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On injective objects and existence of injective hulls in Q-TOP$/(Y, \sigma)$

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Abstract. In this paper, motivated by Cagliari and Mantovani, we have obtained a characterization of injective objects (with respect to the class of embeddings in the category Q-**TOP** of Q-topological spaces) in the comma category Q-**TOP**$/(Y, \sigma)$, when (Y, σ) is a stratified Q-topological space, with the help of their T_0-reflection. Further, we have proved that for any Q-topological space (Y, σ), the existence of an injective hull of $((X, \tau), f)$ in the comma category Q-**TOP**$/(Y, \sigma)$ is equivalent to the existence of an injective hull of its T_0-reflection $((\tilde{X}, \tilde{\tau}), \tilde{f})$ in the comma category Q-**TOP**$/(\tilde{Y}, \tilde{\sigma})$ (and in the comma category Q-**TOP**$_0/(\tilde{Y}, \tilde{\sigma})$, where Q-**TOP**$_0$ denotes the category of T_0-Q-topological spaces).

1 Introduction

Injectivity and projectivity are important concepts of mathematics and play a fundamental role in various fields of mathematics, in particular in commutative and homological algebra, algebraic geometry and topology. Injective objects and injective hulls have been investigated for a long time in various categories. There

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are many results in this direction, even for more general case of $\mathcal{H}$-injectivity and $\mathcal{H}$-injective hulls, for $\mathcal{H}$ an arbitrary class of morphisms. For example, in [3] there are results for $\mathcal{H}$-injective objects and $\mathcal{H}$-injective hulls in the category **Field** of field and their homomorphisms, for $\mathcal{H}$ the class of algebraic extensions. In [2] we can find results related to $\mathcal{H}$-injective objects in the category **Pos** of partially ordered sets and isotone maps, for $\mathcal{H}$ the class of embeddings. In [14], we have results related to $\mathcal{H}$-injectivity and $\mathcal{H}$-injective hulls in the category **TOP** of topological spaces, for $\mathcal{H}$ the class of embeddings.

The importance of $\mathcal{H}$-injective objects and $\mathcal{H}$-injective hulls in the comma category $\mathbf{C}/B$ for B in a given category **C** has also been recognized as they are related with the weak factorization system in **C** (cf. [2, 13]).

Cagliari and Mantovani studied injective objects and injective hulls in the comma category $\mathbf{TOP}/B$ (cf. [4–6]). In particular, they gave a characterization of injective objects (with respect to the class of embeddings in the category **TOP** of topological spaces) in the comma category $\mathbf{TOP}/B$ (cf. [5]). In [6], they gave a result related to the existence of an injective hull of an object in the comma category $\mathbf{TOP}_0/B$ ($B \in \mathbf{TOP}_0$ and $\mathbf{TOP}_0$ is the category of T_0-topological spaces).

It is well known that the category $\mathbf{TOP}_0$ is a reflective subcategory of the category **TOP**. In [5] (see also [14]), Cagliari and Mantovani considered the reflector $\pi : \mathbf{TOP} \to \mathbf{TOP}_0$ and they mentioned that for any topological space Y, by virtue of the reflector π, corresponding to each object (X, f) of the comma category $\mathbf{TOP}/Y$ we have the object (X_0, f_0), where $X_0 = \pi(X)$ and $f_0 = \pi(f)$, of the category $\mathbf{TOP}_0/Y_0$, which is called as the T_0-reflection of (X, f). Cagliari and Mantovani [5] gave a characterization of injective objects (with respect to the class of embeddings in **TOP**) in the comma category $\mathbf{TOP}/Y$ with the help of their T_0-reflection. Cagliari and Mantovani [5] also proved that the existence of an injective hull of (X, f) in the comma category $\mathbf{TOP}/Y$ is equivalent to the existence of an injective hull of its T_0-reflection (X_0, f_0) in the comma category $\mathbf{TOP}/Y_0$ (and in the comma category $\mathbf{TOP}_0/Y_0$).

Solovyov [11] introduced the notion of Q-topological spaces and Q-continuous maps and studied the category Q-**TOP** of Q-topological spaces (where Q is a fixed member of a fixed variety of Ω-algebras). Many familiar categories such as the category L-**TOP** of lattice valued topological spaces of Goguen [8] (and hence the category **TOP** of topological spaces and the category **FTOP** of fuzzy topological spaces (in the sense of Chang [7])) are examples of the category Q-**TOP**. Solovyov [11] also introduced the concept of stratified Q-topological spaces and T_0-

Q-topological spaces. Singh and Srivastava [10] proved that the category Q-**TOP**$_0$ of T_0-Q-topological spaces is a reflective subcategory of Q-**TOP**. In [10], for a given Q-topological space (X, τ), Singh and Srivastava defined an equivalence relation $\sim$ on X as, for every $x_1, x_2 \in X$, $x_1 \sim x_2$ if $\alpha(x_1) = \alpha(x_2)$, for every $\alpha \in \tau$. By taking $\tilde{X} = X/\sim$, the set of equivalence classes, and $\tilde{\tau}$ to be the corresponding quotient Q-topology on $\tilde{X}$ induced by the quotient map $q_X : X \to \tilde{X}$, $q_X(x) = [x]$ (where $[x]$ is the equivalence class of x), and τ, they proved that $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is a Q-**TOP**$_0$-reflection for (X, τ) and as a result of this Q-**TOP**$_0$ is a reflective subcategory of Q-**TOP** (cf. Theorem 4.1 in [10]). Consequently, we have the reflector (cf. Proposition 4.22 and Definition 4.23 in [1]) R : Q-**TOP**$\to$ Q-**TOP**$_0$ give by $R((X, \tau)) = (\tilde{X}, \tilde{\tau})$ and if $f : (X, \tau) \to (Y, \sigma)$ is a Q-continuous map, then $R(f) = \tilde{f}$, where $\tilde{f} : (\tilde{X}, \tilde{\tau}) \to (\tilde{Y}, \tilde{\sigma})$ is the unique Q-continuous map such that $q_Y \circ f = \tilde{f} \circ q_X$. Thus for a given Q-topological space (Y, σ), corresponding to each object $((X, \tau), f)$ of the comma category Q-**TOP**$/(Y, \sigma)$, we have the object $((\tilde{X}, \tilde{\tau}), \tilde{f})$ of the comma category Q-**TOP**$_0/(\tilde{Y}, \tilde{\sigma})$, which is called as the T_0-reflection of $((X, \tau), f)$.

Thus keeping in the mind that in Q-topology, good, 'categorically correct', notions of 'To-ness' and 'T_0-reflection' is already available, it is natural to try to know if the injective objects can be characterized in Q-**TOP**$/(Y, \sigma)$ in a similar fashion as in [5]. We do this investigation in this paper and in the process, we have obtained a characterization of injective objects (with respect to the class of embeddings in Q-**TOP**) in the comma category Q-**TOP**$/(Y, \sigma)$, when (Y, σ) is a stratified Q-topological space, with the help of their T_0-reflection, motivated by Cagliari and Mantovani [5]. Further, we have proved that for any Q-topological space (Y, σ), the existence of an injective hull of $((X, \tau), f)$ in the comma category Q-**TOP**$/(Y, \sigma)$ is equivalent to the existence of an injective hull of its T_0-reflection $((\tilde{X}, \tilde{\tau}), \tilde{f})$ in the comma category Q-**TOP**$/(\tilde{Y}, \tilde{\sigma})$ (and in the comma category Q-**TOP**$_0/(\tilde{Y}, \tilde{\sigma})$).

2 Preliminaries

For categorical notions and results used in this paper but not defined here, we refer to [1].

Definition 2.1. [1] Let **C** be a category and B be an object of **C**. Then objects of the *comma category* **C**$/B$ are pairs (X, f), where X is a **C**-object and $f : X \to B$ is a **C**-morphism. Given any two objects (X, f) and (Y, g) of **C**$/B$, a ***C**/B-morphism*

$h:(X,f)\to(Y,g)$ is a $\mathbf{C}$-morphism $h:X\to Y$ such that $g\circ h=f$.

Let $\mathcal{H}$ be a class of morphisms in a category $\mathbf{C}$.

Definition 2.2. [5] An object I is *$\mathcal{H}$-injective* if for all $h:X\to Y$ in $\mathcal{H}$ and a morphism $f:X\to I$, there exists a morphism $g:Y\to I$ such that $g\circ h=f$.

Definition 2.3. [5] A morphism $h:X\to I$ in $\mathcal{H}$ is *$\mathcal{H}$-essential* if for every morphism $k:I\to Y$, the composite $k\circ h:X\to Y$ lies in $\mathcal{H}$ only if $k:I\to Y$ does; if, in addition, I is $\mathcal{H}$-injective, then $h:X\to I$ is an *$\mathcal{H}$-injective hull* of X.

An object (X,f) of the comma category $\mathbf{C}/B$ is *$\mathcal{H}$-injective* if for any commutative diagram in $\mathbf{C}$

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array} \tag{2.1}$$

with $h:U\to V$ in $\mathcal{H}$, there exists a morphism $s:V\to X$ for which the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array} \tag{2.2}$$

Furthermore, a $\mathbf{C}/B$-morphism $j:(Y,g)\to(X,f)$ with $j:Y\to X$ in $\mathcal{H}$ is *$\mathcal{H}$-essential* if for any $\mathbf{C}/B$-morphism $k:(X,f)\to(Z,h)$ such that $k\circ j:Y\to Z$ is in $\mathcal{H}$, necessarily $k:X\to Z$ is in $\mathcal{H}$ follows; if in addition (X,f) is $\mathcal{H}$-injective, then $j:(Y,g)\to(X,f)$ is an *$\mathcal{H}$-injective hull* of (Y,g) in $\mathbf{C}/B$.

The following definitions are from [11].

Definition 2.4. Let $\Omega=(n_\lambda)_{\lambda\in K}$ be a class of cardinal numbers.

- A pair $(Z,(\omega_\lambda^Z)_{\lambda\in K})$, where Z is a set and $(\omega_\lambda^Z)_{\lambda\in K}$ is a family of maps $\omega_\lambda^Z:Z^{n_\lambda}\to Z$, is called an *$\Omega$-algebra*. A subset M of Z is called a *subalgebra* of the Ω-algebra $(Z,(\omega_\lambda^Z)_{\lambda\in K})$ if $\omega_\lambda^Z((m_j)_{j\in n_\lambda})\in M$, for every $\lambda\in K$ and for every $(m_j)_{j\in n_\lambda}\in M^{n_\lambda}$.

- Let $(Z,(\omega_\lambda^Z)_{\lambda\in K})$ and $(S,(\omega_\lambda^S)_{\lambda\in K})$ be Ω-algebras. A map $g : Z \to S$ is said to be an Ω-*homomorphism* if the diagram

$$\begin{array}{ccc} Z^{n_\lambda} & \xrightarrow{g^{n_\lambda}} & S^{n_\lambda} \\ {\scriptstyle \omega_\lambda^Z}\downarrow & & \downarrow{\scriptstyle \omega_\lambda^S} \\ Z & \xrightarrow[g]{} & S \end{array} \tag{2.3}$$

  commutes for every $\lambda \in K$. $\mathbf{Alg}(\Omega)$ will denote the category of Ω-algebras and Ω-homomorphisms.

- Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety* of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras), and $\mathcal{E}$-quotients (homomorphic images).

From now onwards $(Q,(\omega_\lambda^Q)_{\lambda\in K})$ will denote a fixed member of a fixed variety of Ω-algebras.

