu, f)(y_1)) = col(G(\mu, f)(x_2), G(\mu, f)(y_2)) = \mu(s)$. Thus $G(\mu, f)$ is a color sensitive homomorphism. □

The restricted converse of Proposition 3.13 is now presented below.

Theorem 4.15. *Let $(S, a, A), (T, b, B)$ be strongly faithful semigroup acts and $f : G(S, a, A) \longrightarrow G(T, b, B)$ be a color sensitive homomorphism. Then there exists a semigroup homomorphism $\mu : S \longrightarrow T$ such that (μ, f) is a morphism of acts and $G(\mu, f) = f$.*

Proof. At first, we define a mapping

$$Col_f : S \longrightarrow T,\ Col_f(s) := col(f(x), f(sx))$$

(cf. Definition 4.2) for all $s \in S$ and for any $x \in A$, that is $Col_f(s)f(x) = f(sx)$ for all $x \in A$. Since f is color sensitive (cf. Definition 4.13), the choice of x does not affect the definition of Col_f. Let $s, t \in S$. Take any $x \in A$. Then $(x, tx), (tx, s(tx)) \in E(G(S, A))$. So $(f(x), f(tx)), (f(tx), f(s(tx))) \in E(G(T, B))$ and $f((st)x) = f(s(tx)) = Col_f(s)f(tx) = Col_f(s)Col_f(t)f(x)$. Again, $f((st)x) = Col_f(st)f(x)$. Hence using the fact that (T, b, B) is a strongly faithful act, we obtain $Col_f(st) = Col_f(s)Col_f(t)$. So Col_f is a semigroup homomorphism. Since $f(sx) = Col_f(s)f(x)$, so (Col_f, f) is a morphism of acts and clearly $G(Col_f, f) = f$. □

In view of Proposition 4.14 and Theorem 4.15 we obtain the following.

Theorem 4.16. *Let $(S, a, A), (T, b, B)$ be strongly faithful semigroup acts and $f : G(S, a, A) \longrightarrow G(T, b, B)$ be a digraph homomorphism. f can be induced by a morphism of acts from (S, a, A) to (T, b, B) if and only if f is color sensitive.*

Remark 4.17. Let $\mathbb{SF}$-$\mathbb{S}$gr$\mathbb{A}$ct$_l$ be the full subcategory of $\mathbb{S}$gr$\mathbb{A}$ct$_l$ consisting of all strongly faithful left semigroup acts. Then the functor $\mathcal{F}$ obtained in Proposition 3.14 restricts to a functor:

$$\mathbb{SF}\text{-}\mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l \xrightarrow{\mathcal{F}} \mathbb{S}\mathrm{imp}\text{-}\mathbb{TD}\mathbb{G}\mathrm{raph}$$

Proposition 4.14 tells us that the image of this restricted functor is not a full subcategory of $\mathbb{S}$imp-$\mathbb{TD}\mathbb{G}$raph as the restriction of $\mathcal{F}$ is not in general full (see Example 4.12). In Section 6, we focus on this restricted functor in more details.

In Proposition 3.12 (1), we have observed that the digraph $G(S, a, A)$ is always strongly transitive. But the C-induced action graph $G(S, a, C; A)$ is not necessarily strongly transitive (for example consider the Cayley graph $Cay(\mathbb{Z}_5, \{\bar{1}\}) = G(\mathbb{Z}_5, +_5, \{\bar{1}\}; \mathbb{Z}_5)$ (cf. Example 7.3.3, Page 146 of [22])). In case of strongly faithful act, we precisely answer when $G(S, a, C; A)$ is strongly transitive.

Proposition 4.18. *Let (S, a, A) be a strongly faithful semigroup act where S is a semigroup and C is a nonempty subset of S. Then $G(S, a, C; A)$ is a strongly transitive digraph if and only if C is a subsemigroup of S.*

Proof. Let $s, t \in C$ and $x \in A$. Then $(x, sx), (sx, t(sx)) \in E(G(S, a, C; A))$. Since $G(S, a, C; A)$ is strongly transitive, $(x, t(sx)) \in E(G(S, a, C; A))$, that is, there exists $c \in C$ such that $t(sx) = cx$ which implies that $ts = c$ since (S, a, A) is a strongly faithful semigroup act. Hence C is a subsemigroup of S. Converse follows verbatim from the proof of Proposition 3.12 (1) and using the fact that C is closed under multiplication. □

Suppose (S, a, A) is a semigroup act. For any $s \in S$, let us denote the $\{s\}$-induced action graph of (S, a, A) by $H_s(A)$ and call it the s-colored action sub-digraph of $G(S, a, A)$. The following is an easy observation.

Observation 4.19. Let (S, a, A) be a semigroup act and $C(\neq \emptyset) \subseteq S$. Then

$$G(S, a, C; A) = \bigoplus_{c \in C} H_c(A)$$

where $\bigoplus$ denotes the edge sum of the digraphs as defined in Definition 3.18.

Definition 4.20. Let G be a digraph (with or without loops). The *loop free copy* of G is a graph G^*, obtained from G just by removing (or ignoring) all the loops of G (if exist) and if G has no loop then $G^* = G$.

It is well known that idempotents play an important role in Semigroup theory. The following graph theoretic result gives a characterization of idempotents in a semigroup S, when it is acting strongly faithfully on a set A.

