



Formal balls of \mathbf{Q} -categories

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Communicated by Professor Naser Hosseini

Abstract. The construction of the formal ball model for metric spaces due to Edalat and Heckmann was generalized to \mathbf{Q} -categories by Kostanek and Waszkiewicz, where \mathbf{Q} is a commutative and unital quantale. This paper concerns the influence of the structure of the quantale \mathbf{Q} on the connection between Yoneda completeness of \mathbf{Q} -categories and directed completeness of their sets of formal balls. In the case that \mathbf{Q} is the unit interval $[0, 1]$ equipped with a continuous t-norm $\&$, it is shown that in order that Yoneda completeness of each \mathbf{Q} -category be equivalent to directed completeness of its set of formal balls, a necessary and sufficient condition is that the t-norm $\&$ is Archimedean.

1 Introduction

The formal ball construction is a basic tool for quasi-metric spaces, as demonstrated in [5–7]. In [7] Goubault-Larrecq and Ng argued that “formal balls are the essence of quasi-metric space”. Formal balls were first introduced in [23] for metric spaces, later extended to quasi-metric spaces

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Keywords: \mathbf{Q} -category, formal ball, continuous t-norm, quantale.
[2010]: 18D20, 06B99.

Received: 18 July 2022, Accepted: 9 November 2022.

ISSN: Print 2345-5853, Online 2345-5861.

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in [18, 20], then to the general setting of \mathbf{Q} -categories in [10], where \mathbf{Q} is a commutative and unital quantale.

Metric properties of a quasi-metric space X are closely related with the order structure of its set $\mathbf{B}X$ of formal balls. A typical example says, a quasi-metric space X is continuous and Yoneda complete in the sense of [10, 22] if and only if $\mathbf{B}X$ is a continuous dcpo (directed complete partially ordered set) [5, 7]. Another example in this vein, which motivates this paper, asserts that a quasi-metric space X is Yoneda complete if and only if $\mathbf{B}X$ is a dcpo. This result was first proved in [2] for metric spaces, then in [1, 10] for quasi-metric spaces. In fact, Kostanek and Waszkiewicz [10] proved the conclusion for \mathbf{Q} -categories with \mathbf{Q} being a special kind of *value quantale*, not only for quasi-metric spaces. But, the requirements imposed on the quantale in [10] are so strong that if \mathbf{Q} is the interval $[0, 1]$ together with a continuous t-norm, up to isomorphism there is only one t-norm, namely the product t-norm, that satisfies the requirements.

Thus, for a general quantale \mathbf{Q} , even for the quantale obtained by endowing the interval $[0, 1]$ with a continuous t-norm, the question remains open whether we have an equivalence between Yoneda completeness of \mathbf{Q} -categories and directed completeness of their sets of formal balls.

In this paper we show that the answer is negative. Actually, the answer depends on the structure of the quantale \mathbf{Q} , as we shall see. Corollary 4.9 shows that, in the case that \mathbf{Q} is the unit interval $[0, 1]$ equipped with a continuous t-norm, the equivalence holds if and only if the t-norm is Archimedean.

2 Preliminaries

A commutative and unital quantale (a quantale for short) [17]

$$\mathbf{Q} = (\mathbf{Q}, \&, k)$$

is a commutative monoid with k being the unit, such that the underlying set \mathbf{Q} is a complete lattice (with a top element 1 and a bottom element 0), and that the multiplication $\&$ distributes over arbitrary joins in the sense that

$$p \& \left(\bigvee_{i \in I} q_i \right) = \bigvee_{i \in I} p \& q_i$$

for all $p, q_i \in \mathbf{Q}$ ($i \in I$). If the unit k is the top element, then we say that \mathbf{Q} is *integral*. For all elements p, q of an integral quantale, it is clear that $p \& q \leq p \wedge q$.

Given a quantale $\mathbf{Q} = (\mathbf{Q}, \&, k)$, the multiplication $\&$ determines a binary operator \rightarrow , called the implication operator of $\&$, via the adjoint property:

$$p \& q \leq r \iff q \leq p \rightarrow r.$$

Typical examples of quantales include (i) $(H, \wedge, 1)$, where H is a complete Heyting algebra; (ii) Lawvere's quantale $([0, \infty]^{\text{op}}, +, 0)$; and (iii) $([0, 1], \&, 1)$, where $\&$ is a left continuous t-norm. Actually, a left continuous t-norm on $[0, 1]$ [9] is just a binary operation $\&: [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that $([0, 1], \&, 1)$ is a quantale.