Let Z be a set and Q^Z be the set of all functions from Z to Q. All operations on Q lift point-wise to Q^Z as:

$$(\omega_\lambda^{Q^Z}(\langle p_j\rangle_{j\in n_\lambda}))(z) = \omega_\lambda^Q(\langle p_j(z)\rangle_{j\in n_\lambda}), \text{ for every } \langle p_j\rangle_{j\in n_\lambda} \in (Q^Z)^{n_\lambda} \text{ and every } z \in Z\,.$$

In particular $(Q^Z,(\omega_\lambda^{Q^Z})_{\lambda\in K})$ is an Ω-algebra.

From now onwards, the Ω-algebra $(Q,(\omega_\lambda^Q)_{\lambda\in K})$ and its underlying set, both will be denoted by Q.

Definition 2.5. Let Z be a set. A subset η of Q^Z is called a Q-*topology* on Z if η is a subalgebra of the Ω-algebra $(Q^Z,(\omega_\lambda^{Q^Z})_{\lambda\in K})$. A pair (Z,η), where Z is a set and η is a Q-topology on Z, is called a Q-*topological space*. Let (Z,η) and (X,τ) be Q-topological spaces and $h : Z \to X$ be a function. Then we say that $h : (Z,\eta) \to (X,\tau)$ is Q-*continuous* if $\alpha \circ h \in \eta$, for every $\alpha \in \tau$.

Q-**TOP** will denote the category of Q-topological spaces and Q-continuous maps. Q-**TOP** is a construct via the obvious forgetful functor $|-|$: Q-**TOP**$\to$ **Set** (**Set** is the category of sets and maps).

We mention here that Q-**TOP** is a topological category over **Set** (cf. [9], Remark 2.2) and hence Q-**TOP** is complete and as a result of this the category Q-**TOP** has pullbacks (cf. [1]).

Definition 2.6. A Q-topological space (X, τ) is called T_0 if for every $x_1, x_2 \in X$ such that $x_1 \neq x_2$, there exists $\alpha \in \tau$ such that $\alpha(x_1) \neq \alpha(x_2)$.

Q-**TOP**$_0$ will denote the full subcategory of Q-**TOP** consisting of T_0-Q-topological spaces. It can be easily seen that Q-**TOP**$_0$ is an isomorphism closed subcategory of Q-**TOP**.

Definition 2.7. A Q-topological space (X, τ) is said to be *stratified* if $\underline{q} \in \tau$, for every $q \in Q$, where $\underline{q} : X \to Q$ is defined as $\underline{q}(x) = q$, for every $x \in X$.

Definition 2.8. [1] Let **C** be a concrete category over **X** and $|-| : \mathbf{C} \to \mathbf{X}$ be the corresponding faithful functor.

1. A **C**-morphism $f : A \to B$ is called *initial* provided that for any **C**-object D, an **X**-morphism $g : |D| \to |A|$ is a **C**-morphism whenever $f \circ g : |D| \to |B|$ is a **C**-morphism.
2. An initial morphism $f : A \to B$ that has a monomorphic underlying **X**-morphism is called an *embedding*.
3. A **C**-morphism $f : A \to B$ is called *final* provided that for any **C**-object D, an **X**-morphism $g : |B| \to |D|$ is a **C**-morphism whenever $g \circ f : |A| \to |D|$ is a **C**-morphism.

Remark 2.9. Let (X, τ) and (Y, σ) be Q-topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be a Q-continuous map. Then

1. $f : (X, \tau) \to (Y, \sigma)$ is initial in Q-**TOP** if and only if $\tau = \{\beta \circ f \mid \beta \in \sigma\}$.
2. $f : (X, \tau) \to (Y, \sigma)$ is an embedding in Q-**TOP** if and only if $f : (X, \tau) \to (Y, \sigma)$ is initial and f is one-one.
3. $f : (X, \tau) \to (Y, \sigma)$ is final in Q-**TOP** if and only if $\sigma = \{\nu \in Q^Y \mid \nu \circ f \in \tau\}$.

From now onwards, injective, essential, injective hull in Q-**TOP** (Q-**TOP**$_0$) and in any comma category Q-**TOP**$/(Y, \sigma)$ (Q-**TOP**$_0/(Z, \eta)$) will denote respectively $\mathcal{H}$-injective, $\mathcal{H}$-essential and $\mathcal{H}$-injective hull for $\mathcal{H}$ the class of embeddings in Q-**TOP** (Q-**TOP**$_0$).

3 T_0-reflection

Let (X,τ) be a Q-topological space. Singh and Srivastava [10] defined a relation $\sim$ on X as, for every $x_1, x_2 \in X$, $x_1 \sim x_2$ if $\alpha(x_1) = \alpha(x_2)$, for every $\alpha \in \tau$. Then it can be easily proved that $\sim$ is an equivalence relation on X. Let $\tilde{X} = X/\sim$, the set of equivalence classes, and let $q_X : X \to \tilde{X}$ be defined as, $q_X(x) = [x]$, for every $x \in X$, where $[x]$ is the equivalence class of x. Let $\tilde{\tau} = \{\beta \in Q^{\tilde{X}} \mid \beta \circ q_X \in \tau\}$. Then $(\tilde{X},\tilde{\tau})$ is a T_0-Q-topological space. It can also be easily verified that for a given T_0-Q-topological space (Z,η) and a Q-continuous map $f : (X,\tau) \to (Z,\eta)$, there exists a unique Q-continuous map $f' : (\tilde{X},\tilde{\tau}) \to (Z,\eta)$ such that $f' \circ q_X = f$. Hence $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is a Q-**TOP**$_0$-reflection for (X,τ) and as a result of this Q-**TOP**$_0$ is a reflective subcategory of Q-**TOP** (cf. Theorem 4.1 in [10]). Consequently, we have the reflector (cf. Proposition 4.22 and Definition 4.23 in [1]) $R : Q$-**TOP**$\to Q$-**TOP**$_0$ give by $R((X,\tau)) = (\tilde{X},\tilde{\tau})$ and if $f : (X,\tau) \to (Y,\sigma)$ is a Q-continuous map, then $R(f) = \tilde{f}$, where $\tilde{f} : (\tilde{X},\tilde{\tau}) \to (\tilde{Y},\tilde{\sigma})$ is the unique Q-continuous map such that the following diagram commutes:

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{f} & (Y,\sigma) \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ (\tilde{X},\tilde{\tau}) & \xrightarrow[\tilde{f}]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{3.1}$$

Thus corresponding to each object $((X,\tau), f)$ of the category Q-**TOP**$/(Y,\sigma)$, we have the object $((\tilde{X},\tilde{\tau}),\tilde{f})$ of the category Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$. $((\tilde{X},\tilde{\tau}),\tilde{f})$ is called the T_0-reflection of $((X,\tau), f)$.
We mention here that if (X,τ) is a T_0-Q-topological space, then $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is an isomorphism in Q-**TOP**.

Proposition 3.1. *Let (X,τ) be a Q-topological space. Then $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is initial and final in Q-**TOP**.*

Proof. By the definition of $\tilde{\tau}$, it follows that $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is final. Now let $\alpha \in \tau$. Define $\beta : \tilde{X} \to Q$ as $\beta([x]) = \alpha(x)$. Then it can be easily proved that β is well defined and $\beta \circ q_X = \alpha$. Thus $\beta \circ q_X \in \tau$ and this implies that $\beta \in \tilde{\tau}$. Thus $\alpha = \beta \circ q_X$, where $\beta \in \tilde{\tau}$. Therefore $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is initial in Q-**TOP**. □

Proposition 3.2. *Let (X,τ) and (Y,σ) be Q-topological spaces. A Q-continuous map $f : (X,\tau) \to (Y,\sigma)$ is an embedding in Q-**TOP** if and only if f is one-one and $\tilde{f} : (\tilde{X},\tilde{\tau}) \to (\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**.*

Proof. Suppose first that the map $f:(X,\tau)\longrightarrow(Y,\sigma)$ is an embedding in Q-**TOP**. Then f is one-one and $f:(X,\tau)\longrightarrow(Y,\sigma)$ is initial in Q-**TOP**. Now we have to prove that $\tilde{f}:(\tilde{X},\tilde{\tau})\longrightarrow(\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**. Let $\tilde{f}([x_1])=\tilde{f}([x_2])\Rightarrow(\tilde{f}\circ q_X)(x_1)=(\tilde{f}\circ q_X)(x_2)\Rightarrow(q_Y\circ f)(x_1)=(q_Y\circ f)(x_2)\Rightarrow[f(x_1)]=[f(x_2)]\Rightarrow u(f(x_1))=u(f(x_2))$, for every $u\in\sigma\Rightarrow(u\circ f)(x_1)=(u\circ f)(x_2)$, for every $u\in\sigma\Rightarrow[x_1]=[x_2]$ (as $f:(X,\tau)\longrightarrow(Y,\sigma)$ is initial, so $\tau=\{u\circ f\mid u\in\sigma\}$) $\Rightarrow\tilde{f}$ is one-one. Now let $\beta\in\tilde{\tau}$, then $\beta\circ q_X\in\tau$ and so $\beta\circ q_X=u\circ f$, for some $u\in\sigma$. Also $u=v\circ q_Y$, for some $v\in\tilde{\sigma}$. Thus $\beta\circ q_X=u\circ f=v\circ q_Y\circ f=v\circ(q_Y\circ f)=v\circ(\tilde{f}\circ q_X)=(v\circ\tilde{f})\circ q_X\Rightarrow\beta=v\circ\tilde{f}$ (as q_X is onto). Hence $\tilde{f}:(\tilde{X},\tilde{\tau})\longrightarrow(\tilde{Y},\tilde{\sigma})$ is initial in Q-**TOP**. Therefore $\tilde{f}:(\tilde{X},\tilde{\tau})\longrightarrow(\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**.

Conversely, suppose that f is one-one and $\tilde{f}:(\tilde{X},\tilde{\tau})\longrightarrow(\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**. We have to show that $f:(X,\tau)\longrightarrow(Y,\sigma)$ is an embedding in Q-**TOP**. Since f is one-one, it is sufficient to show that $f:(X,\tau)\longrightarrow(Y,\sigma)$ is initial. Let $\alpha\in\tau$, then $\alpha=\beta\circ q_X$, for some $\beta\in\tilde{\tau}$. Then since $\tilde{f}:(\tilde{X},\tilde{\tau})\longrightarrow(\tilde{Y},\tilde{\sigma})$ is initial and $\beta\in\tilde{\tau}$, $\beta=v\circ\tilde{f}$, for some $v\in\tilde{\sigma}$. So $\alpha=\beta\circ q_X=v\circ\tilde{f}\circ q_X=v\circ(\tilde{f}\circ q_X)=v\circ(q_Y\circ f)=(v\circ q_Y)\circ f=u\circ f$, where $u=v\circ q_Y\in\sigma$. Thus $\alpha=u\circ f$, where $u\in\sigma$. Hence $f:(X,\tau)\longrightarrow(Y,\sigma)$ is initial in Q-**TOP**. Therefore $f:(X,\tau)\longrightarrow(Y,\sigma)$ is an embedding in Q-**TOP**.

