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On nominal sets with support-preorder

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Abstract. Each nominal set X can be equipped with a preorder relation $\preceq$ defined by the notion of support, so-called support-preorder. This preorder also leads us to the support topology on each nominal set. We study support-preordered nominal sets and some of their categorical properties in this paper. We also examine the topological properties of support topology, in particular separation axioms.

1 Introduction

Nominal set theory provides a mathematical framework for studying semantics, modifying variables, and much more in computer science. Indeed, Fraenkel presented nominal sets in [3] as an alternative model of set theory in 1922. In this context Mostowski studied further, which is why nominal sets are sometimes referred to as Fraenkel-Mostowski sets. In the 1990s, Gabbay and Pitts [6] rediscovered nominal sets for the computer science community, and this notion sparked a lot of interest in semantics [1, 2, 4, 5].

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Every nominal set can be viewed as a preordered set equipped with the support preorder relation based on the notion of support. Here considering support-preordered nominal sets, briefly sp-nominal sets, and the category **spNom** of sp-nominal sets and sp-preserving equivariant maps between them, some categorical properties in **spNom** including monics, epics, products and coproducts are investigated. In particular, we find some conditions under which products and coproducts in **spNpm** do exist. This preorder also provides a topological structure on a nominal set which converts nominal sets to nominal spaces. Some topological properties of nominal spaces such as separation axioms and compactness are also studied.

2 Preliminaries

This section contains some necessary notions on nominal sets and topological spaces needed throughout the paper from [8] and [7] respectively. For further information on category theory, one may consult [1].

2.1 Nominal sets Suppose $\mathbb{D}$ is a set, then a permutation π of $\mathbb{D}$ is a bijective map from $\mathbb{D}$ to itself. The permutations of $\mathbb{D}$ with composition and identity form a group, called the *symmetric group* on the set $\mathbb{D}$ and denoted by $\mathrm{Sym}\mathbb{D}$. A permutation $\pi \in \mathrm{Sym}\mathbb{D}$ is finitary if the set $\{d \in \mathbb{D} \,|\, \pi d \neq d\}$ is a finite subset of $\mathbb{D}$. It is clear that $\mathrm{id} \in \mathrm{Sym}\mathbb{D}$ is finitary and that the composition and the inverse of finitary permutations are finitary. Therefore, we get a subgroup of $\mathrm{Sym}\mathbb{D}$ of finitary permutations, denoted by $\mathrm{Perm}(\mathbb{D})$. We fix a countable infinite set $\mathbb{D}$ whose elements are denoted by $a, b, c, \ldots$ and called *atomic names*. Let X be a set equipped with an action of the group $\mathrm{Perm}(\mathbb{D})$, $\mathrm{Perm}(\mathbb{D}) \times X \to X$ mapping (π, x) to πx. We call X a $\mathrm{Perm}(\mathbb{D})$-*set*, whenever for every $\pi_1, \pi_2 \in \mathrm{Perm}(\mathbb{D})$ and every $x \in X$ we have:

(1) $\pi_1(\pi_2 x) = (\pi_1 o \pi_2)x$

(2) $\mathrm{id}\, x = x$.

A subset Y of a $\mathrm{Perm}(\mathbb{D})$-set X is called *equivariant* if $\pi y \in Y$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and $y \in Y$. $\mathrm{Perm}(\mathbb{D})$-sets are the objects of a category, denoted by $\mathrm{Perm}(\mathbb{D})$-**Set** whose morphisms are equivariant maps, i.e. maps subject to the rule $f(\pi x) = \pi f(x)$, for all $x \in X, \pi \in \mathrm{Perm}(\mathbb{D})$, whose compositions and identities are as in the category **Set** of sets and maps.

An element x of a $\mathrm{Perm}(\mathbb{D})$-set X is called a *zero* element if $\pi x = x$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$. The set of all zero elements of the $\mathrm{Perm}(\mathbb{D})$-set X is denoted by $\mathcal{Z}(X)$. A $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are zero is called *discrete*.

Given a $\mathrm{Perm}(\mathbb{D})$-set X, a set of atomic names $D \subseteq \mathbb{D}$ is a *support* for an element $x \in X$ if for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and for every $d \in D$,

$$\pi(d) = d \Rightarrow \pi x = x.$$

Given a $\mathrm{Perm}(\mathbb{D})$-set X, we say an element $x \in X$ is finitely supported, if there is some finite set of atomic names that is, a support for the element x.

Example 2.1. Given a $\mathrm{Perm}(\mathbb{D})$-set X, the power set of X, $\mathcal{P}(X)$, with the action

$$\mathrm{Perm}(\mathbb{D}) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

$$(\pi, S) \rightsquigarrow \{\pi x : x \in S\}$$

is a $\mathrm{Perm}(\mathbb{D})$-set. A set of atomic names D supports $S \in \mathcal{P}(X)$ if and only if

$$(\forall \pi \in \mathrm{Perm}(\mathbb{D}))((\forall d \in D)\ \pi(d) = d) \Rightarrow (\forall\, x \in S)\ \ \pi x \in S.$$

Definition 2.2. [8] A *nominal set* is a $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are finitely supported. Nominal sets are the objects of a category, denoted by **Nom**, whose morphisms are equivariant maps and whose compositions and identities are as in the category of $\mathrm{Perm}(\mathbb{D})$-**Set**.

Remark 2.3. Suppose X is a nominal set and $x \in X$. Intersection of two finite supports of x is a (finite) support of x, [8, Propositions 2.1 and 2.3]. So each $x \in X$ has the least (finite) support which is denoted by $\mathrm{supp}_X\, x$, and when there is no possibility of error, we denote it by $\mathrm{supp}\, x$. In fact, $\mathrm{supp}\, x = \bigcap\{C : C \text{ is a finite support of } x\}$.

Definition 2.4. [8] We say that a set of atomic names $A \subseteq \mathbb{D}$ *strongly supports* an element x of a nominal set X if and only if

$$(\forall \pi \in \mathrm{Perm}(\mathbb{D}))((\forall a \in A)\pi a = a) \Leftrightarrow \pi x = x.$$

A strong nominal set is a $\mathrm{Perm}(\mathbb{D})$-set in which every element is strongly supported by a finite set of atomic names.

Example 2.5. (i) The set $\mathbb{D}$ is a nominal set with the natural action $\pi d = \pi(d)$.

(ii) The action of Perm($\mathbb{D}$) on $\mathbb{D}$ extends pointwise to action of Perm($\mathbb{D}$) on tuples $\mathbb{D}^n$ and $\mathbb{D}^{(n)}$. So, the sets $\mathbb{D}^n = \{(d_1, d_2, \dots, d_n) \in \mathbb{D}^n \mid d_i \in \mathbb{D}\}$ and $\mathbb{D}^{(n)} = \{(d_1, d_2, \dots, d_n) \in \mathbb{D}^n \mid d_i \neq d_j \text{ for } i \neq j\}$ are nominal sets.

Proposition* 2.6*: [8] Suppose X is a Perm($\mathbb{D}$)-set and $x \in X$. A subset $A \subseteq \mathbb{D}$ supports x if and only if, for all $d_1, d_2 \in \mathbb{D} \setminus A$, we have $(d_1\ d_2) \cdot x = x$.

Notation 2.7. We will frequently write $\mathcal{P}_{\text{fs}}(X)$ for the set consisting of all finitely supported subsets of a given nominal set X. By $\operatorname{Fix} C$ we mean the set $\{\pi \in \operatorname{Perm}(\mathbb{D}) \mid \pi a = a, \textit{ for every } a \in C\}$, where $C \subseteq \mathbb{D}$. We also denote by $\mathcal{P}_{\text{f}}(\mathbb{D})$ the set consisting of all finite subsets of $\mathbb{D}$, and by $\mathcal{P}_{\text{cof}}(\mathbb{D})$ the set consisting of all subsets of $\mathbb{D}$ with finite complement.

Lemma 2.8. *Let X be a nominal set and $Y, Z \in \mathcal{P}_{\text{fs}}(X)$. Then, $Y \cup Z$ and $Y \cap Z$ are finitely supported subsets of X.*

Proof. Suppose A is a finite support of Y and B is a finite support of Z. Take $\pi \in \operatorname{Fix}(A \cup B)$. Then, $\pi Y = Y$ and $\pi Z = Z$. So, $\pi(Y \cap Z) = Y \cap Z$ and $\pi(Y \cup Z) = Y \cup Z$. □

Lemma 2.9. *Let X be a nominal set . Then, the following statements are equivalent.*

(i) *X is discrete.*

(ii) *For all $x, y \in X$,* $\operatorname{supp} x = \operatorname{supp} y$.

Proof. (i) ⇒ (ii) It is clear.

(ii) ⇒ (i) On the contrary, suppose there exists $x \in X$ with $\operatorname{supp} x = \{d_1, d_2, \dots, d_k\} \neq \emptyset$. Take distinct elements $d'_1, d'_2, \dots, d'_k \in \mathbb{D}$ with $\{d_1, d_2, \dots, d_k\} \cap \{d'_1, d'_2, \dots, d'_k\} = \emptyset$ and $\pi = (d_1\ d'_1)(d_2\ d'_2) \cdots (d_k\ d'_k)$. Then, we have $\pi x \in X$ with $\operatorname{supp} \pi x \neq \operatorname{supp} x$ which is a contradiction. □

Remark 2.10. Every finite nominal set is discrete.

3 Support-Preordered nominal sets

Every nominal set can be considered as a preordered set, see Definition 3.1. We direct our attention to the category **spNom** of support-preordered nominal sets, in this section, with a view to investigating the properties of its objects and morphisms.

Definition 3.1. By the *support-preorder* on a nominal set X, we mean the binary relation $\preceq$ on X defined by:

$$x \preceq y \Leftrightarrow \operatorname{supp} x \subseteq \operatorname{supp} y.$$

Since $\preceq$ is a preorder (i.e. reflexive and transitive), the pair $(X, \preceq)$ is called a *support-preordered nominal set* or briefly *sp-nominal set*.

It can be easily seen that the support-preorder is equivariant (or action preserving); meaning that:

$$x_1 \preceq x_2 \Rightarrow \pi x_1 \preceq \pi x_2,$$

for each $x_1, x_2 \in X, \pi \in \mathrm{Perm}(\mathbb{D})$.

Example 3.2. The support-preorder on

(i) the nominal set $\mathbb{D}$ is equality. Indeed, since $\operatorname{supp} d = \{d\}$, for every $d \in \mathbb{D}$, we have

$$d \preceq d' \Leftrightarrow \{d\} \subseteq \{d'\} \text{ (or equivalently } d = d').$$

(ii) the nominal set $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ is $\subseteq$. Indeed, since $\operatorname{supp} A = A$, for every $A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, we have

$$A_1 \preceq A_2 \Leftrightarrow A_1 \subseteq A_2,$$

for $A_1, A_2 \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$.

(iii) the nominal set $\mathcal{P}_{\mathrm{cof}}(\mathbb{D})$ is $\supseteq$. Indeed, since $\operatorname{supp} A = A^c$, for every $A \in \mathcal{P}_{\mathrm{cof}}(\mathbb{D})$, we have

$$A_1 \preceq A_2 \Leftrightarrow A_1 \supseteq A_2,$$

for $A_1, A_2 \in \mathcal{P}_{\mathrm{cof}}(\mathbb{D})$.

(iv) the nominal set $\mathcal{P}_{\mathrm{fs}}(\mathbb{D})$ is defined as follows.

$$A_1 \preceq A_2 \Leftrightarrow A_1 \subseteq A_2 \text{ or } A_1 \supseteq A_2 \text{ or } A_1 \cap A_2 = \emptyset \text{ or } A_1 \cup A_2 = \mathbb{D}$$

Indeed, when $A_1 \preceq A_2$ in $\mathcal{P}_{\mathrm{cof}}(\mathbb{D})$, one of the following four items may occur.

- If both A_1, A_2 are finite, then since $\operatorname{supp} A_1 = A_1$ and $\operatorname{supp} A_2 = A_2$, $A_1 \preceq A_2$ if and only if $A_1 \subseteq A_2$.
- If both A_1, A_2 are cofinite, then since $\operatorname{supp} A_1 = A_1^c$, $\operatorname{supp} A_2 = A_2^c$, $A_1 \preceq A_2$ if and only if $A_1 \supseteq A_2$.
- If A_1 is finite and A_2 is cofinite, then since $\operatorname{supp} A_1 = A_1$ and $\operatorname{supp} A_2 = A_2^c$, $A_1 \preceq A_2$ if and only if $A_1 \cap A_2 = \emptyset$.

- If A_1 is cofinite and A_2 is finite, then since $\operatorname{supp} A_1 = A_1^c$ and $\operatorname{supp} A_2 = A_2$, $A_1 \preceq A_2$ if and only if $(A_1^c) \cap (A_2^c) = \emptyset$ if and only if $A_1 \cup A_2 = \mathbb{D}$.

