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On saturated prefilter monads

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Abstract. In this paper we show that the prime saturated prefilter monads are sup-dense and interpolating in saturated prefilter monads. It follows that CNS spaces are the lax algebras for prime saturated prefilter monads. As for the algebraic part, we prove that the Eilenberg-Moore algebras for saturated prefilter monads are exactly continuous I-lattices.

1 Introduction and Preliminaries

It is showed in [23] that every compact Hausdorff topological space can be described as an Eilenberg-Moore algebra for an ultrafilter monad. Later, Barr [1] extended the ultrafilter monad to the category Rel and obtained that every topological space can be described as a lax algebra. There have been extensive study of investigating topological and related structures with lax-algebraic and categorical methods. The primary purpose of this paper is to study so-called prime saturated prefilter monads and the algebras for saturated prefilter monads.

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There are many attempts to extend the notion of filter to an enriched context, such as functional ideals [4, 22], L-filters [7, 10–12], prefilters [20, 21] and $\top$-filters [11, 12, 31]. Saturated prefilters are a class of prefilters, which give rise to monads when the continuous triangular norm satisfies a certain condition, see [18].

In [8] it is proved that the characterization of a topological space in terms of neighborhood system is equivalent to the definition of a Kleisli monoid of a filter monad. Seal [26] found a suitable lax extension of the filter monad for which the lax algebras are precisely topological spaces. The equivalence of these two presentations of topological spaces is derived from the fact that the filter monad is power-enriched. In section 2 we shall see that saturated prefilter monads are power-enriched and that the Kleisli monoids of saturated prefilter monads are CNS spaces which are a special class of fuzzy topological spaces [15]. Then we introduce prime saturated prefilter monads, which are sup-dense and interpolating submonads of saturated prefilter monads. By general results in [9] CNS spaces can be obtained as the lax algebras for prime saturated prefilter monads.

The fact that continuous lattices are the algebras for the filter monad on the category Set or Top_0 was proved by Day [6] and Wyler [29]. There are many counterparts of continuous lattices in an enriched setting. For frame-valued continuous lattices, Yao [30] proved that they are exactly the algebras for the open filter monad on the category of frame-valued T_0 topological spaces. For continuous I-lattices with respect to forward Cauchy ideals, Lai and Zhang [16] identified them as the algebras for composite monad $\mathcal{C}\mathcal{P}^{\dagger}$ on the category $\mathsf{I\text{-}Ord}$, where $\mathcal{C}$ are forward Cauchy ideal monads and $\mathcal{P}^{\dagger}$ are upper-set monads. In section 3 by extending saturated prefilter monads to the category $\mathsf{I\text{-}Ord}$ we shall show that continuous I-lattices with respect to forward Cauchy ideals are exactly the algebras for saturated prefilter monads.

Continuous t-norms and I-ordered sets A triangular norm [13] (t-norm for short) is an associative, commutative and monotone binary operation on the unit interval $[0, 1]$, where the number 1 acts as the identity element. We will denote the unit interval by I throughout the paper.

A t-norm is called continuous if it is continuous as a function from I^2 into I, where I^2 and I are endowed with the standard topology. As &

preserves arbitrary joins in each variable, we obtain a function $\to: I^2 \to I$ via

$$a\&b \leq c \iff a \leq b \to c,$$

which is called the implication with respect to &. Some properties of & and $\to$ are collected below.

Proposition 1.1. *For any $a, b, c \in I$ and $\{a_i\}_i \subset I$, it holds that:*

(1) $1 \to a = a$;

(2) $a \leq b \iff 1 \leq a \to b$;

(3) $a \to (b \to c) = (a\&b) \to c$;

(4) $a\&(a \to b) \leq b$;

(5) $(\bigvee_i a_i) \to b = \bigwedge_i (a_i \to b)$;

(6) $b \to (\bigwedge_i a_i) = \bigwedge_i (b \to a_i)$.

There are three basic continuous t-norms.

(1) The Łukasiewicz t-norm and its implication:

$$a\&b = \max\{0, a + b - 1\}, \quad a \to b = \min\{1, 1 - a + b\}.$$

(2) The product t-norm and its implication:

$$a\&b = ab, \quad a \to b = \min\{1, b/a\}.$$

(3) The Gödel t-norm and its implication:

$$a\&b = \min\{a, b\}, \quad a \to b = \begin{cases} 1, & a \leq b, \\ b, & a > b. \end{cases}$$

A continuous t-norm & is said to satisfy condition (S) if for any $a \in (0, 1]$ the function $a \to (-)$ is continuous on $[0, a)$. Condition (S) first appeared in [24, 25]. With the help of the well-known ordinal sum decomposition theorem, one can construct many continuous t-norms which satisfy condition (S).

An I-order [3, 14, 28, 32] with respect to & on a set X is a function $P\colon X\times X\to I$ such that

$$P(x,x)=1 \quad \text{and} \quad P(x,y)\&P(y,z)\le P(x,z)$$

for any $x,y,z\in X$. The pair (X,P) is called an I-ordered set with respect to &. When there is no danger of confusion, we simply write X for (X,P), $X(-,-)$ for $P(-,-)$ and omit &.

An I-ordered set X is separated if $X(x,y)=X(y,x)=1$ implies that $x=y$. The underlying order of X is defined as $x\le y$ if and only if $X(x,y)=1$.

Suppose that X is an I-ordered set, it is easy to check that $X^{\rm op}(x,y)=X(y,x)$ is also an I-ordered set.

Let X be a set. There exists a natural I-order on the set I^X of all functions from X into I (which is also called fuzzy inclusion order [3]):

$$\mathrm{sub}_X\colon I^X\times I^X\longrightarrow I,\ (\mu,\nu)\longmapsto\bigwedge_{x\in X}\mu(x)\to\nu(x).$$

A function $f\colon X\to Y$ of I-ordered sets is order preserving if

$$X(x,y)\le Y(f(x),f(y))$$

for any $x,y\in X$. I-ordered sets and order preserving functions form a category

$$\mathsf{I\text{-}Ord}.$$

An order preserving function $f\colon X\to Y$ is called left adjoint if there exists an order preserving function $g\colon Y\to X$ such that

$$Y(f(x),y)=X(x,g(y))$$

for any $x\in X$, $y\in Y$. In this case, g is called right adjoint.