A continuous t-norm on $[0, 1]$ is a left continuous t-norm that is continuous with respect to the usual topology. We refer to the monograph [9] for continuous t-norms. Three basic continuous t-norms and their implication operators are listed below:

(i) The Gödel t-norm:

$$x \&_G y = \min\{x, y\}; \quad x \rightarrow y = \begin{cases} 1 & x \leq y, \\ y & x > y. \end{cases}$$

(ii) The product t-norm:

$$x \&_P y = xy; \quad x \rightarrow y = \begin{cases} 1 & x \leq y, \\ y/x & x > y. \end{cases}$$

(iii) The Łukasiewicz t-norm:

$$x \&_L y = \max\{0, x + y - 1\}; \quad x \rightarrow y = \min\{1 - x + y, 1\}.$$

A continuous t-norm on $[0, 1]$ is *Archimedean*, if for all $x, y \in (0, 1)$ there is some integer n such that $x^n < y$, where

$$x^n = \underbrace{x \& x \& \cdots \& x}_{n \text{ times}}.$$

It is not hard to see that a continuous t-norm is Archimedean if and only if it has no idempotent element other than 0 and 1. It is well-known (see e.g. [9]) that if $\&$ is a continuous Archimedean t-norm, then the quantale $([0, 1], \&, 1)$ is either isomorphic to $([0, 1], \&_{\mathcal{L}}, 1)$ or to $([0, 1], \&_{\mathcal{P}}, 1)$. In other words, up to isomorphism there are precisely two Archimedean continuous t-norms on $[0, 1]$: the product t-norm and the Łukasiewicz t-norm.

Definition 2.1. [8, 14, 21] Let \mathbf{Q} be a quantale. A \mathbf{Q} -category consists of a set X and a map $o: X \times X \rightarrow \mathbf{Q}$ such that

$$k \leq o(x, x), \quad o(y, z) \& o(x, y) \leq o(x, z)$$

for all $x, y, z \in X$. As usual, we write X for the pair (X, o) and write $X(x, y)$ for $o(x, y)$ if no confusion would arise.

If \mathbf{Q} is the Boolean algebra $\{0, 1\}$, a \mathbf{Q} -category is exactly a preordered set; that is, a set together with a reflexive and transitive relation. If \mathbf{Q} is Lawvere's quantale $([0, \infty]^{\text{op}}, +, 0)$, a \mathbf{Q} -category is exactly a generalized metric space in the sense of Lawvere [14]; such a \mathbf{Q} -category is also known as a pseudo-quasi-metric space (with distance allowed to be infinite).

Let X be a \mathbf{Q} -category. A *formal ball* of X is a pair (x, r) with $x \in X$ and $r \in \mathbf{Q}$, x is called the center and r the radius. For formal balls (x, r) and (y, s) , define

$$(x, r) \leq_{\mathbf{B}X} (y, s) \quad \text{if } r \leq s \& X(x, y).$$

Then $\leq_{\mathbf{B}X}$ is a reflexive and transitive relation, hence a preorder. We write $\mathbf{B}X$ for the set of formal balls of X endowed with the preorder $\leq_{\mathbf{B}X}$. We often omit the subscript if it causes no confusion. In this paper the radius r of a formal ball (x, r) is allowed to be the bottom element of \mathbf{Q} .

A net $(x_\lambda)_{\lambda \in D}$ in a \mathbf{Q} -category X is *forward Cauchy* [3, 21] if

$$\bigvee_{\lambda} \bigwedge_{\lambda \leq \gamma \leq \mu} X(x_\gamma, x_\mu) \geq k.$$

An element a of X is a *Yoneda limit* [3, 21] of $(x_\lambda)_{\lambda \in D}$ if for all $y \in X$,

$$X(a, y) = \bigvee_{\lambda} \bigwedge_{\mu \geq \lambda} X(x_\mu, y).$$

Yoneda limits of forward Cauchy nets can be characterized as colimits of forward Cauchy weights. A *weight* of a \mathbf{Q} -category X is a map $\phi: X \rightarrow \mathbf{Q}$ such that

$$\phi(y) \& X(x, y) \leq \phi(x)$$

for all $x, y \in X$. For each forward Cauchy net $(x_\lambda)_{\lambda \in D}$ of X , the map

$$\phi := \bigvee_{\lambda} \bigwedge_{\mu \geq \lambda} X(-, x_\mu)$$

is a weight of X ; such a weight is said to be *forward Cauchy* [13]. Forward Cauchy weights of a \mathbf{Q} -category X are also known as *ideals* in the literature, see e.g. [3]. An element a of X is a *colimit* of a weight ϕ [3, 8] if for all $y \in X$,

$$X(a, y) = \bigwedge_{x \in X} (\phi(x) \rightarrow X(x, y)).$$

Proposition 2.2. ([3, Lemma 46]) *Let $(x_\lambda)_{\lambda \in D}$ be a forward Cauchy net of a \mathbf{Q} -category X . Then, an element a of X is a Yoneda limit of $(x_\lambda)_{\lambda \in D}$ if and only if a is a colimit of the forward Cauchy weight*

$$\phi = \bigvee_{\lambda} \bigwedge_{\mu \geq \lambda} X(-, x_\mu).$$

Therefore, a \mathbf{Q} -category X is Yoneda complete if and only if every forward Cauchy weight of X has a colimit.