Remark 3.3. Let X be a nominal set. Then one can define the equivalence relation $\sim$ on X obtained from $\preceq$ to be

$$x \sim x' \iff x \preceq x' \text{ and } x' \preceq x.$$

The quotient set $X/\sim$ together with the canonical action over $\mathrm{Perm}(\mathbb{D})$, $\pi(x/\sim) = (\pi x)/\sim$, is a nominal set and we immediately get the following statements.

(i) $\operatorname{supp}(x/\sim) = \operatorname{supp} x$, for every $x/\sim \in X/\sim$.

(ii) $x/\sim = \{y \in X \mid \operatorname{supp} x = \operatorname{supp} y\}$ is the equivalance class of $x \in X$.

(iii) The support-preoreder is a partial order if and only if $x/\sim = \{x\}$, for all $x \in X$.

Lemma 3.4. *Let X be an sp-nominal set and $x, x' \in X$. Then, there exists π with $\pi x \preceq x'$ or $\pi x' \preceq x$.*

Proof. Let $\operatorname{supp} x = \{d_1, \ldots, d_k\}$ and $\operatorname{supp} x' = \{a_1, \ldots, a_m\}$ with $k \leq m$.

Case (i) If $\operatorname{supp} x \cap \operatorname{supp} x' = \emptyset$, then taking $\pi = (d_1\ a_1) \cdots (d_k\ a_k)$ we obtain $\operatorname{supp} \pi x = \pi \operatorname{supp} x = \{a_1, \ldots, a_k\} \subseteq \operatorname{supp} x'$.

Case (ii) If $\operatorname{supp} x \cap \operatorname{supp} x' = \{a_{j+1}, \ldots, a_k\}$, then taking $\pi = (d_1\ a_1) \cdots (d_j\ a_j)$ we obtain $\operatorname{supp} \pi x = \pi \operatorname{supp} x = \{a_1, \ldots, a_j, \ldots, a_m\} \subseteq \operatorname{supp} x'$. $\square$

Definition 3.5. Suppose X and Y are two sp-nominal sets. An equivariant map $f : X \to Y$ is called *support-preorder preserving* or for convenience *sp-preserving* whenever $f(x_1) \preceq f(x_2)$, for all $x_1 \preceq x_2 \in X$.

Example 3.6. (i) The equivariant map $f : \mathbb{D}^2 \to \mathbb{D}^{(2)} \cup \{\theta\}$ defined by

$$f(d_1, d_2) = \begin{cases} (d_1, d_2) & d_1 \neq d_2 \\ \theta & d_1 = d_2, \end{cases}$$

is sp-preserving.

(ii) The support map $\operatorname{supp} : X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, mapping $x \mapsto \operatorname{supp} x$, is sp-preserving.

It is worth noting that an equivariant map between nominal sets does not necessarily preserve support-preorder, see Example 3.7. So we consider the category of support-preordered nominal sets and sp-preserving maps between them denoted by **spNom**.

Example 3.7. Considering the nominal sets $\mathbb{D}^2$ and $\mathbb{D} \cup \{\theta\}$ we define the equivariant map $f : \mathbb{D}^2 \longrightarrow \mathbb{D} \cup \{\theta\}$ as follows.

$$f(d, d') = \begin{cases} \theta & d \neq d' \\ d & d = d'. \end{cases}$$

For every $d \neq d' \in \mathbb{D}$, we have $(d, d) \preceq (d, d')$ but $f(d, d) \npreceq f(d, d')$.

Definition 3.8. By a downward (upward) directed nominal set we mean a nominal set $(X, \preceq)$ in which each pair of elements has a lower (upper) bound. More explicitly, for every $x_1, x_2 \in X$ there exists $x \in X$ with $x \preceq x_1$ and $x \preceq x_2$ ($x_1 \preceq x$ and $x_2 \preceq x$).

Theorem 3.9. (i) *The sp-nominal set $(X, \preceq)$ is downward directed if and only if $\mathcal{Z}(X) \neq \emptyset$.*

(ii) *The sp-nominal set $(X, \preceq)$ is upward directed if and only if the subset $A = \{|\mathrm{supp}\, x| \mid x \in X\}$ of $\mathbb{N}^0$ has no upper bound.*

Proof. (i) Suppose $(X, \preceq)$ is downward directed. Take $A = \{|\mathrm{supp}\, x| \mid x \in X\} \subseteq \mathbb{N} \cup \{0\}$. By well-ordering principle, A contains a least element and hence, there exists $x_0 \in X$ such that $|\mathrm{supp}\, x_0|$ is infimum in A. If $\mathrm{supp}\, x_0 = \emptyset$, then $x_0 \in \mathcal{Z}(X)$. Otherwise, $\mathrm{supp}\, x_0 = \{d_1, d_2, \ldots, d_n\}$, for some $n \in \mathbb{N}$. Then we take $\{d'_1, d'_2, \ldots, d'_n\} \subseteq \mathbb{D}$ with $\{d_1, d_2, \ldots, d_n\} \cap \{d'_1, d'_2, \ldots, d'_n\} = \emptyset$ and consider the finite permutation $\pi = (d_1\ d'_1)(d_2\ d'_2) \cdots (d_n\ d'_n)$. Since $\mathrm{supp}\, \pi x_0 = \{d'_1, d'_2, \ldots, d'_n\}$, $\mathrm{supp}\, \pi x_0 \cap \mathrm{supp}\, x_0 = \emptyset$, and so $x_0 \neq \pi x_0$. By the assumptions, there exists $x_1 \in X$ with $x_1 \preceq x_0$ and $x_1 \preceq \pi x_0$. Hence, $\mathrm{supp}\, x_1 \subseteq \mathrm{supp}\, x_0 \cap \mathrm{supp}\, \pi x_0 = \emptyset$ and so $\mathcal{Z}(X) \neq \emptyset$.
Conversely, suppose $\theta \in \mathcal{Z}(X) \neq \emptyset$. Then clearly for each pair $x_1, x_2 \in X$, $\theta \preceq x_1$ and $\theta \preceq x_2$.
(ii) Suppose X is upward directed. We show that $A = \{|\mathrm{supp}\, x| \mid x \in X\}$ has no upper bound. On the contrary, let A have an upper bound. Then since the set A^{up}, consisting of the upper bounds of A, is a subset of $\mathbb{N}$, well-ordering principle implies A^{up} has the least element n which is the supremum of A. Now let $n = |\mathrm{supp}\, x_0|$,

for some $x_0 \in X$ and $\operatorname{supp} x_0 = \{d_1, d_2, \dots, d_n\}$. Then, analogous to the proof (i), we take $\{d'_1, d'_2, \dots, d'_n\} \subseteq \mathbb{D}$ with $\{d_1, d_2, \dots, d_n\} \cap \{d'_1, d'_2, \dots, d'_n\} = \emptyset$ and consider $\pi x_0 \neq x_0$ in which $\pi = (d_1\, d'_1)(d_2\, d'_2) \cdots (d_n\, d'_n)$. Now, by the hypothesis, there exists $x' \in X$ such that $x_0 \preceq x'$ and $\pi x_0 \preceq x'$. Therefore, $|\operatorname{supp} x_0| < n$ which is a contradiction.

To prove the converse, suppose $x_1, x_2 \in X$ with $\operatorname{supp} x_1 = \{d_1, d_2, \dots, d_n\}$ and $\operatorname{supp} x_2 = \{d'_1, d'_2, \dots, d'_m\}$. Since A has no upper bound, there is $x' \in X$ with $|\operatorname{supp} x'| \geqslant m + n$. Suppose $\operatorname{supp} x' = \{d''_1, d''_2, \dots, d''_{m+n+r}\}$ in which $r \geqslant 0$. If $\operatorname{supp} x_1 \cup \operatorname{supp} x_2 \subseteq \operatorname{supp} x'$ then $\operatorname{supp} x_1 \subseteq \operatorname{supp} x'$ and $\operatorname{supp} x_2 \subseteq \operatorname{supp} x'$ meaning that $x_1 \preceq x'$ and $x_2 \preceq x'$ which is the desired result. Otherwise, suppose $(\operatorname{supp} x_1 \cup \operatorname{supp} x_2) \cap (\operatorname{supp} x') = \{s_1, s_2, \dots, s_{t-1}\}$ and $(\operatorname{supp} x_1 \cup \operatorname{supp} x_2) \cap (\operatorname{supp} x')^c = \{s_t, s_{t+1}, \dots, s_l\}$ with $s_i \in \operatorname{supp} x_1 \cup \operatorname{supp} x_2$. We take $\pi = (d''_t\; s_t)(d''_{t+1}\; s_{t+1}) \dots (d''_l\; s_l)$. Since $\operatorname{supp} \pi \cap \{d''_1, d''_2, \dots, d''_{t-1}\} = \emptyset$, we have $\pi\{d''_1, d''_2, \dots, d''_l\} \subseteq \operatorname{supp} \pi x'$. Hence, $\operatorname{supp} x_1 \cup \operatorname{supp} x_2 = \{d''_1, d''_2, \dots, d''_{t-1}, s_t, s_{t+1}, \dots, s_l\} \subseteq \operatorname{supp} \pi x'$. That is, $x_1 \preceq \pi x'$ and $x_2 \preceq \pi x'$. □

Definition 3.10. Let X be an sp-nominal set and $Y \in \mathcal{P}_{\mathrm{fs}}(X)$. Then we define $Y_{\downarrow} := \{x \in X \mid x \preceq y,\ \text{for some } y \in Y\}$ and $Y^{\uparrow} := \{x \in X \mid y \preceq x,\ \text{for some } y \in Y\}$. In particular, we write $x_{\downarrow}$ and $x^{\uparrow}$ rather than $Y_{\downarrow}$ and $Y^{\uparrow}$, respectively, when $Y \in \mathcal{P}_{\mathrm{fs}}(X)$ is a singleton set containing x.

Lemma 3.11. *Let X be an sp-nominal set and $Y, Z \in \mathcal{P}_{\mathrm{fs}}(X)$. Then*

(i) $Y_{\downarrow} \cup Z_{\downarrow} = (Y \cup Z)_{\downarrow}$ *and* $Y^{\uparrow} \cup Z^{\uparrow} = (Y \cup Z)^{\uparrow}$.

(ii) $Y_{\downarrow}, Y^{\uparrow} \in \mathcal{P}_{\mathrm{fs}}(X)$, *for every* $Y \in \mathcal{P}_{\mathrm{fs}}(X)$.

(iii) *the set* $L_{\downarrow X} := \{Y_{\downarrow}, \emptyset, X \mid\ Y \in \mathcal{P}_{\mathrm{fs}}(X)\}$ *is a bounded lattice.*

(iv) *the set* $L_{\uparrow X} := \{Y^{\uparrow}, \emptyset, X \mid\ Y \in \mathcal{P}_{\mathrm{fs}}(X)\}$ *is a bounded lattice.*

Proof. (i) One can easily check.

(ii) We show that $\operatorname{supp} Y$ is a support for $Y_{\downarrow}$ and $Y^{\uparrow}$, for each $Y \in \mathcal{P}_{\mathrm{fs}}(X)$. Indeed, if $A = \operatorname{supp} Y$, then for every $\pi \in \operatorname{Fix} A$ we have

$$\begin{aligned}
\pi Y^{\uparrow} &= \pi\{x \in X \mid \operatorname{supp} y \subseteq \operatorname{supp} x,\ \text{for some } y \in Y\} \\
&= \{\pi x \in X \mid \operatorname{supp} \pi y \subseteq \operatorname{supp} \pi x,\ \text{for some } \pi y \in \pi Y\} \\
&= \{x' \in X \mid \operatorname{supp} y' \subseteq \operatorname{supp} x',\ \text{for some } y' \in \pi Y = Y\} \\
&= Y^{\uparrow}.
\end{aligned}$$

Analogously, $\pi Y_{\downarrow} = Y_{\downarrow}$.

(iii) We show that $L_{\downarrow X}$ is closed under finite intersections and unions. Let $Y, Z \in \mathcal{P}_{\mathrm{fs}}(X)$. Then, by Lemma 2.8, $Y \cup Z, Y \cap Z \in \mathcal{P}_{\mathrm{fs}}(X)$. Also, applying (i), we have $(Y \cup Z)_{\downarrow} = Y_{\downarrow} \cup Z_{\downarrow}$ and $Y_{\downarrow} \cap Z_{\downarrow} = (Y \cap Z)_{\downarrow}$.