Power-enriched monads A monad $\mathbb{T}$ on a category A is a triple (T,m,e) consisting of a functor $T\colon\mathsf{A}\to\mathsf{A}$ and two natural transformations $m\colon T^2\to T$, $e\colon\mathrm{id}_{\mathsf{A}}\to T$ satisfying

$$m\cdot eT=m\cdot Te=\mathrm{id}_{\mathsf{A}} \quad \text{and} \quad m\cdot mT=m\cdot Tm.$$

The natural transformation m is called the multiplication of $\mathbb{T}$ and e is called the unit of $\mathbb{T}$.

A monad morphism $\alpha\colon \mathbb{T} \to \mathbb{S}$ is a natural transformation $\alpha\colon T \to S$ satisfying

$$\alpha \cdot e = d \quad \text{and} \quad \alpha \cdot m = n \cdot (\alpha * \alpha),$$

where $\mathbb{S} = (S, n, d)$ and $*$ is the horizontal composition of natural transformations.

An Eilenberg-Moore algebra for $\mathbb{T}$ (or $\mathbb{T}$-algebra) is a pair $(X, t\colon TX \to X)$ such that $t \cdot e_X = 1_X$ and $t \cdot m_X = t \cdot T(t)$. A $\mathbb{T}$-monomorphism $f\colon (X,t) \to (Y,s)$ of $\mathbb{T}$-algebras is a morphism $f\colon X \to Y$ in A such that $f \cdot t = s \cdot Tf$. $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms form a category $\mathsf{A}^{\mathbb{T}}$.

Let $\mathbb{T} = (T, m, e)$ be a monad on the category Set. Let S be a subfunctor of T such that for each $x \in X$ and $\mathcal{F} \in S^2X$ it holds that $e_X(x) \in SX$ and $(m \cdot (i * i))_X(\mathcal{F}) \in SX$, where $i\colon S \to T$ is the inclusion transformation. Then $(S, m \cdot (i * i), e)$ is a monad and is called a submonad of $\mathbb{T}$. To keep the notation simple, we denote the submonad by (S, m, e).

The powerset monad $\mathbb{P} = (P, \bigcup, \{-\})$ on the category Set is defined as follows:

- $P\colon \mathsf{Set} \to \mathsf{Set}$ is the covariant powerset functor;
- $\{-\}_X\colon X \to PX$, $x \mapsto \{x\}$;
- $\bigcup_X\colon P^2X \to PX$, $\mathcal{A} \mapsto \bigcup \mathcal{A}$.

Suppose that $\mathbb{T} = (T, m, e)$ is a monad on the category Set and $\alpha\colon \mathbb{P} \to \mathbb{T}$ is a monad morphism. For each set X there is an order on TX given by:

$$F \le G \iff m_X \cdot \alpha_{TX}(\{F, G\}) = G$$

for any $F, G \in TX$.

A power-enriched monad [9] is a pair $(\mathbb{T}, \alpha\colon \mathbb{P} \to \mathbb{T})$ such that for any set X, Y the function

$$(-)^{\mathbb{T}}\colon \mathsf{Set}(X, TY) \longrightarrow \mathsf{Set}(TX, TY),\ f \longmapsto m_X \cdot T(f)$$

is monotone, where $\mathsf{Set}(-, TY)$ is ordered pointwise.

A morphism $\delta\colon (\mathbb{T}, \alpha) \to (\mathbb{S}, \beta)$ of power-enriched monads is a monad morphism such that $\beta = \delta \cdot \alpha$.

We write 1_A for the function defined by $1_A(x) = 1$ whenever $x \in A$ and $1_A(x) = 0$ whenever $x \notin A$. For each singleton set $\{x\}$ we simply write 1_x for $1_{\{x\}}$.

Example 1.2. The I-powerset monad $\mathbb{P}_I = (P_I, \bigcup, \{-\})$ with respect to & is defined as follows:

- $P_I\colon \mathsf{Set} \to \mathsf{Set}$ sends each set X to I^X and each function $f\colon X \to Y$ to the function $P_I(f)\colon \mu \mapsto \bigvee\{\mu(x) \mid f(x) = (-)\}$;
- $\{-\}_X\colon X \to P_I X$, $x \mapsto 1_x$;
- $\bigcup_X\colon P_I^2 X \to P_I X$, $\Phi \mapsto \bigvee\{\Phi(\mu)\&\mu \mid \mu \in P_I X\}$.

It is power-enriched by $\theta_X\colon PX \longrightarrow P_I X$, $A \longmapsto 1_A$, the order on $P_I X$ induced by θ is pointwise order.

Saturated prefilters

Definition 1.3. A prefilter [20] on a set X is a subset F of I^X subject to the following conditions:

(F1) function 1_X belongs to F;

(F2) $\mu \wedge \nu \in F$ for any $\mu, \nu \in F$;

(F3) if $\mu \geq \nu$ and $\nu \in F$, then $\mu \in F$.

A prefilter is called saturated provided that

$$\forall \nu \in I^X, \quad \bigvee_{\mu \in F} \mathrm{sub}_X(\mu, \nu) = 1 \implies \nu \in F.$$

A prefilter basis on X is a subset B of I^X such that for any $\mu, \nu \in B$ there is some $\lambda \in B$ with $\lambda \leq \mu \wedge \nu$. Let

$$\widehat{B} = \{\nu \mid \bigvee_{\mu \in B} \mathrm{sub}_X(\mu, \nu) = 1\},$$

then $\widehat{B}$ is a saturated prefilter and is called the saturation of B.

For every function $f\colon X\to Y$ and every saturated prefilter F on X, it is easy to check that $f(F)=\{\mu\mid\mu\cdot f\in F\}$ is a prefilter. The saturation of $f(F)$ follows from the saturation of F and the inequality

$$\bigvee_{\mu\cdot f\in F}\mathrm{sub}_X(\mu\cdot f,\nu\cdot f)\geq\bigvee_{\mu\cdot f\in F}\mathrm{sub}_Y(\mu,\nu).$$

We obtain a functor $\mathsf{SPF}\colon\mathsf{Set}\to\mathsf{Set}$ defined on object X by the set of saturated prefilters on X and on morphism f by $\mathsf{SPF}(f)\colon F\mapsto f(F)$.