Given a \mathbf{Q} -category X , Yoneda completeness of X is closely related with directed completeness of its set $\mathbf{B}X$ of formal balls. As mentioned before, when \mathbf{Q} is Lawvere's quantale $([0, \infty]^{\text{op}}, +, 0)$, a \mathbf{Q} -category (i.e., a generalized metric space) is Yoneda complete if and only if its set of formal balls is directed complete [1, 2, 10].

In the case that \mathbf{Q} is a continuous and integral quantale, Proposition 2.2 and Lemma 2.3 below explain to some extent why Yoneda completeness of a \mathbf{Q} -category and directed completeness of its set of formal balls are closely related.

Before proceeding on, we recall the notion of continuous lattices first. Let a, b be elements of a partially ordered set P . We say that a is *way below* b , in symbols $a \ll b$, if for each directed set D of P with a join,

$$b \leq \bigvee D \implies \exists d \in D, a \leq d.$$

A *continuous lattice* [4] is a complete lattice L for which every element is the join of elements way below it; that is, $a = \bigvee\{x \in L \mid x \ll a\}$. The interval $[0, 1]$ is clearly a continuous lattice. A *continuous quantale* is a quantale for which the underlying lattice is continuous.

Lemma 2.3. *Let \mathbf{Q} be a continuous and integral quantale. Then, for each weight ϕ of a \mathbf{Q} -category X the following are equivalent:*

- (1) ϕ is forward Cauchy.
- (2) ϕ satisfies the following conditions:
 - (i) $\bigvee_{x \in X} \phi(x) = 1$;
 - (ii) If $r \ll 1$ and $s_i \ll \phi(x_i)$ ($i = 1, 2$), then there exists $x \in X$ such that $r \ll \phi(x)$ and that $s_i \ll X(x_i, x)$ ($i = 1, 2$).
- (3) There is a directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ such that $\bigvee_{\lambda \in D} r_\lambda = 1$ and that $\phi = \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} X(-, x_\mu)$.

Proof. That (3) implies (1) follows immediately from Lemma 2.4 below. The equivalence (1) \Leftrightarrow (2) is contained in [12, Lemma 6.3]; the implication (2) \Rightarrow (3) is also proved there implicitly. So, here we only write down the construction of the directed subset. Suppose that ϕ satisfies the conditions (i) and (ii). Let

$$\mathbf{B}\phi = \{(x, r) \in \mathbf{B}X \mid r \ll \phi(x)\}.$$

Then $\mathbf{B}\phi$ is a directed subset of $\mathbf{B}X$ that satisfies the requirement. \square

Lemma 2.4. *Let \mathbf{Q} be an integral quantale; let X be a \mathbf{Q} -category and $(x_\lambda, r_\lambda)_{\lambda \in D}$ be a directed subset of $\mathbf{B}X$. If $\bigvee_{\lambda \in D} r_\lambda = 1$, then $(x_\lambda)_{\lambda \in D}$ is a forward Cauchy net in X .*

Proof. Since $r_\lambda \leq r_\mu \& X(x_\lambda, x_\mu)$ whenever $\lambda \leq \mu$, it follows that

$$1 = \bigvee_{\lambda \in D} r_\lambda \leq \bigvee_{\lambda \in D} \bigwedge_{\lambda \leq \mu} r_\mu \leq \bigvee_{\lambda \in D} \bigwedge_{\lambda \leq \mu \leq \gamma} X(x_\mu, x_\gamma),$$

hence $(x_\lambda)_{\lambda \in D}$ is forward Cauchy. \square

3 Counterexamples

In this section we give two examples to show that for a general quantale \mathbf{Q} , Yoneda completeness of a \mathbf{Q} -category may fail to be equivalent to directed completeness of its set of formal balls.

Example 3.1. This example presents a \mathbf{Q} -category that is Yoneda complete, but its set of formal balls is not directed complete.

Let $\&$ be a continuous non-Archimedean t-norm and let \mathbf{Q} be the quantale $([0, 1], \&, 1)$. Since $\&$ is continuous and non-Archimedean, there is some $b \in (0, 1)$ such that $b \& b = b$. Let $X = (0, b)$ and define a \mathbf{Q} -category structure on X by

$$X(x, y) = \begin{cases} 1 & x = y, \\ \min\{x \rightarrow y, y \rightarrow x\} & x \neq y. \end{cases}$$

Since $X(x, y) \leq b$ whenever $x \neq y$, every forward Cauchy net of X is eventually constant, so X is Yoneda complete. It remains to show that $\mathbf{B}X$ is not directed complete. To this end, pick a strictly increasing sequence $(x_n)_{n \geq 1}$ in $(0, b)$ that converges to b . For each n let $r_n = x_n$. We claim that the subset $(x_n, r_n)_{n \geq 1}$ of $\mathbf{B}X$ is directed and has no join.