(iv) The proof is similar to (iii). □

Theorem 3.12. *If the sp-nominal set $(X, \preceq)$ is a lattice, then X is isomorphic to a subnominal set of $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$.*

Proof. Suppose $(X, \preceq)$ is a lattice. Then, for every $x, x' \in X$ with $\operatorname{supp} x = \operatorname{supp} x'$, we have $x \preceq x'$ and $x' \preceq x$. Since X is a lattice, $x = x'$. Hence, the equivariant map $\operatorname{supp} : X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ defined by $x \mapsto \operatorname{supp} x$ is injective. □

Lemma 3.13. *Suppose X and Y are two sp-nominal sets, $f : X \to Y$ is an sp-preserving map, and $x \in X$ with $\operatorname{supp} f(x) \neq \emptyset$. Then, $\operatorname{supp} f(x) = \operatorname{supp} x$.*

Proof. First we note that since f is equivariant, $\operatorname{supp} f(x) \subseteq \operatorname{supp} x$ and hence, $\operatorname{supp} x \neq \emptyset$ follows from $\operatorname{supp} f(x) \neq \emptyset$, for an arbitrary $x \in X$ with $\operatorname{supp} f(x) \neq \emptyset$. One can suppose $\operatorname{supp} x = \{d_1, d_2, \ldots, d_k\}$. Since, by the assumption, $\operatorname{supp} f(x) \neq \emptyset$, we choose an element $d \in \operatorname{supp} f(x) \subseteq \operatorname{supp} x$. Now, for every $d_i \in \operatorname{supp} x$ with $d_i \neq d$, we have $\operatorname{supp} (d_i\ d)x = \operatorname{supp} x$. So, $(d_i\ d)x \preceq x$. Since f is order-preserving, $f((d_i\ d)x) \preceq f(x)$. Thus, $\operatorname{supp} (d_i\ d)f(x) \subseteq \operatorname{supp} f(x)$ and so $d_i \in \operatorname{supp} f(x)$, for all $d_i \in \operatorname{supp} x$. That is, $\operatorname{supp} x \subseteq \operatorname{supp} f(x)$ and so $\operatorname{supp} f(x) = \operatorname{supp} x$. □

Corollary 3.14. (i) *If $(X, \preceq)$ is an sp-nominal set with $\mathcal{Z}(X) = \emptyset$, then id_X is the only sp-preserving map over X.*

(ii) *The category* **spNom** *is not connected.*

(iii) *Let $f, g : X \to A$ be two parallel sp-preserving maps with $\mathcal{Z}(A) = \emptyset$ and the support map $\operatorname{supp}_A : A \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ be injective. Then, $f = g$.*

Proof. (i) Let $f : X \to X$ be an sp-preserving map. Then, since $\mathcal{Z}(X) = \emptyset$, applying Lemma 3.13, $\operatorname{supp} f(x) = \operatorname{supp} x$, for all $x \in X$. Now, since $\preceq$ is antisymmetric, $f(x) = x$ for all $x \in X$.

(ii) By Lemma 3.13, there exists no sp-preserving map from $\mathbb{D}^{(2)}$ to $\mathbb{D}$.

(iii) Let $x \in X$. Then, by Lemma 3.13, $\operatorname{supp} f(x) = \operatorname{supp} x = \operatorname{supp} g(x)$. Now, since supp_A is injective, $f(x) = g(x)$. □

Lemma 3.15. *Let A and X be two sp-nominal sets with $\mathcal{Z}(A) = \emptyset$. Then,*

(i) *given a map $f : X \to A$, if f is an sp-preserving map, then for all $x \in X$ we have* $\operatorname{supp} x = \operatorname{supp} f(x)$. *The converse is stablished if the support map* $\operatorname{supp}_A : A \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ *is injective.*

(ii) *if $f : X \to A$ is an sp-preserving map and* $\operatorname{supp}_X : X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ *is injective, then f is injective.*

Proof. (i) Follows immidiately from Lemma 3.13. For the converse, it is clear that f is sp-preserving. We show that f is equivariant. Indeed, since by the assumption, $\pi\operatorname{supp} f(x) = \pi\operatorname{supp} x = \operatorname{supp} \pi x = \operatorname{supp} f(\pi x)$ and supp_A is injective, $\pi f(x) = f(\pi x)$.

(ii) Let $f(x) = f(x')$ with $x, x' \in X$. Then, by Lemma 3.13, we have $\operatorname{supp} x = \operatorname{supp} f(x) = \operatorname{supp} f(x') = \operatorname{supp} x'$. Now, since supp_X is injective, $x = x'$. □

Theorem 3.16. *Let X be an sp-nominal set and $\mathcal{Z}(X) = \emptyset$. Then any $\rho \in Con(X)$ with $\mathcal{Z}(X/\rho) = \emptyset$ whose canonical map, $\pi : X \to X/\rho$, $x \mapsto x/\rho$, is sp-preserving, is a subset of $\sim$.*

Proof. Suppose $\mathcal{Z}(X) = \emptyset$, and $\rho \in Con(X) \setminus \{\nabla\}$ such that the canonical map $X \to X/\rho$, $x \mapsto x/\rho$ is sp-preserving. Then, by Lemma 3.13, $\operatorname{supp} x = \operatorname{supp} x/\rho$, for every $x \in X$. If $(x, x') \in \rho$ then $x/\rho = x'/\rho$. Hence, $\operatorname{supp} x = \operatorname{supp} x'$; that is, $x \sim x'$. So $\rho \subseteq \sim$. □

Theorem 3.17. *Let X be an sp-nominal set. Then,*

(i) *for each $x \in X$, $x_\downarrow \neq \emptyset$ ($x^\uparrow \neq \emptyset$).*

(ii) *if $y \in x_\downarrow$ ($y \in x^\uparrow$) and $z \preceq y$ ($y \preceq z$), then $z \in x_\downarrow$ ($z \in x^\uparrow$).*

(iii) *for all $x \neq y$, if $x \in y^\uparrow$ ($x \in y_\downarrow$), then $x^\uparrow \subseteq y^\uparrow$ and $x_\downarrow \subseteq y_\downarrow$.*

(iv) *the sets $x_\downarrow$ and $x^\uparrow$ are finitely supported subsets of X and* $\operatorname{supp} x$ *is a finite support for them.*

(v) *for all* $\pi \in \mathrm{Perm}(\mathbb{D})$, *we have $\pi x_\downarrow = (\pi x)_\downarrow$ and $\pi x^\uparrow = (\pi x)^\uparrow$.*

(vi) *the set $\mathcal{S} = \{x_\downarrow, x^\uparrow \mid x \in X\}$ is a nominal subset of $\mathcal{P}_{\mathrm{fs}}(X)$.*

Proof. (i) For each $x \in X$, $x \in x_\downarrow$ ($x \in x^\uparrow$).

(ii) and (iii) follow from the fact that the relation "$\preceq$" is transitive.

(iv) Applying Proposition 2.6, assume $a, b \notin \operatorname{supp} x$. We show $(a\ b)x_\downarrow = x_\downarrow$. Let $y \in x_\downarrow$. Then, $\operatorname{supp} y \subseteq \operatorname{supp} x$ and so, for all $a, b \notin \operatorname{supp} x$, we have $(a\ b)y = y$. Thus, $(a\ b)x_\downarrow = x_\downarrow$. Analogously, $\operatorname{supp} x$ is a finite support for $x^\uparrow$.

(v) Since $\preceq$ is equivariant, we have

$$\begin{aligned} y \in (\pi x)_\downarrow &\Leftrightarrow y \preceq \pi x \\ &\Leftrightarrow \pi^{-1} y \preceq x \\ &\Leftrightarrow y \in \pi x_\downarrow. \end{aligned}$$

Analogously, $\pi x^\uparrow = (\pi x)^\uparrow$.

(vi) By (vi) and (v), $\mathcal{S}$ is an equivariant subset of $\mathcal{P}_{\mathrm{fs}}(X)$ and so it is a nominal set. □

Proposition* 3.18*: Let X be an sp-nominal set and $x \in X$. Then,

(i) $X \setminus x_\downarrow$ is a finitely supported subset of X and $\operatorname{supp}(X \setminus x_\downarrow) \subseteq \operatorname{supp} x$.

(ii) $X \setminus x_\downarrow = \bigcup_{t \in X \setminus x_\downarrow} t^\uparrow$.

(iii) $X \setminus x^\uparrow = \bigcup_{t \in X \setminus x^\uparrow} t_\downarrow$.

Proof. (i) By Proposition 2.6, we show that, for every $a, b \notin \operatorname{supp} x$ and $t \in X \setminus x_\downarrow$, $(a\ b)t \in X \setminus x_\downarrow$. On the contrary, suppose $(a\ b)t \in x_\downarrow$. Then $(a\ b)t \preceq x$ and hence, $t \preceq (a\ b)x$. Since $a, b \notin \operatorname{supp} x$, $t \preceq x$ which is a contradiction.

(ii) For the nontrivial part, let $y \in \bigcup_{t \in X \setminus x_\downarrow} t^\uparrow$. Then, there exists $t \in X \setminus x_\downarrow$ with $y \in t^\uparrow$. Now, if $y \in x_\downarrow$, then $y \preceq x$ and we get $t \preceq y \preceq x$, which contradicts $t \in X \setminus x_\downarrow$.

(iii) The proof is similar to (ii). □

Theorem 3.19. *If $f : X \to Y$ is an sp-preserving map, then*

(i) $f(x_\downarrow) \subseteq f(x)_\downarrow$.

(ii) $f(x^\uparrow) \subseteq f(x)^\uparrow$.

(iii) $\pi f(x_\downarrow) = f(\pi x_\downarrow)$.

(iv) $f(x_\downarrow) = f(x)_\downarrow$, *if f is surjective and $\mathcal{Z}(Y) = \emptyset$.*

(v) $f(x^\uparrow) = f(x)^\uparrow$, *if f is surjective and $f(x) \notin \mathcal{Z}(Y)$.*

Proof. (i) Let $y \in f(x_\downarrow)$. Then, there exists $t \in x_\downarrow$ with $f(t) = y$. Since $t \preceq x$ and f is sp-preseving, $y = f(t) \preceq f(x)$.

(ii) Analogous to (i) one can prove (ii).

(iii) Let $y \in \pi f(x_\downarrow)$. Then, $\pi^{-1} y \in f(x_\downarrow)$ and so there exists $t \in x_\downarrow$ with $f(t) = \pi^{-1} y$. Now, we have $\pi t \preceq \pi x$ and $y = f(\pi t)$. Thus, $y \in f(\pi x_\downarrow)$. Analogously, we have $f(\pi x_\downarrow) \subseteq \pi f(x_\downarrow)$.

(iv) By (i), it is enough to show that $f(x)_{\downarrow} \subseteq f(x_{\downarrow})$. Let $t \in f(x)_{\downarrow}$. Then, there exists $x' \in X$ with $f(x') = t$. So, $f(x') \leq f(x)$. Now since, by Lemma 3.13, $\operatorname{supp} f(x') = \operatorname{supp} x'$ and $\operatorname{supp} f(x) = \operatorname{supp} x$, we have $x' \leq x$.

(v) Analogous to (iv) one can prove (v).

$\square$

Lemma 3.20. *Let $f : X \to Y$ be an sp-preseving equivariant map between sp-nominal sets X and Y, and $a \in Y$. Then,*

(i) $\operatorname{supp} a$ *supports* $f^{-1}(a_{\downarrow})$ *and* $f^{-1}(a_{\downarrow}) = \bigcup_{f(x) \in a_{\downarrow}} x_{\downarrow}$.

(ii) $\operatorname{supp} a$ *supports* $f^{-1}(a^{\uparrow})$ *and* $f^{-1}(a^{\uparrow}) = \bigcup_{f(x) \in a^{\uparrow}} x^{\uparrow}$.

Proof. (i) First, we show that $\operatorname{supp} a$ is a finite support for $f^{-1}(a_{\downarrow})$. Let $d, d' \notin \operatorname{supp} a$. Then,

$$(d\ d')f^{-1}(a_{\downarrow}) = f^{-1}((d\ d')a_{\downarrow}) = f^{-1}(a_{\downarrow}).$$

Now, we prove that $f^{-1}(a_{\downarrow}) = \bigcup_{f(x) \in a_{\downarrow}} x_{\downarrow}$. Let $t \in f^{-1}(a_{\downarrow})$. Then, $f(t) \in a_{\downarrow}$. Since $t \in t_{\downarrow}$, $t \in \bigcup_{f(x) \in a_{\downarrow}} x_{\downarrow}$. To prove the other side, let $x' \in \bigcup_{f(x) \in a_{\downarrow}} x_{\downarrow}$. Then, there exists $t \in X$ with $f(t) \in a_{\downarrow}$ and $x' \leq t$. So, $\operatorname{supp} f(t) \subseteq \operatorname{supp} a$ and $\operatorname{supp} x' \subseteq \operatorname{supp} t$. Since f is order-preserving, $\operatorname{supp} f(x') \subseteq \operatorname{supp} f(t)$ and so $\operatorname{supp} f(x') \subseteq \operatorname{supp} a$. Thus, $f(x') \leq a$ and so $x' \in f^{-1}(a_{\downarrow})$.