Proposition 4.21. *Let (S, a, A) be a strongly faithful semigroup act. Let $s \in S$ be an element such that $sx_0 \neq x_0$ for some $x_0 \in A$. Then s is an idempotent in S if and only if the loop free copy of $H_s(A)$, that is $H_s(A)^*$ is bipartite with bipartition $\{H_1, H_2\}$ where $H_1 = \{x \in A \mid (x, y) \in E(H_s(A)^*)$ for some $y \in A\} \cup \{x \in A \mid x$ is an isolated vertex in $H_s(A)^*\}$ and $H_2 = \{y \in A \mid (x, y) \in E(H_s(A)^*)$ for some $x \in A\}$.*

Proof. Suppose $s \in E(S)$. Clearly $V(H_s(A)^*) = H_1 \cup H_2$. Since there exists $x_0 \in A$ such that $sx_0 \neq x_0$ so $(x_0, sx_0) \in E(H_s(A)^*)$. Then $x_0 \in H_1$, $sx_0 \in H_2$ which shows that both H_1, H_2 are nonempty. Also observe that $H_1 \cap H_2 = \emptyset$ as if $x \in H_1 \cap H_2$ then $(x, y), (z, x) \in E(H_s(A)^*)$ for some $y, z \in A$. But this implies $y = sx = s^2z = sz = x$ which is not possible as $H_s(A)^*$ is loop free. Let $x, y \in H_1$ such that $(x, y) \in E(H_s(A)^*)$ then $y \in H_2$

which implies that $y \in H_1 \cap H_2$, a contradiction. So H_1 is an independent set. Similarly H_2 is also independent. Thus $\{H_1, H_2\}$ is a bipartition of $H_s(A)^*$.

Conversely suppose $H_s(A)^*$ is bipartite with bipartition $\{H_1, H_2\}$ where H_1, H_2 are as mentioned in the statement. Let $x \in A$ be a non-isolated vertex in H_1. Then $(x, sx) \in E(H_s(A)^*)$ and if $sx \neq s^2x$ then $(sx, s^2x) \in E(H_s(A)^*)$. Hence $sx \in H_1 \cap H_2$ which contradicts that $\{H_1, H_2\}$ is a bipartition. So it follows that $sx = s^2x$ which implies that $s = s^2$ as (S, a, A) is a strongly faithful act. □

The case where $sx = x$ for all $x \in A$, is explored in the following proposition. We omit the proof as the first part follows from Lemma 4.6 and the remaining part is self explanatory.

Proposition 4.22. *Suppose (S, a, A) is a strongly faithful semigroup act where $|A| \geqslant 2$ and s is an element of S which fixes every element of A. Then s is idempotent and the graph $H_s(A)^*$ is a null graph, hence bipartite.*

We conclude this section by finding an upper bound for the girth of the s-colored action sub-digraph $H_s(A)$ when s is a periodic element.

Theorem 4.23. *Let (S, a, A) be a semigroup act and $s \in S$ be a periodic element with index m and period t. Then*

(i) $gr(H_s(A)) \leqslant t$

(ii) $gr(H_s(A)) \mid t$

(iii) *the length of any directed path in $H_s(A)$ is $\leqslant m + t - 1$*

Proof. (i) For any $x \in A$, consider the set $P_x = \{n \in \mathbb{N} \mid s^{m+n}x = s^m x\}$. P_x is nonempty as $t \in P_x$. By well ordering principle, P_x has a least element. Let $min(P_x) = k_x$. Obviously $k_x \leqslant t$. We claim that $C:\ s^m x \to s^{m+1}x \to s^{m+2}x \to \ldots \to s^{m+k_x-1}x \to s^{m+k_x}x = s^m x$ is a directed cycle in $H_s(A)$. It suffices to show that there is no vertex repetition (other than end vertices) in C. Suppose $s^{m+i}x = s^{m+j}x$ where $0 \leqslant i < j \leqslant k_x$. Now $s^m x = s^{m+k_x}x = s^{m+j+k_x-j}x = s^{k_x-j}(s^{m+j}x) = s^{k_x-j}(s^{m+i}x) = s^{m+k_x-(j-i)}x = s^{m+l}x$ where $l = k_x - (j - i) < k_x$, which contradicts the minimality of k_x if $l > 0$. Hence $l = 0$ which is possible only if $i = 0$ and $j = k_x$. So C is a directed cycle of length k_x. Therefore $gr(H_s(A)) \leqslant k_x \leqslant t$.

(ii) Let $gr(H_s(A)) = n$ and $x \to sx \to s^2x \to ... \to s^{n-1}x \to s^nx = x$ be a directed cycle of length n. From (i) it follows that $n \leqslant t$. By Euclid's division lemma, there exist nonnegative integers q, r such that $t = nq + r$ where $0 \leqslant r < n$. Now $s^mx = s^{m+t}x = s^{m+nq+r}x = s^{m+r}(s^{nq}x) = s^{m+r}x$ since $s^nx = x$. If $r > 0$ then it follows that $min(P_x) = k_x \leqslant r$ where P_x is as mentioned in the proof of (i) and so $gr(H_s(A)) \leqslant k_x \leqslant r < n$, a contradiction. Hence $r = 0$ and so $gr(H_s(A)) \mid t$.

(iii) Starting with any vertex x in A, we can obtain a directed walk $x \to sx \to s^2x \to ... \to s^mx \to s^{m+1}x \to s^{m+2}x \to ... \to s^{m+t-1}x \to s^{m+t}x = s^mx$ with a vertex repetition occurring at s^mx. This happens irrespective of the choice of the starting vertex x. Thus any directed path must have length $\leqslant m + t - 1$. □

To show that the girth of $H_s(A)$ actually attains the upper bound t (period of s) in case of strongly faithful semigroup action, we present the following Corollary.

Corollary 4.24. *Let (S, A) be a strongly faithful semigroup act and $s \in S$ be a periodic element with index m and period t. Then $m = 1$ and $gr(H_s(A)) = t$.*

Proof. Let $gr(H_s(A)) = n$ and $x \to sx \to s^2x \to ... \to s^{n-1}x \to s^nx = x$ be a directed cycle of length n in $H_s(A)$. Then $s^{n+1}x = sx$. As the action is strongly faithful so $s^{1+n} = s$. By minimality of m and t, $m \leqslant 1$ and $t \leqslant n$. Therefore $m = 1$. Now by Theorem 4.23 (i), $n \leqslant t$ which implies that $gr(H_s(A)) = n = t$. □

5 The reverse question

In this section we pose the very question 'Which digraphs can be the action graph $G(S, a, A)$ corresponding to some semigroup act (S, a, A)?' This question can be rephrased via the functor $\mathcal{F}$ obtained in Proposition 3.14: 'What is the image of $\mathcal{F}$?' The property of being strongly transitive plays a pivotal role in making a digraph, an action graph. Firstly we show that (*cf*. Theorem 5.4) any simple, transitive digraph (*cf*. Definition 5.2) can be seen as a spanning sub-digraph of the action graph of some semigroup act. In fact we can get even more.

