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On some properties of the space of minimal prime ideals of $C_c(X)$

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Abstract. In this article we consider some relations between the topological properties of the spaces X and $Min(C_c(X))$ with algebraic properties of $C_c(X)$. We observe that the compactness of $Min(C_c(X))$ is equivalent to the von-Neumann regularity of $q_c(X)$, the classical ring of quotients of $C_c(X)$. Furthermore, we show that if X is a strongly zero-dimensional space, then each contraction of a minimal prime ideal of $C(X)$ is a minimal prime ideal of $C_c(X)$ and in this case $Min(C(X))$ and $Min(C_c(X))$ are homeomorphic spaces. We also observe that if X is an F_c-space, then $Min(C_c(X))$ is compact if and only if X is countably basically disconnected if and only if $Min(C_c(X))$ is homeomorphic with $\beta_0 X$. Finally, by introducing z_c°-ideals, countably cozero complemented spaces, we obtain some conditions on X for which $Min(C_c(X))$ becomes compact, basically disconnected and extremally disconnected.

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1 Introduction

As usual all rings are commutative with unity and all topological spaces are Hausdorff completely regular (i.e., Tychonoff) and $C(X)$ is a ring of all real valued continuous functions on a space X. We denote by $C_c(X)$ the subring of $C(X)$ consisting of those functions with countable image. The subring $C_c^*(X)$ of $C(X)$ consists of bounded elements of $C_c(X)$, i.e., $C_c^*(X) = C^*(X) \cap C_c(X)$. For more results about $C_c(X)$ and $C_c^*(X)$, the reader is referred to [5, 10, 11, 17, 20–23].

We recall that a zero-dimensional space is a Hausdorff space with a base consisting of clopen sets. It is known that for any topological space X, there exists a zero-dimensional space Y in which $C_c(X) \cong C_c(Y)$, see [11]. Furthermore, X is called a strongly zero-dimensional space if each pair of disjoint zero-sets are contained in disjoint clopen sets. βX is used for the Stone-$\check{C}$ech compactification of X, see [12]. Also, every zero-dimensional space X has a unique zero-dimensional compactification denoted by $\beta_0 X$, the Banaschewski compactification of X, such that each continuous function from X into a compact and zero-dimensional space T, has a continuous extension from $\beta_0 X$ into T. It is known that $\beta X = \beta_0 X$ if and only if X is a strongly zero-dimensional space, for more results, see [5].

The prime spectrum of a ring R, which is denoted by $Spec(R)$, is a space of prime ideals of R with Zariski topology, for which the closed sets are the sets $V(I) = \{P \in Spec(R) : I \subseteq P\}$ where I is an ideal of R and the open sets are $D(I) = Spec(R) \setminus V(I)$, where I is an ideal of R. Also, $Min(R)$ as a dense subspace of $Spec(R)$, is the space of minimal prime ideals of R. The space $Spec(R)$ is a compact and T_0-space whereas $Min(R)$ is a Hausdorff and zero-dimensional space but not necessarily compact, also the properties of $Min(C(X))$ are studied. For more results about the space of prime ideals, see [7, 13, 19]. It is shown that both $C_c(X)$ and $C_c^*(X)$ are clean rings, see [5]. By a similar proof to that of [13, Corollary 5.2], we observe that $Min(C_c(X))$ and $Min(C_c^*(X))$ are homeomorphic spaces.

A reduced ring R satisfies the annihilator condition (or a.c.) if for each $a, b \in R$, there exists $c \in R$ such that $Ann(c) = Ann(a) \cap Ann(b)$. For each $f \in C_c(X)$, the annihilator of f is denoted by $Ann_c(f)$. It is easy to show that $C_c(X)$ has a.c.. Furthermore, a reduced ring R satisfies countable annihilator condition (or c.a.c.) if for every sequence $\{r_n\}_{n\in\mathbb{N}}$ in R, there exists $s \in R$ such that $Ann(s) = \cap_{n\in\mathbb{N}} Ann(r_n)$. $C(X)$ has c.a.c, see [13]. The set of all zero divisors of a ring R is denoted by $Zd(R)$. Also, R has property A if for every finitely generated ideal I in R such that $I \subseteq Zd(R)$, we have $Ann(I) \neq (0)$. The ring $C^F(X)$, the subring of $C_c(X)$

consisting of those functions with finite image, has c.a.c. and property A, so its classical ring of quotients is von-Neumann regular. Thus, $Min(C^F(X))$ is always compact, see [15, Theorem B]. A ring R is called a disconnected ring if it has a nontrivial idempotent. If R is a disconnected ring, there exists $e \in R$ such that $e^2 = e$, $0 \neq e \neq 1$, so $Spec(R) = D(e) \cup D(1-e)$ where $D(e) \cap D(1-e) = \emptyset$. The classical ring of quotients of $C_c(X)$ is denoted by $q_c(X)$ and some properties of $q_c(X)$ are studied; see also [6, 20, 21].