Since $\&$ is a continuous t-norm, for all $x, y \in [0, 1]$ we have

$$x \& (x \rightarrow y) = \min\{x, y\}.$$

For all $n \leq m$, since

$$r_n = x_n = x_m \& (x_m \rightarrow x_n) = r_m \& X(x_n, x_m),$$

then $(x_n, r_n) \leq (x_m, r_m)$, hence $(x_n, r_n)_{n \geq 1}$ is directed.

Next we show that $(x_n, r_n)_{n \geq 1}$ does not have a join. Suppose on the contrary that (x, r) is a join of $(x_n, r_n)_{n \geq 1}$. Since $x < b$, there is some n_0 such that $x < x_m$ for all $m \geq n_0$. Since (x, r) is an upper bound of $(x_n, r_n)_{n \geq 1}$, for each $m \geq n_0$, we have

$$r_m \leq r \& X(x_m, x) \leq x_m \rightarrow x,$$

this is impossible since the left side tends to b , while the right side tends to x .

Example 3.2. This example presents a \mathbf{Q} -category that is not Yoneda complete, but its set of formal balls is directed complete.

Let \mathbf{Q} be the quantale $([0, 1], \&, 1)$, where $\&$ is the Gödel t-norm \min . Let

$$X = \{1\} \cup \{1 - 1/n \mid n \geq 2\}.$$

Define a \mathbf{Q} -category structure on X by

$$X(x, y) = \begin{cases} 1 & x = y, \\ 1/3 & x = 1, y \neq 1, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

We claim that X is not Yoneda complete, but $\mathbf{B}X$ is directed complete.

For each $n \geq 2$, let $x_n = 1 - 1/n$. It is readily verified that the sequence $(x_n)_{n \geq 2}$ is forward Cauchy and has no Yoneda limit, so X is not Yoneda complete. It remains to check that $\mathbf{B}X$ is directed complete. Given a directed subset D of $\mathbf{B}X$, we write D as a net $(x_\lambda, r_\lambda)_{\lambda \in D}$ indexed by itself. Since $r_\lambda \leq r_\mu \& X(x_\lambda, x_\mu)$ whenever $\lambda \leq \mu$, the net $(r_\lambda)_{\lambda \in D}$ is monotone. Let $r = \bigvee_{\lambda \in D} r_\lambda$. We proceed with two cases.

Case 1. The net $(x_\lambda)_{\lambda \in D}$ is eventually constant; that means, there is some $a \in X$ and some $\lambda \in D$ such that $x_\mu = a$ whenever $\mu \geq \lambda$. In this case (a, r) is clearly a join of D .

Case 2. The net $(x_\lambda)_{\lambda \in D}$ is not eventually constant. Then for each λ there is some $\mu \geq \lambda$ such that $x_\lambda \neq x_\mu$, hence

$$r_\lambda \leq r_\mu \& X(x_\lambda, x_\mu) \leq \min\{r_\mu, x_\lambda, x_\mu\} \leq x_\lambda.$$

With help of this fact, one readily verifies that (a, r) is a join of D , where $a = \min\{x \in X \mid r \leq x\}$.

A \mathbf{Q} -category X is *Smyth complete* if it is Cauchy complete (see e.g. [8, 14]) and all forward Cauchy weights of X are Cauchy. The notion of Smyth completeness originated in [19]. For Smyth completeness of quasi-metric spaces we refer to [5, 11, 16]. The postulation of Smyth complete \mathbf{Q} -categories given here is based on the characterization of Smyth complete quasi-metric spaces in [15, Section 6].

Consider the \mathbf{Q} -category X in Example 3.1. Since every forward Cauchy net of X is eventually constant, X is Yoneda complete and Smyth complete,

hence continuous in the sense of [10, 22]. So, in contrast to the situation for quasi-metric spaces [16, Theorem 3.2], for a general quantale \mathbf{Q} , the set of formal balls of a Smyth complete \mathbf{Q} -category may fail to be directed complete.

Example 3.1 also shows that for a directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$, the net $(x_\lambda)_{\lambda \in D}$ of X need not be forward Cauchy.

Proposition 3.3. *Let $\&$ be a continuous t-norm on $[0, 1]$ and let $\mathbf{Q} = ([0, 1], \&, 1)$. The following are equivalent:*

- (1) $\&$ is Archimedean.
- (2) For each \mathbf{Q} -category X and each directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with some $r_\lambda > 0$, $(x_\lambda)_{\lambda \in D}$ is a forward Cauchy net.