In what follows all digraphs are considered to be finite.

Definition 5.1. Suppose D_1, D_2 are two digraphs. D_1 is said to coincide with D_2 '*except possibly at loops*' if D_1^* is isomorphic to D_2^* where D_i^* denotes the loop free copy of D_i for $i = 1, 2$.

Definition 5.2. [3] A simple, uncolored digraph $D = (V(D), h, E(D))$ is called *transitive* if for any three distinct vertices a, b, c, $(a, b), (b, c) \in E(D)$ implies that $(a, c) \in E(D)$.

Every strongly transitive digraph (*cf*. Definition 3.9) is clearly transitive. So following remark is in order as a counterpart of Remark 3.10.

Remark 5.3. If a simple, uncolored digraph is a 1-magma (and consequently equipped with a structure of a semicategory) then it is a transitive digraph.

Theorem 5.4. *Let $D = (V(D), h, E(D))$ be a simple, transitive digraph. Then there exists a left semigroup act (S, a, A) for a semigroup S and a nonempty set A such that $G(S, a, A)$ coincides with D except possibly at loops.*

Proof. At first, we construct a semigroup from the digraph D. Consider the free semigroup generated by the set of arcs $E(D)$ of D and name it as $S_{E(D)}$ (In [11] it is denoted by $< D >$ and referred as 'arc generated semigroup'). Let $A = V(D)$. Define an action of the generating set $E(D)$ on A by a map $a : E(D) \times A \longrightarrow A$ such that

$$a((i, j), x) = (i, j)x := \begin{cases} j & \text{if } x = i \\ x & \text{if } x \neq i \end{cases}$$

Clearly by universal property[11] a can be extended to an action (for simplicity we also denote this by a) of the free semigroup $S_{E(D)}$ on A by defining

$$((i_{n+1}, j_{n+1})(i_n, j_n), ..., (i_1, j_1))(x) :=$$

[11]the action of $E(D)$ on A gives rise to a mapping $\psi : E(D) \longrightarrow Map(A)$ (the semigroup of all self maps on A); the universal property of free semigroup guarantees the existence of a unique semigroup homomorphism $\phi : S_{E(D)} \longrightarrow Map(A)$, induced by ψ which is nothing but an action of $S_{E(D)}$ on A.

$$(i_{n+1}, j_{n+1})(((i_n, j_n)(i_{n-1}, j_{n-1}), ..., (i_1, j_1))(x))$$

for all $n \in \mathbb{N}$ and for all $x \in A$ and so $(S_{E(D)}, a, A)$ becomes a left semigroup act. We claim that $G(S_{E(D)}, a, A)^* = D^*$. It is evident that $V(G(S_{E(D)}, a, A)^*) = A = V(D^*)$.

Let $(x, y) \in E(G(S_{E(D)}, a, A)^*)$. Then $x \neq y$ and by definition there exists $s = (i_1, j_1)(i_2, j_2)...(i_k, j_k) \in S_{E(D)}$ such that $y = sx$, that is $y = ((i_1, j_1)(i_2, j_2)...(i_k, j_k))x$. Now in view of the action of $E(D)$ on A, it follows that either $(x, y) \in \{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ or there exist arcs

$$(i_{\alpha_1}, y), (i_{\alpha_2}, i_{\alpha_1}), (i_{\alpha_3}, i_{\alpha_2}), ..., (i_{\alpha_m}, i_{\alpha_{m-1}}), (x, i_{\alpha_m}) \in \{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$$

with $m \geqslant 1$. In the first case, clearly $(x, y) \in E(D^*)$ and in the later case, since $(i_{\alpha_1}, y), (i_{\alpha_2}, i_{\alpha_1}), (i_{\alpha_3}, i_{\alpha_2}), ..., (i_{\alpha_m}, i_{\alpha_{m-1}}), (x, i_{\alpha_m}) \in E(D)$, so there is a directed walk from x to y in D, containing a directed path from x to y which together with the facts that D is a transitive digraph and $x \neq y$, implies that $(x, y) \in E(D^*)$. So $E(G(S_{E(D)}, a, A)^*) \subseteq E(D^*)$. On the other hand if $(x, y) \in E(D^*)$ then $(x, y) \in S_{E(D)}$. Since $y = (x, y)x$ and $x \neq y$, so by definition $(x, y) \in E(G(S_{E(D)}, a, A)^*)$ and $E(D^*) \subseteq E(G(S_{E(D)}, a, A)^*)$. Therefore $G(S_{E(D)}, a, A)^* = D^*$ and D coincides with $G(S_{E(D)}, a, A)$ except possibly at loops. □

Note 5.5. If $E(D)$ contains at least two arcs with different tails then the digraph $G(S_{E(D)}, a, A)$, constructed in the proof of Theorem 5.4 has a loop at each vertex. Therefore if D is a simple, transitive digraph with a loop at each vertex then we can extend the proof of Theorem 5.4 to show that D is always an action graph. But there also exists action graph which has no loop at some vertex (see Example 3.15).

Definition 5.6. Let $D = (V(D), h, E(D))$ be a finite, simple digraph. A vertex v is called a *universal sink* (see [6], Page 614) if all the other vertices of D are in-neighbours of v and it has no out-neighbour. We say that v is a *looped universal sink* if all vertices are in-neighbours of v and it has only one out-neighbour which is v itself (that is, there is a loop at v).

Note that in Example 3.15, a is a looped universal sink. Also observe that if universal sink or looped universal sink exists in a digraph then these are unique. In the next theorem, we present a class of digraphs which are always action graphs of semigroup acts.