A space X is a P-space (resp., CP-space) if $C(X)$ (resp., $C_c(X)$) is a von-Neumann regular ring. Every P-space is a CP-space, but the converse is not necessarily true. For instance let $X = [0, 1] \cup \mathbb{N}$, then X is a CP-space but not a P-space. It is known that P-spaces and CP-spaces coincide when X is zero-dimensional. For more results, see [11].

For each $f \in C_c(X)$, the zero-set (cozero-set) of f is denoted by $Z(f)$ ($coz(f)$) and $\mathrm{S}(f) = \mathrm{cl}(coz(f))$ is the support of f. The set of all zero-sets of members of $C_c(X)$ is denoted by $Z_c(X)$. Also, $Z_c(X)$ is closed under countable intersection property. Furthermore, $Z_c(X) = Z(X)$ if and only if X is strongly zero-dimensional, see [5, Proposition 2.4]. A space X is an almost P-space if and only if every nonempty zero-set has nonempty interior. It is shown that if X is an almost P-space, then it is a basically disconnected space if and only if X is a P-space, equivalently $Min(C(X))$ is a compact space, see [1, Proposition 2.8].

We recall that an ideal I in $C(X)$ (resp., $C_c(X)$) is a z-ideal (resp., z_c-ideal) if for each $f \in I$, $g \in C(X)$ (resp., $g \in C_c(X)$) and $Z(f) = Z(g)$ we have $g \in I$. If I is a z-ideal, then $I^c = I \cap C_c(X)$ is a z_c-ideal. Also, if I is a z_c-ideal and P is a prime ideal minimal over I, i.e., $P \in Min(I)$ (the set prime ideals minimal over I), then P is a z_c-ideal. For more results of z_c-ideals, see [11]. Similar to the concept of M_p in $C(X)$, the fixed maximal ideals in $C_c(X)$ is denoted by M_{cp} $(p \in X)$. By [5] we have:

$$M_{cp} = \{f \in C_c(X) : p \in Z(f)\} = M_p \cap C_c(X).$$

Also, if X is a zero-dimensional space, the set of all maximal ideals in $C_c(X)$ is denoted by M_c^p $(p \in \beta_0 X)$, which is defined as follows:

$$M_c^p = \{f \in C_c(X) : p \in \mathrm{cl}_{\beta_0 X} Z(f)\}.$$

Moreover, $M_c^p = M_{cp}$ if $p \in X$. For more details see [5].
As in $C(X)$, similar to the concept of the ideal O^p, $p \in \beta X$, for a zero-dimensional

space X, we have the ideal O_c^p in $C_c(X)$:

$$O_c^p = \{f \in C_c(X) : p \in \mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X} Z(f)\}, \quad (p \in \beta_0 X).$$

Furthermore, $O_c^p = O_{cp} = \{f \in C_c(X) : p \in int_X Z(f)\}$ if $p \in X$. For a zero-dimensional space X, the ideal O_c^p is a z_c-ideal. Moreover, if X is a strongly zero-dimensional space, then for each $p \in \beta_0 X$ we have $O_c^p = O^p \cap C_c(X)$. We recall from [5] that for every zero-dimensional space X, the spaces $\beta_0 X$ and $\mathfrak{M}_c(X)$ are homeomorphic in which $\mathfrak{M}_c(X) = \{M_c^p : p \in \beta_0 X\}$, with Zariski topology. It is known that every prime ideal P in $C_c(X)$ contains O_c^p for a unique $p \in \beta_0 X$, and M_c^p is the only maximal ideal containing P, see [5, Lemma 4.11]. Similar to the definition of F-spaces that is considered in [12], a zero-dimensional space X is said to be an F_c-space if O_c^p is a prime ideal in $C_c(X)$, for each $p \in \beta_0 X$. Every F-space is an F_c-space whereas $\mathbb{R}$ is an F_c-space but it is not an F-space. However the converse is true if X is strongly zero-dimensional. Furthermore, X is an F_c-space if and only if $\beta_0 X$ is an F_c-space. For more results about F_c-spaces, see [5].

We recall that X is a basically disconnected (resp., extremally disconnected) space if every cozero-set (resp., open set) has an open closure. Every basically disconnected space is strongly zero-dimensional, see [12, 14O.3]. Also, a zero-dimensional space X is extremally disconnected if and only if $\beta_0 X$ is extremally disconnected, see [17]. It is shown that every basically disconnected space is an F_c-space. The converse is not generally true; for example $\beta\mathbb{N} \setminus \mathbb{N}$ is an F_c-space which is not basically disconnected, see [5, Remark 6.7].