□

In view of Proposition 8.14 in [1], we have the following result:

Proposition 3.3. *Let (X,τ) and (Y,σ) be Q-topological spaces and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be an initial map in Q-**TOP** such that f is bijective. Then $f:(X,\tau)\longrightarrow(Y,\sigma)$ is an isomorphism in Q-**TOP**.*

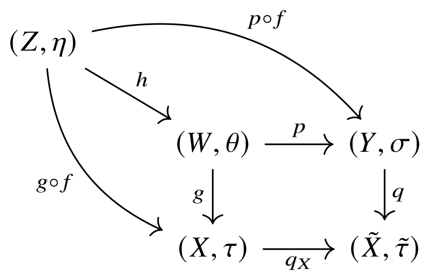
Proposition 3.4. [1] *In any category, monomorphisms, regular monomorphisms and retractions are pullback stable.*

Proposition 3.5. *Let (X,τ) and (Y,σ) be Q-topological spaces and let $q:(Y,\sigma)\longrightarrow(\tilde{X},\tilde{\tau})$ be a Q-continuous map. Let $p:(W,\theta)\longrightarrow(Y,\sigma)$ be a pullback of $q_X:(X,\tau)\longrightarrow(\tilde{X},\tilde{\tau})$ along $q:(Y,\sigma)\longrightarrow(\tilde{X},\tilde{\tau})$ in the category Q-**TOP***

$$\begin{array}{ccc}(W,\theta) & \xrightarrow{p} & (Y,\sigma)\\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle q}\\ (X,\tau) & \xrightarrow[q_X]{} & (\tilde{X},\tilde{\tau})\end{array}\tag{3.2}$$

Then $\tilde{p} : (\tilde{W}, \tilde{\theta}) \to (\tilde{Y}, \tilde{\sigma})$ *is an isomorphism in* Q-***TOP***.

Proof. First we will prove that the map $p : (W, \theta) \to (Y, \sigma)$ is initial in Q-**TOP**. Let (Z, η) be a Q-topological space and let $f : Z \to W$ be a map such that $p \circ f : (Z, \eta) \to (Y, \sigma)$ is Q-continuous. Then, $q_X \circ (g \circ f) = q \circ (p \circ f)$. So $q_X \circ (g \circ f) : (Z, \eta) \to (\tilde{X}, \tilde{\tau})$ is Q-continuous, but since $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is initial in Q-**TOP**, $g \circ f : (Z, \eta) \to (X, \tau)$ is Q-continuous. Now since the diagram 3.2 is a pullback, there exists a unique Q-continuous map $h : (Z, \eta) \to (W, \theta)$ such that the following diagram commutes:



(3.3)

Now if $h \neq f$, then if we consider the diagram 3.3 in the category **Set**, then in **Set** we have two maps $f, h : Z \to W$ for which the diagram 3.3 commutes, but this will be a contradiction because the diagram 3.2, if considered in the category **Set**, is a pullback square in **Set** also. Thus $h = f$ and hence $f : (Z, \eta) \to (W, \theta)$ is Q-continuous. Therefore $p : (W, \theta) \to (Y, \sigma)$ is initial in Q-**TOP**. Now consider the following commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (W, \theta) & \xrightarrow{p} & (Y, \sigma) \\ {\scriptstyle q_W}\downarrow & & \downarrow{\scriptstyle q_Y} \\ (\tilde{W}, \tilde{\theta}) & \xrightarrow[\tilde{p}]{} & (\tilde{Y}, \tilde{\sigma}) \end{array} \tag{3.4}$$

Since $p : (W, \theta) \to (Y, \sigma)$ is initial in Q-**TOP**, as in the proof of Proposition 3.2, we can prove that $\tilde{p} : (\tilde{W}, \tilde{\theta}) \to (\tilde{Y}, \tilde{\sigma})$ is initial in Q-**TOP**. Now since q_X is onto, i.e. q_X is a retraction in **Set** and since the diagram 3.2, if considered in the category **Set**, is a pullback square in **Set** also, by Proposition 3.4, p is a retraction in **Set**, i.e. p is onto. Since $q_Y \circ p = \tilde{p} \circ q_W$ and both p and q_Y are onto, $\tilde{p} \circ q_W$ is onto. This implies that $\tilde{p}$ is onto. Next, we will prove that $\tilde{p}$ is one-one. Let $\tilde{p}([w_1]) = \tilde{p}([w_2]) \Rightarrow (\tilde{p} \circ q_W)(w_1) = (\tilde{p} \circ q_W)(w_2) \Rightarrow$

$(q_Y \circ p)(w_1) = (q_Y \circ p)(w_2) \Rightarrow [p(w_1)] = [p(w_2)] \Rightarrow u(p(w_1)) = u(p(w_2))$, for every $u \in \sigma \Rightarrow (u \circ p)(w_1) = (u \circ p)(w_2)$, for every $u \in \sigma \Rightarrow [w_1] = [w_2]$ (since $\theta = \{u \circ p \mid u \in \sigma\}$ as $p : (W, \theta) \to (Y, \sigma)$ is initial in Q-**TOP**). This implies that $\tilde{p}$ is one-one. Thus $\tilde{p} : (\tilde{W}, \tilde{\theta}) \to (\tilde{Y}, \tilde{\sigma})$ is initial in Q-**TOP** and $\tilde{p}$ is bijective. Therefore by Proposition 3.3, $\tilde{p} : (\tilde{W}, \tilde{\theta}) \to (\tilde{Y}, \tilde{\sigma})$ is an isomorphism in Q-**TOP**. □

4 A characterization of injective objects in Q-TOP$/(Y, \sigma)$

Proposition 4.1. *Let (X, τ) and (Y, σ) be Q-topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be an initial map in Q-**TOP** such that f is onto, then $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$.*

Proof. Let following be a commutative square in Q-**TOP**:

$$\begin{array}{ccc} (W, \theta) & \xrightarrow{l} & (X, \tau) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ (Z, \eta) & \xrightarrow[g]{} & (Y, \sigma) \end{array} \tag{4.1}$$

where $h : (W, \theta) \to (Z, \eta)$ is an embedding in Q-**TOP**.
Now since in the category **Set**$/Y$, injective objects are surjective maps over Y, there exists a function $k : Z \to X$ such that $k \circ h = l$ and $f \circ k = g$. Now let $\alpha \in \tau$, then $\alpha = \beta \circ f$, for some $\beta \in \sigma$ as $f : (X, \tau) \to (Y, \sigma)$ is initial in Q-**TOP**. Then $\alpha \circ k = \beta \circ f \circ k = \beta \circ (f \circ k) = \beta \circ g \in \eta$ as $g : (Z, \eta) \to (Y, \sigma)$ is Q-continuous. Thus $k : (Z, \eta) \to (X, \tau)$ is Q-continuous. Hence we have a Q-continuous map $k : (Z, \eta) \to (X, \tau)$ such that $k \circ h = l$ and $f \circ k = g$. Therefore $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$. □

Corollary 4.2. *Let (X, τ) be a Q-topological space, then $((X, \tau), q_X)$ is injective in Q-**TOP**$/(\tilde{X}, \tilde{\tau})$.*

Proof. It immediately follows from Proposition 3.1 and Proposition 4.1. □

Proposition 4.3. *Let $((X, \tau), f)$ be injective in Q-**TOP**$/(Y, \sigma)$ and $((Y, \sigma), g)$ be injective in Q-**TOP**$/(Z, \eta)$. Then $((X, \tau), g \circ f)$ is injective in Q-**TOP**$/(Z, \eta)$.*

Proof. It easily follows from the definition of injective objects in comma categories. □

The following Lemma 4.4 can be easily verified:

Lemma 4.4. *Let* (X,τ) *and* (Y,σ) *be* T_0*-*Q*-topological spaces and let* $f:(X,\tau)\to(Y,\sigma)$ *be a* Q*-continuous map. Then* $f:(X,\tau)\to(Y,\sigma)$ *is an embedding in* Q*-**TOP** if and only if it is an embedding in* Q*-**TOP***$_0$.

Proposition 4.5. *Let* (X,τ) *and* (Y,σ) *be* T_0*-*Q*-topological spaces and let* $f:(X,\tau)\to(Y,\sigma)$ *be a* Q*-continuous map. Then* $((X,\tau),f)$ *is injective in* Q*-**TOP***$/(Y,\sigma)$ *if and only if it is injective in* Q*-**TOP***$_0/(Y,\sigma)$.

Proof. Suppose first that $((X,\tau),f)$ is injective in Q-**TOP**$/(Y,\sigma)$. Let following be a commutative diagram in Q-**TOP**$_0$:

$$\begin{array}{ccc}(A,\tau_A) & \xrightarrow{l} & (X,\tau)\\ h\downarrow & & \downarrow f\\ (B,\tau_B) & \xrightarrow[k]{} & (Y,\sigma)\end{array} \tag{4.2}$$

where $h:(A,\tau_A)\to(B,\tau_B)$ is an embedding in Q-**TOP**$_0$.
Then by Lemma 4.4, $h:(A,\tau_A)\to(B,\tau_B)$ is an embedding in Q-**TOP** and then since $((X,\tau),f)$ is injective in Q-**TOP**$/(Y,\sigma)$, there exists a Q-continuous map $s:(B,\tau_B)\to(X,\tau)$ such that $s\circ h=l$ and $f\circ s=k$. Therefore $((X,\tau),f)$ is injective in Q-**TOP**$_0/(Y,\sigma)$.

Conversely, assume that $((X,\tau),f)$ is injective in Q-**TOP**$_0/(Y,\sigma)$. Let following be a commutative diagram in Q-**TOP**:

$$\begin{array}{ccc}(W,\theta) & \xrightarrow{g} & (X,\tau)\\ h\downarrow & & \downarrow f\\ (Z,\eta) & \xrightarrow[k]{} & (Y,\sigma)\end{array} \tag{4.3}$$

where $h:(W,\theta)\to(Z,\eta)$ is an embedding in Q-**TOP**.
We note that since (X,τ) and (Y,σ) are T_0-Q-topological spaces, $q_X:(X,\tau)\to(\tilde{X},\tilde{\tau})$ and $q_Y:(Y,\sigma)\to(\tilde{Y},\tilde{\sigma})$ are isomorphisms in Q-**TOP**. Now we will first prove that $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$. Let following be a commutative diagram in Q-**TOP**$_0$:

$$\begin{array}{ccc} (N,\tau_N) & \xrightarrow{\;l\;} & (\tilde{X},\tilde{\tau}) \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ (M,\tau_M) & \xrightarrow[n]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{4.4}$$

where $m : (M,\tau_M) \to (N,\tau_N)$ is an embedding in Q-$\mathbf{TOP}_0$.
Now since $((X,\tau), f)$ is injective in Q-$\mathbf{TOP}_0/(Y,\sigma)$, there exists a Q-continuous map $s : (M,\tau_M) \to (X,\tau)$ such that the following diagram commutes:

$$\begin{array}{ccccc} (N,\tau_N) & \xrightarrow{\;l\;} & (\tilde{X},\tilde{\tau}) & \xrightarrow{\;q_X^{-1}\;} & (X,\tau) \\ {\scriptstyle m}\big\downarrow & & {\scriptstyle s}\nearrow & & \big\downarrow{\scriptstyle f} \\ (M,\tau_M) & \xrightarrow[n]{} & (\tilde{Y},\tilde{\sigma}) & \xrightarrow[q_Y^{-1}]{} & (Y,\sigma) \end{array} \tag{4.5}$$

Thus we have a Q-continuous map $q_X \circ s : (M,\tau_M) \to (\tilde{X},\tilde{\tau})$ and it can be easily verified that the following is a commutative diagram in Q-$\mathbf{TOP}_0$:

$$\begin{array}{ccc} (N,\tau_N) & \xrightarrow{\;l\;} & (\tilde{X},\tilde{\tau}) \\ {\scriptstyle m}\big\downarrow & {\scriptstyle q_X\circ s}\nearrow & \big\downarrow{\scriptstyle \tilde{f}} \\ (M,\tau_M) & \xrightarrow[n]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{4.6}$$