It is easy to check that

$$\mathsf{e}_X\colon X\longrightarrow\mathsf{SPF}X,\ x\longmapsto\{\mu\mid\mu(x)=1\}$$

defines the components of a natural transformation from $\mathrm{id}_{\mathsf{Set}}$ to SPF.

For each $\mu\in I^X$, let

$$\mu^\sharp\colon\mathsf{SPF}X\longrightarrow I,\ F\longmapsto\bigvee_{\nu\in F}\mathrm{sub}_X(\nu,\mu).$$

It is easy to check that

$$\mu^\sharp(F)\to\nu^\sharp(F)\geq\mathrm{sub}_X(\mu,\nu)$$

for any μ, $\nu\in I^X$ and saturated prefilter F. With the help of $(-)^\sharp$, we can obtain a natural transformation:

$$\mathsf{m}_X\colon\mathsf{SPF}^2X\longrightarrow\mathsf{SPF}X,\ \mathcal{F}\longmapsto\{\mu\mid\mu^\sharp\in\mathcal{F}\}.$$

Proposition 1.4. ([18, Theorem 5.10]) *The following statements are equivalent:*

(1) *the continuous t-norm & satisfies condition (S);*

(2) *the triple* $(\mathsf{SPF},\mathsf{m},\mathsf{e})$ *is a monad.*

Because of Proposition 1.4, from now on, we always assume that the continuous t-norm & satisfies conditions (S) unless otherwise specified.

2 Descriptions of CNS spaces

An I-topology [19] on a set X is a subset σ of I^X, satisfying the following conditions:

(O1) for every $a \in I$, constant function $a_X \equiv a$ belongs to σ;

(O2) $\mu \wedge \nu \in \sigma$ for any $\mu, \nu \in \sigma$;

(O3) $\bigvee_i \mu_i \in \sigma$ for any subset $\{\mu_i\}_i \subset \sigma$;

An I-topological space is a pair (X, σ), where σ is an I-topology on X. The function $\mathfrak{N}_x$ given by

$$\mathfrak{N}_x \colon I^X \longrightarrow I \colon \mu \longmapsto \bigvee_{\substack{\nu \in \sigma \\ \nu \leq \mu}} \nu(x)$$

is called the neighborhood system of (X, σ) at x. I-topological spaces are determined by their neighborhood system, see [12].

A CNS space [15] is an I-topological space provided that for each $x \in X$

$$\mathfrak{N}_x = \bigvee_{\mathfrak{N}_x(\mu)=1} \mathrm{sub}_X(\mu, -).$$

The prefilter

$$\mathcal{N}_x = \{\mu \mid \mathfrak{N}_x(\mu) = 1\}$$

is called neighborhood prefilter at x. A CNS space is an I-topological space whose I-neighborhood system $\mathfrak{N}$ is determined by its neighborhood prefilter $\mathcal{N}$.

Proposition 2.1. ([15, Corollary 5.9.]) *A family $\{\mathcal{N}_x\}_{x \in X}$ of saturated prefilters is the neighborhood prefilter of a CNS space X if and only if it satisfies:*

(CN1) *$\mu(x) = 1$ holds for each $\mu \in \mathcal{N}_x$;*

(CN2) *for each $\mu \in \mathcal{N}_x$, there exists some $\nu \in \mathcal{N}_x$ such that $\nu \leq \mu$ and $a \to \nu \in \mathcal{N}_y$ for any $y \in X$ and $a < \nu(y)$.*

CNS spaces as Kleisli monoids For each set X let

$$\kappa_X\colon PX \longrightarrow \mathsf{SPF}X\colon A \longmapsto \{\mu \mid \mu \geq 1_A\}.$$

One can verify that κ is a monad morphism $\kappa\colon \mathbb{P} \to \mathbb{SPF}$. For any $F, G \in \mathsf{SPF}X$, since $m_X \cdot \kappa_{\mathsf{SPF}(X)}(\{F, G\}) = \{\mu \mid \mu^\sharp(F) = 1 \text{ and } \mu^\sharp(G) = 1\}$, the order on $\mathsf{SPF}X$ induced by κ is the reverse inclusion order, i.e.

$$F \leq G \iff F \supset G.$$

It is easy to show that

$$f \leq g \implies f^{\mathbb{SPF}} \leq g^{\mathbb{SPF}}$$

for any $f, g\colon X \to \mathsf{SPF}Y$. Thus, $(\mathbb{SPF}, \kappa)$ is a power-enriched monad.

Let $\mathbb{T} = (T, m, e)$ be a monad power-enriched by α. A $\mathbb{T}$-monoid is a pair $(X, \rho\colon X \to TX)$ such that

$$\rho \circ \rho \leq \rho, \ e_X \leq \rho,$$

where $\circ$ is the Kleisli composition, i.e. $g \circ f = g^{\mathbb{T}} \cdot f$ for any $f\colon X \to TY$, $g\colon Y \to TZ$.

A morphism $f\colon (X, \rho) \to (Y, \rho')$ of $\mathbb{T}$-monoids is a function $f\colon X \to Y$ subject to

$$Tf \cdot \rho \leq \rho' \cdot f.$$

$\mathbb{T}$-monoids give rise to a category

$$\mathbb{T}\text{-}\mathsf{Mon}.$$

Proposition 2.2. *There is an isomorphism:*

$$\mathbb{SPF}\text{-}\mathsf{Mon} \cong \mathsf{CNS}.$$

Proof. It is sufficient to show that for any function $\rho\colon X \to \mathsf{SPF}X$ the family $\{\rho(x)\}_{x \in X}$ satisfies the conditions (CN1) and (CN2) if and only if ρ satisfies

$$\rho \circ \rho \leq \rho, \ \mathsf{e}_X \leq \rho.$$

The condition (CN1) is equivalent to $\mathsf{e}_X \leq \rho$. As for the equivalence of the condition (CN2) and $\rho \circ \rho \leq \rho$, let $x \in X$

$$(\rho \circ \rho)(x) \leq \rho(x) \iff (\rho \circ \rho)(x) \supset \rho(x)$$

$$\begin{aligned}&\iff (\mathsf{m}_X \cdot \mathsf{SPF}(\rho) \cdot \rho)(x) \supset \rho(x)\\&\iff \{\mu \mid \mu^\sharp \cdot \rho \in \rho(x)\} \supset \rho(x)\\&\iff (\forall \mu,\ \mu \in \rho(x) \implies \mu^\sharp \cdot \rho \in \rho(x)).\end{aligned}$$