In this paper, we consider some topological properties of $Min(C_c(X))$ as a dense subspace of $Spec(C_c(X))$. In Section 2, we consider some special properties of $Min(C_c(X))$, also its relations with algebraic properties of $C_c(X)$ and topological properties of X. The properties of $Min(C(X))$ is studied in some articles, see [7, 13]. We show that $Min(C(X))$ and $Min(C_c(X))$ are homeomorphic when X is a strongly zero-dimensional space. It is proved that X is an F_c-space and $Min(C_c(X))$ is compact if and only if X is a c-basically disconnected space. Section 3 is devoted to the compactness of the space $Min(C_c(X))$. We introduce z_c°-ideals in $C_c(X)$ and observe that X is strongly zero-dimensional if and only if every prime z_c°-ideal in $C_c(X)$ is a contraction of a unique z°-ideal of $C(X)$. Also, we show that whenever X is an almost CP-space then $Min(C_c(X))$ is a compact space if and only if every prime z_c°-ideal in $C_c(X)$ is a minimal prime ideal. We introduce some special spaces X for which $Min(C_c(X))$ becomes a compact, basically disconnected and extremally disconnected space. Similar to the concept of cozero complemented spaces or $c.c$-spaces which is first introduced in [14], we introduce countably cozero complemented spaces or $c.c.c$-spaces which is equivalent to the compactness of $Min(C_c(X))$. We compare $c.c$-spaces and $c.c.c$-spaces and give some examples. Also, according to the definition of perfectly normal spaces in which $Min(C(X))$ is compact and extremally disconnected, see [13], we introduce countably perfectly normal spaces for which $Min(C_c(X))$ is compact and extremally disconnected.

2 The space $Min(C_c(X))$

For each $f \in C_c(X)$, the set $V_c(f) = V_c((f)) = \{P \in Spec(C_c(X)) : f \in P\}$ (resp., $D_c(f) = D_c((f)) = \{P \in Spec(C_c(X)) : f \notin P\}$) contains closed (resp., open) set for the Zariski topology on $Spec(C_c(X))$. We denote $V_{cm}(f) = V_c(f) \cap Min(C_c(X))$ (resp., $D_{cm}(f) = D_c(f) \cap Min(C_c(X))$ and $V_M(f) = V_c(f) \cap \mathfrak{M}_c(X)$ (resp., $D_M(f) = D_c(f) \cap \mathfrak{M}_c(X)$). Clearly, the set $\{D_{cm}(f) : f \in C_c(X)\}$ (resp., $\{D_M(f) : f \in C_c(X)\}$) is also a base for open subsets of the subspace $Min(C_c(X))$ (resp., $\mathfrak{M}_c(X)$) of $Spec(C_c(X))$. Since $C_c(X)$ is a clean ring, then $Spec(C_c(X))$ is strongly zero-dimensional, see [5, 18]. Also, similar to [12, 1B], $Spec(C_c(X))$ is a disconnected space if and only if $C_c(X)$ is a disconnected ring (has a nontrivial idempotent) if and only if X is a disconnected space.

Proposition 2.1. *Let X is a CP-space, then for each $f, g \in C_c(X)$, $Z(f) = Z(g)$ if and only if $V_{cm}(f) = V_{cm}(g)$.*

Proof. It is clear. □

Proposition 2.2. *A space X is a CP-space if and only if for each $f \in C_c(X)$, $V_c(f)$ is open.*

Proof. First let $Z(f) = \mathrm{int}_X Z(f)$, so $Z(f^2) = \mathrm{int}_X Z(f)$. By [12, Exercise 1D.1], there exists $g \in C_c(X)$ such that $f = f^2 g$, so the set $D_c(f) = Spec(C_c(X)) \setminus D_c(1 - fg)$ is closed. Conversely, let $p \in Z(f)$, so we have $f \in M_{cp}$, i.e., $M_{cp} \notin D_c(f)$. Since $D_c(f)$ is a closed set, there exists $g \in C_c(X)$ such that:

$$M_{cp} \in D_c(g) \subseteq Spec(C_c(X)) \setminus D_c(f) = V_c(f).$$

Thus, $p \in coz(g) \subseteq Z(f)$, so $p \in \mathrm{int}_X Z(f)$. □

Definition 2.3. A space X is an almost CP-space if for each nonempty $Z(f) \in Z_c(X)$, we have $\mathrm{int}_X Z(f) \neq \emptyset$ or equivalently, $Z(f)$ is a regular closed set, that is $Z(f) = \mathrm{cl}_X \mathrm{int}_X Z(f)$.

Clearly, every almost P-space is an almost CP-space. The converse is true if X is zero-dimensional. The space $\mathbb{R}$ of real numbers with usual topology is an almost CP-space but not almost P-space.