Theorem 5.7. *Let $D = (V(D), h, E(D))$ be a strongly transitive, simple digraph with a looped universal sink. Then D is the action graph of a semigroup act.*

Proof. Let ξ be the looped universal sink of D. Since D is simple, every arc in D can be expressed as a unique ordered pair of vertices of D. We adjoin an element θ with $E(D)$ and denote the resulting set by S. Define a multiplication on S as follows:

$$(a,b)(c,d) := \begin{cases} (c,b) & \text{if } d = a \\ \theta & \text{if } d \neq a \end{cases}$$

$$\theta(a,b) := (a,b)\theta := \theta \text{ for all } (a,b) \in E(D)$$
$$\theta\theta := \theta$$

Since D is strongly transitive, this multiplication is well defined on S. To show that this is associative let $s, t, u \in S$. If at least one of s, t and u is θ, then clearly $(st)u = s(tu)$. If none of them is θ then let $s = (a,b), t = (c,d)$ and $u = (e,f)$. Four cases may arise namely Case-(i) $f = c$ and $d = a$, Case-(ii) $f \neq c$ and $d = a$, Case-(iii) $f = c$ and $d \neq a$ and Case-(iv) $f \neq c$ and $d \neq a$. Now it is a matter of routine verification that $(st)u = s(tu)$ in each of these cases. Therefore S becomes a semigroup under this multiplication. We define a left action of S on $V(D)$ by a map $\nu : S \times V(D) \longrightarrow V(D)$ such that

$$\nu((a,b),x) = (a,b)x := \begin{cases} b & \text{if } x = a \\ \xi & \text{if } x \neq a \end{cases}$$

$$\nu(\theta, x) = \theta x := \xi \text{ for all } x \in V(D)$$

Now we show that this is actually a semigroup action. Let $s, t \in S$ and $x \in V(D)$. If at least one of s, t is θ, then it is trivial. So let $s = (a,b)$ and $t = (c,d)$.

Case-(i) $d = a$ and $c = x$. Then $(st)x = ((a,b)(c,d))x = (c,b)x = b$ and $s(tx) = (a,b)((c,d)x) = (a,b)d = b$.

Case-(ii) $d = a$ and $c \neq x$. Then $(st)x = ((a,b)(c,d))x = (c,b)x = \xi$ and $s(tx) = (a,b)((c,d)x) = (a,b)\xi = \xi$ since ξ is a looped universal sink.

Case-(iii) $d \neq a$ and $c = x$. It follows that $(st)x = ((a,b)(c,d))x = \theta x = \xi$ and $s(tx) = (a,b)((c,d)x) = (a,b)d = \xi$.

Case-(iv) $d \neq a$ and $c \neq x$. Then $(st)x = ((a,b)(c,d))x = \theta x = \xi$ and $s(tx) = (a,b)((c,d)x) = (a,b)\xi = \xi$ since ξ is the looped universal sink.

Therefore $(S, \nu, V(D))$ becomes a left semigroup act. We claim that $G(S, \nu, V(D)) = D$. Clearly both have same set of vertices. Let $(x,y) \in E(G(S, \nu, V(D)))$. If $y = \xi$ then $(x,y) = (x,\xi) \in E(D)$ since ξ is the looped universal sink. If $y \neq \xi$ then $y = sx$ for some $s = (a,b) \in E(D)$. So $x = a$ and $y = b$ whence $(x,y)(= (a,b)) \in E(D)$. Conversely if $(a,b) \in E(D)$ then $(a,b) \in S$ and since $b = (a,b)a$ so $(a,b) \in E(G(S, \nu, V(D)))$. Hence both the digraphs have same set of arcs which implies that $D = G(S, \nu, V(D))$. □

Remark 5.8. The semigroup constructed in the above proof is significant in the sense that it is the opposite semigroup of a Brandt semigroup [20].

The question which immediately comes in mind is that, 'Do the graphs of Theorem 5.7 cover all action graphs of semigroup act?' The answer is 'No' as illustrated by the following example.

Example 5.9. Let $S = \{s,t\}$ be a two element right zero semigroup and $A = \{a,b,c\}$. Define an action ν of S on A by $\nu(s,a) := b$, $\nu(s,b) := b$, $\nu(s,c) := c$ and $\nu(t,a) := c$, $\nu(t,b) := b$ and $\nu(t,c) := c$. It can be verified easily that (S, ν, A) is a semigroup act. The corresponding action graph is shown below (Fig-5.1). Note that it has no looped universal sink.

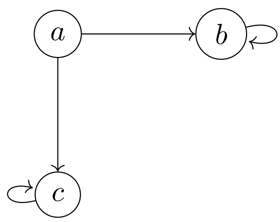


Fig-5.1

Now we try to refine the result obtained in Theorem 5.4, with the help of the following observation. Note that for any semigroup act (S, a, A), Proposition 3.12 assures that (i) the action graph $G(S, a, A)$ is a strongly transitive digraph, (ii) every vertex of $G(S, a, A)$ has positive out-degree and (iii) if A is finite then $G(S, a, A)$ must have a loop. Therefore it is clear that digraphs which are not strongly transitive or has no loop or has a vertex with out-degree 0 can not be an action graph. We establish below that these obvious necessary conditions are also sufficient for a digraph to be the action graph of a semigroup act.

Theorem 5.10. *Let $D = (V(D), h, E(D))$ be a simple digraph. Then D is the action graph of a semigroup act if and only if D is strongly transitive and every vertex of D has positive out-degree.*

Proof. Clearly the conditions are necessary. For the converse suppose D is strongly transitive and every vertex has positive out-degree. Therefore $N_D^+(x) \neq \emptyset$ for all $x \in V(D)$. By axiom of choice, for every vertex x we can select $n_x \in N_D^+(x)$. Then $(x, n_x) \in E(D)$ for all $x \in V(D)$. Now consider the free semigroup $S_{E(D)}$ generated by $E(D)$ and define an action of $E(D)$ on $V(D)$ by a map $a : E(D) \times V(D) \longrightarrow V(D)$ such that

$$a((i,j),x) = (i,j)x := \begin{cases} j & \text{if } x = i \\ n_x & \text{if } x \neq i \end{cases}$$