For each $\mu \in \rho(x)$ we have that

$$\mu^\sharp \cdot \rho \leq \mu^\sharp(\mathsf{e}_X) = \mu.$$

For each $y \in X$ and $a < (\mu^\sharp \cdot \rho)(y)$, $a \to (\mu^\sharp \cdot \rho) \in \rho(y)$ follows from the saturation of $\rho(y)$ and

$$\bigvee_{\nu \in \rho(y)} \mathrm{sub}_X(\nu, a \to (\mu^\sharp \cdot \rho)) = \bigvee_{\nu \in \rho(y)} a \to \mathrm{sub}_X(\nu, \mu^\sharp \cdot \rho) = 1.$$

The functoriality of this correspondence is trivial. □

CNS spaces as lax algebras Given a monad $\mathbb{T} = (T, m, e)$ on the category Set, a lax extension [9] $\check{\mathbb{T}}$ of $\mathbb{T}$ to the category Rel is given by a family of functions $T_{X,Y} \colon \mathsf{Rel}(X, Y) \to \mathsf{Rel}(TX, TY)$ satisfying the following conditions:

- $r \leq r' \implies \check{T}r \leq \check{T}r'$;
- $(1_{TX})_\circ \leq \check{T}1_X$;
- $\check{T}r \cdot \check{T}s \leq \check{T}(r \cdot s)$;
- $(Tf)_\circ \leq \check{T}f_\circ$ and $(Tf)^\circ \leq \check{T}f^\circ$;
- $(e_Y)_\circ \cdot r \leq \check{T}r \cdot (e_X)_\circ$;
- $(m_Y)_\circ \cdot \check{T}\check{T}r \leq \check{T}r \cdot (m_X)_\circ$

for any function $f \colon X \to Y$ and relations $r, r' \colon X \nrightarrow Y$, $s \colon Y \nrightarrow Z$, where $f_\circ$ denotes the graph of f and f° denotes the cograph of f. To keep notation simple, we usually write f instead of $f_\circ$ in the remainder of the paper.

A lax algebra for $\check{\mathbb{T}}$, also referred as $(\mathbb{T}, 2, \check{\mathbb{T}})$-algebra, is a pair $(X, r \colon TX \nrightarrow X)$ such that

$$r \cdot \check{T}r \leq r \cdot m_X \quad \text{and} \quad 1_X \leq r \cdot e_X.$$

A morphism $f \colon (X, r) \to (Y, s)$ of lax algebras is a function $f \colon X \to Y$ such that

$$f \cdot r \leq s \cdot Tf.$$

Lax algebras and morphisms of lax algebras assemble into a category

$$(\mathbb{T}, 2)\text{-}\mathsf{Cat}.$$

The Kleisli extension $\overline{\mathbb{SPF}}$ of the power-enriched monad $(\mathbb{SPF}, \kappa)$ is given as follows:

$$F\,(\overline{\mathsf{SPF}}r)\,G \iff r^\kappa(G) \leq F$$

for any relation $r\colon X \nrightarrow Y$ and $F \in \mathsf{SPF}X$, $G \in \mathsf{SPF}Y$, where

$$r^\kappa(G) = \Big\{\mu \Big| G \ni \mu^r : y \mapsto \bigwedge_{x \in r^\flat(y)} \mu(x)\Big\},$$

in which $r^\flat\colon Y \to PX$, $y \mapsto \{x \mid x\,r\,y\}$. Thanks to Theorem IV.1.5.3 in [9], there is an isomorphism:

$$(\mathbb{SPF}, 2)\text{-}\mathsf{Cat} \cong \mathbb{SPF}\text{-}\mathsf{Mon}.$$

Definition 2.3. A saturated prefilter F is said to be *prime* if $\mu \vee \nu \in F$ implies that $\mu \in F$ or $\nu \in F$ for any $\mu, \nu \in I^X$.

Lemma 2.4. *Let F be a prime saturated prefilter. Then, $(\mu \vee \nu)^\sharp(F) = \mu^\sharp(F) \vee \nu^\sharp(F)$ holds for any $\mu, \nu \in I^X$.*

Proof. For each $\lambda \in F$ let $a_\lambda = \mathrm{sub}_X(\lambda, \mu \vee \nu)$. Then we have that $(a_\lambda \to (\mu \vee \nu)) \in F$. Since F is prime, we can assume without loss of generality that $a_\lambda \to \mu \in F$ for all $\lambda \in F$. By

$$\mu^\sharp(F) = (a_\lambda \to \mu)^\sharp(F) \to \mu^\sharp(F) \geq \mathrm{sub}_X(a_\lambda \to \mu, \mu) \geq a_\lambda,$$

we have that $\mu^\sharp(F) = (\mu \vee \nu)^\sharp(F)$. □

Assigning to each set X the set of all prime saturated prefilters on X gives rise to a functor $\mathsf{PSF}\colon \mathsf{Set} \to \mathsf{Set}$, which is a subfunctor of SPF.

For any $x \in X$ the saturated prefilter $\mathsf{e}_X(x)$ is prime. For any $\mathcal{U} \in \mathsf{PSF}^2X$

$$(\mathsf{m} \cdot (\mathsf{i} * \mathsf{i}))_X(\mathcal{U}) = \{\mu \mid \mu^\sharp \cdot \mathsf{i}_X \in \mathcal{U}\},$$

where i is the inclusion transformation from PSF to SPF. It follows from Lemma 2.4 that $(\mathsf{m} \cdot (\mathsf{i} * \mathsf{i}))_X(\mathcal{U})$ is prime. Hence, we have the following proposition.

Proposition 2.5. $\mathbb{PSF} = (\mathsf{PSF}, \mathsf{m}, \mathsf{e})$ *is a submonad of* $(\mathsf{SPF}, \mathsf{m}, \mathsf{e})$.

The initial lax extension of $\mathbb{PSF}$ induced by i and $\overline{\mathbb{SPF}}$ is given as follows:

$$F\ (\overline{\mathsf{PSF}}r)\ G \iff \mathsf{i}_X(F)\ (\overline{\mathsf{SPF}}r)\ \mathsf{i}_X(G)$$

for each relation $r\colon X \nrightarrow Y$ and each $F \in \mathsf{PSF}X,\ G \in \mathsf{PSF}Y$.