Proof. (1) \Rightarrow (2) We'll make use of the following fact about continuous Archimedean t-norms:

$$0 < r \leq s \& t \implies t \geq s \rightarrow r.$$

Assume that $(x_\lambda, r_\lambda)_{\lambda \in D}$ is a directed subset of $\mathbf{B}X$ and, without loss of generality, assume that $r_\lambda > 0$ for all $\lambda \in D$. Since $r_\lambda \leq r_\mu \& X(x_\lambda, x_\mu)$ whenever $\lambda \leq \mu$, then $X(x_\lambda, x_\mu) \geq r_\mu \rightarrow r_\lambda$ whenever $\lambda \leq \mu$. Since $(r_\lambda)_{\lambda \in D}$ converges to its join and the implication operator of an Archimedean continuous t-norm is continuous except possibly at $(0, 0)$, it follows that $X(x_\lambda, x_\mu)$ tends to 1, so $(x_\lambda)_{\lambda \in D}$ is forward Cauchy.

(2) \Rightarrow (1) Suppose on the contrary that $\&$ is non-Archimedean. Consider the \mathbf{Q} -category X in Example 3.1. Then the subset $(x_n, r_n)_{n \geq 1}$ of $\mathbf{B}X$ given there is directed, but $(x_n)_{n \geq 2}$ is not forward Cauchy, a contradiction. \square

4 The main result

In order to state the main result, we still need two notions.

Let $\mathbf{Q} = (\mathbf{Q}, \&, k)$ be a quantale. We say that $\&$ *distributes over non-empty meets* if

$$p \& \left(\bigwedge_{i \in I} q_i \right) = \bigwedge_{i \in I} p \& q_i$$

for any $p \in \mathbf{Q}$ and any non-empty subset $(q_i)_{i \in I}$ of \mathbf{Q} . It is clear that any continuous t-norm on $[0, 1]$ distributes over non-empty meets.

Definition 4.1. Let \mathbf{Q} be a continuous and integral quantale. We say that a \mathbf{Q} -category X has property (R), if for each pair (s, t) of elements of \mathbf{Q} with $0 < s \leq t$, there is some $r \ll 1$ such that for all $x, y \in X$ and all $r' \geq r$, we always have

$$(x, t \& r') \leq (y, s) \iff (x, r') \leq (y, t \rightarrow s).$$

Corollary 4.7 below provides a characterization of \mathbf{Q} -categories with property (R) in the case that the quantale \mathbf{Q} is the unit interval $[0, 1]$ together with a continuous t-norm. By this characterization it is easy to find \mathbf{Q} -categories with or without property (R). Now we present the main result of this paper.

Theorem 4.2. *Let \mathbf{Q} be a continuous and integral quantale such that $\&$ distributes over non-empty meets; let X be a \mathbf{Q} -category.*

- (i) *If X is Yoneda complete, then each directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\bigvee_{\lambda \in D} r_\lambda = 1$ has a join.*
- (ii) *If X has property (R) and $\mathbf{B}X$ is directed complete, then X is Yoneda complete.*

Before proving Theorem 4.2, we make some preparations.

Lemma 4.3. *Let \mathbf{Q} be a continuous and integral quantale such that $\&$ distributes over non-empty meets; let X be a \mathbf{Q} -category and let $(x_\lambda, r_\lambda)_{\lambda \in D}$ be a directed subset of $\mathbf{B}X$. If $(x_\lambda)_{\lambda \in D}$ is a forward Cauchy net with x being a Yoneda limit, then (x, r) is a join of the directed set $(x_\lambda, r_\lambda)_{\lambda \in D}$, where $r = \bigvee_{\lambda \in D} r_\lambda$.*

Proof. The proof is a slight improvement of that for Lemma 7.7 in [10].

First, we show that (x, r) is an upper bound of $(x_\lambda, r_\lambda)_{\lambda \in D}$; that is, $r_\lambda \leq r \& X(x_\lambda, x)$ for all $\lambda \in D$.

For each $\lambda \in D$ and each $\epsilon \ll r_\lambda$, since $\&$ distributes over non-empty meets and

$$1 = X(x, x) = \bigvee_{\delta} \bigwedge_{\mu \geq \delta} X(x_\mu, x),$$

there is some $\delta \in D$ such that $\epsilon \leq r_\lambda \& X(x_\mu, x)$ whenever $\mu \geq \delta$. Thus, for all $\mu \geq \lambda, \delta$, we have

$$\epsilon \leq r_\lambda \& X(x_\mu, x) \leq r_\mu \& X(x_\lambda, x_\mu) \& X(x_\mu, x) \leq r \& X(x_\lambda, x).$$

By arbitrariness of ϵ we obtain that $r_\lambda \leq r \& X(x_\lambda, x)$.