Proposition 2.4. *Let X be a zero-dimensional space and $f \in C_c(X)$. Then $\mathrm{int}_X Z(f) \neq \emptyset$ if and only if $V_c(f)$ has a non-empty interior.*

Proof. Let $x \in \mathrm{int}_X Z(f)$. Then $f \in O_{cx}$ and hence there is $g \notin M_{cx}$ such that $fg = 0$ by Lemma 4.11 in [5]. Now $D_c(g) \subseteq V_c(f)$ for if $Q \in D_c(g)$, then $g \notin Q$ implies that $f \in Q$, i.e., $Q \in V_c(f)$. This means that $V_c(f)$ has a non-empty interior. For the converse, let $\mathrm{int}_X V_c(f) \neq \emptyset$ and $P \in \mathrm{int}_X V_c(f)$. Then $P \in D_c(g) \subseteq V_c(f)$ for some $g \in C_c(X)$. Take a minimal prime ideal $P_\circ$ contained in P. Thus $P_\circ \in D_c(g) \subseteq V_c(f)$ implies that $f \in P_\circ$, whence f is a zero-divisor and hence $\mathrm{int}_X Z(f) \neq \emptyset$. $\square$

Corollary 2.5. *Let X be a zero-dimensional space. Then every member of the base $Z_c(X)$ of X is regular closed if and only if every member of the base $\{V_c(f) : f \in C_c(X)\}$ of Spec(X) is.*

Proposition 2.6. *Let X be a strongly zero-dimensional space. If $P \in Min(C(X))$, then $P \cap C_c(X) \in Min(C_c(X))$.*

Proof. Clearly, $P \cap C_c(X) \in Spec(C_c(X))$. Let $f \in P \cap C_c(X)$, we show there exists $g \in C_c(X)$ such that $g \notin P \cap C_c(X)$, $fg = 0$. Since $f \in P$, $P \in Min(C(X))$, there exists $h \in C(X)$ such that $h \notin P$, $fh = 0$. Since X is strongly zero-dimensional, there exists $g \in C_c(X)$ such that $Z(g) = Z(h)$, by Proposition 2.4 in [5]. Clearly, $fg = 0$, $g \notin P \cap C_c(X)$ and we are done. $\square$

Remark 2.7. The converse of Proposition 2.5 is not necessarily true, in the sense that if the contraction of each minimal prime ideal of $C(X)$ to $C_c(X)$ is minimal prime, then X is not necessarily a strongly zero-dimensional (even a zero-dimentional) space. For example, let $X = \mathbb{R}$ and $P \in Min(C(\mathbb{R}))$. Since $\mathbb{R}$ is connected, then $C_c(\mathbb{R}) = \mathbb{R}$, so $P \cap C_c(\mathbb{R}) = \{0\}$. Moreover, $Min(C_c(\mathbb{R})) = Min(\mathbb{R}) = \{0\}$. Thus, $P \in Min(C(\mathbb{R}))$ and $P \cap C_c(\mathbb{R}) \in Min(C_c(\mathbb{R}))$ but $\mathbb{R}$ is not a strongly zero-dimensional space, see [8].

Let us note that if $Q \in Min(C_c(X))$, then there exists $P_Q \in Min(C(X))$ such that $Q = P_Q \cap C_c(X)$, see [11, comment preceding Corollarly 3.4]. Furthermore, if X is strongly zero-dimensional and $Q \in Min(C_c(X))$, then there is a unique $P_Q \in Min(C(X))$ such that $Q = P_Q \cap C_c(X)$, see [16, Theorem 5.19]. Moreover, an ideal J in $C_c(X)$ is a z_c-ideal if and only if it is a contraction of a z-ideal in $C(X)$, see [5, Proposition 4.3].

In the next theorem using the argument of Remark 2.7, we show that the spaces $Min(C(X))$ and $Min(C_c(X))$ are homeomorphic, in case X is strongly zero-dimensional.

Theorem 2.8. *Let X be a strongly zero-dimensional space, then $Min(C(X))$ and $Min(C_c(X))$ are homeomorphic spaces.*

Proof. We define the mapping $\varphi : Min(C(X)) \longrightarrow Min(C_c(X))$ where $\varphi(P) = P \cap C_c(X)$. We show φ is a homeomorphism. Obviously, φ is a bijective function. Also, φ is open. To see this let $D_m(f)$ be a member of the base of $Min(C(X))$, where $f \in C(X)$. We show that $\varphi(D_m(f))$ is open in $Min(C_c(X))$. Since X is strongly zero-dimensional, there is $g \in C_c(X)$ such that $Z(f) = Z(g)$ by Proposition 2.4 in [6] and it is enough to prove that $\varphi(D_m(f)) = D_{cm}(g)$. First suppose that $P \in \varphi(D_m(f))$. Then $P = \varphi(Q)$ for some $Q \in D_m(f)$. Hence $f \notin Q$ and since $Z(g) = Z(f)$ and Q is a z-ideal, we have $g \notin Q$. This follows that $Q \cap C_c(X) \in D_{cm}(g)$. Thus $P = \varphi(Q) = Q \cap C_c(X) \in D_{cm}(g)$ implies that $\varphi(D_m(f)) \subseteq D_{cm}(g)$. Next suppose that $T \in D_{cm}(g)$. Then there exists $P \in Min(C(X))$ such that $T = P \cap C_c(X)$. Since $g \notin T$ we have $g \notin P$. But P is a z-ideal and $Z(f) = Z(g)$, so $f \notin P$ and this follows that $P \in D_m(f)$. Now $T = P \cap C_c(X) = \varphi(P) \in \varphi(D_m(f))$, i.e., $D_{cm}(g) \subseteq \varphi(D_m(f))$ and we are done. $\square$

The following proposition and its Corollary 2.10 are immediate consequence of Lemma 5.7 and Corollary 5.8 in [5] which are counterparts of the fact that $Min(C(X)) = \{O^p : p \in \beta X\}$, where X is a completely regular Hausdorff F-space.