(ii) Is analogous to (i). $\square$

4 Some categorical properties of the category spNom

In the category **spNom** of sp-nominal sets the class of monics (left cancellable sp-preserving maps) and the class of monomorphisms (injective sp-preserving maps) do not coincide, see the following example, while epics are exactly surjectives, by Theorem 4.2.

Example 4.1. The sp-preserving map $f : \mathbb{D}^{2} \to \mathbb{D}^{(2)} \cup \{\theta\}$ in Example 3.6 is monic while it is not injective. Indeed, since f is identity on $\mathbb{D}^{(2)}$ and $f(d, d) = \theta$, f is not injective. We show that f is monic. To do so, take $g_1, g_2 : X \to \mathbb{D}^2$ to be sp-preserving maps with $fg_1 = fg_2$. Since $\mathcal{Z}(\mathbb{D}^2) = \emptyset$, $\operatorname{supp} g_1(x) \neq \emptyset$ and $\operatorname{supp} g_2(x) \neq \emptyset$, for every $x \in X$. So, by Lemma 3.13, we have $\operatorname{supp} g_1(x) = \operatorname{supp} x = \operatorname{supp} g_2(x)$. Notice that, $g_i(x) \in \mathbb{D}^2$ implies that $g_i(x) = (d, d)$ or $g_i(x) = (d, d')$, where $i = 1, 2$ and $d \neq d'$. We have the following cases;

Case (1): If $fg_1(x) = fg_2(x) = \theta$, then $g_1(x) = (d, d)$ and $g_2(x) = (d', d')$. Since $\{d\} = \operatorname{supp} g_1(x) = \operatorname{supp} g_2(x) = \{d'\}$, $d = d'$. So, in this case, $g_1(x) = g_2(x)$.
Case (2): If $fg_1(x) = fg_2(x) \neq \theta$, then $g_1(x), g_2(x) \in \mathbb{D}^{(2)}$. Now since $fg_1(x) = g_1(x)$ and $fg_2(x) = g_2(x)$, $g_1(x) = g_2(x)$. Thus, $g_1 = g_2$.

Theorem 4.2. *In the category* **spNom** *epics are exactly surjectives.*

Proof. Let $f : X \longrightarrow Y$ be an epic sp-preserving map. We show f is surjective. On the contrary suppose f is not surjective. Hence, $Y \setminus Im(f) \neq \emptyset$. We define the sp-preserving maps $g_1, g_2 : Y \longrightarrow Y \cup \{\theta_1, \theta_2, \theta_3\}$ to be

$$g_1(y) = \begin{cases} y & \text{when } y \in Im(f) \text{ and } y^{\uparrow} \subseteq Im(f) \\ \theta_3 & \text{when } y \in Im(f) \text{ and } y^{\uparrow} \nsubseteq Im(f) \\ \theta_1 & \text{otherwise} \end{cases}$$

$$g_2(y) = \begin{cases} y & \text{when } y \in Im(f) \text{ and } y^{\uparrow} \subseteq Im(f) \\ \theta_3 & \text{when } y \in Im(f) \text{ and } y^{\uparrow} \nsubseteq Im(f) \\ \theta_2 & \text{otherwise} \end{cases}$$

Since $g_1 of = g_2 of$, $g_1 = g_2$ and hence, for each $y \in Y \setminus Im(f)$, $g_1(y) = g_2(y)$. Therefore, $\theta_1 = \theta_2$, which is a contradiction. □

Theorem 4.3. *The category* **spNom** *is not regular.*

Proof. We show that the sp-preserving map $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^{(2)} \cup \{\theta\}$, given in Example 3.6, is not an equalizer while, by Example 4.1, it is monic. On the contrary, suppose there exist $Y \in$ **spNom** and two parallel sp-preserving maps g_1, g_2 such that f is an equalizer of $g_1, g_2 : \mathbb{D}^{(2)} \cup \{\theta\} \longrightarrow Y$. Since $g_1 of = g_2 of$, we have $g_1 of(d, d) = g_2 of(d, d)$, for every $d \in \mathbb{D}$. Therefore, $g_1(\theta) = g_2(\theta)$. Now we consider the zero map $h : \{\theta_1\} \longrightarrow \mathbb{D}^{(2)} \cup \{\theta\}$, $\theta_1 \mapsto \theta$. Since $g_1 oh = g_2 oh$, by universal property of equalizer, there is a unique sp-preserving map $\varphi : \{\theta_1\} \longrightarrow \mathbb{D}^2$, which commutes the desired diagrams and this contradicts the fact that $\mathbb{D}^2$ has no zero element. □

Corollary 4.4. *The category* **spNom** *is not balanced.*

Proof. Consider $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^{(2)} \cup \{\theta\}$, given in Example 3.6. By Example 4.1, f is monic. Since f is surjective, f is epic. Therefore, f is a bimorphism. But, since f is not injective, it is not an isomorphism. □

In general the category **spNom** does not contains all of the products and coproducts as seen in Examples 4.5 and 4.18. Here we charactrize conditions under which products and coproducts exist.

Example 4.5. Let $X_1 = X_2 = \mathbb{D}$. We show that there exists no coproduct of X_1 and X_2 in the category **spNom**. On the contrary, suppose that (X, α_1, α_2) is the coproduct of X_1 and X_2. Take the singleton sp-nominal set $Z = \{\theta\}$ in which $\theta \notin \mathbb{D}$ and consider the sp-preserving maps $z, \iota : \mathbb{D} \to Z\dot{\cup}\mathbb{D}$, and $\iota_1, \iota_2 : \mathbb{D} \to \mathbb{D} \times \{1, 2\}$ defined by $z(d) := \theta$, $\iota(d) := d$, $\iota_1(d) := (d, 1)$, and $\iota_2(d) := (d, 2)$, for all $d \in \mathbb{D}$. Then, since X is coproduct, there exist unique sp-preserving maps φ and ψ such that the following diagrams commute.

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "\alpha_1"] \arrow[dr, "\iota_1"'] & X \arrow[d, "\varphi"] & \mathbb{D} \arrow[l, "\alpha_2"'] \arrow[dl, "\iota_2"] \\
& \mathbb{D} \times \{1, 2\} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D} \arrow[r, "\alpha_1"] \arrow[dr, "\iota"'] & X \arrow[d, "\psi"] & \mathbb{D} \arrow[l, "\alpha_2"'] \arrow[dl, "z"] \\
& Z\dot{\cup}\mathbb{D} &
\end{tikzcd}$$

$(*)$ $(**)$

According to Diagram $(*)$, $\varphi(\alpha_2(d)) = \iota_2(d) = (d, 2)$ and $\varphi(\alpha_1(d)) = \iota_1(d) = (d, 1)$, for every $d \in \mathbb{D}$, meaning that α_1 and α_2 are non-zero sp-preserving maps. So, by Lemma 3.13, $\operatorname{supp}\alpha_1(d) = \operatorname{supp}\alpha_2(d) = \{d\}$. By Diagram $(**)$, we have $\psi(\alpha_2(d)) = \theta$ and $\psi(\alpha_1(d)) = \iota(d) = d$. Now, $\operatorname{supp}\alpha_1(d) = \operatorname{supp}\alpha_2(d)$ implies that $\alpha_1(d) \leq \alpha_2(d)$. Now, since ψ is order preserving, $d = \psi(\alpha_1(d)) \leq \psi(\alpha_2(d)) = \theta$ which is a contradiction.

The following theorem determines which family of sp-nominal sets has coproduct.

Theorem 4.6. *The coproduct of a family of sp-nominal sets $(X_i)_{i \in I}$ exists if and only if all X_i's are discrete except probably one.*

Proof. $(\Leftarrow)$ If X_i's are all discrete, then one can easily see that the coproduct is the disjoint union of X_i's. Now let X_t be the non-discrete member of the family. Then we have $X_j = \mathcal{Z}(X_j)$, for all $j \neq t$ and for every sp-nominal set

$(B, (\beta_i : X_i \to B)_{i \in I})$, we have the following commutative diagram

$$\begin{tikzcd}
X_t \arrow[r, "\iota_t"] \arrow[dr, "\beta_t"'] & \bigcup_{i\in I}(X_i \times \{i\}) \arrow[d, "\varphi"] & X_j \arrow[l, "\iota_j"'] \arrow[dl, "\beta_j"] \\
& B &
\end{tikzcd}$$

in which $\iota_i : X_i \to \bigcup_{i\in I}(X_i \times \{i\})$ maps every $x \in X_i$ to (x, i), for every $i \in I$, and φ is uniquely defined by $\varphi((x, i)) = \beta_i(x)$, for every $(x, i) \in \bigcup_{i\in I}(X_i \times \{i\})$. So $\bigcup_{i\in I}(X_i \times \{i\})$ is the coproduct.

($\Rightarrow$) Suppose $(X, (\alpha_i)_{i\in I})$ is the coproduct of $(X_i)_{i\in I}$ and there exists some $t \in I$ such that X_t is non-discrete. We show that X_i's are discrete, for all $i \neq t$. Since X_t is non-discrete, there exists a non-zero element $x_t \in X_t$. On the contrary, suppose X_j is non-discrete, for some $j \in I$, with $j \neq t$. Suppose $x_j \in X_j \setminus \mathcal{Z}(X_j)$. Since $(X, (\alpha_i)_{i\in I})$ is coproduct, we have the following commutative diagrams;

$$\begin{tikzcd}
X_t \arrow[r, "\alpha_t"] \arrow[dr, "\iota_t"'] & X \arrow[d, "\varphi"] & X_i \arrow[l, "\alpha_i"'] \arrow[dl, "\iota_i"] \\
& \bigcup_{i\in I}(X_i \times \{i\}) & (*)
\end{tikzcd}$$

$$\begin{tikzcd}
X_t \arrow[r, "\alpha_t"] \arrow[dr, "h_t"'] & X \arrow[d, "\psi"] & X_i \arrow[l, "\alpha_i"'] \arrow[dl, "h_i"] \\
& \bigcup_{i\in I}(X_i \times \{i\}) \cup \{\theta\} & (**)
\end{tikzcd}$$

where ι_i's are inclusions, $h_i = \iota_i$, for all $i \neq t$, and h_t is the zero map. According to Diagram $(*)$, $\varphi(\alpha_t(x_t)) = (x_t, t)$ and $\varphi(\alpha_j(x_j)) = (x_j, j)$. So, $\alpha_t(x_t)$ and $\alpha_j(x_j)$ are non-zero. Applying Lemma 3.13, $\operatorname{supp} \alpha_t(x_t) = \operatorname{supp} x_t$ and $\operatorname{supp} \alpha_j(x_j) = \operatorname{supp} x_j$. By Diagram $(**)$, we have $\psi(\alpha_t(x_t)) = \theta$ and $\psi(\alpha_j(x_j)) = (x_j, j)$. By Lemma 3.4, there exists π with $\pi\alpha_j(x_j) \preceq \alpha_t(x_t)$ or $\pi\alpha_t(x_t) \preceq \alpha_j(x_j)$. If $\alpha_j(\pi x_j) \preceq \alpha_t(x_t)$, then we have $(\pi x_j, j) = h_j(\pi x_j) = \psi(\alpha_j(\pi x_j)) \preceq \psi(\alpha_t(x_t)) = h_t(x_t) = \theta$ which is a contradiction. If $\pi\alpha_t(x_t) \preceq \alpha_j(x_j)$, we can then make a similar diagram to $(**)$, by exchanging the definitions of h_j and h_t in Diagram $(**)$, and get a contradiction using a similar argument. $\square$

Now we examine the existence of products, but first take note the following corollary of Lemma 3.13.

Corollary 4.7. *If $f : X \to A$ is an sp-preserving map between sp-nominal sets with $\mathcal{Z}(A) = \emptyset$, then the following diagram is commutative.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{supp}_X"'] & A \arrow[dl, "\mathrm{supp}_A"] \\ \mathcal{P}_{\mathrm{f}}(\mathbb{D}) & \end{tikzcd}$$

Lemma 4.8. *Let $(P, (p_i : P \to A_i)_{i\in I})$ be a product of a family $(A_i)_{i\in I}$ with $\mathcal{Z}(A_i) = \emptyset$ in the category* **spNom** *which is not empty. Then*

(i) *for all i, the following diagram is commutative.*

$$\begin{tikzcd} P \arrow[r, "p_i"] \arrow[d, "\mathrm{supp}_P"'] & A_i \arrow[dl, "\mathrm{supp}_{A_i}"] \\ \mathcal{P}_{\mathrm{f}}(\mathbb{D}) & \end{tikzcd}$$

(ii) *for every sp-nominal set X with the commutative diagram*

$$\begin{tikzcd} X \arrow[r, "q_i"] \arrow[d, "\mathrm{supp}_X"'] & A_i \arrow[dl, "\mathrm{supp}_{A_i}"] \\ \mathcal{P}_{\mathrm{f}}(\mathbb{D}) & \end{tikzcd}$$

there exists a unique sp-preserving $h : X \to P$ with the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\mathrm{supp}_X"'] & P \arrow[dl, "\mathrm{supp}_P"] \\ \mathcal{P}_{\mathrm{f}}(\mathbb{D}) & \end{tikzcd}$$

Proof. (i) Follows by Corollary 4.7.