Given a set X and a saturated prefilter F on X, with the help of a Prime Ideal Theorem [5] it is easy to show that

$$F = \bigcap\{U \in \mathsf{PSF}X \mid F \subset U\}.$$

Hence, the morphism $\mathsf{i}\colon \mathbb{PSF} \to \mathbb{SPF}$ is sup-dense in the sense of [9]. The following proposition shows that i is interpolating in the sense of [9]. Given a $\mu \in I^X$ we simply write $\mu^\sharp$ for $(\mu^\sharp \cdot \mathsf{i}_X)\colon \mathsf{PSF}X \to I$ if no confusion would arise.

Proposition 2.6. *For any relation* $r\colon \mathsf{SPF}X \nrightarrow X$ *and* $x \in X$ *let* U *be a prime saturated prefilter on* X *such that* $\{\mu \mid \mu^\sharp \geq 1_{r^\flat(x)}\} \subset U$. *Then there exists a prime saturated prefilter* $\mathcal{U}$ *on* $\mathsf{PSF}X$ *such that*

$$\mathsf{m}_X(\mathcal{U}) \subset U \quad \textit{and} \quad 1_{r^\flat(x)} \in \mathcal{U}.$$

Proof. Let $\{\mathcal{U}_i\}_{i\in I}$ denote the set of prime saturated prefilters on $\mathsf{PSF}X$ containing $1_{r^\flat(x)}$. Assume for a contradiction that for each $i \in I$ there exists some μ_i such that $\mu_i^\sharp \in \mathcal{U}_i$ and $\mu_i \notin U$.

There is some finite J_0 such that $\bigvee_{i\in J_0} \mu_i^\sharp \geq 1_{r^\flat(x)}$, otherwise by the Prime Ideal Theorem we can find a prime saturated prefilter containing $1_{r^\flat(x)}$ and missing the directed set $\{\bigvee_{i\in J} \mu_i^\sharp \mid J \subset I,\ J \text{ is finite}\}$. By Lemma 2.4 one has that $(\bigvee_{i\in J_0} \mu_i)^\sharp = \bigvee_{i\in J_0} \mu_i^\sharp \geq 1_{r^\flat(x)}$, hence $\bigvee_{i\in J_0} \mu_i \in U$. U is prime, a contradiction. □

Since the morphism $\mathsf{i}\colon \mathbb{PSF} \to \mathbb{SPF}$ is sup-dense and interpolating, there exists an isomorphism between $(\mathbb{PSF}, 2)$-Cat and $\mathbb{SPF}$-Mon (Theorem IV.2.3.3 in [9]).

Combining both the preceding discussion, we have the following result.

Theorem 2.7. *There is an isomorphism:*

$$(\mathbb{PSF}, 2)\text{-}\mathsf{Cat} \cong (\mathbb{SPF}, 2)\text{-}\mathsf{Cat} \cong \mathbb{SPF}\text{-}\mathsf{Mon} \cong \mathsf{CNS}.$$

3 Algebras for saturated prefilter monads

Monads on I-Ord Let X be an I-ordered set. A function $\mu\colon X \to I$ is called an upper set [14] if $\mu(a)\&X(a,b) \le \mu(b)$ holds for each $a, b \in X$. Endowing the set of all upper sets of X with I-order $\mathrm{sub}_X^{\mathrm{op}}$, we obtain an I-ordered set and denote it by $\mathcal{P}^\dagger X$.

For each order preserving function $f\colon X \to Y$ and each upper set μ of X, let

$$\mathcal{P}^\dagger f(\mu) = \bigvee_{x\in X} \mu(x)\&Y(f(x), -).$$

It is easy to check that $\mathcal{P}^\dagger f(\mu)$ is an upper set of Y, so we obtain a functor:

$$\mathcal{P}^\dagger\colon I\text{-}\mathsf{Ord} \longrightarrow I\text{-}\mathsf{Ord}.$$

The contravariant Yoneda embedding

$$\mathbf{y}_X^{\mathrm{op}}\colon X \longrightarrow \mathcal{P}^\dagger X,\ x \longmapsto X(x, -)$$

defines the components of a natural transformation from identical functor to $\mathcal{P}^\dagger$.

Given an upper set μ of X, an infimum of μ is an element $\inf_X \mu \in X$ such that

$$\mathrm{sub}_X^{\mathrm{op}}(\mathbf{y}_X^{\mathrm{op}}(x), \mu) = X(x, \inf_X \mu)$$

for any $x \in X$. If every upper set μ of X has an infimum, then it is called complete.

The I-ordered set $\mathcal{P}^\dagger X$ is complete and $\inf_{\mathcal{P}^\dagger X}(\phi) = \bigvee\{\phi(\mu)\&\mu \mid \mu \in \mathcal{P}^\dagger X\}$ for each $\phi \in {\mathcal{P}^\dagger}^2 X$. By routine checking, upper sets give rise to monads $(\mathcal{P}^\dagger, \inf_{\mathcal{P}^\dagger}, \mathbf{y}^{\mathrm{op}})$ on the category I-Ord, which are called upper-set monads.

Since the upper-set monads are dual Kock-Zöberlein type, (X, t) is a $\mathcal{P}^\dagger$-algebra if and only if X is complete, separated, and $t = \inf_X$. An order preserving function $f\colon X \to Y$ is a $\mathcal{P}^\dagger$-monomorphism if f preserves the infima of every upper set of X.

A lower set of X is a function $\mu\colon X \to I$ such that $X(a,b)\&\mu(b) \le \mu(a)$. Lower sets give rise to a functor:

$$\mathcal{P}\colon \mathsf{I}\text{-}\mathsf{Ord} \longrightarrow \mathsf{I}\text{-}\mathsf{Ord},$$

where $\mathcal{P}X$ is endowed with the I-order sub_X and

$$\mathcal{P}f(\mu) = \bigvee_{x\in X} \mu(x)\&Y(-, f(x))$$

for each order preserving function $f\colon X \to Y$ and each lower set μ of X.

The Yoneda embedding

$$\mathbf{y}_X\colon X \longrightarrow \mathcal{P}X,\ x \longmapsto X(-, x)$$

defines the components of a natural transformation from identical functor to $\mathcal{P}$.