Next we show that $(x, r) \leq (y, s)$ for any upper bound (y, s) of $(x_\lambda, r_\lambda)_{\lambda \in D}$. Since (y, s) is an upper bound of $(x_\lambda, r_\lambda)_{\lambda \in D}$, then

$$r_\lambda \leq r_\mu \leq s \& X(x_\mu, y)$$

whenever $\lambda \leq \mu$, hence

$$r = \bigvee_{\lambda \in D} r_\lambda \leq \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} s \& X(x_\mu, y) = s \& X(x, y),$$

which shows that $(x, r) \leq (y, s)$, as desired. \square

Proposition 4.4. *Let \mathbf{Q} be a continuous and integral quantale such that $\&$ distributes over non-empty meets. If X is a Yoneda complete \mathbf{Q} -category, then every directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\bigvee_{\lambda \in D} r_\lambda = 1$ has a join.*

Proof. This follows directly from Lemma 2.4 and Lemma 4.3. \square

Lemma 4.5. *Let \mathbf{Q} be a continuous and integral quantale. Then for all x and y of a \mathbf{Q} -category X with property (R), the following conditions are equivalent:*

- (1) $(x, 1) \leq (y, 1)$.
- (2) $(x, s) \leq (y, s)$ for all $s \neq 0$.
- (3) $(x, s) \leq (y, s)$ for some $s \neq 0$.

Proof. It suffices to check (3) \Rightarrow (1). Since $0 < s \leq s$, there is some $r \ll 1$ such that

$$(x, s \& r') \leq (y, s) \iff (x, r') \leq (y, s \rightarrow s)$$

for all $r' \geq r$. Putting $r' = 1$ gives that $(x, 1) \leq (y, 1)$. \square

Proposition 4.6. *Let $\&$ be a continuous t -norm on $[0, 1]$ and let $\mathbf{Q} = ([0, 1], \&, 1)$. Then, $\&$ is Archimedean if and only if every \mathbf{Q} -category has property (R).*

Proof. For sufficiency we need to show that $\&$ has no nontrivial idempotent element. For this it suffices to show that for all $s \neq 0$ and $q \in [0, 1]$,

$$s \leq s \& q \implies q = 1.$$

Consider the \mathbf{Q} -category $X = \{x, y\}$ with $X(x, x) = X(y, y) = 1$ and $X(x, y) = q = X(y, x)$. Since $(x, s) \leq (y, s)$ and X has property (R), then $(x, 1) \leq (y, 1)$, hence $1 \leq 1 \& X(x, y) = q$.

As for necessity, assume that $\&$ is a continuous Archimedean t-norm. Then $\&$ is either isomorphic to the Łukasiewicz t-norm or to the product t-norm. In the following we check the conclusion for the case that $\&$ is isomorphic to the Łukasiewicz t-norm, leaving the other case to the reader. Without loss of generality, we assume that $\&$ is, not only isomorphic to, the Łukasiewicz t-norm; that is,

$$x \& y = \max\{0, x + y - 1\}.$$

Suppose that X is a \mathbf{Q} -category and $0 < s \leq t$. Pick $r \in (1 - t, 1)$. Then $r \ll 1$ and for all $r' \geq r$,

$$\begin{aligned} (x, t \& r') \leq (y, s) &\iff r' + t - 1 \leq s + X(x, y) - 1 \\ &\iff r' \leq s - t + X(x, y) \\ &\iff (x, r') \leq (y, t \rightarrow s). \end{aligned}$$

This completes the proof. \square

By the ordinal sum decomposition theorem of continuous t-norms [9] and the argument of Proposition 4.6, one readily verifies the following conclusion.

Corollary 4.7. *Let $\mathbf{Q} = ([0, 1], \&, 1)$, where $\&$ is a continuous t-norm on $[0, 1]$. Then, a \mathbf{Q} -category X has property (R) if and only if it satisfies the following condition: for all $x, y \in X$, if $X(x, y) \geq p$ for some idempotent element $p > 0$, then $X(x, y) = 1$.*

Lemma 4.8. *Suppose that $\mathbf{Q} = (\mathbf{Q}, \&, k)$ is a continuous and integral quantale such that $\&$ distributes over non-empty meets. Let X be a \mathbf{Q} -category; let a be an element of X and let $(x_\lambda, r_\lambda)_{\lambda \in D}$ be a directed subset of $\mathbf{B}X$ for which $\bigvee_{\lambda \in D} r_\lambda = 1$. Consider the statements:*

- (1) a is a Yoneda limit of $(x_\lambda)_{\lambda \in D}$.
- (2) $(a, 1)$ is a join of $(x_\lambda, r_\lambda)_{\lambda \in D}$.

Then, (1) implies (2). Further, if X has property (R) and $\mathbf{B}X$ is directed complete, (2) also implies (1).