Clearly this can be extended to an action (which we denote by the same a) of $S_{E(D)}$ on $V(D)$ defined by

$$((i_1, j_1)(i_2, j_2)...(i_k, j_k))x := ((i_1, j_1)(i_2, j_2)...(i_{k-1}, j_{k-1}))(i_k, j_k)x.$$

Hence $(S_{E(D)}, a, V(D))$ is a left semigroup act. Both $G(S_{E(D)}, a, V(D))$ and D have same set of vertices. Let $(x, y) \in E(G(S_{E(D)}, a, V(D)))$. Then there exists $s = (i_m, j_m)(i_{m-1}, j_{m-1})...(i_1, j_1) \in S_{E(D)}$ such that $y = sx$. Let $x_0 := x$ and $x_k := (i_k, j_k)x_{k-1}$ for all $k = 1, 2, ..., m$. Then $x_m = y$ and by definition of the action, it follows that $(x_{k-1}, x_k) \in E(D)$ for all $k = 1, 2, ..., m$. Therefore $x = x_0 \to x_1 \to x_2 \to ... \to x_{m-1} \to x_m = y$ is a directed walk from x to y in D which assures that there is a directed path from x to y in D. Since D is strongly transitive so $(x, y) \in E(D)$. Conversely if $(x, y) \in E(D)$ then $(x, y) \in S_{E(D)}$ and since $y = (x, y)x$ so $(x, y) \in E(G(S_{E(D)}, a, V(D)))$. Thus $D = G(S_{E(D)}, a, V(D))$. □

Corollary 5.11. *Let $D = (V(D), h, E(D))$ be a simple digraph. Then D is equal to $G(S, a, C; A)$ for a semigroup act (S, a, A) and a nonempty subset C of S if and only if every vertex of D has positive out-degree.*

Proof. Clearly the condition is necessary. Conversely suppose every vertex of D has positive out-degree. We consider the transitive closure[22] of D and denote it by $\overline{D}^t$. By the hypothesis we can choose $n_x \in N_D^+(x)$ for all $x \in V(\overline{D}^t)$. Using these choices of out-neighbours of vertices and following the proof of Theorem 5.10, we can show that $G(S_{E(\overline{D}^t)}, a, V(\overline{D}^t))$ is equal to $\overline{D}^t$. Let $C = E(D) \subseteq S_{E(\overline{D}^t)}$. Then C is nonempty. Let $(x, y) \in E(G(S_{E(\overline{D}^t)}, a, C; V(D)))$. Then there exists $c = (x', y') \in C$ such that $y = cx = (x', y')x$. If $x = x'$ then $y = y'$ and so $(x, y) = (x', y') \in C = E(D)$. If $x \neq x'$ then $y = n_x \in N_D^+(x)$ and so $(x, y) = (x, n_x) \in E(D)$. Conversely if $(x, y) \in E(D) = C$ then $(x, y) \in E(G(S_{E(\overline{D}^t)}, a, C; V(D)))$ as $y = (x, y)x$. Hence $D = G(S_{E(\overline{D}^t)}, a, C; V(D))$. □

Remark 5.12. Theorem 5.10 characterizes the image of the functor $\mathcal{F}$ (*cf.* Proposition 3.14) on objects, precisely it shows that $\mathcal{F}$ is surjective on objects. A functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is called *essentially surjective* [24] or, *dense* [22] if for every $D \in Obj(\mathbb{D})$, there exists $C \in Obj(\mathbb{C})$ such that $F(C)$ is isomorphic to D in $\mathbb{D}$. The functor which are surjective on objects are clearly dense. Therefore $\mathcal{F}$ is also a dense functor.

6 More functorial properties of action graph construction

This final section is devoted to the study of more properties of the functor $\mathcal{F} : \mathbb{S}\text{gr}\mathbb{A}\text{ct}_l \longrightarrow \mathbb{S}\text{imp-}\mathbb{T}\mathbb{D}\mathbb{G}\text{raph}$ and its restriction to the full subcategory $\mathbb{S}\mathbb{F}\text{-}\mathbb{S}\text{gr}\mathbb{A}\text{ct}_l$ (*cf.* Remark 4.17). The categorical terminologies and results used here are mainly taken from [1], [21] and [24].

We now recall the definitions of *comorphism* and *strong homomorphism* in the category $\mathbb{D}\mathbb{G}\text{raph}$ of digraphs.

Definition 6.1. [22] Let $G_1 = (V(G_1), h_1, E(G_1))$, $G_2 = (V(G_2), h_2, E(G_2))$ be simple, uncolored digraphs. A mapping $f : V(G_1) \longrightarrow V(G_2)$ is called a *comorphism (continuous graph homomorphism)* if $(f(x), f(y)) \in E(G_2)$ implies that $(x, y) \in E(G_1)$. f is called a *strong homomorphism* if it

[22]the transitive closure of a simple digraph G is a digraph G' such that $V(G) = V(G')$ and $(i, j) \in E(G')$ if and only if there exists a directed path from i to j in G.

preserves as well as reflects arcs that is, $(x, y) \in E(G_1)$ if and only if $(f(x), f(y)) \in E(G_2)$.

At this point we may ask a natural question, 'When will the functor $\mathcal{F}$ produce strong homomorphism?' Following proposition gives an answer.

Proposition 6.2. *Let $(\mu, f) : (S, a, A) \longrightarrow (T, b, B)$ be a morphism of acts such that μ is surjective and f is injective. Then $\mathcal{F}(\mu, f)$ is a strong homomorphism in $\mathbb{DG}$raph.*

Proof. We just need to show that $\mathcal{F}(\mu, f) : \mathcal{F}(S, a, A) \longrightarrow \mathcal{F}(T, b, B)$ is a comorphism. Assume that $(\mathcal{F}(\mu, f)(x), \mathcal{F}(\mu, f)(y)) \in E(\mathcal{F}(T, b, B))$. Then $(f(x), f(y)) \in E(G(T, b, B))$ which implies $f(y) = tf(x)$ for some $t \in T$. Since μ is surjective, there exists $s \in S$ such that $t = \mu(s)$ and so $f(y) = \mu(s)f(x) = f(sx)$. Finally applying injectivity of f we obtain $y = sx$. Therefore $(x, y) \in E(G(S, a, A)) = E(\mathcal{F}(S, a, A))$ and so $\mathcal{F}(\mu, f)$ is a strong homomorphism. □

The conditions (i) μ is surjective, (ii) f is injective in Proposition 6.2 are sufficient but not necessary for $\mathcal{F}(\mu, f)$ to be a strong homomorphism in $\mathbb{DG}$raph. This fact is illustrated in the following example.