Given a lower set μ of X, a supremum of μ is an element $\sup_X \mu \in X$ such that

$$\mathrm{sub}_X(\mu, \mathbf{y}_X(x)) = X(\sup_X \mu, x)$$

for any $x \in X$. If every lower set μ of X has a supremum, then it is called cocomplete. It is well-known that an I-ordered set is complete if and only if it is cocomplete [27, Proposition 5.10.].

The I-ordered set $\mathcal{P}X$ is cocomplete and $\sup_{\mathcal{P}X}(\phi) = \bigvee_{\mu\in\mathcal{P}X} \phi(\mu)\&\mu$ for each $\phi \in \mathcal{P}^2X$. Similarly, lower sets give rise to monads $(\mathcal{P}, \sup_{\mathcal{P}}, \mathbf{y})$ on the category I-Ord, which are called lower-set monads.

A net $\{x_i\}_i$ on an I-ordered set X is called forward Cauchy [28] if

$$\bigvee_i \bigwedge_{k\geq j\geq i} X(x_j, x_k) = 1.$$

A lower set μ is a forward Cauchy ideal if

$$\mu = \bigvee_i \bigwedge_{j\geq i} X(-, x_j)$$

for some forward Cauchy net $\{x_i\}_i$. An ideal of the underlying ordered set of X is a forward Cauchy net on X.

Proposition 3.1. ([16, Proposition 4.8.]) *Let μ be a forward Cauchy ideal on a complete, separated I-ordered set X. Then $\{x \mid \mu(x) = 1\}$ is a forward Cauchy net on X and*

$$\mu = \bigvee_{\mu(x)=1} X(-, x).$$

Forward Cauchy ideals are an extension of ideals to an enriched setting, they give rise to submonads $(\mathcal{C}, \sup_{\mathcal{C}}, \mathbf{y})$ of the lower-set monads [17].

Dually to the upper-set monads, the lower-set monads are Kock-Zöberlein type. Hence, (X, t) is a $\mathcal{P}$-algebra ($\mathcal{C}$-algebra) if and only if X is separated, every lower set (forward Cauchy ideal) of X has a supremum, and $t = \sup_X$.

Let $\mathbb{T} = (T, m, e)$ and $\mathbb{S} = (S, n, d)$ be monads on $\mathcal{A}$. According to [2], if there is a lifting $\widetilde{\mathbb{S}}$ of $\mathbb{S}$ through the forgetful functor $G^{\mathbb{T}} \colon \mathcal{A}^{\mathbb{T}} \to \mathcal{A}$, then one can obtain a composite monad $\mathbb{S}\mathbb{T} = (ST, w, d * e)$, where w is given by

$$w_X = (n * m)_X \cdot S\widetilde{S}(m_X) \cdot STSe_{TX},$$

in which $\widetilde{S}(m_X)$ is the structural function of $\widetilde{S}(TX, m_X)$. $\mathbb{S}\mathbb{T}$-algebras correspond bijectively to the pairs $\{(t, s)\}$ with the property that (X, t) is a $\mathbb{T}$-algebra, (X, s) is an $\mathbb{S}$-algebra and $s \colon \widetilde{S}(X, t) \to (X, t)$ is a $\mathbb{T}$-homomorphism.

There is a lifting of $\mathcal{C}$ through $G^{\mathcal{P}^\dagger} \colon \mathsf{I\text{-}Ord}^{\mathcal{P}^\dagger} \to \mathsf{I\text{-}Ord}$ if and only if every upper set of $\mathcal{C}X$ has an infimum. Thanks to Theorem 6.4. in [16] we have the following proposition.

Proposition 3.2. *The I-ordered set $\mathcal{C}X$ is complete if and only if the continuous t-norm satisfies condition (S).*

It is easy to check that the multiplication of the composite monads $\mathcal{C}\mathcal{P}^\dagger$ is given by

$$\mathbf{n}_X(\mathbf{F}) \colon \mathcal{P}^\dagger X \longrightarrow I,\ \mu \longmapsto \mathbf{F}(\mu^\natural)$$

for each I-ordered set X and $\mathbf{F} \in \mathcal{C}{\mathcal{P}^\dagger}^2 X$, where $\mathcal{P}^\dagger\mathcal{C}\mathcal{P}^\dagger X \ni \mu^\natural \colon \mathfrak{F} \mapsto \mathfrak{F}(\mu)$. The following proposition follows from the preceding discussion about the algebras for composite monads.

Proposition 3.3. ([16, Proposition 5.5.]) *Let X be an I-ordered set. The following are equivalent:*

(1) *X is complete, separated and $\sup_X \colon \mathcal{C}X \to X$ preserves the infima of every upper set of $\mathcal{C}X$;*

(2) *X is a $\mathcal{C}\mathcal{P}^\dagger$-algebra.*

Given a $\mathcal{C}\mathcal{P}^\dagger$-algebra X, since $\mathcal{C}X$ is complete and $\sup_X \colon \mathcal{C}X \to X$ preserves the infima of every upper set of $\mathcal{C}X$, then $\sup_X \colon \mathcal{C}X \to X$ is right adjoint. Thus, X is a continuous I-lattice.

The saturated prefilter monads over $\mathsf{I\text{-}Ord}$ Let $\gamma\colon (\mathbb{S},\alpha) \to (\mathbb{T},\beta)$ be a morphism of power-enriched monads $\mathbb{S} = (S, n, d)$ and $\mathbb{T} = (T, m, e)$.

Following [9], we can construct a new monad $\mathbb{T}'$ on $\mathbb{S}\text{-}\mathsf{Mon}$ as follows:

Functor: for each $\mathbb{S}$-monoid (X, ρ) the underlying set of $T'((X, \rho))$ denoted by $T'X$ is defined as the equalizer $q_X\colon T'X \to TX$ of the pair $((\gamma_X \cdot \rho)^{\mathbb{T}}, 1_{TX})$.

Denoting the factorization of $(\gamma_X \cdot \rho)^{\mathbb{T}}$ through q_X by p_X, the structure function of $T'((X, \rho))$ is defined as

$$S(p_X) \cdot w_X \cdot q_X\colon T'X \longrightarrow ST'X,$$

where w_X is the right adjoint of $m_X \cdot \gamma_{TX}$.

Unit: $e'_{(X,\rho)}$ is the factorization of $\gamma_X \cdot \rho$ through q_X.

Multiplication: $m'_{(X,\rho)} = p_X \cdot q_X^{\mathbb{T}} \cdot q_{T'X}$.