Proposition 2.9. *Let X be a zero-dimensional F_c-space, then*

$$Min(C_c(X)) = \{O_c^p : p \in \beta_0 X\}.$$

Corollary 2.10. *Let X be a compact F_c-space, then*

$$Min(C_c(X)) = \{O_{cp} : p \in X\}.$$

From [17], a space X is countably basically disconnected or briefly c-basically disconnected if for each $f \in C_c(X)$, $\mathrm{int}_X Z(f)$ is closed. Every basically disconnected space is c-basically disconnected. The converse is not necessarily true, for example the space of real numbers with usual topology is a c-basically disconnected space which is not basically disconnected. In view of [17] every c-basically disconnected space is an F_c-space. Furthermore, if X is a c-basically disconnected and strongly zero-dimensional space, then X is basically disconnected. Moreover, if X is a zero-dimensional space, then X is c-basically disconnected if and only if $\beta_0 X$ is c-basically disconnected, see [17].

Remark 2.11. Let X be a zero-dimensional space. If $P \in Min(C_c(X))$, then there exists a unique $p \in \beta_0 X$ for which $O_c^p \subseteq P \subseteq M_c^p$, see [5, Lemma 4.11]. Thus, the mapping φ from $Min(C_c(X))$ into $\beta_0 X$ defined by $\varphi(P) = p$ is well-defined.

The next theorem is the counterpart of [13, Theorem 5.3].

Theorem 2.12. *Let X be a zero-dimensional space and φ be a mapping from $Min(C_c(X))$ into $\beta_0 X$ by $\varphi(P) = p$, where P is contained in M_c^p. Then the following statements hold.*

(1) φ is a continuous mapping of $Min(C_c(X))$ onto $\beta_0 X$.

(2) φ is one-to-one if and only if O_c^p is a prime ideal for each $p \in \beta_0 X$.

(3) φ is a homeomorphism if and only if X is c-basically disconnected.

(4) $Min(C_c(X))$ is compact and X is an F_c-space if and only if X is a c-basically disconnected space.

Proof. (1) Since $\beta_0 X$ is zero-dimensional, it has a base $\mathcal{B}$ consisting clopen sets. Let $V \in \mathcal{B}$, and define $F : \beta_0 X \longrightarrow \mathbb{R}$, such that $F(\beta_0 X \setminus V) = 0$, $F(V) = 1$. Put $f = F|_X$. Clearly, $F \in C_c(\beta_0 X)$, and therefore $f \in C_c(X)$. We show that $\varphi^{-1}(V) = D_{cm}(f)$. To see this let $P \in \varphi^{-1}(V)$, so $\varphi(P) = p \in V$. Since $V \cap Z(f) = \emptyset$, we infer that $p \notin \mathrm{cl}_{\beta_0 X} Z(f)$, so $p \notin M_c^p$. Therefore $p \notin P$, i.e., $f \notin P$ which implies that $P \in D_{cm}(f)$. Hence $\varphi^{-1}(V) \subseteq D_{cm}(f)$. Conversely, let $Q \in D_{cm}(f)$, hence $f \notin Q$. On the other hand we have $O_c^q \subseteq Q \subseteq M_c^q$, for some $q \in \beta_0 X$. Hence $f \notin O_c^q$, i.e., $q \notin \mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X} Z(f)$. It is easy to see that $\beta_0 X \setminus V \subseteq \mathrm{cl}_{\beta_0 X} Z(f)$ and consequently, $\beta_0 X \setminus V \subseteq \mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X} Z(f)$. Therefore $q \notin \beta_0 X \setminus V$, i.e., $q \in V$. But we have $\varphi(Q) = q$, so $Q \in \varphi^{-1}(V)$; therefore $D_{cm}(f) \subseteq \varphi^{-1}(V)$. Hence φ is continuous. To see this φ is onto, let

$P \in Min(C_c(X))$. Clearly, M_c^p contains a minimal prime ideal $P \in Min(C_c(X))$. Hence $\varphi(P) = p$, and we are done.