Example 6.3. Consider the semigroup act $(\mathbb{N}, \nu, \mathbb{Z}_6)$ of Example 3.16. We define $\mu : \mathbb{N} \longrightarrow \mathbb{N}$ by $\mu(n) := n^2$ for all $n \in \mathbb{N}$ and $f : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_6$ by $f(a) := a^2$ for all $a \in \mathbb{Z}_6$. Then μ is a semigroup homomorphism and $f(na) = (na)^2 = n^2a^2 = \mu(n)f(a)$ that is, (μ, f) is a morphism of acts. It can be easily verified that $\mathcal{F}(\mu, f) : \mathcal{F}(\mathbb{N}, \nu, \mathbb{Z}_6) \longrightarrow \mathcal{F}(\mathbb{N}, \nu, \mathbb{Z}_6)$ is a strong endomorphism. But clearly μ is not surjective and f is not injective as $f(\bar{1}) = f(\bar{5}) = \bar{1}$.

Definition 6.4. [24] A category $\mathbb{C}$ is called *monoidal* if it is equipped with

(i) a product (precisely a bifunctor) $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ which is associative up to a natural isomorphism α;

(ii) an object I which is both left and right identity for $\otimes$ up to natural isomorphisms λ and ρ respectively, such that all diagrams involving α, λ and ρ commute.

For the most common example we refer $\mathbb{S}$et which is a monoidal category with cartesian product of sets playing the role of a bifunctor and any 1-element set as an identity.

Lemma 6.5. *The category* $\mathbb{S}gr\mathbb{A}ct_l$ *is monoidal.*

Proof. Define $\otimes : \mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l \times \mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l \longrightarrow \mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l$ by $(S, a, A) \otimes (T, b, B) := (S \times T, a \times b, A \times B)$ (*cf.* Observation 3.17). By [24, Proposition III.5.1] we just need to show that $\otimes$ defines a product in the category $\mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l$ and there is a terminal object in this category. Note that (π_S, π_A) and (π_T, π_B) are projections from $(S \times T, a \times b, A \times b)$ onto (S, a, A) and (T, b, B) respectively where π_S, π_T are respective projections of $S \times T$ onto S and T in $\mathbb{S}\mathrm{emi}\mathbb{G}\mathrm{rp}$ and π_A, π_B are projections of $A \times B$ on A and B respectively in the category $\mathbb{S}\mathrm{et}$. Let (U, c, C) be any semigroup act and $(\mu, f) : (U, c, C) \longrightarrow (S, a, A)$, $(\eta, g) : (U, c, C) \longrightarrow (T, b, B)$ be any two morphisms of acts. Now it can be easily shown that $(\mu \times \eta, f \times g) : (U, c, C) \longrightarrow (S \times T, a \times b, A \times B)$ is the unique morphism of acts for which the following diagram commutes.

$$\begin{array}{ccccc} & & (U,c,C) & & \\ & \swarrow^{(\mu,f)} & \downarrow{\scriptstyle (\mu\times\eta, f\times g)} & \searrow^{(\eta,g)} & \\ (S,a,A) & \xleftarrow[(\pi_S,\pi_A)]{} & (S\times T, a\times b, A\times B) & \xrightarrow[(\pi_T,\pi_B)]{} & (T,b,B) \end{array}$$

Hence $\otimes$ is a product in $\mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l$. If $\mathbf{1_S}$ be the one element semigroup and $\{x\}$ be any singleton set then with respect to the trivial action ι of $\mathbf{1_S}$ on $\{x\}$, $(\mathbf{1_S},\ \iota, \{x\})$ is a terminal object in the category $\mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l$. Hence the lemma follows. □

We note that the category of digraphs $\mathbb{D}\mathbb{G}\mathrm{raph}$ as well as the full subcategory $\mathbb{S}\mathrm{imp}$-$\mathbb{T}\mathbb{D}\mathbb{G}\mathrm{raph}$ are also monoidal categories. The cross product of digraphs '$\times$' (*cf.* Definition 3.18 (*ii*)) is easily seen to be a bifunctor and associative on objects up to digraph isomorphisms and any digraph containing one vertex and one directed loop, commonly known as *rose with one petal* (R_1), is a both sided identity for the cross product.

Definition 6.6. Let $(\mathbb{C}, \otimes_{\mathbb{C}}, I_{\mathbb{C}}), (\mathbb{D}, \otimes_{\mathbb{D}}, I_{\mathbb{D}})$ be two monoidal categories. A *monoidal functor* between these categories is a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ together with

(i) a morphism $\epsilon : I_{\mathbb{D}} \longrightarrow F(I_{\mathbb{C}})$ and

(ii) natural transformations $\mu_{x,y} : F(x) \otimes_{\mathbb{D}} F(y) \longrightarrow F(x \otimes_{\mathbb{C}} y)$ for all $x, y \in Obj(\mathbb{C})$ satisfying the following conditions:

(1) For all objects x, y, z of $\mathbb{C}$, the following diagram commutes.