Thus $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$. Now since $h : (W,\theta) \to (Z,\eta)$ is an embedding in Q-$\mathbf{TOP}$, by Proposition 3.2, $\tilde{h} : (\tilde{W},\tilde{\theta}) \to (\tilde{Z},\tilde{\eta})$ is an embedding in Q-$\mathbf{TOP}$ and then by Lemma 4.4, $\tilde{h} : (\tilde{W},\tilde{\theta}) \to (\tilde{Z},\tilde{\eta})$ is an embedding in Q-$\mathbf{TOP}_0$. Then since $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$, there exists a Q-continuous map $s' : (\tilde{Z},\tilde{\eta}) \to (\tilde{X},\tilde{\tau})$ such that the following diagram commutes:

$$\begin{array}{ccc} (\tilde{W},\tilde{\theta}) & \xrightarrow{\;\tilde{g}\;} & (\tilde{X},\tilde{\tau}) \\ {\scriptstyle \tilde{h}}\big\downarrow & {\scriptstyle s'}\nearrow & \big\downarrow{\scriptstyle \tilde{f}} \\ (\tilde{Z},\tilde{\eta}) & \xrightarrow[\tilde{k}]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{4.7}$$

Let $p = q_X^{-1} \circ s' \circ q_Z$. Then $p \circ h = q_X^{-1} \circ s' \circ q_Z \circ h = q_X^{-1} \circ s' \circ (q_Z \circ h) = q_X^{-1} \circ s' \circ (\tilde{h} \circ q_W) = q_X^{-1} \circ (s' \circ \tilde{h}) \circ q_W = q_X^{-1} \circ \tilde{g} \circ q_W = q_X^{-1} \circ (\tilde{g} \circ q_W) = q_X^{-1} \circ (q_X \circ g) = g$ and $f \circ p = f \circ q_X^{-1} \circ s' \circ q_Z = (f \circ q_X^{-1}) \circ s' \circ q_Z = (q_Y^{-1} \circ \tilde{f}) \circ s' \circ q_Z =$

$q_Y^{-1} \circ (\tilde{f} \circ s') \circ q_Z = q_Y^{-1} \circ \tilde{k} \circ q_Z = q_Y^{-1} \circ (\tilde{k} \circ q_Z) = q_Y^{-1} \circ (q_Y \circ k) = k$. Thus we have a Q-continuous map $p : (Z,\eta) \to (X,\tau)$ such that $p \circ h = g$ and $f \circ p = k$. Therefore $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$. □

Proposition 4.6. *Let (X,τ) be a Q-topological space and (Y,σ) be a stratified Q-topological space. If $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$, then $f : (X,\tau) \to (Y,\sigma)$ is a retraction in Q-**TOP**. In particular, for any $x \in X$ there exists a section $s_x : (Y,\sigma) \to (X,\tau)$ of $f : (X,\tau) \to (Y,\sigma)$ with $s_x(f(x)) = x$.*

Proof. Consider the following commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (\{x\},\delta) & \xrightarrow{i_x} & (X,\tau) \\ {\scriptstyle f_x}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (Y,\sigma) & \xrightarrow[id_Y]{} & (Y,\sigma) \end{array} \qquad (4.8)$$

where $\delta = \{\underline{q} \mid q \in Q\}$, $i_x : \{x\} \to X$ is the inclusion map and $f_x : \{x\} \to Y$ is defined as $f_x(x) = f(x)$.
It can be easily seen that $f_x : (\{x\},\delta) \to (Y,\sigma)$ is an embedding in Q-**TOP**. Then since $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$, there exists a Q-continuous map $s_x : (Y,\sigma) \to (X,\tau)$ such that $s_x \circ f_x = i_x$ and $f \circ s_x = id_Y$. Therefore $s_x(f(x)) = x$ and $f : (X,\tau) \to (Y,\sigma)$ is a retraction in Q-**TOP**. □

Let (X,τ) be a Q-topological space and let $x \in X$. Consider the subset $\{x' \in X \mid x' \sim x\}$ of X (note that $\sim$ is the equivalence relation on X defined in the starting of the section 3). Clearly this subset gives the equivalence class $[x]$ of x. We will denote this subset of X by C_x where it is considered as a subset of X and it will be denoted by $[x]$ where it is considered as an element of $\tilde{X}$, to avoid confusions.

Proposition 4.7. *Let (X,τ) and (Y,σ) be Q-topological spaces and let $f : (X,\tau) \to (Y,\sigma)$ be a Q-continuous map. Then $f(C_x) \subseteq C_{f(x)}$, for every $x \in X$.*

Proof. Let $y \in f(C_x)$, then $y = f(x')$, for some $x' \in C_x$. Now let $\beta \in \sigma$. Then since $f : (X,\tau) \to (Y,\sigma)$ is Q-continuous, $\beta \circ f \in \tau$ and then since $x' \sim x$, $(\beta \circ f)(x') = (\beta \circ f)(x) \Rightarrow \beta(f(x')) = \beta(f(x))$. Thus $f(x') \sim f(x)$. This implies that $y = f(x') \in C_{f(x)}$. Therefore $f(C_x) \subseteq C_{f(x)}$. □

Proposition 4.8. [1] *Every retract of an injective object is injective.*

Proposition 4.9. *Let (X,τ) and (Y,σ) be Q-topological spaces. If $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$, then $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$.*

Proof. We know that in the category **Set**, retractions are precisely onto maps and since q_X is onto, q_X is a retraction in **Set**. So there exists a map $g : \tilde{X} \to X$ such that $q_X \circ g = id_{\tilde{X}}$. This implies that $(q_X \circ g)([x]) = [x]$, for every $x \in X \Rightarrow [g([x])] = [x]$, for every $x \in X \Rightarrow \alpha(g[x]) = \alpha(x)$, for every $\alpha \in \tau$ and for every $x \in X$. Now we will show that $g : (\tilde{X},\tilde{\tau}) \to (X,\tau)$ is Q-continuous. Let $\alpha \in \tau$. Then $((\alpha \circ g) \circ q_X)(x) = \alpha(g([x])) = \alpha(x)$, for every $x \in X$. Thus $(\alpha \circ g) \circ q_X = \alpha \in \tau$. Hence $\alpha \circ g \in \tilde{\tau}$. So $g : (\tilde{X},\tilde{\tau}) \to (X,\tau)$ is Q-continuous. Now $q_Y \circ f \circ g = (q_Y \circ f) \circ g = (\tilde{f} \circ q_X) \circ g = \tilde{f} \circ (q_X \circ g) = \tilde{f} \circ id_{\tilde{X}} = \tilde{f}$. Thus we have morphisms $g : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((X,\tau), q_Y \circ f)$ and $q_X : ((X,\tau), q_Y \circ f) \to ((\tilde{X},\tilde{\tau}),\tilde{f})$ in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ such that $q_X \circ g = id_{\tilde{X}}$. Thus $((\tilde{X},\tilde{\tau}),\tilde{f})$ is a retract of $((X,\tau), q_Y \circ f)$ in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$. By Proposition 4.3 and Corollary 4.2, $((X,\tau), q_Y \circ f)$ is injective in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ and then by Proposition 4.8, $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$. Therefore by Proposition 4.5, $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$.

□

The following Theorem 4.10 is concerned with the extension of Theorem 2.6 of [5], in the category Q-**TOP**$/(Y,\sigma)$.

Theorem 4.10. *Let (X,τ) be a Q-topological space and (Y,σ) be a stratified Q-topological space. Then $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$ if and only if*

1. *$f(C_x) = C_{f(x)}$, for every $x \in X$.*
2. *Its T_0-reflection $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$.*

Proof. Let $((X,\tau), f)$ be injective in Q-**TOP**$/(Y,\sigma)$.

(1) Let $x \in X$. Then by Proposition 4.7, $f(C_x) \subseteq C_{f(x)}$. Now we have to show that $C_{f(x)} \subseteq f(C_x)$. By Proposition 4.6, there exists a section $s_x : (Y,\sigma) \to (X,\tau)$ of $f : (X,\tau) \to (Y,\sigma)$ such that $s_x(f(x)) = x$ and $f \circ s_x = id_Y$. Then since by Proposition 4.7, $s_x(C_{f(x)}) \subseteq C_{(s_x \circ f)(x)}$, $s_x(C_{f(x)}) \subseteq C_x$ (as $s_x(f(x)) = x$). Then $f(s_x(C_{f(x)})) \subseteq f(C_x) \Rightarrow (f \circ s_x)(C_{f(x)}) \subseteq f(C_x) \Rightarrow id_Y(C_{f(x)}) \subseteq f(C_x) \Rightarrow C_{f(x)} \subseteq f(C_x)$. Therefore $f(C_x) = C_{f(x)}$.

(2) Follows from Proposition 4.9.

Conversely, assume that (1) and (2) hold. Let following be a commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (W,\theta) & \xrightarrow{m} & (X,\tau) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ (Z,\eta) & \xrightarrow[n]{} & (Y,\sigma) \end{array} \tag{4.9}$$

where $h:(W,\theta)\to(Z,\eta)$ is an embedding in Q-**TOP**.

Now since $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$, by Proposition 4.5, $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-$\mathbf{TOP}/(\tilde{Y},\tilde{\sigma})$. So there exists a Q-continuous map $s:(Z,\eta)\to(\tilde{X},\tilde{f})$ such that the following diagram commutes:

$$\begin{array}{ccccc} (W,\theta) & \xrightarrow{m} & (X,\tau) & \xrightarrow{q_X} & (\tilde{X},\tilde{\tau}) \\ {\scriptstyle h}\downarrow & & \nearrow{\scriptstyle s} & & \downarrow{\scriptstyle \tilde{f}} \\ (Z,\eta) & \xrightarrow[n]{} & (Y,\sigma) & \xrightarrow[q_Y]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{4.10}$$

Let $\tilde{X}=\{[x_j]\mid j\in J\}$. Now let $w\in m^{-1}(C_{x_j})$, then $(s\circ h)(w)=(q_X\circ m)(w)=[m(w)]=[x_j]$ (as $m(w)\in C_{x_j}$, $[m(w)]=[x_j]$). So $h(w)\in s^{-1}(\{[x_j]\})$. Thus for each $j\in J$, we have a map $h_j:m^{-1}(C_{x_j})\to s^{-1}(\{[x_j]\})$ defined as $h_j(w)=h(w)$, for every $w\in m^{-1}(C_{x_j})$. Next let $z\in s^{-1}(\{[x_j]\})$, then $(q_Y\circ n)(z)=(\tilde{f}\circ s)(z)=\tilde{f}(s(z))=\tilde{f}([x_j])=(\tilde{f}\circ q_X)(x_j)=(q_Y\circ f)(x_j)$. This implies that $[n(z)]=[f(x_j)]$. So $n(z)\sim f(x_j)$ and hence $n(z)\in C_{f(x_j)}$. Thus for each $j\in J$, we have a map $n_j:s^{-1}(\{[x_j]\})\to C_{f(x_j)}$ defined as $n_j(z)=n(z)$, for every $z\in s^{-1}(\{[x_j]\})$. Now consider the following commutative diagram in category **Set**:

$$\begin{array}{ccc} m^{-1}(C_{x_j}) & \xrightarrow{m_j} & C_{x_j} \\ {\scriptstyle h_j}\downarrow & & \downarrow{\scriptstyle f_j} \\ s^{-1}(\{[x_j]\}) & \xrightarrow[n_j]{} & C_{f(x_j)} \end{array} \tag{4.11}$$

where $f_j:C_{x_j}\to C_{f(x_j)}$ is defined as $f_j(x)=f(x)$, for every $x\in C_{x_j}$ and $m_j:m^{-1}(C_{x_j})\to C_{x_j}$ is defined as $m_j(w)=m(w)$, for every $w\in m^{-1}(C_{x_j})$. Now by (1), $f_j:C_{x_j}\to C_{f(x_j)}$ is onto and so (C_{x_j},f_j) is injective in the category $\mathbf{Set}/C_{f(x_j)}$. Also since h is one-one, h_j is one-one. So there exists a map $g_j:s^{-1}(\{[x_j]\})\to C_{x_j}$ such that $g_j\circ h_j=m_j$ and $f_j\circ g_j=n_j$. Thus for each $j\in J$, we have a map $g_j:s^{-1}(\{[x_j]\})\to C_{x_j}$ such that $g_j\circ h_j=m_j$ and