(ii) First we note that since $\mathcal{Z}(A_i) = \emptyset$, for each $i \in I$, by (i), $\mathcal{Z}(P) = \emptyset$. Now for every sp-nominal set X with a family of sp-preserving maps $(q_i : X \to A_i)$, by the universal property of product, one can get a unique sp-preserving map $h : X \to P$ with $p_i h = q_i$, for every $i \in I$. Now the result follows from (i). $\square$

Theorem 4.9. *Let A be an sp-nominal set with $\mathcal{Z}(A) = \emptyset$ and* supp_A *is injetive. Then, product of A and A is* $(A, \mathrm{id}_A, \mathrm{id}_A)$.

Proof. Consider $X \in$ **spNom** together with the sp-preserving maps $f_1, f_2 : X \to A$. By Corollary 3.14 (iii), we have $f_1 = f_2$. So we get the unique sp-preserving map $f = f_1 : X \to A$ with $f \circ \mathrm{id}_A = f_1 = f_2$. □

Example 4.10. The product of the sp-nominal sets $\mathbb{D}$ and $\mathbb{D}$ is $(\mathbb{D}, \mathrm{id}_{\mathbb{D}}, \mathrm{id}_{\mathbb{D}})$.

Lemma 4.11. *Let $f : X \to \mathbb{D}^{(n)}$ be an sp-preserving map. Then,*
(i) *for every $x \in X$, $|\mathrm{supp}\, x| = n$.*
(ii) *X is isomorphic to $\mathbb{D}^{(n)}$, if $X = \mathrm{Perm}(\mathbb{D})x$, for some $x \in X$.*
(iii) *X is isomorphic to a disjoint union of $\mathbb{D}^{(n)}$.*

Proof. (i) Since $\mathcal{Z}(\mathbb{D}^{(n)}) = \emptyset$, by Lemma 3.13, $\mathrm{supp}\, x = \mathrm{supp}\, f(x) = \mathrm{supp}\,(d_1, \ldots, d_n) = \{d_1, \ldots, d_n\}$.

(ii) We show that f is bijective. But first we note that $f(\pi x) = (\pi d_1, \ldots, \pi d_n)$, for every $\pi \in \mathrm{Perm}(\mathbb{D})$, in which $(d_1, \ldots, d_n) = f(x)$. Now let $f(\pi x) = f(\delta x)$. Then $(\pi d_1, \ldots, \pi d_n) = (\delta d_1, \ldots, \delta d_n)$, and hence, $\pi^{-1}\delta \in \mathrm{Fix}\,(\{d_1, \ldots, d_n\}) = \mathrm{Fix}\,(\mathrm{supp}\, x)$. Therefore, $\pi^{-1}\delta x = x$ and hence, $\delta x = \pi x$. The map f is also onto, since for every $\{b_1, \ldots, b_n\} \in \mathbb{D}^{(n)}$, $f((d_1 b_1)x, \cdots, (d_n b_n)x) = (b_1, \ldots, b_n)$.

(iii) Since X as a nominal set is the disjoint union of its orbits, by (ii), we are done. □

We mention the following remark and terminology used in Theorem 4.13 with considering $\dot{\bigcup}_{i\in I}\mathbb{D}^{(n)} = \mathbb{D}^{(n)} \times I$.

Remark 4.12. (i) If P is the product of a family of $(\mathbb{D}^{(n)})_{i\in I}$, then P is a disjoint union of $\mathbb{D}^{(n)}$'s, by Lemma 4.11 (iii).

(ii) Since the nominal set $\mathbb{D}^{(n)}$ is transitive, for every two elements $(d_1, d_2, \ldots, d_n), (b_1, b_2, \ldots, b_n) \in \mathbb{D}^{(n)}$ we have $(b_1, b_2, \ldots, b_n) = (d_1\ b_1)(d_2\ b_2)\cdots(b_n\ d_n)(d_1, d_2, \ldots, d_n)$, every equivariant map $\sigma : \mathbb{D}^{(n)} \to \mathbb{D}^{(n)}$ is bijective and, by [8, Lemma $2 \cdot 12$], it is sp-preserving.

(iii) Since the nominal set $\mathbb{D}^{(n)}$ is cyclic, the set $S = \{\sigma : \mathbb{D}^{(n)} \to \mathbb{D}^{(n)} : \sigma \text{ is equivariant}\}$ has $n!$ elements, and so one can cosider S=$\{\sigma_1, \ldots, \sigma_{n!}\}$.

(iv) We define $\varphi : \dot{\bigcup}_{i\in I}\mathbb{D}^{(n)} \to \mathbb{D}^{(n)}$ by $\varphi((d_1, \ldots, d_n), i) := (d_{\sigma_i(1)}, \ldots, d_{\sigma_i(n)})$ and denote it by $\varphi((d_1, \ldots, d_n), i) = \sigma_i(d_1, \ldots, d_n)$, where $i \in I = \{1, 2, \ldots, n!\}$ and $\sigma_i \in S$. One can easily check that φ is sp-preserving map.

(v) It is clear that $\pi_1 : \mathbb{D}^{(n)} \times I \to \mathbb{D}^{(n)}$ by $\pi_1((d_1, \dots, d_n), i) = (d_1, \dots, d_n)$, for all $i \in I$, is an sp-preserving map.

(vi) The map $f : \mathbb{D}^{(n)} \times J \to \mathbb{D}^{(n)}$ is an sp-preserving if and only if there exists $\sigma \in S$ with $f((d_1, \dots, d_n), j) = \sigma(d_1, \dots, d_n)$.

Theorem 4.13. *The triple* $(P = \dot{\bigcup}_{i\in I} \mathbb{D}^{(n)}, \pi_1, \varphi)$ *is the product of* $\mathbb{D}^{(n)}$ *and* $\mathbb{D}^{(n)}$, *where* φ, π_1 *are defined in Remark* 4.12 (iv, v) *and* $I = \{1, 2, \dots, n!\}$.

Proof. Consider (X, f, g), in which X is an sp-nominal set and $f, g : X \to \mathbb{D}^{(n)}$ are sp-preserving maps. Then, by Lemma 4.11 (iii) and since $\mathbb{D}^{(n)}$ is cyclic, $X = \dot{\bigcup}_J \mathrm{Perm}(\mathbb{D})x_j$. If $f(x_j) = (b_1, \dots, b_n)$ and $g(x_j) = (c_1, \dots, c_n)$, for each $j \in J$, then since, by Lemma 3.13, we have $\{b_1, \cdots, b_n\} = \operatorname{supp} f(x_j) = \operatorname{supp} x_j = \operatorname{supp} g(x_j) = \{c_1, \dots, c_n\}$, there exists $\sigma_{k_j} \in S$ with $\sigma_{k_j} f|_{\mathrm{Perm}(\mathbb{D})x_j} = g|_{\mathrm{Perm}(\mathbb{D})x_j}$. Now we consider $h : X \to P$ to be the equivariant map defined by $h(x_j) = (f(x_j), k_j)$. Then we have $f = \pi_1 h$ and $\varphi h = g$, means the desired diagrams commutes. Also uniqueness follows from the definition of h. □

Remark 4.14. (i) For given sp-nominal sets X and Y with $\mathcal{Z}(X) = \mathcal{Z}(Y) = \emptyset$, if P is the non-empty product of X and Y then applying Lemma 3.13, for every $t \in P$, there exist $x \in X$ and $y \in Y$ with $\operatorname{supp} t = \operatorname{supp} x = \operatorname{supp} y$. Note that, the product of cyclic nominal sets $\mathrm{Perm}(\mathbb{D})x$ and $\mathrm{Perm}(\mathbb{D})x'$ with $|\operatorname{supp} x| \neq |\operatorname{supp} x'|$ is empty nominal set.

(ii) Conisder sp-nominal sets $X = \mathrm{Perm}(\mathbb{D})x$ and $X' = \mathrm{Perm}(\mathbb{D})x' \cup \{\theta\}$ where x, x' are non-zero and $|\operatorname{supp} x| \neq |\operatorname{supp} x'|$. If $f : Y \to X$ and $g : Y \to X'$ are sp-preserving maps, then applying Lemma 3.13, one can see that g must be a zero map.

(iii) Suppose X and Y are two non-discrete sp-nominal sets with $|\operatorname{supp} x| \neq |\operatorname{supp} y|$, for all $x \in X$ and $y \in Y$. Let $p, q : Z \to X \dot{\cup} Y$ be two sp-preserving maps. Then, $p(z) \in X \iff q(z) \in X$.

Example 4.15. The product of $\mathbb{D}^{(n)}$ and $\mathbb{D}^{(k)} \cup \{\theta\}$, with $k \neq n$, is $(\mathbb{D}^{(n)}, z, \mathrm{id}_{\mathbb{D}^{(n)}})$, in which z : $\mathbb{D}^{(n)} \to \mathbb{D}^{(k)} \cup \{\theta\}$ defined by $z((d_1, d_2, \dots, d_n)) = \theta$, for all $(d_1, d_2, \dots, d_n) \in \mathbb{D}^{(n)}$. Since, for any $X \in$ **spNom** together with sp-preserving maps $f_1 : X \to \mathbb{D}^{(n)}$, and $f_2 : X \to \mathbb{D}^{(k)} \cup \{\theta\}$, by Remark 4.14 (ii), f_2 is a zero map, and we have the following commutative diagram.

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, dashed, "f_1"] \arrow[dr, "f_2"] & \\ \mathbb{D}^{(n)} & \mathbb{D}^{(n)} \arrow[l, "\mathrm{id}_{\mathbb{D}^{(n)}}"] \arrow[r, "z"'] & \mathbb{D}^{(k)} \cup \{\theta\} \end{tikzcd}$$

Theorem 4.16. *Suppose X and Y are two sp-nominal sets with $|\operatorname{supp} x| \neq |\operatorname{supp} y|$, for all $x \in X$ and $y \in Y$. Let $\mathcal{Z}(X) = \mathcal{Z}(Y) = \emptyset$ and (P, p_1, p_2) be the product of X and X and $(Q, q_1.q_2)$ be the product of Y and Y. Then, $P\dot{\cup}Q$ is the product of $X\dot{\cup}Y$ and $X\dot{\cup}Y$.*

Proof. Consider the diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X\dot{\cup}Y & P\dot{\cup}Q \arrow[l, "q"] \arrow[r, "p"'] & X\dot{\cup}Y, \end{tikzcd}$$

in which $p(a) = \begin{cases} p_1(a) & a \in P \\ q_1(a) & a \in Q \end{cases}$ and $q(a) = \begin{cases} p_2(a) & a \in P \\ q_2(a) & a \in Q \end{cases}$. Then the following cases may occur.