Proposition 3.4. [9, Theorem IV.4.3.2] *The triple $\mathbb{T}' = (T', m', e')$ is a monad on $\mathbb{S}\text{-}\mathsf{Mon}$ and there exists an isomorphism:*

$$\mathsf{Set}^{\mathbb{T}} \cong \mathbb{S}\text{-}\mathsf{Mon}^{\mathbb{T}'}.$$

For each set X, let

$$\tau_X\colon P_I X \longrightarrow \mathsf{SPF}X,\ \mu \longmapsto \{\nu \mid \nu \geq \mu\},$$

it is easy to check that $\tau\colon (\mathbb{P}_I, \theta) \to (\mathbb{SPF}, \kappa)$ is morphism of power-enriched monads.

Given a $\mathbb{P}_I$-monoid (X, ρ), the condition $e_X \leq \rho$ is equivalent to that $\rho(x)(x) = 1$ for any $x \in X$. For each $x \in X$ it holds that

$$\begin{aligned} \rho \circ \rho(x) \leq \rho(x) &\iff \textstyle\bigcup_X (P_I(\rho)(\rho(x))) \leq \rho(x) \\ &\iff \textstyle\bigcup_X \left(\displaystyle\bigvee_{\rho(y)=(-)} \rho(x)(y) \right) \leq \rho(x) \\ &\iff \bigvee_{y \in X} \rho(x)(y) \& \rho(y) \leq \rho(x). \end{aligned}$$

It is easy to check that the morphisms of $\mathbb{P}_I$-monoids are exactly the order preserving functions. Hence we have the following isomorphism:

$$\mathsf{I\text{-}Ord} \cong \mathbb{P}_I\text{-}\mathsf{Mon}.$$

From now on, we always treat a $\mathbb{P}_I$-monoid (X,ρ) as a set X ordered by

$$X(x,y)=\rho(x)(y)$$

and do not distinguish $\mathbb{P}_I$-monoids and I-ordered sets.

Now, we come to the main result of this section.

Theorem 3.5. *The monad $\mathbb{SPF}'$ on $\mathbb{P}_I$-Mon is isomorphic to the monad $(\mathcal{CP}^{\dagger},\mathbf{n},(\mathbf{y}*\mathbf{y}^{\mathrm{op}}))$;*

In order to prove this theorem, we make some preparations. Given a $\mathbb{P}_I$-monoid X, let

$$\mu^{\uparrow}(x)=\mathrm{sub}_X(X(x,-),\mu)$$

for each $\mu\in I^X$ and let

$$F^{\uparrow}=\{\mu^{\uparrow}\mid\mu\in F\}$$

for each saturated prefilter F on X. Some basic properties of $(-)^{\uparrow}$ are collected bellow.

Proposition 3.6. *Let (X,ρ) be a $\mathbb{P}_I$-monoid and F be a saturated prefilter on X. Then*

(1) *$\mu^{\uparrow}\in\mathcal{P}^{\dagger}X$ and $\mu\geq\mu^{\uparrow}$ for any $\mu\in P_IX$;*

(2) *$\mu^{\uparrow}\wedge\nu^{\uparrow}=(\mu\wedge\nu)^{\uparrow}$ holds for any $\mu,\nu\in P_IX$;*

(3) *$\mathrm{sub}_X(\nu,\mu^{\uparrow})=\mathrm{sub}_X(\nu,\mu)$ holds for any $\nu\in\mathcal{P}^{\dagger}X$ and $\mu\in P_IX$;*

(4) *$\mathrm{sub}_X(\mu^{\uparrow},\nu^{\uparrow})\geq\mathrm{sub}_X(\mu,\nu)$ for any $\mu,\nu\in P_IX$;*

(5) *$(\tau_X\cdot\rho)^{\mathbb{SPF}}(F)=F$ holds if and only if $\mu^{\uparrow}\in F$ for any $\mu\in F$, in this case $F^{\uparrow}$ is a forward Cauchy net on $\mathcal{P}^{\dagger}X$.*

Proof. For (1), $\mu^{\uparrow}\leq\mu$ is trivial and $\mu^{\uparrow}(x)\&X(x,y)\leq\mu^{\uparrow}(y)$ follows from $(X(x,t)\to\mu(t))\&X(x,y)\&X(y,t)\leq\mu(t)$ for any $t\in X$.

(2) is trivial.

For (3), it holds that

$$\bigwedge_{x\in X}\nu(x)\to\left(\bigwedge_{t\in X}X(x,t)\to\mu(t)\right)=\bigwedge_{\substack{x\in X\\t\in X}}\nu(x)\&X(x,t)\to\mu(t)$$

$$= \bigwedge_{t\in X}\left(\bigvee_{x\in X}\nu(x)\&X(x,t)\right)\to\mu(t)$$
$$= \bigwedge_{t\in X}\nu(t)\to\mu(t).$$

(4) follows immediately from (1) and (3).

To see (5), since $(\tau_X\cdot\rho)^{\mathbb{SPF}}(F)=\{\mu\mid\mu^{\uparrow}\in F\}$ we have that $(\tau_X\cdot\rho)^{\mathbb{SPF}}(F)=F$ if and only if $\mu^{\uparrow}\in F$ for any $\mu\in F$. It follows from (2) that $F^{\uparrow}$ is an ideal of the underlying ordered set of $\mathcal{P}^{\dagger}X$. □

Lemma 3.7. *Let (X,ρ) be a $\mathbb{P}_I$-monoid. There is a bijective correspondence*

$$\mathcal{CP}^{\dagger}\underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}}\{F\in\mathsf{SPF}X\mid(\tau_X\cdot\rho)(F)=F\}$$

where $\Lambda(F)$ is the forward Cauchy ideal generated by $F^{\uparrow}$ and $\Gamma(\mathfrak{F})$ is the saturation of $\{\mu\mid\mathfrak{F}(\mu)=1\}$.

Proof. Given a $\mathfrak{F}\in\mathcal{CP}^{\dagger}X$, since $\mathfrak{F}$ is a forward Cauchy ideal then $\{\mu\mid\mathfrak{F}(\mu)=1\}$ is a prefilter basis. For any $\mu\in\Gamma(\mathfrak{F})$, since

$$\bigvee_{\nu\in\Gamma(\mathfrak{F})}\mathrm{sub}_X(\nu,\mu^{\uparrow})\geq\bigvee_{\mathfrak{F}(\nu)=1}\mathrm{sub}_X(\nu,\mu)=1$$

we have that $\mu^{\uparrow}\in\Gamma(\mathfrak{F})$. Thus, Γ is well-defined.