$f_j \circ g_j = n_j$. Note that since $\tilde{X} = \bigcup_{j\in J}\{[x_j]\}$, $Z = \bigcup_{j\in J} s^{-1}(\{[x_j]\})$. Thus we can define a map $g : Z \to X$ as $g(z) = g_j(z)$, if $z \in s^{-1}(\{[x_j]\})$. Now we have to prove that $f \circ g = n$, $g \circ h = m$ and $g : (Z, \eta) \to (X, \tau)$ is Q-continuous. Let $z \in Z$, then there exists a unique $j \in J$ such that $z \in s^{-1}(\{[x_j]\})$. Then $(f \circ g)(z) = f(g(z)) = f(g_j(z)) = f_j(g_j(z)) = (f_j \circ g_j)(z) = n_j(z) = n(z)$. This implies that $f \circ g = n$. Now consider $(g \circ h)(w) = g(h(w)) = g_j(h(w))$, if $h(w) \in s^{-1}(\{[x_j]\})$. Now if $h(w) \in s^{-1}(\{[x_j]\})$, then $(s \circ h)(w) = [x_j] \Rightarrow (q_X \circ m)(w) = [x_j] \Rightarrow [m(w)] = [x_j] \Rightarrow m(w) \sim x_j \Rightarrow m(w) \in C_{x_j} \Rightarrow w \in m^{-1}(C_{x_j})$ and so $h(w) = h_j(w)$. Thus if $h(w) \in s^{-1}(\{[x_j]\})$, then $(g \circ h)(w) = g_j(h(w)) = g_j(h_j(w)) = (g_j \circ h_j)(w) = m_j(w) = m(w)$. Thus $g \circ h = m$. Now we show that $g : (Z, \eta) \to (X, \tau)$ is Q-continuous. Let $\alpha \in \tau$, then $\alpha = \beta \circ q_X$, for some $\beta \in \tilde{\tau}$. Now let $z \in Z$, then there exists a unique $j \in J$ such that $z \in s^{-1}(\{[x_j]\})$. Now consider $(\alpha \circ g)(z) = (\beta \circ q_X \circ g)(z) = (\beta \circ q_X)(g(z)) = (\beta \circ q_X)(g_j(z)) = \beta([g_j(z)]) = \beta([x_j])$ (since $g_j(z) \in C_{x_j}$, $[g_j(z)] = [x_j]$). Thus $(\alpha \circ g)(z) = \beta([x_j]) = \beta(s(z)) = (\beta \circ s)(z)$. Thus $\alpha \circ g = \beta \circ s \in \eta$ as $s : (Z, \eta) \to (\tilde{X}, \tilde{\tau})$ is Q-continuous. Hence $g : (Z, \eta) \to (X, \tau)$ is Q-continuous. Therefore $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$. □

5 Existence of injective hulls in Q-TOP$/(Y, \sigma)$

Proposition 5.1. *Let (X, τ) and (Y, σ) be T_0-Q-topological spaces. Then $((X, \tau), f)$ has an injective hull in Q-**TOP**$_0/(Y, \sigma)$ if and only if it has an injective hull in Q-**TOP**$/(Y, \sigma)$ and in this case injective hulls coincide.*

Proof. Suppose first that $((X, \tau), f)$ has an injective hull $j : ((X, \tau), f) \to ((Z, \eta), g)$ in Q-**TOP**$_0/(Y, \sigma)$. Then $((Z, \eta), g)$ is injective in Q-**TOP**$_0/(Y, \sigma)$ and then by Proposition 4.5, $((Z, \eta), g)$ is injective in Q-**TOP**$/(Y, \sigma)$. Furthermore, $j : (X, \tau) \to (Z, \eta)$ is an embedding in Q-**TOP**$_0$ and then by Lemma 4.4, $j : (X, \tau) \to (Z, \eta)$ is an embedding in Q-**TOP**. Now we have to prove that $j : ((X, \tau), f) \to ((Z, \eta), g)$ is essential in Q-**TOP**$/(Y, \sigma)$. We note that since $j : (X, \tau) \to (Z, \eta)$ is an embedding in Q-**TOP**, by Proposition 3.2, $\tilde{j} : (\tilde{X}, \tilde{\tau}) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP** and hence $\tilde{j} : (\tilde{X}, \tilde{\tau}) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP**$_0$ by Lemma 4.4. Now we will first prove that $\tilde{j} : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((\tilde{Z}, \tilde{\eta}), \tilde{g})$ is essential in Q-**TOP**$_0/(\tilde{Y}, \tilde{\sigma})$. Let $h : ((\tilde{Z}, \tilde{\eta}), \tilde{g}) \to ((A, \tau_A), m)$ be a morphism in Q-**TOP**$_0/(\tilde{Y}, \tilde{\sigma})$ such that $h \circ \tilde{j} : (\tilde{X}, \tilde{\tau}) \to (A, \tau_A)$ is an embedding in Q-**TOP**$_0$. Since composition of embeddings is an embedding, $h \circ \tilde{j} \circ q_X : (X,\tau) \to (A,\tau_A)$ is an embedding in Q-**TOP**$_0$ and then since $h \circ \tilde{j} \circ q_X = (h \circ q_Z) \circ j$, $(h \circ q_Z) \circ j : (X,\tau) \to (A,\tau_A)$ is an embedding in Q-**TOP**$_0$. Thus we have a morphism $h \circ q_Z : ((Z,\eta),g) \to ((A,\tau_A), q_Y^{-1} \circ m)$ in Q-**TOP**$_0/(Y,\sigma)$ such that $(h \circ q_Z) \circ j : (X,\tau) \to (A,\tau_A)$ is an embedding in Q-**TOP**$_0$. Then since $j : ((X,\tau),f) \to ((Z,\eta),g)$ is essential in Q-**TOP**$_0/(Y,\sigma)$, $h \circ q_Z : (Z,\eta) \to (A,\tau_A)$ is an embedding in Q-**TOP**$_0$. Now since $(Z,\eta) \in Q$-**TOP**$_0$, $q_Z : (Z,\eta) \to (\tilde{Z},\tilde{\eta})$ is an isomorphism in Q-**TOP**$_0$ and so it is essential in Q-**TOP**$_0$ and then since $h \circ q_Z : (Z,\eta) \to (A,\tau_A)$ is an embedding in Q-**TOP**$_0$, $h : (\tilde{Z},\tilde{\eta}) \to (A,\tau_A)$ is an embedding in Q-**TOP**$_0$. Thus $\tilde{j} : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((\tilde{Z},\tilde{\eta}),\tilde{g})$ is essential in Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$. Next let $k : ((Z,\eta),g) \to ((W,\theta),l)$ be a morphism in Q-**TOP**$/(Y,\sigma)$ such that $k \circ j : (X,\tau) \to (W,\theta)$ is an embedding in Q-**TOP**. Then by Proposition 3.2, $\tilde{k} \circ \tilde{j} : (\tilde{X},\tilde{\tau}) \to (\tilde{W},\tilde{\theta})$ is an embedding in Q-**TOP** and then by Lemma 4.4, it is an embedding in Q-**TOP**$_0$. Thus we have a morphism $\tilde{k} : ((\tilde{Z},\tilde{\eta}),\tilde{g}) \to ((\tilde{W},\tilde{\theta}),\tilde{l})$ in Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$ such that $\tilde{k} \circ \tilde{j} : (\tilde{X},\tilde{\tau}) \to (\tilde{W},\tilde{\theta})$ is an embedding in Q-**TOP**$_0$. Then since $\tilde{j} : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((\tilde{Z},\tilde{\eta}),\tilde{g})$ is essential in Q-**TOP**$_0/(\tilde{Y},\tilde{\sigma})$, $\tilde{k} : (\tilde{Z},\tilde{\eta}) \to (\tilde{W},\tilde{l})$ is an embedding in Q-**TOP**$_0$ and hence it is an embedding in Q-**TOP** by Lemma 4.4. Now we will prove that k is one-one. Let $k(z_1) = k(z_2) \Rightarrow (q_W \circ k)(z_1) = (q_W \circ k)(z_2) \Rightarrow (\tilde{k} \circ q_Z)(z_1) = (\tilde{k} \circ q_Z)(z_2) \Rightarrow z_1 = z_2$ (since $\tilde{k} \circ q_Z$ is one-one as both $\tilde{k}$ and q_Z are one-one). Hence k is one-one. Thus by Proposition 3.2, $k : (Z,\eta) \to (W,\theta)$ is an embedding in Q-**TOP**. Hence $j : ((X,\tau),f) \to ((Z,\eta),g)$ is essential in Q-**TOP**$/(Y,\sigma)$. Therefore $j : ((X,\tau),f) \to ((Z,\eta),g)$ is an injective hull of $((X,\tau),f)$ in Q-**TOP**$/(Y,\sigma)$.

Conversely, let $((X,\tau),f)$ have an injective hull $j : ((X,\tau),f) \to ((Z,\eta),g)$ in Q-**TOP**$/(Y,\sigma)$. Now since $(X,\tau) \in Q$-**TOP**$_0$, $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is an isomorphism in Q-**TOP** and hence $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is an embedding in Q-**TOP**. Also since $j : (X,\tau) \to (Z,\eta)$ is an embedding in Q-**TOP**, by Proposition 3.2, $\tilde{j} : (\tilde{X},\tilde{\tau}) \to (\tilde{Z},\tilde{\eta})$ is an embedding in Q-**TOP**. Thus $\tilde{j} \circ q_X : (X,\tau) \to (\tilde{Z},\tilde{\eta})$ is an embedding in Q-**TOP** and since $\tilde{j} \circ q_X = q_Z \circ j$, $q_Z \circ j : (X,\tau) \to (\tilde{Z},\tilde{\eta})$ is an embedding in Q-**TOP**. Thus we have a morphism $q_Z : ((Z,\eta),g) \to ((\tilde{Z},\tilde{\eta}), q_Y^{-1} \circ \tilde{g})$ in Q-**TOP**$/(Y,\sigma)$ such that $q_Z \circ j : (X,\tau) \to (\tilde{Z},\tilde{\eta})$ is an embedding in Q-**TOP** and then since $j : ((X,\tau),f) \to ((Z,\eta),g)$ is essential in Q-**TOP**$/(Y,\sigma)$, $q_Z : (Z,\eta) \to (\tilde{Z},\tilde{\eta})$ is an embedding in Q-**TOP**. Hence $q_Z : (Z,\eta) \to (\tilde{Z},\tilde{\eta})$ is initial in Q-**TOP** and q_Z is bijective and thus by Proposition 3.3, $q_Z : (Z,\eta) \to (\tilde{Z},\tilde{\eta})$ is an isomorphism in Q-**TOP**. Now since Q-**TOP**$_0$ is an isomorphism closed subcategory of Q-**TOP**, $(Z,\eta) \in Q$-

$\mathbf{TOP}_0$. Thus $j : ((X, \tau), f) \to ((Z, \eta), g)$ is a morphism in $Q\text{-}\mathbf{TOP}_0/(Y, \sigma)$. Now since $j : (X, \tau) \to (Z, \eta)$ is an embedding in Q-**TOP**, by Lemma 4.4, $j : (X, \tau) \to (Z, \eta)$ is an embedding in $Q\text{-}\mathbf{TOP}_0$. It can also be easily verified that $j : ((X, \tau), f) \to ((Z, \eta), g)$ is essential in $Q\text{-}\mathbf{TOP}_0/(Y, \sigma)$. We also note that since $((Z, \eta), g)$ is injective in $Q\text{-}\mathbf{TOP}/(Y, \sigma)$ by Proposition 4.5, it is injective in $Q\text{-}\mathbf{TOP}_0/(Y, \sigma)$. Therefore $j : ((X, \tau), f) \to ((Z, \eta), g)$ is an injective hull of $((X, \tau), f)$ in $Q\text{-}\mathbf{TOP}_0/(Y, \sigma)$. □

Definition 5.2. [12] Let $m : U \to B$ and $e : A \to U$ be morphisms in a category **C**. Then a *pullback complement* of the pair (m, e) in the category **C** is a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & U \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ P & \xrightarrow[\bar{e}]{} & B \end{array} \tag{5.1}$$

such that, given any pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{d} & U \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[g]{} & B \end{array} \tag{5.2}$$

and a morphism $h : X \to A$ with $e \circ h = d$, there is a unique morphism $h' : Y \to P$ with $\bar{e} \circ h' = g$ and $h' \circ k = \bar{m} \circ h$.