(2) Let φ be one-to-one and $p \in \beta_0 X$. If P_1, P_2 are two minimal prime ideals in $C_c(X)$ containing O_c^p, then $\varphi(P_1) = \varphi(P_2) = p$. Therefore $P_1 = P_2$. Now, since O_c^p is the intersection of all minimal prime ideals of $C_c(X)$ containing O_c^p, we infer that O_c^p is prime. Conversely, if O_c^p is prime for each $p \in \beta_0 X$, then X is an F_c-space and by Proposition 2.9 we have $Min(C_c(X)) = \{O_c^p | p \in \beta_0 X\}$, and this shows that φ is one-to-one.

(3) Let φ be a homeomorphism. It is sufficient to show that $\beta_0 X$ is c-basically disconnected. To see this let $F \in C_c(\beta_0 X)$ and $f = F|_X$. Since φ is one-to-one, by part (2) and Proposition 2.9, we have $Min(C_c(X)) = \{O_c^p | p \in \beta_0 X\}$, and $\varphi(V_{cm}(f)) = \{\varphi(P) | P \in V_{cm}(f)\} = \{p \in \beta_0 X | f \in O_c^p\} = \mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X} Z(f)$. But by [17, Lemma 2.3] we have $\mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X} Z(f) = \mathrm{int}_{\beta_0 X} Z(F)$. Hence $\varphi(V_{cm}(f)) = \mathrm{int}_{\beta_0 X} Z(F)$. Since $V_{cm}(f)$ is closed in $Min(C_c(X))$ and φ is homeomorphism, we infer that $\mathrm{int}_{\beta_0 X} Z(F)$ is closed in $\beta_0 X$, which means that $\beta_0 X$ is c-basically disconnected. Conversely, by part (1) φ is continuous and since X is c-basically disconnected, it is an F_c-space and hence φ is one-to-one, by part (2). Now let $f \in C_c(X)$. By [17, Lemma 2.3], there exists $F \in C_c(\beta_0 X)$ such that $\mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X} Z(f) = \mathrm{int}_{\beta_0 X} Z(F)$, hence $\varphi(V_{cm}(f)) = \mathrm{int}_{\beta_0 X} Z(F)$. Since X is c-basically disconnected, we infer that $\beta_0 X$ is so, and consequently $\varphi(V_{cm}(f)) = \mathrm{int}_{\beta_0 X} Z(F)$ is closed in $\beta_0 X$. This means that φ is closed and it is a homeomorphism.

(4) Let $Min(C_c(X))$ be compact and X be an F_c-space. Then by (2) φ is one-to-one. Hence φ is a continuous one-one mapping from the compact space $Min(C_c(X))$ onto $\beta_0 X$. Therefore φ^{-1} is continuous. This means that φ is a homeomorphism and by (3), X is a c-basically disconnected space. Conversely, if X is c-basically disconnected space, then X is an F_c-space and φ is a homeomorphism by (3). This implies that $Min(C_c(X))$ is compact, for $\beta_0 X$ is compact. □

Using Theorem 2.12, if we apply the proof of [13, Corolary 5.5] word for word, we obtain the following result.

Corollary 2.13. *Let X be a zero-dimensional space. Then the space $Min(C_c(X))$ is compact if and only if X is c.c.c. space (a space X such that for each $f \in C_c(X)$, there is $g \in C_c(X)$ with $Z(f) \cup Z(g) = X$ and $int_X Z(f) \cap int_X Z(g) = \emptyset$).*

Example 2.14. The space $Min(C_c(X))$ is not always compact. For instance using Theorem 2.12 and [12, 6M], the space $Min(C_c(\beta\mathbb{N} \setminus \mathbb{N}))$ is not compact, while

$Min(C_c(\beta\mathbb{N}))$ is a compact space.

3 Spaces X for which $Min(C_c(X))$ is compact

We recall that a proper ideal I in a ring R is a z°-ideal if for each $a \in I$, we have $P_a \subseteq I$ in which $P_a = \bigcap\{P : P \in V(a)\}$. Furthermore, whenever a is a zero divisor, then P_a is a proper z°-ideal that is called a basic z°-ideal. We denote P^c_f as a basic z°_c-ideal in $C_c(X)$ for each $f \in C_c(X)$. For more details of z°-ideals in $C(X)$, see [3, 4].

Definition 3.1. A proper ideal I in $C_c(X)$ is called a z°_c-ideal if for each $f \in I$, we have $P^c_f \subseteq I$.