Proof. That (1) implies (2) follows from Lemma 2.4 and Lemma 4.3. Now, assume that X has property (R), $\mathbf{B}X$ is directed complete, and that $(a, 1)$ is a join of $(x_\lambda, r_\lambda)_{\lambda \in D}$. We show that a is a Yoneda limit of $(x_\lambda)_{\lambda \in D}$; that means, for all $y \in X$,

$$X(a, y) = \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} X(x_\mu, y).$$

Fix $\lambda \in D$. Since for all $\mu \geq \lambda$,

$$r_\lambda \& X(a, y) \leq r_\mu \& X(a, y) \leq X(x_\mu, a) \& X(a, y) \leq X(x_\mu, y),$$

it follows that

$$r_\lambda \& X(a, y) \leq \bigwedge_{\mu \geq \lambda} X(x_\mu, y),$$

hence

$$X(a, y) = \bigvee_{\lambda \in D} r_\lambda \& X(a, y) \leq \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} X(x_\mu, y).$$

For the converse inequality, let

$$t = \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} X(x_\mu, y).$$

We wish to show that $t \leq X(a, y)$. We may assume that $t > 0$.

It is clear that $(x_\lambda, t \& r_\lambda)_{\lambda \in D}$ is a directed subset of $\mathbf{B}X$, hence has a join, say (z, s) . We claim that $(z, s) \cong (a, t)$; that is, $(z, s) \leq (a, t)$ and $(a, t) \leq (z, s)$. Since (a, t) is an upper bound of the directed set $(x_\lambda, t \& r_\lambda)_{\lambda \in D}$, it follows that $(z, s) \leq (a, t)$; in particular $0 < s \leq t$. Since $\bigvee_{\lambda \in D} r_\lambda = 1$, we may assume that all r_λ are large enough. Since (z, s) is a join of $(x_\lambda, t \& r_\lambda)_{\lambda \in D}$, then $(x_\lambda, t \& r_\lambda) \leq (z, s)$ for all $\lambda \in D$, then $(x_\lambda, r_\lambda) \leq (z, t \rightarrow s)$ for all $\lambda \in D$ because X has property (R), and then $(a, 1) \leq (z, t \rightarrow s)$ because $(a, 1)$ is a join of $(x_\lambda, r_\lambda)_{\lambda \in D}$. Therefore, $t = s$ and $(a, 1) \leq (z, 1)$, hence $(a, t) \leq (z, t) = (z, s)$. This proves that $(z, s) \cong (a, t)$.

For each $\lambda \in D$, since

$$r_\lambda \& t = r_\lambda \& \bigvee_{\gamma \in D} \bigwedge_{\mu \geq \gamma} X(x_\mu, y)$$

$$\begin{aligned}
&= r_\lambda \& \bigvee_{\gamma \geq \lambda} \bigwedge_{\mu \geq \gamma} X(x_\mu, y) \quad (D \text{ is directed}) \\
&\leq \bigvee_{\gamma \geq \lambda} \bigwedge_{\mu \geq \gamma} r_\lambda \& X(x_\mu, y) \\
&\leq \bigvee_{\gamma \geq \lambda} \bigwedge_{\mu \geq \gamma} r_\mu \& X(x_\lambda, x_\mu) \& X(x_\mu, y) \\
&\leq X(x_\lambda, y),
\end{aligned}$$

it follows that $(x_\lambda, t \& r_\lambda) \leq (y, 1)$. Thus, $(y, 1)$ is an upper bound of the set $(x_\lambda, t \& r_\lambda)_{\lambda \in D}$, hence $(a, t) \leq (y, 1)$, and then $t \leq X(a, y)$. \square

Proof of Theorem 4.2. (i) This is $(1) \Rightarrow (2)$ in Lemma 4.8.

(ii) We show that every forward Cauchy weight ϕ of X has a colimit. By Lemma 2.3, there is a directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ such that $\bigvee_{\lambda \in D} r_\lambda = 1$ and that

$$\phi = \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} X(-, x_\mu).$$

By assumption, $(x_\lambda, r_\lambda)_{\lambda \in D}$ has a join, say $(a, 1)$. By Lemma 4.8, a is a Yoneda limit of the forward Cauchy net $(x_\lambda)_{\lambda \in D}$, hence a colimit of ϕ by Proposition 2.2. \square

Corollary 4.9. *Let $\&$ be a continuous t -norm on $[0, 1]$ and $\mathbf{Q} = ([0, 1], \&, 1)$. Then the following are equivalent:*

- (1) $\&$ is Archimedean.
- (2) For each \mathbf{Q} -category X , X is Yoneda complete if and only if $\mathbf{B}X$ is directed complete.

Proof. (1) \Rightarrow (2) If X is Yoneda complete, then $\mathbf{B}X$ is directed complete by Proposition 3.3 and Lemma 4.3. Conversely, if $\mathbf{B}X$ is directed complete, then X is Yoneda complete by Proposition 4.6 and Theorem 4.2.