$$\begin{array}{ccc}
(F(x) \otimes_{\mathbb{D}} F(y)) \otimes_{\mathbb{D}} F(z) & \xrightarrow{\cong} & F(x) \otimes_{\mathbb{D}} (F(y) \otimes_{\mathbb{D}} F(z)) \\
{\scriptstyle \mu_{x,y} \otimes id}\Big\downarrow & & \Big\downarrow{\scriptstyle id \otimes \mu_{y,z}} \\
F(x \otimes_{\mathbb{C}} y) \otimes_{\mathbb{D}} F(z) & & F(x) \otimes_{\mathbf{D}} F(y \otimes_{\mathbb{C}} z) \\
{\scriptstyle \mu_{x \otimes_{\mathbb{C}} y, z}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu_{x, y \otimes_{\mathbb{C}} z}} \\
F((x \otimes_{\mathbb{C}} y) \otimes_{\mathbb{C}} z) & \xrightarrow[\cong]{} & F(x \otimes_{\mathbb{C}} (y \otimes_{\mathbb{C}} z))
\end{array}$$

(2) For all $x \in Obj(\mathbb{C})$, the following diagrams commute.

$$\begin{array}{ccc}
I_{\mathbb{D}} \otimes_{\mathbb{D}} F(x) & \xrightarrow{\epsilon \otimes id} & F(I_{\mathbb{D}}) \otimes_{\mathbb{D}} F(x) \\
{\scriptstyle \cong}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu_{I_{\mathbb{D}}, x}} \\
F(x) & \xleftarrow[\cong]{} & F(I_{\mathbb{D}} \otimes_{\mathbb{D}} x)
\end{array}$$

and

$$\begin{array}{ccc}
F(x) \otimes_{\mathbb{D}} I_{\mathbb{D}} & \xrightarrow{id \otimes \epsilon} & F(x) \otimes_{\mathbb{D}} F(I_{\mathbb{D}}) \\
{\scriptstyle \cong}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu_{x, I_{\mathbb{D}}}} \\
F(x) & \xleftarrow[\cong]{} & F(x \otimes_{\mathbb{D}} I_{\mathbb{D}})
\end{array}$$

Moreover if ϵ and all $\mu_{x,y}$ are isomorphisms then F is called a *strong monoidal functor.*

Theorem 6.7. *Let* $(\mathbb{S}gr\mathbb{A}ct_l, \otimes, (\mathbf{1_S}, \iota, \{x\}))$ *and* $(\mathbb{S}imp - \mathbb{TDG}raph, \times, R_1)$ *be the respective monoidal categories. Then* $\mathcal{F}$ *is a strong monoidal functor.*

Proof. Clearly $\mathcal{F}(\mathbf{1_S}, \iota, \{x\}) (= G(\mathbf{1_S}, \iota, \{x\}))$ is a digraph with only one vertex x and a loop on it. Hence it is isomorphic to R_1, the identity with respect to the bifunctor $\times$ in Simp-$\mathbb{TDG}$raph. We denote this isomorphism by ϵ. Suppose (S, a, A) and (T, b, B) are any two semigroup acts. Proposition 3.19 (i) implies that the digraphs $\mathcal{F}(S, a, A) \times \mathcal{F}(T, b, B)$ and $\mathcal{F}((S, a, A) \otimes (T, b, B))$ are indeed isomorphic (in fact identical). The identity graph isomorphism between $\mathcal{F}(S, a, A) \times \mathcal{F}(T, b, B)$ and $\mathcal{F}((S, a, A) \otimes (T, b, B))$ plays the role of natural transformation $\mu_{(S,a,A),(T,b,B)}$. Now it becomes a routine matter of verification to show that the diagrams of Definition 6.6 involving ϵ and μ are commutative. □

We conclude this section with some results on the restriction of the functor $\mathcal{F}$ to the full subcategory $\mathbb{SF}$-$\mathbb{S}$gr$\mathbb{A}$ct$_l$. By virtue of Example 4.12 and Proposition 4.14 it is clear that the image of this restriction is not a full subcategory of $\mathbb{S}$imp-$\mathbb{TDG}$raph. Let $\mathbb{CS}$-$\mathbb{TDG}$raph be the subcategory of $\mathbb{S}$imp-$\mathbb{TDG}$raph whose objects are of the form $\mathcal{F}(S, a, A)$ for some $(S, a, A) \in Obj(\mathbb{SF}\text{-}\mathbb{S}\mathrm{gr}\mathbb{A}\mathrm{ct}_l)$ and morphisms are color sensitive digraph homomorphisms (*cf*. Definition 4.13).

The graph theoretical result obtained in Theorem 4.16 is connected with the functor $\mathcal{F}$ via the following result.

Theorem 6.8. *The restricted functor* $\mathcal{F} : \mathbb{SF}\text{-}\mathbb{S}gr\mathbb{A}ct_l \longrightarrow \mathbb{CS}\text{-}\mathbb{TDG}raph$ *is an equivalence of categories and hence these two categories are equivalent.*

Proof. By definition of the category $\mathbb{CS}$-$\mathbb{TDG}$raph, $\mathcal{F}$ is surjective on objects. We just need to show that it is faithful and full. Let $(S, a, A), (T, b, B)$ be any two strongly faithful semigroup acts and $(\mu, f), (\eta, g) : (S, a, A) \longrightarrow (T, b, B)$ be two morphism of acts such that $\mathcal{F}(\mu, f) = \mathcal{F}(\eta, g)$. This implies $f = g$. Now for any $s \in S$ and $a \in A$, $\mu(s)f(a) = f(sa) = g(sa) = \eta(s)g(a) = \eta(s)f(a)$. Hence $\mu(s) = \eta(s)$ since (T, b, B) is a strongly faithful act. Therefore $(\mu, f) = (\eta, g)$ and $\mathcal{F}$ is faithful. Finally from Theorem 4.16, it follows that $\mathcal{F}$ is surjective on morphisms, that is a full functor. □

Note 6.9. Since $\mathcal{F}$ is an equivalence between the categories $\mathbb{SF}$-$\mathbb{S}$gr$\mathbb{A}$ct$_l$ and $\mathbb{CS}$-$\mathbb{TDG}$raph hence by [24, Theorem IV.4.1] it follows that $\mathcal{F}$ is a part of an adjoint equivalence and has a left adjoint namely the functor $\mathcal{G}$: $\mathbb{CS}$-$\mathbb{TDG}$raph $\longrightarrow$ $\mathbb{SF}$-$\mathbb{S}$gr$\mathbb{A}$ct$_l$ which maps each action graph to its corresponding strongly faithful semigroup act and each color sensitive digraph homomorphism to the morphism of acts by which it is induced.