Theorem 3.2. *Let X be a zero-dimensional almost CP-space. The following statements are equivalent.*

(1) *X is c-basically disconnected.*

(2) *X is a CP-space.*

(3) *$q_c(X)$ is a* von-Neumann *regular ring.*

(4) *$Min(C_c(X))$ is a compact space.*

(5) *Every prime z°_c-ideal in $C_c(X)$ is a minimal prime ideal.*

(6) *Every prime z_c-ideal in $C_c(X)$ is a minimal prime ideal.*

Proof. (1) $\Longrightarrow$ (2) The proof is evident.
(2)$\Longrightarrow$ (3) Since X is a CP-space, then $C_c(X)$ is a von-Neumann regular ring, so $q_c(X)$ is a von-Neumann regular ring.
(3)$\Longrightarrow$ (4) By [15, Theorem B] , $Min(C_c(X))$ is compact.
(4) $\Longrightarrow$ (1) By Corollary 2.13, for each $f \in C_c(X)$, there exists $g \in C_c(X)$ such that $Z(f) \cup Z(g) = X$ and $\mathrm{int}_X(Z(f) \cap Z(g)) = \emptyset$. Since X is an almost CP-space, we have $Z(f) \cap Z(g) = \emptyset$. Consequently, $Z(f) = X \setminus Z(g)$ is open and so $\mathrm{int}_X Z(f) = Z(f)$ is closed, i.e., X is c-basically disconnected.
(3) $\Longleftrightarrow$ (5) It is clear by [4, Proposition 1.26].
(3) $\Longleftrightarrow$ (6) It is evident by [4, Proposition 1.26]. □

Proposition 3.3. *Let X be a zero-dimensional F_c-space. The following statements are equivalent.*

(1) $Min(C_c(X))$ *is a compact space.*

(2) X *is* c*-basically disconnected.*

(3) $Min(C_c(X))$ *and* $\beta_0 X$ *are homeomorphic.*

(4) $q_c(X)$ *is a* von-Neumann *regular ring.*

(5) *Every prime* z_c°*-ideal in* $C_c(X)$ *is a minimal prime ideal.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) It is valid by using Theorem 2.12.
(1)$\Longleftrightarrow$ (4) By [15, Theorem B].
(4) $\Longleftrightarrow$(5) Using [4, Proposition 1.26] it is evident. □

By [14], we recall that X is a cozero complemented space or briefly $c.c$-space if for each $f \in C(X)$, there exists $g \in C(X)$ such that the union of their cozero-sets is dense and intersection of their cozero-sets is empty. It is shown in [13] that X is a $c.c$-space if and only if $Min(C(X))$ is a compact space. X is called a perfectly normal space if X is a normal space and every closed subset of X is a G_δ-set (equivalently, every closed subset is a zero-set). Also, X is a perfectly normal space if and only if for every disjoint closed sets A and B in X, there exists $f \in C(X)$ such that $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$, see [8].

Definition 3.4. A subset $\mathcal{A}$ of a space X is called a CP-set (resp. almost CP-set) of X if whenever $\mathcal{A} \subseteq Z(f)$ for some $f \in C_c(X)$, then $\mathcal{A} \subseteq int_X Z(f)$ (resp. $\mathcal{A} \subseteq cl_X int_X Z(f)$). If $\{p\}$ is a CP-set (resp. almost CP-set), then p is called a CP-point (resp. almost CP-point). Similar to [12, 4L], if $M_{cp} = O_{cp}$ for $p \in X$, then p is a CP-point of X. Thus, X is a CP-space if and only if every point is a CP-point. Also, if every point of X is an almost CP-point, then X is an almost CP-space.

Lemma 3.5. *If* X *is a* $c.c.c$*-space and* p *is an almost* CP*-point, then* p *is a* CP*-point of* X*.*

Proof. Let $p \in Z(f)$ for some $f \in C_c(X)$. Since X is a $c.c.c$-space, there exists $g \in C_c(X)$ such that $fg = 0$ and $\mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g) = \emptyset$. Clearly, $coz(g) \subseteq \mathrm{int}_X Z(f)$. We claim that $p \in coz(g)$. If not, then $p \in Z(g)$. Thus, $p \in Z(f^2 + g^2)$, that is $Z(f^2 + g^2) \neq \emptyset$. Since p is an almost CP-point, we infer that $\mathrm{int}_X Z(f^2 + g^2) \neq \emptyset$ which is a contradiction. □

The following lemma is the counterpart of [2, Proposition 2.8].

Lemma 3.6. *A space X is a CP-space if and only if X is both almost CP-space and $c.c.c$-space.*

Proof. If X is a CP-space, then $Z(f)$=$\text{int}_X Z(f)$. Also, if $Z(f) \neq \emptyset$, then $\text{int}_X Z(f) \neq \emptyset$, i.e., X is an almost CP-space. Furthermore, for each $f \in C_c(X)$ we have $Z(f) = Z(f^{\frac{1}{3}})$, so there exists $g \in C_c(X)$ such that $f^{\frac{1}{3}} = f.g$, so $f^{\frac{1}{3}}(1 - f^{\frac{2}{3}}.g) = 0$. We set $1 - f^{\frac{2}{3}}.g = h \in C_c(X)$, then $Z(f^{\frac{1}{3}}) \cup Z(h) = X$. Thus, $Z(f) \cup Z(h) = X$ and $Z(f) \cap Z(h) = \emptyset$, for which $h \in C_c(X)$. Consequently, $\text{int}_X Z(f) \cap \text{int}_X Z(g) = \emptyset$, so X is a $c.c.c$-space. The converse is clear by applying Lemma 3.5. □

Definition 3.7. A space X is a countably perfectly normal space or c-perfectly normal space if for every disjoint closed sets A and B in X, there exists $f \in C_c(X)$ such that $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$. Clearly, if X is a c-perfectly normal space, then every closed subset of X is a zero-set, whence each open subset of X is a cozero-set.