In the category **Set**, pullback complement of the pair (m, e), where $m : Z \to Y$ is one-one and $e : X \to Z$ is a map, always exists and given by

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & Z \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ (Y \setminus m(Z)) + X & \xrightarrow[\bar{e}]{} & Y \end{array} \tag{5.3}$$

where $(Y \setminus m(Z)) + X = \{(y, 1) \mid y \in Y \setminus m(Z)\} \cup \{(x, 2) \mid x \in X\}$, $\bar{m} : X \to (Y \setminus m(Z)) + X$ is defined as $\bar{m}(x) = (x, 2)$ and $\bar{e} : (Y \setminus m(Z)) + X \to Y$ is defined as $\bar{e}(y, 1) = y$, $\bar{e}(x, 2) = (m \circ e)(x)$ (cf. [5, 12]). It can also be easily verified that if e is onto, then $\bar{e}$ is onto.

Proposition 5.3. *Let $m : (Z, \eta) \to (Y, \sigma)$ be an embedding in Q-**TOP** and let $e : (X, \tau) \to (Z, \eta)$ be initial in Q-**TOP**. Then there exists a pullback complement of (m, e) in Q-**TOP**:*

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{e} & (Z,\eta) \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ (W,\theta) & \xrightarrow[\bar{e}]{} & (Y,\sigma) \end{array} \tag{5.4}$$

*where $\bar{e} : (W, \theta) \to (Y, \sigma)$ is initial in Q-**TOP**.*

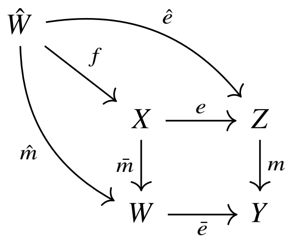
Proof. Let us consider the pullback complement of the pair (m, e) in **Set** given by the following:

$$\begin{array}{ccc} X & \xrightarrow{e} & Z \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ W & \xrightarrow[\bar{e}]{} & Y \end{array} \tag{5.5}$$

where $W = (Y \setminus m(Z)) + X$, $\bar{m} : X \to (Y \setminus m(Z)) + X$ is defined as $\bar{m}(x) = (x, 2)$ and $\bar{e} : (Y \setminus m(Z)) + X \to Y$ is defined as $\bar{e}(y, 1) = y$, $\bar{e}(x, 2) = (m \circ e)(x)$. Now let $\theta = \{\beta \circ \bar{e} \mid \beta \in \sigma\}$. Let $\beta \circ \bar{e} \in \theta$, where $\beta \in \sigma$. Then $\beta \circ \bar{e} \circ \bar{m} = \beta \circ m \circ e = \beta \circ (m \circ e) \in \tau$ (as $m \circ e : (X, \tau) \to (Y, \sigma)$ is Q-continuous). Thus $\bar{m} : (X, \tau) \to (W, \theta)$ is Q-continuous. Next let following be a commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (\hat{W},\hat{\theta}) & \xrightarrow{\hat{e}} & (Z,\eta) \\ {\scriptstyle \hat{m}}\downarrow & & \downarrow{\scriptstyle m} \\ (W,\theta) & \xrightarrow[\bar{e}]{} & (Y,\sigma) \end{array} \tag{5.6}$$

Since the diagram 5.5 is a pullback square in **Set**, there exists a unique map $f : \hat{W} \to X$ such that the following diagram commutes:



(5.7)

Now we will show that $f : (\hat{W}, \hat{\theta}) \to (X, \tau)$ is Q-continuous. Let $\beta \circ e \in \tau$, where $\beta \in \eta$. Then $\beta \circ e \circ f = \beta \circ \hat{e} \in \hat{\theta}$ (as $\hat{e} : (\hat{W}, \hat{\theta}) \to (Z, \eta)$ is Q-continuous). Thus $f : (\hat{W}, \hat{\theta}) \to (X, \tau)$ is Q-continuous. Hence the diagram 5.4 is a pullback square in Q-**TOP**. Now let following be a pullback square in Q-**TOP**:

$$\begin{array}{ccc} (A, \tau_A) & \xrightarrow{d} & (Z, \eta) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle m} \\ (B, \tau_B) & \xrightarrow[g]{} & (Y, \sigma) \end{array} \tag{5.8}$$

and let $h : (A, \tau_A) \to (X, \tau)$ be a Q-continuous map such that $e \circ h = d$. Then if we consider the diagram 5.8 in **Set**, then it is a pullback square in **Set** also and since the diagram 5.5 is a pullback complement diagram in **Set**, there exists a unique map $h' : B \to W$ such that $\bar{e} \circ h' = g$ and $h' \circ k = \bar{m} \circ h$. So now it is sufficient to show that $h' : (B, \tau_B) \to (W, \theta)$ is Q-continuous. Let $\beta \circ \bar{e} \in \theta$, where $\beta \in \sigma$. Then $\beta \circ \bar{e} \circ h' = \beta \circ g \in \tau_B$ (as $g : (B, \tau_B) \to (Y, \sigma)$ is Q-continuous). Thus $h' : (B, \tau_B) \to (W, \theta)$ is Q-continuous. Therefore the diagram 5.4 gives a pullback complement of (m, e) in Q-**TOP**. □

Proposition 5.4. [4] *Let **C** be a category and let following be a pullback square in **C***

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array} \tag{5.9}$$

If (X, f) is injective in $\mathbf{C}/Y$, then (W, q) is injective in $\mathbf{C}/Z$.

The following Theorem 5.5 is concerned with the extension of Theorem 2.11 of [5], in the category Q-$\mathbf{TOP}/(Y, \sigma)$.

Theorem 5.5. *Let* (X,τ) *and* (Y,σ) *be Q-topological spaces and let* $f:(X,\tau)\to(Y,\sigma)$ *be a Q-continuous map. Then following statements are equivalent:*

1. $((X,\tau),f)$ *has an injective hull in* Q-***TOP***$/(Y,\sigma)$.
2. $((\tilde{X},\tilde{\tau}),\tilde{f})$ *has an injective hull in* Q-***TOP***$_0/(\tilde{Y},\tilde{\sigma})$.
3. $((\tilde{X},\tilde{\tau}),\tilde{f})$ *has an injective hull in* Q-***TOP***$/(\tilde{Y},\tilde{\sigma})$.

Proof. (1) $\Rightarrow$ (2) Let $((X,\tau),f)$ have an injective hull $j:((X,\tau),f)\to((Z,\eta),g)$ in Q-**TOP**$/(Y,\sigma)$. We will show that $\tilde{j}:((\tilde{X},\tilde{\tau}),\tilde{f})\to((\tilde{Z},\tilde{\eta}),\tilde{g})$ is an injective hull of $((\tilde{X},\tilde{\tau}),\tilde{f})$ in Q-**TOP$_0$**$/(\tilde{Y},\tilde{\sigma})$. Now since $((Z,\eta),g)$ is injective in Q-**TOP**$/(Y,\sigma)$, by Proposition 4.9 $((\tilde{Z},\tilde{\eta}),\tilde{g})$ is injective in Q-**TOP$_0$**$/(\tilde{Y},\tilde{\sigma})$. Next since $j:(X,\tau)\to(Y,\sigma)$ is an embedding in Q-**TOP**, $\tilde{j}:(\tilde{X},\tilde{\tau})\to(\tilde{Z},\tilde{\eta})$ is an embedding in Q-**TOP** and then by Lemma 4.4, $\tilde{j}:(\tilde{X},\tilde{\tau})\to(\tilde{Z},\tilde{\eta})$ is an embedding in Q-**TOP$_0$**. Next let $k:((\tilde{Z},\tilde{\eta}),\tilde{g})\to((W,\theta),l)$ be a morphism in Q-**TOP$_0$**$/(\tilde{Y},\tilde{\sigma})$ such that $k\circ\tilde{j}:(\tilde{X},\tilde{\tau})\to(W,\theta)$ is an embedding in Q-**TOP$_0$**. We have to prove that $k:(\tilde{Z},\tilde{\eta})\to(W,\theta)$ is an embedding in Q-**TOP$_0$**. Consider the Q-topology $\{\alpha\circ p_1\mid\alpha\in\theta\}$ on $W\times Z$, where $p_1:W\times Z\to W$ is the first projection map. Let $T=W\times Z$ and $\rho=\{\alpha\circ p_1\mid\alpha\in\theta\}$. Now we will prove that $(\tilde{T},\tilde{\rho})$ and (W,θ) are isomorphic. Define a map $h:\tilde{T}\to W$ as $h([(w,z)])=w$. Let $[(w_1,z_1)]=[(w_2,z_2)]\Rightarrow(\alpha\circ p_1)(w_1,z_1)=(\alpha\circ p_1)(w_2,z_2)$, for every $\alpha\in\theta\Rightarrow\alpha(w_1)=\alpha(w_2)$, for every $\alpha\in\theta\Rightarrow w_1=w_2$ (since $(W,\theta)\in Q$-**TOP$_0$** and if $w_1\neq w_2$, then there exists $\alpha\in\theta$ such that $\alpha(w_1)\neq\alpha(w_2)$). Thus the map h is well-defined. Now let $h([(w_1,z_1)])=h([(w_2,z_2)])\Rightarrow w_1=w_2\Rightarrow \alpha(w_1)=\alpha(w_2)$, for every $\alpha\in\theta\Rightarrow(\alpha\circ p_1)(w_1,z_1)=(\alpha\circ p_1)(w_2,z_2)$, for every $\alpha\in\theta\Rightarrow[(w_1,z_1)]=[(w_2,z_2)]$. Thus h is one-one and hence h is bijective. Now let $\alpha\in\theta$. Then $(\alpha\circ h\circ q_{W\times Z})(w,z)=\alpha(w)=(\alpha\circ p_1)(w,z)\Rightarrow\alpha\circ h\circ q_{W\times Z}=\alpha\circ p_1\in\rho\Rightarrow\alpha\circ h\in\tilde{\rho}$. Thus $h:(\tilde{T},\tilde{\rho})\to(W,\theta)$ is Q-continuous. Now let $\beta\in\tilde{\rho}$, then $\beta\circ q_{W\times Z}\in\rho$ and so $\beta\circ q_{W\times Z}=\alpha\circ p_1$, for some $\alpha\in\theta$. Then $(\beta\circ q_{W\times Z})(w,z)=(\alpha\circ p_1)(w,z)=\alpha(w)=\alpha(h([(w,z)]))=(\alpha\circ h)([(w,z)])=(\alpha\circ h\circ q_{W\times Z})(w,z)\Rightarrow\beta\circ q_{W\times Z}=\alpha\circ h\circ q_{W\times Z}$ and so $\beta=\alpha\circ h$ (as $q_{W\times Z}$ is onto). Thus $h:(\tilde{T},\tilde{\rho})\to(W,\theta)$ is initial in Q-**TOP** and also h is bijective. Hence by Proposition 3.3, $h:(\tilde{T},\tilde{\rho})\to(W,\theta)$ is an isomorphism in Q-**TOP**. Now it can be easily verified that the map $k'=(k\circ q_Z,id_Z):(Z,\eta)\to(W\times Z,\rho)$ is Q-continuous and the following diagram commutes:

$$\begin{array}{ccc} (Z,\eta) & \xrightarrow{k'} & (W\times Z,\rho) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle l\circ h\circ q_{W\times Z}} \\ (Y,\sigma) & \xrightarrow[q_Y]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{5.10}$$