In Remark 5.12, we recalled the definition of essentially surjective (dense) functors. Analogous to this notion, we can define a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ to be *essentially injective*[3] if for any $A, B \in Obj(\mathbb{C})$, $F(A) \cong F(B)$ in $\mathbb{D}$ implies that $A \cong B$ in $\mathbb{C}$. Example 3.15 indicates that in general the functor $\mathcal{F}$ defined from $\mathbb{S}$gr$\mathbb{A}$ct$_l$ to $\mathbb{S}$imp-$\mathbb{TDG}$raph is not an essentially injective functor. Suppose (S, a, A) and (T, b, B) are two strongly faithful semigroup acts such that the corresponding action graphs are isomorphic in the category $\mathbb{CS}$-$\mathbb{TDG}$raph. Then there exist a color sensitive digraph isomorphism $f : G(S, a, A) \longrightarrow G(T, b, B)$. By Theorem 4.15, there is a semigroup homomorphism $\mu : S \longrightarrow T$ such that (μ, f) is a morphism of acts and $G(\mu, f) = f$. Now Since the restricted functor $\mathcal{F} : \mathbb{SF}$-$\mathbb{S}gr\mathbb{A}ct_l \longrightarrow$ $\mathbb{CS}$-$\mathbb{TDG}$raph is fully faithful so by [1, Corollary 3.32] $\mathcal{F}$ reflects isomorphisms and hence $(\mu, f) : (S, a, A) \longrightarrow (T, b, B)$ is an isomorphism of acts. We summarize this discussion in terms of the functor $\mathcal{F}$ in the following result.

Proposition 6.10. *The functor $\mathcal{F} : \mathbb{SF}$-$\mathbb{S}gr\mathbb{A}ct_l \longrightarrow \mathbb{CS}$-$\mathbb{TDG}$raph is essentially injective.*

7 Concluding remarks

(1) In this paper we have studied some basic graph theoretic properties of the action graph in connection with the algebraic properties of a semigroup act. Vertex transitivity of the action graph of a semigroup act can be an interesting topic for future study. Other symmetry conditions like arc transitivity, distance transitivity may also lead to further research. On the other hand various types of regular and inverse semigroup acts and corresponding action graphs may provide scope for interesting future study. In every section we have tried to establish the categorical connection between the category of semigroup acts and category of certain digraphs via several results (*cf*. Remark 5.12, Theorem 6.8). These interconnections may enrich each other's theory.

(2) A left semigroup act (S, a, A) gives rise to another directed graph $G_C(S, a, A)$ in a canonical way whose set of vertices is A and the set of arcs consists of triples (x, s, y) such that $a(s, x) = y$. This graph may have multiple arcs and so may not be simple. To study the relationship of semigroup act (S, a, A) and this canonical digraph $G_C(S, a, A)$ can be a good direction for future work.

(3) A digraph can alternatively be represented as follows:

Let $\mathbb{G}$ be the finite category with two objects:

$$0 \underset{t}{\overset{s}{\rightrightarrows}} 1$$

the category of directed graphs is the category of set-valued presheaves $\mathbb{DG}\text{raph} = [\mathbb{G}^{op}, \mathbb{S}\text{et}]$; thus a digraph X is given by two sets: X_0, the set of vertices and X_1, the set of arcs and two maps from X_1 to X_0:

$$\begin{array}{c} X_1 \\ s \downarrow\downarrow t \\ X_0 \end{array}$$

and a morphism of directed graphs is a natural transformation $f = (f_0, f_1)$

$$X \xrightarrow{f=(f_0,f_1)} Y$$

which is described by a diagram in $\mathbb{S}$et:

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ s \downarrow\downarrow t & & s \downarrow\downarrow t \\ X_0 & \xrightarrow[f_0]{} & Y_0 \end{array}$$

such that $s \circ f_1 = f_0 \circ s$ and $t \circ f_1 = f_0 \circ t$. This presentation of the category $\mathbb{DG}$raph of directed graphs has the advantage to show that digraphs and their morphisms do form a *Grothendieck topos* and consequently, in order to develop the theory of digraphs, one can apply all the concepts related with topos . Moreover if we replace the category $\mathbb{S}$et by an elementary topos $\mathcal{E}$, then a functor:

$$\mathbb{G}^{op} \xrightarrow{F} \mathcal{E}$$

is an internal digraph in the topos $\mathcal{E}$. It will be interesting to investigate as to how the present work can be applied in the study of such an internal digraph. Keeping in mind this abstraction and the fact that fuzzy sets form a quasitopos (a category which is similar to topos but need not be balanced), it is natural to ask if one can apply some technologies of the present work to understand what are fuzzy digraphs, what are semigroup acts for fuzzy sets etc.

(4) Though the category S-$\mathbb{M}$od (*cf*. Remark 3.4) and $\mathbb{S}$gr$\mathbb{A}$ct$_l$ are not the same but both come from semigroup acts. So it is natural to investigate the faithfulness, fullness, essentially surjectivity etc. for the counterpart of the functor $\mathcal{F}$ (*cf*. Proposition 3.14) from S-$\mathbb{M}$od to $\mathbb{D}\mathbb{G}$raph.

It is relevant to note here that S-$\mathbb{M}$od has some nice categorical properties viz., it is locally finitely presentable because there is a projective sketch whose set-models are S-acts. Also there is a semimonad on $\mathbb{S}$et whose algebras are exactly the S-acts.

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Promit Mukherjee *Department of Mathematics, Jadavpur University, Kolkata-700032, India.*
Email: promitmukherjeejumath@gmail.com

Rajlaxmi Mukherjee *Department of Mathematics, Garhbeta College, Paschim Medinipur-721127, India.*
Email: ju.rajlaxmi@gmail.com

Sujit Kumar Sardar *Department of Mathematics, Jadavpur University, Kolkata-700032, India.*
Email: sksardarjumath@gmail.com