Case (1): If $f(Z) \subseteq X$, then by Remark 4.14 (iii), $g(Z) \subseteq X$ and we get the commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[d, dashed, "\exists! h_1"] \arrow[dr, "f"] & \\ X & P \arrow[l, "q_1"] \arrow[r, "p_1"'] & X \end{tikzcd}$$

by the universal property of product which implies the commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "f"] & \\ X\dot{\cup}Y & P\dot{\cup}Q \arrow[l, "q"] \arrow[r, "p"'] & X\dot{\cup}Y. \end{tikzcd}$$

in which $h = h_1$.
Case (2): If $f(Z) \subseteq Y$, then $g(z) \subseteq Y$, by Remark 4.14 (iii), and the result is proved analogous to Case (1).
Case (3): If $f(Z) \cap X \neq \emptyset$ and $f(Z) \cap Y \neq \emptyset$, then we have $Z = Z_1 \dot{\cup} Z_2$ in which $Z_1 = \{z \in Z \mid f(z) \in X\}$ and $Z_2 = \{z \in Z \mid f(z) \in Y\}$. Therefore, we get $h_1 : Z_1 \to P$, by Case (1) and $h_2 : Z_2 \to Q$ by Case (2). Now we define the sp-preserving map. Therefore, $h : Z \to P\dot{\cup}Q$ by

$$h(z) = \begin{cases} h_1(z) & z \in Z_1 \\ h_2(z) & z \in Z_2, \end{cases}$$

which commutes the desired diagram. □

Corollary 4.17. (i) *The product of* $\mathbb{D}^2$ *and* $\mathbb{D}^2$ *exists.*
(ii) *The product of* $\mathbb{D}^3$ *and* $\mathbb{D}^3$ *exists.*
(iii) *The product of* $\mathbb{D}^n$ *and* $\mathbb{D}^n$ *exists.*
(iv) *The product of* $\mathbb{D}^n$ *and* $\mathbb{D}^{(k)}$ *with* $k \leq n$ *exists.*
(v) *The product of* $\mathbb{D}^n$ *and* $\mathbb{D}^k$ *with* $k \leq n$ *exists.*

Proof. (i) Notice that, $\mathbb{D}^2 = \{(d,d) \mid d \in \mathbb{D}\}\dot{\cup}\mathbb{D}^{(2)}$ where $\{(d,d) \mid d \in \mathbb{D}\} \cong \mathbb{D}$. Let $X = \{(d,d) \mid d \in \mathbb{D}\}$ and $Y = \mathbb{D}^{(2)}$. By Theorem 4.13, the product of X and X, and the product of Y and Y exist. So, applying Theorem 4.16, the product of $\mathbb{D}^2$ and $\mathbb{D}^2$ exists. Indeed, the product $\mathbb{D}^2$ and $\mathbb{D}^2$ is $(\mathbb{D}^{(2)} \times \{1,2\} \cup \mathbb{D}, \rho_1, \rho_2)$ in which $\rho_j(d) = (d,d)$, for $j = 1,2$ and

$$\rho_j((d_1,d_2),i) = \begin{cases} (d_1,d_2) & i = 1,2, j = 1 \\ (d_1,d_2) & i = 1, j = 2 \\ (d_2,d_1) & i = 2, j = 2. \end{cases}$$

(ii) We have

$$\mathbb{D}^3 = \{(d,d,d) \mid d \in \mathbb{D}\} \dot{\cup} \{(d,d,d') \mid d \neq d' \in \mathbb{D}\} \dot{\cup} \{(d,d',d) \mid d \neq d' \in \mathbb{D}\}$$

$$\dot{\cup} \{(d',d,d) \mid d \neq d' \in \mathbb{D}\} \dot{\cup} \mathbb{D}^{(3)}.$$

So, $\mathbb{D}^3 \cong \mathbb{D} \dot{\cup} (\mathbb{D}^{(2)} \times \{1,2,3\}) \dot{\cup} \mathbb{D}^{(3)}$. By Theorem 4.13, the product of $\mathbb{D}^{(i)}$ and $\mathbb{D}^{(i)}$ exists, for $i = 1,2,3$. So, applying Theorem 4.16, the product of $\mathbb{D}^3$ and $\mathbb{D}^3$ exists. Indeed, the product of $\mathbb{D}^3$ and $\mathbb{D}^3$ is $((\dot{\cup}_{i=1}^{6} \mathbb{D}^{(3)} \times \{i\}) \cup (\dot{\cup}_{i=1}^{9} \mathbb{D}^{(2)} \times \{i\}) \cup \mathbb{D}, \rho_1, \rho_2)$, in which

$$\begin{cases} \rho_1(d) = (d,d,d) & \\ \rho_1((d_1,d_2),i) = ((d_1,d_2),j) & j \in \{1,2,3\}, i \in \{1,\dots,6\} \\ \rho_1((d_1,d_2,d_3),i) = (d_1,d_2,d_3) & i \in \{1,\dots,6\} \end{cases}$$

and

$$\begin{cases} \rho_2(d) = (d,d,d) & \\ \rho_2((d_1,d_2),i) = ((d_{\sigma_i(1)}, d_{\sigma_i(2)}),j) & j \in \{1,2,3\}, \sigma_i \in S_2 \\ \rho_2((d_1,d_2,d_3),i) = (d_{\sigma_i(1)}, d_{\sigma_i(2)}, d_{\sigma_i(3)}) & \sigma_i \in S_3 \end{cases}$$

(iii) Similar to (i) and (ii), follows by Theorems 4.13 and 4.16.

(iv) Note that, $\mathbb{D}^n$ is isomorphic to a disjoint union of $\mathbb{D}^{(i)}$'s where $i = 1,2,3,\dots,n$. Let $\mathbb{D}^n = \dot{\bigcup}_i (\mathbb{D}^{(i)} \times I_i)$. By Remark 4.14(i), $\mathbb{D}^{(k)}$ and $\mathbb{D}^{(i)}$ when $i \neq k$ have no product. So, the product of $\mathbb{D}^{(k)}$ and $\mathbb{D}^n$ is equal to the product of $\mathbb{D}^{(k)}$ and $\mathbb{D}^{(k)} \times I_k$ which exists by Theorems 4.16 and 4.13.

(v) Suppose $\mathbb{D}^k = \dot{\bigcup}_i (\mathbb{D}^{(i)} \times I_i)$. By (iv), the product of $\mathbb{D}^n$ and $\mathbb{D}^{(i)}$ exists. So, applying Theorems 4.16 and 4.13 we get the result. □

Example 4.18. Let $X_1 = \mathbb{D}$ and $X_2 = \mathbb{D} \cup \{\theta\}$. We show that there exists no product of X_1 and X_2 in the category **spNom**. On the contrary, suppose that (P, ρ_1, ρ_2) is the product of X_1 and X_2. Then, by the universal property of product, we have the following commutative diagrams

$$\begin{tikzcd} & \mathbb{D} \arrow[dl, "\mathrm{id}"'] \arrow[d, "\varphi"] \arrow[dr, "\iota"] & \\ \mathbb{D} & P \arrow[l, "\rho_1"] \arrow[r, "\rho_2"'] & \mathbb{D} \cup \{\theta\} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{D} \arrow[dl, "\mathrm{id}"'] \arrow[d, "\psi"] \arrow[dr, "\mathrm{z}"] & \\ \mathbb{D} & P \arrow[l, "\rho_1"] \arrow[r, "\rho_2"'] & \mathbb{D} \cup \{\theta\} \end{tikzcd}$$

$(*)$ $(**)$

in which ι is inclusion and z is the zero sp-preserving map. By Diagram $(*)$ we have $\rho_1(\varphi(d)) = \mathrm{id}(d) = d$ and $\rho_2(\varphi(d)) = \iota(d) = d$, for every $d \in \mathbb{D}$, meaning that ρ_1 and ρ_2 are non-zero sp-preserving maps. So, by Lemma 3.13, $\mathrm{supp}\,\rho_1(\varphi(d)) = \mathrm{supp}\,\rho_2(\varphi(d)) = \{d\}$. By Diagram $(**)$, we have $\rho_2(\psi(d)) = \theta$ and $\rho_1(\psi(d)) = \mathrm{id}(d) = d$. Since, $\mathrm{supp}\,\varphi(d) = \mathrm{supp}\,\psi(d)$, $\varphi(d) \preceq \psi(d)$. But $\rho_2(\varphi(d)) \npreceq \rho_2(\psi(d))$ which is a contradiction.

Theorem 4.19. *Let X and Y be strong nominal sets and $\mathcal{Z}(X) = \mathcal{Z}(Y) = \emptyset$. Also let $X = \dot{\bigcup}_{i\in I}\mathrm{Perm}(\mathbb{D})x_i$ and $Y = \dot{\bigcup}_{j\in J}\mathrm{Perm}(\mathbb{D})y_j$. Then $P = ((\dot{\bigcup}_{i\in I}(\mathrm{Perm}(\mathbb{D})x_i \times \{y \in Y|\ \mathrm{supp}\,y = \mathrm{supp}\,x_i\}), \rho_1, \rho_2)$, with the action $\pi(x, y) = (\pi x, y)$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and $(x, y) \in P$, is the product of X and Y in* **spNom**, *in which ρ_1 is projection map on the first component and $\rho_2 : P \to Y$ is defined by $\rho_2(\pi x_i, y) = \pi y$.*

Proof. First we note that $\mathrm{supp}\,(x, y) = \mathrm{supp}\,x$, for every $(x, y) \in P$. Hence, P is a nominal set and ρ_1 is an sp-preserving map. Also ρ_2 is well-defined, since if $(\pi x_i, y) = (\pi_1 x_i, y)$ with $\mathrm{supp}\,x_i = \mathrm{supp}\,y$. Hence, $\pi_1^{-1}\pi x_i = x_i$. Since $\mathrm{supp}\,x_i = \mathrm{supp}\,y$, by [8, Theorem 2.7], $\pi_1^{-1}\pi y = y$. So ρ_2 is well-defined. The map ρ_2 is also sp-preserving. Indeed, if $(\pi x_i, y) \preceq (\pi_1 x_j, y')$, for some $(\pi x_i, y), (\pi_1 x_j, y') \in P$, then $\mathrm{supp}\,\pi x_i \subseteq \mathrm{supp}\,\pi_1 x_j$. Since $\mathrm{supp}\,x_i = \mathrm{supp}\,y$ and $\mathrm{supp}\,x_j = \mathrm{supp}\,y'$, $\pi\mathrm{supp}\,y \subseteq \pi_1\mathrm{supp}\,y'$ and we get the result.

Now consider $N \in$**spNom** together with sp-preserving maps $f_1 : N \to X$ and $f_2 : N \to Y$. Since $\mathcal{Z}(X) = \mathcal{Z}(Y) = \emptyset$, by Lemma 3.13, $\mathrm{supp}\,n = \mathrm{supp}\,f_1(n) = \mathrm{supp}\,f_2(n)$, for all $n \in N$. Define $\varphi : N \to P$ by $n \mapsto (f_1(n), \pi^{-1}f_2(n))$ in which $f_1(n) = \pi x_i$, for some $\pi \in \mathrm{Perm}(\mathbb{D})$ and $i \in I$. Since $\mathrm{supp}\,x_i = \pi^{-1}\mathrm{supp}\,f_2(n)$, we have $(x_i, \pi^{-1}f_2(n)) \in P$, and hence $\varphi(n) = (\pi x_i, \pi^{-1}f_2(n)) = \pi(x_i, \pi^{-1}f_2(n)) \in P$. Since f_1 preserves support-preorder, so is φ. The map φ is equivariant, because for every $\pi \in \mathrm{Perm}(\mathbb{D})$ and $n \in N$ we have

$$\begin{aligned}
\varphi(\pi_1 n) &= (f_1(\pi_1 n), (\pi_1\pi)^{-1} f_2(\pi_1 n))\\
&= (\pi_1 f_1(n), \pi^{-1}\pi_1^{-1} f_2(\pi_1 n))\\
&= (\pi_1 f_1(n), \pi^{-1}\pi_1^{-1}\pi_1 f_2(n))\\
&= (\pi_1 f_1(n), \pi^{-1} f_2(n))\\
&= \pi_1(f_1(n), \pi^{-1} f_2(n))\\
&= \pi_1\varphi(n).
\end{aligned}$$

Also $\rho_1 o\varphi(n) = f_1(n)$ and $\rho_2 o\varphi(n) = \rho_2(f_1(n), \pi^{-1} f_2(n)) = \pi\pi^{-1} f_2(n) = f_2(n)$. One can easily check that φ is the unique sp-preserving map with $\rho_1\varphi = f_1$ and $\rho_2\varphi = f_2$. □

Given arbitrary $X_1, X_2 \in$ **spNom**, if at least one of X_1 or X_2 is discrete then one can easily see the product of X_1 and X_2 is $(X_1 \times X_2, \pi_1, \pi_2)$. In the following we characterize conditions under which the product of non-discrete sp-nominal sets exists.

Theorem 4.20. *The product of non-discrete nominal sets X and Y exists if and only if at least one of X or Y has no zero element, and if one of X or Y has some zero element(s), the condition* $\{\mathrm{supp}\, x|\ x \in X\} \cap \{\mathrm{supp}\, y|\ y \in Y\} = \emptyset$ *is required for the product to exist.*

Proof. ($\Rightarrow$) Suppose $\mathcal{Z}(X) \neq \emptyset$. We show that the existence of product implies $\mathcal{Z}(Y) = \emptyset$. On the contrary, suppose that $\mathcal{Z}(Y) \neq \emptyset$ and $\theta_1 \in \mathcal{Z}(X)$ and $\theta_2 \in \mathcal{Z}(Y)$, and (P, ρ_1, ρ_2) is the product of X and Y in **spNom**. Consider the sp-preserving maps $\mathrm{z}_1 : X \longrightarrow Y$ defined by $\mathrm{z}_1(x) = \theta_2$, for all $x \in X$, and $\mathrm{z}_2 : Y \longrightarrow X$ defined by $\mathrm{z}_2(y) = \theta_1$, for all $y \in Y$. Then, by the universal property of product, we have the following commutative diagrams.