As $\mathcal{P}^{\dagger}X$ is complete and separated, by Proposition 3.1 we have that

$$\mathfrak{F}=\bigvee_{\mathfrak{F}(\mu)=1}\mathcal{P}^{\dagger}X(-,\mu).$$

For any $\nu\in\Gamma(\mathfrak{F})$ we have that

$$\mathfrak{F}(\nu^{\uparrow})=\bigvee_{\mathfrak{F}(\mu)=1}\mathrm{sub}_X(\mu,\nu^{\uparrow})=\bigvee_{\mathfrak{F}(\mu)=1}\mathrm{sub}_X(\mu,\nu)=1,$$

hence $\Gamma(\mathfrak{F})^{\uparrow}=\{\mu\mid\mathfrak{F}(\mu)=1\}$. Thus,

$$\mathfrak{F}=\bigvee_{\mathfrak{F}(\mu)=1}\mathcal{P}^{\dagger}X(-,\mu)$$

$$= \bigvee_{\mu \in \Gamma(\mathfrak{F})^{\uparrow}} \mathcal{P}^{\dagger}X(-,\mu)$$
$$= \Lambda\Gamma(\mathfrak{F}).$$

The equality $\Gamma\Lambda(F) = F$ is trivial. □

Now, we prove Theorem 3.5.

Proof of Theorem 3.5. Let (X, ρ) be a $\mathbb{P}_I$-monoid. By Lemma 3.7, we have that the function

$$q_X \colon \mathcal{C}\mathcal{P}^{\dagger}X \longrightarrow \mathsf{SPF}X,\ \mathfrak{F} \longmapsto \Gamma(\mathfrak{F})$$

is the equalizer of the pair $((\tau_X \cdot \rho)^{\mathbb{SPF}}, 1_{\mathsf{SPF}X})$, and for each function $f \colon A \to \mathsf{SPF}X$ with $f = (\tau_X \cdot \rho)^{\mathbb{SPF}} \cdot f$,

$$\overline{f} \colon A \longrightarrow \mathcal{C}\mathcal{P}^{\dagger}X,\ x \longmapsto \Lambda(f(x))$$

is the unique factorization of f through q_X.

The p_X is given by

$$p_X \colon \mathsf{SPF}X \longrightarrow \mathcal{C}\mathcal{P}^{\dagger}X,\ F \longmapsto \Lambda(((\tau_X \cdot \rho)(F))^{\uparrow}).$$

Since

$$\begin{aligned}
(m_X \cdot \tau_{\mathsf{SPF}X})(\phi) \supset F &\iff \{\mu \mid \mu^{\sharp} \geq \phi\} \supset F \\
&\iff \phi \leq \bigwedge_{\mu \in F} \mu^{\sharp} \\
&\iff \phi \leq \bigwedge_{\substack{\mu \in F \\ \nu \in P_I X}} \mathrm{sub}_X(\mu,\nu) \to \nu^{\sharp} \\
&\iff \phi \leq \bigwedge_{\nu \in P_I X} \nu^{\sharp}(F) \to \nu^{\sharp},
\end{aligned}$$

we obtain the right adjoint w_X of $m_X \cdot \tau_{\mathsf{SPF}X}$. So the structural function of $\mathsf{SPF}'X$ is given by

$$(P_I(p_X) \cdot w_X \cdot q_X)(\mathfrak{F})(\mathfrak{G}) = \bigvee_{p_X(F) = \mathfrak{G}} \bigwedge_{\mu \in P_I X} \mu^{\sharp}(q_X(\mathfrak{F})) \to \mu^{\sharp}(F)$$

$$
\begin{aligned}
&= \bigwedge_{\mu\in P_I X} \mu^\sharp(q_X(\mathfrak{F})) \to \mu^\sharp(q_X(\mathfrak{G})) && (F\subset q_X(p_X(F)))\\
&= \bigwedge_{\mu\in P_I X} \mathfrak{F}(\mu^\uparrow) \to \mathfrak{G}(\mu^\uparrow) && (\mu^\sharp\cdot q_X = (\mu^\uparrow)^\natural)\\
&= \bigwedge_{\mu\in \mathcal{P}^\dagger X} \mathfrak{F}(\mu) \to \mathfrak{G}(\mu),
\end{aligned}
$$

for any $\mathfrak{F},\mathfrak{G}\in\mathcal{CP}^\dagger X$. Thus, the $\mathbb{P}_I$-monoid structure of $\mathsf{SPF}'X$ is $\mathrm{sub}_{\mathcal{P}^\dagger X}$. Therefore, $\mathsf{SPF}'X = \mathcal{CP}^\dagger X$.

By routine computing, we have that $e'_{(X,\rho)} = (\mathbf{y} * \mathbf{y}^{\mathrm{op}})_X$.

For each $\mathbf{F}\in(\mathcal{CP}^\dagger)^2X$, it holds that

$$
\begin{aligned}
(p_X\cdot q_X^{\mathbb{SPF}}\cdot q_{\mathcal{CP}^\dagger X})(\mathbf{F}) &= (p_X\cdot q_X^{\mathbb{SPF}})(\Gamma(\mathbf{F}))\\
&= p_X(\{\mu \mid (\mu^\sharp\cdot q_X)\in\Gamma(\mathbf{F})\})\\
&= \bigvee_{(\mu^\sharp\cdot q_X)\in\Gamma(\mathbf{F})} \mathrm{sub}_X(\mu^\uparrow,-)\\
&= \bigvee_{(\mu^\uparrow)^\natural\in\Gamma(\mathbf{F})} \mathrm{sub}_X(\mu^\uparrow,-) && (\mu^\sharp\cdot q_X = (\mu^\uparrow)^\natural)\\
&= \bigvee_{\mathbf{F}((\mu^\uparrow)^\natural)=1} \mathrm{sub}_X(\mu^\uparrow,-)\\
&= \mathbf{F}((-)^\natural) && \text{(Proposition 3.1)}\\
&= \mathbf{n}(\mathbf{F})(-). && \square
\end{aligned}
$$

Corollary 3.8. *The $\mathbb{SPF}$-algebras are exactly continuous I-lattices.*

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