(2) \Rightarrow (1) Example 3.1. \square

The following example shows that in Theorem 4.2(ii), the requirement that $\mathbf{B}X$ is directed complete cannot be weakened to that every directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\bigvee_{\lambda \in D} r_\lambda = 1$ has a join.

Example 4.10. Let \mathbf{Q} be the quantale $\mathbf{Q} = ([0, 1], \&_P, 1)$, where $\&_P$ is the product t-norm. By Proposition 4.6 every \mathbf{Q} -category has property (R). We claim that there is a \mathbf{Q} -category X such that every directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\bigvee_{\lambda \in D} r_\lambda = 1$ has a join, but X is not Yoneda complete. Since $([0, 1], \&_P, 1)$ is isomorphic to Lawvere's quantale $([0, \infty]^{\text{op}}, +, 0)$, it suffices to construct a quasi-metric space (X, d) such that (X, d) is not Yoneda complete, but every directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\inf_{\lambda \in D} r_\lambda = 0$ has a join.

Let $X = \{0\} \cup \{1/n \mid n \geq 2\}$. Define a quasi-metric d on X by

$$d(x, y) = \begin{cases} 1/2 & x = 0, y \neq 0, \\ \max\{0, y - x\} & \text{otherwise.} \end{cases}$$

In the following we show that (X, d) satisfies the requirements.

(i) (X, d) is not Yoneda complete, since the sequence $(1/n)_{n \geq 2}$ is forward Cauchy but has no Yoneda limit.

(ii) We show that every directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\inf_{\lambda \in D} r_\lambda = 0$ has a join. Since $\inf_{\lambda \in D} r_\lambda = 0$, it follows from Lemma 2.4 that $(x_\lambda)_{\lambda \in D}$ is a forward Cauchy net of (X, d) , hence either $(x_\lambda)_{\lambda \in D}$ is eventually constant or $(x_\lambda)_{\lambda \in D}$ converges to 0 (in the usual sense).

Case 1. $(x_\lambda)_{\lambda \in D}$ converges to 0. In this case we show that $(0, 0)$ is a join of $(x_\lambda, r_\lambda)_{\lambda \in D}$. First, since $d(z, 0) = 0$ for all $z \in X$, then $r_\lambda \geq 0 + d(x_\lambda, 0)$ for all $\lambda \in D$, hence $(0, 0)$ is an upper bound of $(x_\lambda, r_\lambda)_{\lambda \in D}$. Next, assume that (y, s) is an upper bound of $(x_\lambda, r_\lambda)_{\lambda \in D}$. It is clear that $s = 0$, so $r_\lambda \geq d(x_\lambda, y)$ for all $\lambda \in D$. Since r_λ converges to 0 and $(x_\lambda)_{\lambda \in D}$ converges to 0, it follows that $y = 0$. This shows that $(0, 0)$ is the only upper bound, hence a join, of $(x_\lambda, r_\lambda)_{\lambda \in D}$.

Case 2. $(x_\lambda)_{\lambda \in D}$ is eventually constant. By assumption there is some $a \in X$ and some $\lambda \in D$ such that $x_\mu = a$ whenever $\mu \geq \lambda$. It is trivial that $(a, 0)$ is a join of $(x_\lambda, r_\lambda)_{\lambda \in D}$.

However, for standard quasi-metric spaces (see [7, Definition 2.1]), we have the following conclusion.

Corollary 4.11. *Let X be a standard quasi-metric spaces. Then, X is Yoneda complete if and only if every directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\inf_{\lambda \in D} r_\lambda = 0$ has a join.*

Proof. Proposition 2.4 in [7] shows that if X is a standard quasi-metric space and every directed subset $(x_\lambda, r_\lambda)_{\lambda \in D}$ of $\mathbf{B}X$ with $\inf_{\lambda \in D} r_\lambda = 0$ has a join, then $\mathbf{B}X$ is directed complete. Thus, the conclusion follows from Theorem 4.2 (ii) immediately. \square

5 Conclusion

Following Lawvere [14], the study of \mathbf{Q} -categories is part of a *generalized pure logic* with \mathbf{Q} as the set of *truth-values*. The connection between categorical properties of a \mathbf{Q} -category and order-theoretic properties of its set of formal balls has received much attention both in mathematics and theoretic computer science. Corollary 4.9 in this paper shows that the structure of the truth-values, i.e., the structure of the quantale \mathbf{Q} , also interacts with this connection. This kind of interaction deserves further investigation.

Acknowledgement

The authors are thankful to the referee for the valuable comments and suggestions that lead to a significant improvement of the paper and acknowledge gratefully the support of the National Natural Science Foundation of China (No. 11871358).

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