Now since q_Y is onto, (Y, q_Y) is injective in the category $\mathbf{Set}/\tilde{Y}$ and since k' is one-one, there exists a map $m : W \times Z \to Y$ such that $m \circ k' = g$ and $q_Y \circ m = l \circ h \circ q_{W\times Z}$. Let $v \circ q_Y \in \sigma$, where $v \in \tilde{\sigma}$, then $v \circ q_Y \circ m = v \circ l \circ h \circ q_{W\times Z} \in \rho$ (as $l \circ h \circ q_{W\times Z} : (W\times Z, \rho) \to (\tilde{Y}, \tilde{\sigma})$ is Q-continuous). Thus $m : (W\times Z, \rho) \to (Y, \sigma)$ is Q-continuous. Hence $((W \times Z, \rho), m) \in Q$-$\mathbf{TOP}/(Y, \sigma)$. Now it can be easily verified that the following diagram commutes:

$$\begin{array}{ccc} (Z,\eta) & \xrightarrow{k'} & (W\times Z,\rho) \\ {\scriptstyle q_Z}\big\downarrow & & \big\downarrow{\scriptstyle q_{W\times Z}} \\ (\tilde{Z},\tilde{\eta}) & \xrightarrow[h^{-1}\circ k]{} & (\tilde{T},\tilde{\rho}) \end{array} \tag{5.11}$$

Thus $\tilde{k'} = h^{-1} \circ k$. Let $p = k' \circ j$. Then $\tilde{p} = \tilde{k'} \circ \tilde{j} = (h^{-1} \circ k) \circ \tilde{j} = h^{-1} \circ (k \circ \tilde{j})$. Now since $h^{-1} : (W, \theta) \to (\tilde{T}, \tilde{\rho})$ is an isomorphism in Q-$\mathbf{TOP}$, it is an embedding in Q-$\mathbf{TOP}$. We know that composition of embeddings is an embedding and since $\tilde{p} = h^{-1} \circ (k \circ \tilde{j})$, $\tilde{p} : (\tilde{X}, \tilde{\tau}) \to (\tilde{T}, \tilde{\rho})$ is an embedding in Q-$\mathbf{TOP}$. Also $p = k' \circ j$ is one-one. Thus by Proposition 3.2, $p = k' \circ j : (X, \tau) \to (W \times Z, \rho)$ is an embedding in Q-$\mathbf{TOP}$. Thus $k' : ((Z, \eta), g) \to ((W \times Z, \rho), m)$ is a morphism in Q-$\mathbf{TOP}/(Y, \sigma)$ such that $k' \circ j : (X, \tau) \to (W \times Z, \rho)$ is an embedding in Q-$\mathbf{TOP}$ and then since $j : ((X, \tau), f) \to ((Z, \eta), g)$ is essential in Q-$\mathbf{TOP}/(Y, \sigma)$, $k' : (Z, \eta) \to (W \times Z, \rho)$ is an embedding in Q-$\mathbf{TOP}$. Then by Proposition 3.2, $\tilde{k'} : (\tilde{Z}, \tilde{\eta}) \to (\tilde{T}, \tilde{\rho})$ is an embedding in Q-$\mathbf{TOP}$. Now since $\tilde{k'} = h^{-1} \circ k$, so $h^{-1} \circ k : (\tilde{Z}, \tilde{\eta}) \to (\tilde{T}, \tilde{\rho})$ is an embedding in Q-$\mathbf{TOP}$. This implies that $k : (\tilde{Z}, \tilde{\eta}) \to (W, \theta)$ is an embedding in Q-$\mathbf{TOP}$ and then by Lemma 4.4, $k : (\tilde{Z}, \tilde{\eta}) \to (W, \theta)$ is an embedding in Q-$\mathbf{TOP}_0$. Thus $\tilde{j} : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((\tilde{Z}, \tilde{\eta}), \tilde{g})$ is essential in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$. Therefore $\tilde{j} : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((\tilde{Z}, \tilde{\eta}), \tilde{g})$ is an injective hull of $((\tilde{X}, \tilde{\tau}), \tilde{f})$ in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$.

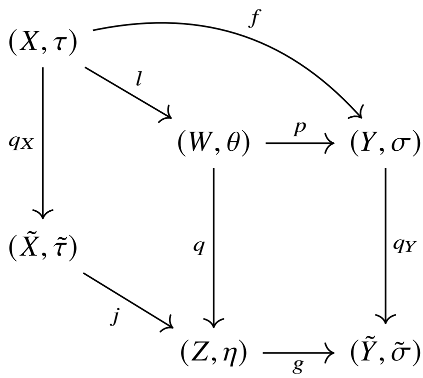
(2) $\Rightarrow$ (3) Follows from Proposition 5.1.

(3) $\Rightarrow$ (1) Let $j : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((Z, \eta), g)$ be an injective hull of $((\tilde{X}, \tilde{\tau}), \tilde{f})$ in Q-$\mathbf{TOP}/(\tilde{Y}, \tilde{\sigma})$. Then by Proposition 5.1, $j : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((Z, \eta), g)$ is an injective hull of $(\tilde{X}, \tilde{\tau}), \tilde{f})$ in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$. Thus clearly $(Z, \eta) \in Q$-$\mathbf{TOP}_0$.

Now let $q : (W, \theta) \to (Z, \eta)$ be a pullback of $q_Y : (Y, \sigma) \to (\tilde{Y}, \tilde{\sigma})$ along $g : (Z, \eta) \to (\tilde{Y}, \tilde{\sigma})$ in Q-**TOP**:

$$\begin{array}{ccc} (W,\theta) & \xrightarrow{q} & (Z,\eta) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ (Y,\sigma) & \xrightarrow[q_Y]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{5.12}$$

We note that by Proposition 3.5, $\tilde{q} : (\tilde{W}, \tilde{\theta}) \to (\tilde{Z}, \tilde{\eta})$ is an isomorphism in Q-**TOP**. Now since $g \circ j \circ q_X = (g \circ j) \circ q_X = \tilde{f} \circ q_X = q_Y \circ f$ and the diagram 5.12 is a pullback, there exists a unique Q-continuous map $l : (X, \tau) \to (W, \theta)$ for which the following diagram commutes:



(5.13)

Now since $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ and $j : (\tilde{X}, \tilde{\tau}) \to (Z, \eta)$ both are initial in Q-**TOP** and composition of initial maps is initial, $j \circ q_X : (X, \tau) \to (Z, \eta)$ is initial in Q-**TOP** and since $j \circ q_X = q \circ l$, $q \circ l : (X, \tau) \to (Z, \eta)$ is initial in Q-**TOP** and then it can be easily proved that $l : (X, \tau) \to (W, \theta)$ is initial in Q-**TOP**. Now let $e : X \to l(X)$ defined as $e(x) = l(x)$ and let $m : l(X) \to W$ be the inclusion map. Next if we take the Q-topology $\theta' = \{\beta \circ m \mid \beta \in \theta\}$ on $l(X)$, then we have a factorization of $l : (X, \tau) \to (W, \theta)$ given by $l = m \circ e$ such that $e : (X, \tau) \to (l(X), \theta')$ is onto and $m : (l(X), \theta') \to (W, \theta)$ is an embedding in Q-**TOP**. Then since $l : (X, \tau) \to (W, \theta)$ is initial in Q-**TOP** and $l = m \circ e$, $e : (X, \tau) \to (l(X), \theta')$ is initial in Q-**TOP**. Thus by Proposition 5.3, there exists a pullback complement of (m, e) in Q-**TOP** given by:

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{\ e\ } & (l(X),\theta') \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ (E,\tau_E) & \xrightarrow[\bar{e}]{} & (W,\theta) \end{array} \tag{5.14}$$

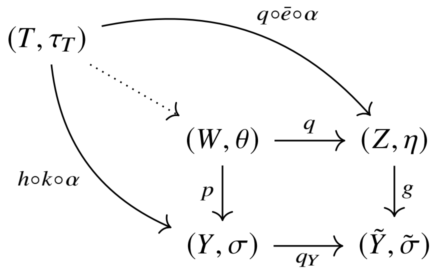
where $E = (W \setminus m(l(X))) + X$, $\bar{m} : X \to E$ is defined as $\bar{m}(x) = (x,2)$, $\bar{e} : E \to W$ is defined as $\bar{e}(w,1) = w$, $\bar{e}(x,2) = (m \circ e)(x)$ and $\bar{e} : (E,\tau_E) \to (W,\theta)$ is initial in Q-**TOP**. Also since e is onto, $\bar{e}$ is onto. Now since $\bar{e}$ is onto and $\bar{e} : (E,\tau_E) \to (W,\theta)$ is initial in Q-**TOP**, as in the proof of the Proposition 3.5, we can prove that $\tilde{\bar{e}}$ is bijective and $\tilde{\bar{e}} : (\tilde{E},\tilde{\tau_E}) \to (\tilde{W},\tilde{\theta})$ is initial in Q-**TOP** and hence by Proposition 3.3, $\tilde{\bar{e}} : (\tilde{E},\tilde{\tau_E}) \to (\tilde{W},\tilde{\theta})$ is an isomorphism in Q-**TOP**. Also by Proposition 4.1, $((E,\tau_E),\bar{e})$ is injective in Q-**TOP**$/(W,\theta)$. Now since $((Z,\eta),g)$ is injective in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ and the diagram 5.12 is a pullback, by Proposition 5.4, $((W,\theta),p)$ is injective in Q-**TOP**$/(Y,\sigma)$. Thus by Proposition 4.3, $((E,\tau_E),p\circ\bar{e})$ is injective in Q-**TOP**$/(Y,\sigma)$. Now since $p\circ\bar{e}\circ\bar{m} = p\circ l = f$, $\bar{m} : ((X,\tau),f) \to ((E,\tau_E),p\circ\bar{e})$ is a morphism in Q-**TOP**$/(Y,\sigma)$. Now we will prove that $\bar{m} : ((X,\tau),f) \to ((E,\tau_E),p\circ\bar{e})$ is an injective hull of $((X,\tau),f)$ in Q-**TOP**$/(Y,\sigma)$. We know that regular monomorphisms in Q-**TOP** are precisely embeddings in Q-**TOP** and since the diagram 5.14 is a pullback and $m : (l(X),\theta') \to (W,\theta)$ is an embedding in Q-**TOP**, by Proposition 3.4, $\bar{m} : (X,\tau) \to (E,\tau_E)$ is an embedding in Q-**TOP**. Next let $k : ((E,\tau_E),p\circ\bar{e}) \to ((G,\tau_G),h)$ be a morphism in Q-**TOP**$/(Y,\sigma)$ such that $k\circ\bar{m} : (X,\tau) \to (G,\tau_G)$ is an embedding in Q-**TOP**. We have to show that $k : (E,\tau_E) \to (G,\tau_G)$ is an embedding in Q-**TOP**. To prove this, in view of Proposition 3.2 it is sufficient to prove that $\tilde{k} : (\tilde{E},\tilde{\tau_E}) \to (\tilde{G},\tilde{\tau_G})$ is an embedding in Q-**TOP** and k is one-one. Now since $k\circ\bar{m} : (X,\tau) \to (G,\tau_G)$ is an embedding in Q-**TOP**, by Proposition 3.2, $\tilde{k}\circ\tilde{\bar{m}} : (\tilde{X},\tilde{\tau}) \to (\tilde{G},\tilde{\tau_G})$ is an embedding in Q-**TOP**. Now $(\tilde{q})^{-1}\circ q_Z\circ j\circ q_X = (\tilde{q})^{-1}\circ q_Z\circ(j\circ q_X) = (\tilde{q})^{-1}\circ q_Z\circ(q\circ l) = (\tilde{q})^{-1}\circ(q_Z\circ q)\circ l = (\tilde{q})^{-1}\circ(\tilde{q}\circ q_W)\circ l = q_W\circ l$. Thus the following diagram commutes:

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{\ l\ } & (W,\theta) \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_W} \\ (\tilde{X},\tilde{\tau}) & \xrightarrow[(\tilde{q})^{-1}\circ q_Z\circ j]{} & (\tilde{W},\tilde{\theta}) \end{array} \tag{5.15}$$

and hence $\tilde{l} = (\tilde{q})^{-1} \circ q_Z \circ j$. We also have $g \circ q_Z^{-1} \circ \tilde{q} \circ q_W = g \circ q_Z^{-1} \circ (\tilde{q} \circ q_W) = g \circ q_Z^{-1} \circ (q_Z \circ q) = g \circ q = q_Y \circ p$. Thus the following diagram commutes:

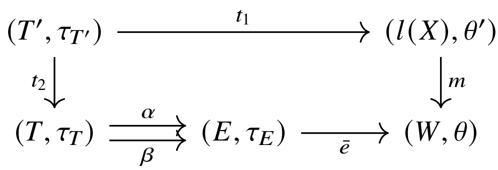
$$\begin{array}{ccc} (W, \theta) & \xrightarrow{\;p\;} & (Y, \sigma) \\ {\scriptstyle q_W}\downarrow & & \downarrow{\scriptstyle q_Y} \\ (\tilde{W}, \tilde{\theta}) & \xrightarrow[g \circ q_Z^{-1} \circ \tilde{q}]{} & (\tilde{Y}, \tilde{\sigma}) \end{array} \tag{5.16}$$

and so $\tilde{p} = g \circ q_Z^{-1} \circ \tilde{q}$. Now since $(Z, \eta) \in Q$-**TOP**$_0$, $q_Z : (Z, \eta) \to (\tilde{Z}, \tilde{\eta})$ is an isomorphism in Q-**TOP** and since $(\tilde{q})^{-1} : (\tilde{Z}, \tilde{\eta}) \to (\tilde{W}, \tilde{\theta})$ is also an isomorphism in Q-**TOP**, $(\tilde{q})^{-1} \circ q_Z : (Z, \eta) \to (\tilde{W}, \tilde{\theta})$ is an isomorphism in Q-**TOP** and so it is an essential embedding in Q-**TOP** and then it can be easily seen that $(\tilde{q})^{-1} \circ q_Z : ((Z, \eta), g) \to ((\tilde{W}, \tilde{\theta}), \tilde{p})$ is a morphism in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$ which is essential in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$. Thus $\tilde{l} = (\tilde{q})^{-1} \circ q_Z \circ j : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((\tilde{W}, \tilde{\theta}), \tilde{p})$ is essential in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$. Now since $l = \bar{e} \circ \bar{m}$, $\tilde{l} = \tilde{\bar{e}} \circ \tilde{\bar{m}}$ and so $\tilde{\bar{m}} = (\tilde{\bar{e}})^{-1} \circ \tilde{l}$. Now since $(\tilde{\bar{e}})^{-1} : (\tilde{W}, \tilde{\theta}) \to (\tilde{E}, \tilde{\tau_E})$ is an isomorphism in Q-**TOP**, it is an essential embedding in Q-**TOP**. Thus we have a morphism $(\tilde{\bar{e}})^{-1} : ((\tilde{W}, \tilde{\theta}), \tilde{p}) \to ((\tilde{E}, \tilde{\tau_E}), \tilde{p} \circ \tilde{\bar{e}})$ in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$ which is essential in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$. Then since $\tilde{\bar{m}} = (\tilde{\bar{e}})^{-1} \circ \tilde{l}$, $\tilde{\bar{m}} : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((\tilde{E}, \tilde{\tau_E}), \tilde{p} \circ \tilde{\bar{e}})$ is essential in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$. Now since $\tilde{k} : ((\tilde{E}, \tilde{\tau_E}), \tilde{p} \circ \tilde{\bar{e}}) \to ((\tilde{G}, \tilde{\tau_G}), \tilde{h})$ is a morphism in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$ such that $\tilde{k} \circ \tilde{\bar{m}} : (\tilde{X}, \tilde{\tau}) \to (\tilde{G}, \tilde{\tau_G})$ is an embedding in Q-**TOP**, and $\tilde{\bar{m}} : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((\tilde{E}, \tilde{\tau_E}), \tilde{p} \circ \tilde{\bar{e}})$ is essential in Q-**TOP**/$(\tilde{Y}, \tilde{\sigma})$, $\tilde{k} : (\tilde{E}, \tilde{\tau_E}) \to (\tilde{G}, \tilde{\tau_G})$ is an embedding in Q-**TOP**. Thus by Proposition 3.2, to prove that $k : (E, \tau_E) \to (G, \tau_G)$ is an embedding in Q-**TOP**, now it is sufficient to prove that k is one-one. Let (T, τ_T) be a Q-topological space and let $\alpha, \beta : (T, \tau_T) \to (E, \tau_E)$ be Q-continuous maps such that $k \circ \alpha = k \circ \beta$. This implies that $q_G \circ k \circ \alpha = q_G \circ k \circ \beta \Rightarrow (q_G \circ k) \circ \alpha = (q_G \circ k) \circ \beta \Rightarrow (\tilde{k} \circ q_E) \circ \alpha = (\tilde{k} \circ q_E) \circ \beta \Rightarrow \tilde{k} \circ (q_E \circ \alpha) = \tilde{k} \circ (q_E \circ \beta)$ and since $\tilde{k}$ is one-one, $q_E \circ \alpha = q_E \circ \beta$. Now since $g \circ q \circ \bar{e} \circ \alpha = q_Y \circ p \circ \bar{e} \circ \alpha = q_Y \circ h \circ k \circ \alpha$ and the diagram 5.12 is a pullback, there exists a unique Q-continuous map from (T, τ_T) to (W, θ) making the following diagram commutative



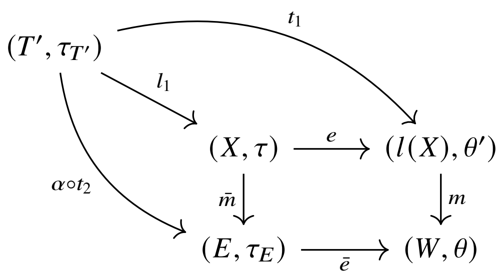
(5.17)

Now we have $p\circ\bar{e}\circ\beta=(p\circ\bar{e})\circ\beta=(h\circ k)\circ\beta=h\circ(k\circ\beta)=h\circ k\circ\alpha$ and $q\circ\bar{e}\circ\beta=q_Z^{-1}\circ\tilde{q}\circ q_W\circ\bar{e}\circ\beta=q_Z^{-1}\circ\tilde{q}\circ(q_W\circ\bar{e})\circ\beta=q_Z^{-1}\circ\tilde{q}\circ(\tilde{\bar{e}}\circ q_E)\circ\beta=q_Z^{-1}\circ\tilde{q}\circ\tilde{\bar{e}}\circ(q_E\circ\beta)=q_Z^{-1}\circ\tilde{q}\circ\tilde{\bar{e}}\circ(q_E\circ\alpha)=q_Z^{-1}\circ\tilde{q}\circ(\tilde{\bar{e}}\circ q_E)\circ\alpha=q_Z^{-1}\circ\tilde{q}\circ(q_W\circ\bar{e})\circ\alpha=(q_Z^{-1}\circ\tilde{q}\circ q_W)\circ\bar{e}\circ\alpha=q\circ\bar{e}\circ\alpha$. Also we have $p\circ\bar{e}\circ\alpha=h\circ k\circ\alpha$. Thus here we have two Q-continuous maps $\bar{e}\circ\alpha,\bar{e}\circ\beta:(T,\tau_T)\to(W,\theta)$ making the diagram 5.17 commutative. So $\bar{e}\circ\alpha=\bar{e}\circ\beta$. Now consider a pullback of $\bar{e}\circ\alpha=\bar{e}\circ\beta:(T,\tau_T)\to(W,\theta)$ along $m:(l(X),\theta')\to(W,\theta)$ in Q-**TOP** given by the following:



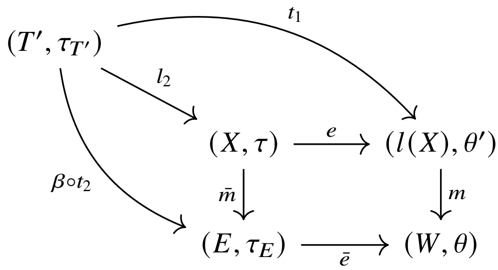
(5.18)

Now since the diagram 5.14 is a pullback square, there exists a unique Q-continuous map $l_1:(T',\tau'_T)\to(X,\tau)$ for which the following diagram commutes:



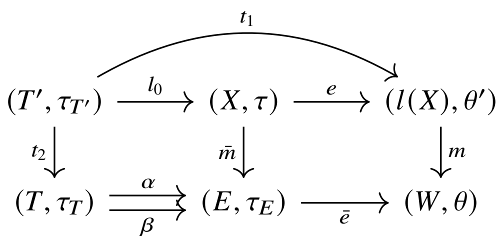
(5.19)

Similarly we have a unique Q-continuous map $l_2:(T',\tau'_T)\to(X,\tau)$ for which the following diagram commutes:



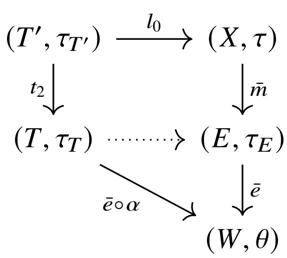
(5.20)

Then we have $\beta \circ t_2 = \bar{m} \circ l_2 \Rightarrow k \circ \beta \circ t_2 = k \circ \bar{m} \circ l_2 \Rightarrow k \circ \alpha \circ t_2 = k \circ \bar{m} \circ l_2$ and $\bar{m} \circ l_1 = \alpha \circ t_2 \Rightarrow k \circ \bar{m} \circ l_1 = k \circ \alpha \circ t_2$. Thus $k \circ \bar{m} \circ l_1 = k \circ \bar{m} \circ l_2$ and since $k \circ \bar{m}$ is one-one, $l_1 = l_2$. Let $l_1 = l_2 = l_0$. Thus the following diagram commutes:



(5.21)

Now since the diagram 5.14 is a pullback complement diagram, with respect to the pullback diagram 5.18 and Q-continuous map $l_0 : (T', \tau_{T'}) \to (X, \tau)$, there exists a unique Q-continuous map from (T, τ_T) to (E, τ_E) making the following diagram commutative



(5.22)

But here we have two Q-continuous maps $\alpha, \beta : (T, \tau_T) \to (E, \tau_E)$ making the diagram 5.22 commutative. So $\alpha = \beta$ and hence k is one-one. Therefore $\bar{m} : ((X, \tau), f) \to ((E, \tau_E), p \circ \bar{e})$ is an injective hull of $((X, \tau), f)$ in Q-**TOP**$/(Y, \sigma)$.
□

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