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}"'] \arrow[d, dashed, "\exists!\varphi"] \arrow[dr, "\mathrm{z}_1"] & \\
X & P \arrow[l, "\rho_1"] \arrow[r, "\rho_2"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[dl, "\mathrm{z}_2"'] \arrow[d, dashed, "\exists!\psi"] \arrow[dr, "\mathrm{id}"] & \\
X & P \arrow[l, "\rho_1"] \arrow[r, "\rho_2"'] & Y
\end{tikzcd}$$

Then since $\rho_1(\varphi(x)) = \mathrm{id}(x) = x$, for every $x \in X$, we have ρ_1 is a non-zero sp-preserving map. Analogously, one can see that ρ_2 is a non-zero sp-preserving map. By the assumption we can take $x \in X \setminus \mathcal{Z}(X)$ and $y \in Y \setminus \mathcal{Z}(Y)$.

Suppose $|\operatorname{supp} x| \leq |\operatorname{supp} y|$. It can be assumed $\operatorname{supp} x \subseteq \operatorname{supp} y$ without loss of generality. So, by Lemma 3.13, $\operatorname{supp} \rho_1(\varphi(x)) = \operatorname{supp} \varphi(x) = \operatorname{supp} x$ and $\operatorname{supp} \rho_2(\psi(y)) = \operatorname{supp} \psi(y) = \operatorname{supp} y$. Hence, $\varphi(x) \leq \psi(y)$. But, by the above commutative diagrams, $\rho_1(\varphi(x) \not\leq \rho_1(\psi(y))$ which is a contradiction. Hence, $\mathcal{Z}(Y) = \emptyset$.

Now we show that in the case $\mathcal{Z}(X) \neq \emptyset$, $\mathcal{Z}(Y) = \emptyset$, the existence of product implies $\{\operatorname{supp} x|\ x \in X\} \cap \{\operatorname{supp} y|\ y \in Y\} = \emptyset$. On the contrary, suppose that there are $x_1 \in X$ and $y_1 \in Y$ with $\operatorname{supp} x_1 = \operatorname{supp} y_1$. Since $\mathcal{Z}(Y) = \emptyset$, $x_1 \notin \mathcal{Z}(X)$. Consider the sp-preserving maps $z, f : Y \to X$ defined by $z(y) = \theta_1$, for all $y \in Y$, and

$$f(y) = \begin{cases} \pi x_1 & \text{when } y = \pi y_1 \in \operatorname{Perm}(\mathbb{D}) y_1 \\ \theta_1 & \text{otherwise} \end{cases}$$

Then since P is product, we get the following commutative diagrams

$$\begin{tikzcd} & Y \arrow[dl, "z"'] \arrow[d, dashed, "\exists!\varphi"] \arrow[dr, "\mathrm{id}"] & \\ X & P \arrow[l, "\rho_1"] \arrow[r, "\rho_2"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[d, dashed, "\exists!\psi"] \arrow[dr, "\mathrm{id}"] & \\ X & P \arrow[l, "\rho_1"] \arrow[r, "\rho_2"'] & Y \end{tikzcd}$$

which implies $\rho_2(\varphi(y_1)) = \mathrm{id}(y_1) = y_1$ and $\rho_1(\psi(y_1)) = f(y_1) = x_1$. Since $\rho_1(\varphi(y_1)) = \theta_1$ and $\rho_1(\psi(y_1)) = x_1$, $\psi(y_1) \neq \varphi(y_1)$. So, by Lemma 3.13, $\operatorname{supp} \rho_2(\varphi(y_1)) = \operatorname{supp} \varphi(y_1) = \operatorname{supp} y_1$ and $\operatorname{supp} \rho_1(\psi(y_1)) = \operatorname{supp} \psi(y_1) = \operatorname{supp} y_1$. Hence, $\psi(y_1) \leq \varphi(y_1)$ but $\rho_1(\psi(y_1)) \not\leq \rho_1(\varphi(y_1))$ which is a contradiction.

($\Leftarrow$) If $\mathcal{Z}(X) = \mathcal{Z}(Y) = \emptyset$, then Theorem 4.19 implies the result. Now let $\mathcal{Z}(Y) = \emptyset$ and $\{\operatorname{supp} x|\ x \in X\} \cap \{\operatorname{supp} y|\ y \in Y\} = \emptyset$. Then we show that $(\bigcup_{\theta_i \in \mathcal{Z}(X)} (Y \times \{i\}), \pi, z)$, in which $\pi : \bigcup_{\theta_i \in \mathcal{Z}(X)} (Y \times \{i\}) \to Y$ defined by $\pi((y, i)) := y$ and $z : \bigcup_{\theta_i \in \mathcal{Z}(X)} (Y \times \{i\}) \to X$ defined by $z((y, i)) := \theta_i$, for every $(y, i) \in \bigcup_{\theta_i \in \mathcal{Z}(X)} (Y \times \{i\})$, is the product of X_1 and X_2. To do so, consider $A \in$ **spNom** together with sp-preserving maps $f : A \to X$ and $g : A \to Y$. Since $\mathcal{Z}(Y) = \emptyset$, by Lemma 3.13, $\operatorname{supp} a \neq \emptyset$ and $\operatorname{supp} a = \operatorname{supp} g(a)$, for all $a \in A$. Also since $\{\operatorname{supp} x|\ x \in X\} \cap \{\operatorname{supp} y|\ y \in Y\} = \emptyset$, Lemma 3.13 implies that $f(a) \in \mathcal{Z}(X)$, for all $a \in A$. We define $\varphi : A \to \dot{\bigcup}_{i \in \mathcal{Z}(X)} (Y \times \{i\})$ to be $\varphi(a) = (g(a), i)$, in which $f(a) = \theta_i \in \mathcal{Z}(X)$, for every $a \in A$. Since g is an sp-preserving map and $f(a) \in \mathcal{Z}(X)$, for all $a \in A$, the map φ is sp-preserving making the desired diagram commute. $\square$

5 Nominal space

Each nominal set X can be considered as a topological space with the *support segment topology* (*or simply, support topology*) arised from $S = \{x_{\downarrow}, x^{\uparrow} \mid x \in X\}$ as the subbasis. The nominal set with the support topology, $(X, \mathcal{S})$, is called a *nominal space*. This section is devoted to study the topological properties of nominal spaces.

Example 5.1. According to Example 3.2,

(i) the support topology on $\mathbb{D}$ is discrete.

(ii) the support topologies on $\mathcal{P}_{\mathrm{cof}}(\mathbb{D})$ and $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ are also discrete. Indeed, for each $A \in \mathcal{P}_{\mathrm{cof}}(\mathbb{D})$, $A^{\uparrow} \cap A_{\downarrow} = \{A' \in \mathcal{P}_{\mathrm{cof}}(\mathbb{D}) \mid \operatorname{supp} A' = \operatorname{supp} A\} = \{A\}$. Similarly one can show that the nominal space $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ is discrete.

(iii) the support topology on $\mathcal{P}_{\mathrm{fs}}(\mathbb{D})$ is non-discrete. Because for each $A \in \mathcal{P}_{\mathrm{fs}}(\mathbb{D})$, $A^{\uparrow} \cap A_{\downarrow} = \{A' \in \mathrm{P}_{\mathrm{fs}}(\mathbb{D}) \mid \operatorname{supp} A' = \operatorname{supp} A\} = \{A, A^c\}$.

Definition 5.2. A congruence relation ρ on X *saturates* $L \subseteq X$ if the condition $u \in L$ and $u \rho v$ imply $v \in L$.

Lemma 5.3. *Let X be a nominal space and $U \in \mathcal{S}$. Then, $\sim$ saturates U.*

Proof. Let $x \in U$ and $x \sim y$. Then, since $U = \bigcup_{i \in I} \bigcap_{j \in J} V_{i_j}$, in which J is finite, there exists $i \in I$ such that for all $j \in J$ we have $x \in V_{i_j}$. Notice that, $V_{i_j} = x_{i_j}\downarrow$ or $V_{i_j} = x_{i_j}{}^{\uparrow}$. Assume $V_{i_j} = x_{i_j}\downarrow$ ($V_{i_j} = x_{i_j}^{\uparrow}$). We show that $y \in V_{i_j}$. Indeed, since $y \preceq x$ and $x \preceq y$, we have $y \preceq x \preceq x_{i_j}$ ($x_{i_j} \preceq x \preceq y$). So $y \preceq x_{i_j}$ ($x_{i_j} \preceq y$) and so $y \in V_{i_j}$ ($y \in V_{i_j}$). Thus, $y \in U$. □

Theorem 5.4. *Let X be a nominal space . Then,*

(i) *if $x \in U \in \mathcal{S}$, then $x^{\uparrow} \cap x_{\downarrow} \subseteq U$.*

(ii) *if $x \in F$ and F is closed, then $x^{\uparrow} \cap x_{\downarrow} \subseteq F$.*

Proof. (i) Let $y \in x^{\uparrow} \cap x_{\downarrow}$. Then, $y \preceq x$ and $x \preceq y$ and so $x \sim y$. Now, applying Lemma 5.3, $y \in U$.

(ii) Let $y \in x^{\uparrow} \cap x_{\downarrow}$. Then, $y \sim x$. Assume $y \notin F$. So, $y \in X \setminus F$. Since $X \setminus F$ is open and $x \sim y$, by (i) $x \in X \setminus F$. Which is a contradiction. □

Corollary 5.5. *Let X be a nominal set. Then $x/{\sim} = x^{\uparrow} \cap x_{\downarrow}$ is the smallest open set containing x.*

Corollary 5.6. (i) *If X is a nominal space, then $x_\downarrow$ ($x^\uparrow$) is clopen.*

(ii) *If U is clopen, then $U = \bigcup_{y\in U}(y^\uparrow \cap y_\downarrow)$.*

(iii) *If $U \in \mathcal{S}$, then U is clopen.*

Proof. (i) Follows from Proposition 3.18(ii, iii).

(ii) Follows from Theorem 5.4(i, ii).

(iii) If $U \in \mathcal{S}$, then $X \setminus U$ is closed and so by (ii) it is a (finitely supported) union of open subsets of X. Thus, $X \setminus U$ is open and so U is closed. □

Theorem 5.7. *Let X and Y be two sp-nominal sets. Then every sp-preserving map $f : X \to Y$ is continuous.*

Proof. Applying Lemma 3.20, we have

$$f^{-1}(a^\uparrow \cap a_\downarrow) = f^{-1}(a^\uparrow) \cap f^{-1}(a_\downarrow) = [\bigcup_{f(x)\in a^\uparrow} x^\uparrow] \cap [\bigcup_{f(x)\in a_\downarrow} x_\downarrow].$$

So, $f^{-1}(a^\uparrow \cap a_\downarrow)$ is open in X, for all $a \in Y$. Now, by Corollary 5.6, we get the result. □

The following example shows that the converse of Theorem 5.7 does not hold.

Example 5.8. Take $f : \mathbb{D}^2 \to \mathbb{D}\dot\cup\{\theta\}$ to be the equivariant map defined by

$$f(d, d') = \begin{cases} d & d = d' \\ \theta & d \neq d'. \end{cases}$$

Since support topology of $\mathbb{D}$ is discrete, the least open sets of $\mathbb{D}\dot\cup\{\theta\}$ are singleton sets. Now we have

$$f^{-1}(\{d\}) = (d, d)_\downarrow, \quad f^{-1}(\{\theta\}) = \mathbb{D}^{(2)} = \bigcup_{d\neq d'} (d, d')^\uparrow,$$

and hence, f is continuous. On the other hand, we have $(d, d) \preceq (d, d_1)$ while $f(d, d) \npreceq f(d, d_1)$, that is, f is not an sp-preserving map.

Example 5.9. Let X be a nominal space. Then, by applying Example 3.6 (ii) and Theorem 5.7, the support map $\mathrm{supp} : X \to \mathcal{P}_\mathrm{f}(\mathbb{D})$, mapping $x \mapsto \mathrm{supp}\, x$, is continuous.

In the sequal of this section we examine separation axioms and describe compact nominal spaces. Among many separation axioms that can be imposed on topological spaces, here we discuss the "Hausdorff condition" (T_2). Because it implies the uniqueness of limits of sequences, nets, and filters. We first note that nominal spaces can be Hausdorff or not, see Examples 5.11 and 5.12. Therefore, we seek to characterize those nominal spaces that are Hausdorff. To do so, we first recall the following definition.

Definition 5.10. A topological space X is called

- T_0 if for every pair of points, there exists at least one open set that contains one but not the other; that is, if $x_1 \neq x_2 \in X$ then there is an open set U with $x_1 \in U$ and $x_2 \notin U$.
- T_1 if for every pair of points, there exist open sets that each of which contains one but not the other; that is, if $x_1 \neq x_2 \in X$ then there are open sets U_1 and U_2 with $x_1 \in U_1, x_2 \notin U_1$, and $x_2 \in U_2, x_1 \notin U_2$.
- T_2 or *Hausdorff* if every pair of points can be separated by open sets; that is, if $x_1 \neq x_2 \in X$ then there are disjoint open sets U_1 and U_2 with $x_1 \in U_1$ and $x_2 \in U_2$.
- *normal* if every disjoint pair of closed sets can be separated by open sets; that is, if A_1 and A_2 are disjoint closed subsets of X then there are disjoint open sets U_1 and U_2 with $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$.
- *regular* if any closed set and any point can be separated by open sets; that is, if A is closed set and $x \in X$ then there exist disjoint open sets U_1 and U_2 with $A \subseteq U_1$ and $x \in U_2$.
- T_3 or *regular Hausdorff* if it is a topological space that is, both regular and a Hausdorff space.
- T_4 *Space* or *normal Hausdorff* if X is both a normal space and a T_1 space.
- a *separatory for each pair of subsets* if every disjoint pair of subsets can be separated by open sets; that is, if $A_1, A_2 \in \mathcal{P}(X)$ are disjoint then there are disjoint open sets U_1 and U_2 with $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$.

Example 5.11. (i) Considering the nominal space $\mathbb{D}^2$, we have

$$\begin{aligned}(d_1, d_2)_\downarrow &= \{(d, d') \in \mathbb{D}^2 \mid \operatorname{supp}(d, d') \subseteq \operatorname{supp}(d_1, d_2)\}\\ &= \{(d_1, d_1), (d_2, d_2), (d_1, d_2), (d_2, d_1)\}, \text{and}\\ (d_1, d_2)^\uparrow &= \{(d, d') \in \mathbb{D}^2 \mid \operatorname{supp}(d_1, d_2) \subseteq \operatorname{supp}(d, d')\}\\ &= \{(d_1, d_2), (d_2, d_1)\},\end{aligned}$$

for every $d_1 \neq d_2 \in \mathbb{D}$. Now since, for every $d_1 \neq d_2 \in \mathbb{D}$, Theorem 5.4 implies $((d_1, d_2)_\downarrow) \cap ((d_1, d_2)^\uparrow) = \{(d_1, d_2), (d_2, d_1)\}$ is the smallest open set, contains the points (d_1, d_2) and (d_2, d_1) in $\mathbb{D}^2$, in which $d_1 \neq d_2$, can not be separated by disjoint open sets. Hence, this space is neither Hausdorff nor T_1.

(ii) Considering the nominal space $\mathbb{D}^{(k)}$ with $k \geq 2$, since the cardinality of the support of each element equals k, we have

$$\begin{aligned}(d_1, d_2, \ldots, d_k)_\downarrow &= \{(d'_1, d'_2, \ldots, d'_k) \mid \{d'_1, d'_2, \ldots, d'_k\} = \{d_1, d_2, \ldots, d_k\}\}\\ &= (d_1, d_2, \ldots, d_k)^\uparrow.\end{aligned}$$

Hence, the poins $(d_1, d_2, \ldots, d_k) \neq (d_2, d_1, d_3, \ldots, d_k) \in \mathbb{D}^{(k)}$ can not be separated by disjoint open sets, because $(d_1, d_2, \ldots, d_k), (d_2, d_1, d_3, \ldots, d_k) \in (d_1, d_2, \ldots, d_k)^\uparrow \cap (d_1, d_2, \ldots, d_k)_\downarrow$ and, by Theorem 5.4, $(d_1, d_2, \ldots, d_k)^\uparrow \cap (d_1, d_2, \ldots, d_k)_\downarrow$ is the smallest open sets containing $(d_1, d_2, \ldots, d_k)$ and $(d_2, d_1, d_3, \ldots, d_k)$, meaning that this space is neither Hausdorff nor T_1.

(iii) Using Example 5.1(iii), one can easily see that the points $\{d\}$ and $\mathbb{D} \setminus \{d\}$ in the nominal space $\mathcal{P}_{\mathrm{fs}}(\mathbb{D})$ can not be separated by disjoint open sets and hence, $\mathcal{P}_{\mathrm{fs}}(\mathbb{D})$ is not Hausdorff.

Analogously, one can see that the nominal space $\mathcal{P}_{\mathrm{fs}}(\mathbb{D})$ is neither Hausdorff nor T_1.

Example 5.12. Using Example 5.1(ii), since the nominal space $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ contains singleton element hence, it is disceret. Therefore, it is Hausdorff.

Theorem 5.13. *Let $(X, \mathcal{S})$ be a nominal space. Then, X is Hausdorff if and only if the support map,* supp : $X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$*, separates the elements of X.*

Proof. ($\Rightarrow$) Let X be Hausdorff and $x \neq y \in X$. Then, there exist $U, V \in \mathcal{S}$ with $x \in U, y \in V$ and $U \cap V = \emptyset$. Then, by Theorem 5.4 and Corollary 5.5, $x/\!\sim \cap\, y/\!\sim = \emptyset$. Hence, $\operatorname{supp} x \neq \operatorname{supp} y$.

($\Leftarrow$) Let $y \neq x \in X$. Then, by the hypothesis, we have $\operatorname{supp} x \neq \operatorname{supp} y$ and hence, $x/\sim \cap\, y/\sim = \emptyset$. Now Corollary 5.5 implies the result. $\square$

Lemma 5.14. *Any nominal space is a regular space.*

Proof. Suppose X is a nominal space. Take a closed set F and $x_1 \in X$ with $x_1 \notin F$. By Corollary 5.6 (iii), F is open. Thus, there are two open sets $x_1^{\uparrow} \cap x_{1\downarrow}$ and F with $(x_1^{\uparrow} \cap x_{1\downarrow}) \cap F = \emptyset$. $\square$

Corollary 5.15. *Any nominal space is a normal space.*

Proof. By Corollary 5.6 (iii), since each closed set is open, we get the result. $\square$

Theorem 5.16. *Let X be a nominal space. Then the following statements are equivalent:*

(i) *The relation $\preceq$ is a partially order on X.*
(ii) $x^{\uparrow} \cap x_{\downarrow} = \{x\}$, *for every $x \in X$.*
(iii) X *is* T_0.
(iv) X *is* T_1.
(v) X *is* T_2 *(or Hausdorff space).*
(vi) X *is* T_3.
(vii) X *is* T_4.
(viii) X *is a separator for each* $A, B \in P(X)$ *with* $A \cap B = \emptyset$.
(ix) *The support map* $\operatorname{supp} : X \to P_f(\mathbb{D})$ *is injective.*

Proof. (i) $\Rightarrow$ (ii) Let $t \in x^{\uparrow} \cap x_{\downarrow}$. Then, $t \preceq x$ and $x \preceq t$. Since $\preceq$ is antisymmetric, $t = x$.

(ii) $\Rightarrow$ (iii) Follows by taking open sets $x_{\downarrow} \cap x^{\uparrow}$ and $x'_{\downarrow} \cap x'^{\uparrow}$ for each $x \neq x'$.

(iii) $\Rightarrow$ (iv), and (iv $\Rightarrow$ v) follow from Theorem 5.4(i) and Corollary 5.5.

Lemma 5.14 implies (v $\Rightarrow$ vi).

Corollary 5.15 implies (vi $\Rightarrow$ vii).

(vii) $\Rightarrow$ (viii) For each $A, B \in \mathcal{P}(X)$ such that $A \cap B = \emptyset$ we show $(\bigcup_{a \in A}(a^{\uparrow} \cap a_{\downarrow})) \cap (\bigcup_{b \in B}(b^{\uparrow} \cap b_{\downarrow})) = \emptyset$. On the contrary, suppose $x \in (\bigcup_{a \in A}(a^{\uparrow} \cap a_{\downarrow})) \cap (\bigcup_{b \in B}(b^{\uparrow} \cap b_{\downarrow}))$. Hence, there are $a \in A$ and $b \in B$ such that $x \in (a^{\uparrow} \cap a_{\downarrow})$ and $x \in (b^{\uparrow} \cap b_{\downarrow})$. Therefore, $\operatorname{supp} x = \operatorname{supp} a = \operatorname{supp} b$ hence, $a \in (b^{\uparrow} \cap b_{\downarrow})$. Since $A \cap B = \emptyset$ hence, $a \neq b$, which is a contradiction with X is T_4. Obviously $A \subseteq (\bigcup_{a \in A}(a^{\uparrow} \cap a_{\downarrow}))$ and $B \subseteq (\bigcup_{b \in B}(b^{\uparrow} \cap b_{\downarrow}))$, we get the result.

(viii) $\Rightarrow$ (ix) For each $x_1 \neq x_2$ we consider $\{x_1\} = A$ and $B = \{x_2\}$. Now, by assumption, we have $(x_1^{\uparrow} \cap x_{1\downarrow}) \cap (x_2^{\uparrow} \cap x_{2\downarrow}) = \emptyset$. Hence, $\operatorname{supp} x_1 \neq \operatorname{supp} x_2$ for each $x_1 \neq x_2$.

(ix) $\Rightarrow$ (i) Follows from Corollary 5.5. □

Theorem 5.17. *A nominal space X is compact if and only if the set*

$$A = \{(x, x') \mid \operatorname{supp} x \neq \operatorname{supp} x'\} = \nabla/\sim,$$

is finite.

Proof. ($\Rightarrow$) Suppose X is compact. Then, by Corollary 5.5, one can consider the open cover $X \subseteq \bigcup_{x \in X}(x^{\uparrow} \cap x_{\downarrow})$ of X. So there exist $x_1, x_2, \cdots, x_n \in X$ such that $X \subseteq \bigcup_{1 \leq i \leq n}(x_i^{\uparrow} \cap x_{i\downarrow})$; meaning that X only contains a finite number of elements with different supports, and hence, A is a finite set.

($\Leftarrow$) Suppose A is finite and $\{U_j\}_{j \in I}$ is an arbitrary open cover for X. Since A is finite, there are finitely many elements of X, such as $x_1, \ldots, x_n$, each pair of which have different supports. Since, by Corollary 5.5, $x_i^{\uparrow} \cap x_{i\downarrow}$ is the smallest open subset of X containing x_i, for every $1 \leq i \leq n$, there exists $j_i \in I$ such that $x_i^{\uparrow} \cap x_{i\downarrow} \subseteq U_{j_i}$, for evey $1 \leq i \leq n$. Hence, we have $X \subseteq \bigcup_{1 \leq i \leq n}(x_i^{\uparrow} \cap x_{i\downarrow}) \subseteq \bigcup_{1 \leq i \leq n} U_{j_i}$. □

Theorem 5.18. *A nominal space X is compact if and only if $X/\sim$ is finite.*

Proof. ($\Rightarrow$) Let X be compact and $X = \bigcup_{x \in X}(x/\sim)$. Then, there exist $x_1, x_2, \cdots, x_n \in X$ with $X \subseteq \bigcup_{1 \leq i \leq n}(x_i/\sim)$. So, $X/\sim$ is finite.

($\Leftarrow$) Let $X/\sim$ be finite. Then, $X = \bigcup_{1 \leq i \leq n} x_i/\sim$. Suppose $X = \bigcup_{\alpha} U_\alpha$. By Theorem 5.4, there exists $\alpha_{x_i} \in I$ such that $x_i/\sim \subseteq U_{\alpha_{x_i}}$, for every $1 \leq i \leq n$. Hence, $X = \bigcup_{1 \leq i \leq n} x_i/\sim \subseteq \bigcup_{1 \leq i \leq n} U_{\alpha_{x_i}} \subseteq X$ and so $X = \bigcup_{1 \leq i \leq n} U_{\alpha_{x_i}}$. □

Corollary 5.19. *Let X be a nominal space. The following statements are equivalent:*

(i) *X is compact.*
(ii) *X is a discrete nominal set.*
(iii) $\mathcal{S} = \{\emptyset, X\}$.

Proof. (i) $\Rightarrow$ (ii) By Remark 3.3, $X/\sim$ is a nominal set. If X is compact then, by Theorem 5.18, $X/\sim$ is finite. So, by Remark 2.10, $X/\sim$ is a discrete nominal set and $\operatorname{supp}(x/\sim) = \emptyset$, for every $x/\sim \in X/\sim$. Now since for every $t \in X$, there exists

$x/{\sim} \in X/{\sim}$ with $t \in x/{\sim}$ and $\operatorname{supp} x/{\sim} = \operatorname{supp} x = \operatorname{supp} t$, we have $\operatorname{supp} t = \emptyset$, for all $t \in X$, and we are done.

(ii) $\Rightarrow$ (iii) Suppose $\operatorname{supp} x = \emptyset$, for each $x \in X$. Then $x^{\uparrow} = x_{\downarrow} = x^{\uparrow} \cap x_{\downarrow} = X$, for each $x \in X$. Therefore, $\mathcal{S} = \{\emptyset, X\}$.

(iii) $\Rightarrow$ (i) This is clear. $\square$

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