By [13], if X is a metric space, then $Min(C(X))$ is compact and extremally disconnected. Also, if a ring R satisfies c.a.c. and $Min(R)$ is locally compact, then $Min(R)$ is basically disconnected.
The next result is the counterpart of [13, Theorem 5.6].

Theorem 3.8. *The following statements hold.*

(1) *If for each $f \in C_c(X)$, $\mathrm{S}(f)$ is a zero-set, then $Min(C_c(X))$ is compact and basically disconnected.*
(2) *If X is a c-perfectly normal space, then $Min(C_c(X))$ is compact and extremally disconnected.*

Remark 3.9. Whenever every closed set in a space X is a zero-set in $Z_c(X)$, then X is a c-perfectly normal space. Also, every c-perfectly normal space is a perfectly normal space. The converse is not necessarily true. The space $\mathbb{R}$ of real numbers is perfectly normal but not c-perfectly normal. Obviously, perfectly normal spaces and c-perfectly normal spaces coincide when X is strongly zero-dimensional.

Lemma 3.10. *For each $f, g \in C_c(X)$, the following statements are valid.*

(1) $V_{cm}(Ann_c(f)) \subseteq V_{cm}(g)$ *if and only if* $Z(f) \cup Z(g) = X$
(2) $V_{cm}(g) \subseteq V_{cm}(Ann_c(f))$ *if and only if* $\text{int}_X(Z(f) \cap Z(g)) = \emptyset$

(3) $Ann_c(Ann_c(f)) = Ann_c(g)$ *if and only if* $V_{cm}(f) = V_{cm}(Ann_c(g))$

(4) $V_{cm}(f) = V_{cm}(Ann_c(Ann_c(f)))$

Proof. By [13, Lemmas 3.1 and 5.4] the proof is evident. $\square$

The proof of the next theorem is clear, see Lemma 3.10, Corollary 2.13 and [13, Theorem 3.4].

Theorem 3.11. *The following statements for a space X are equivalent.*

(1) $Min(C_c(X))$ *is a compact space.*

(2) *For each* $f \in C_c(X)$*, there exists* $g \in C_c(X)$ *such that* $Ann_c(Ann_c(f)) = Ann_c(g)$.

(3) *For each* $f \in C_c(X)$*, there exists* $g \in C_c(X)$ *such that* $V_{cm}(f) = V_{cm}(Ann_c(g))$.

(4) *For each* $f \in C_c(X)$*, there exists* $g \in C_c(X)$ *such that* $\mathrm{S}(f) \cup \mathrm{S}(g) = X$ *and* $\mathrm{int}_X(\mathrm{S}(f) \cap \mathrm{S}(g)) = \emptyset$.

(5) *X is a* $c.c.c$*-space.*

Recall that X is a realcompact space if every real maximal ideal in $C(X)$ is fixed, so $\mathbb{R}$ and all subspaces are realcompact, see [12, Chapter 8]. Also, by [9, Lemma 3.1], if X is locally compact and realcompact, then $\beta X \setminus X$ is an almost P-space. Furthermore, by [14, 1.6 (e)], X is P-space if and only if X is both almost P-space and $c.c$-space.

Example 3.12. In the following examples, we may have a space that is neither a $c.c$-space nor a $c.c.c$-space. Also, a $c.c$-space is not necessarily a $c.c.c$-space. Furthermore, a $c.c.c$-space may not be a $c.c$-space. Note that in the strongly zero-dimensional space, two concepts of $c.c$-space and $c.c.c$-space coincide.

(1) The space $X = \beta\mathbb{N} \setminus \mathbb{N}$ is a strongly zero-dimensional and almost CP-space which is not a CP-space, then X is neither a $c.c.c$-space nor a $c.c-$space.

(2) Let $X = [-1, 0] \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Clearly X is a $c.c$-space. We show that X is not a $c.c.c$-space. To see this, we define $f : X \longrightarrow \mathbb{R}$ with:

$$f(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ \\ \dfrac{1}{n} & x = \dfrac{1}{n} \end{cases}$$

Obviously, $f \in C_c(X)$. If there exists $g \in C_c(X)$ such that $fg = 0$ and $\mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g) = \emptyset$, then for $r \neq 0$, g must be

$$g(x) = \begin{cases} r & -1 \leq x \leq 0 \\ \\ 0 & x = \dfrac{1}{n} \end{cases}$$

The image of g is countable. Since $\lim\limits_{x \longrightarrow 0^-} g(x) = r$, $\lim\limits_{x \longrightarrow 0^+} g(x) = 0$, then g is not continuous at zero, so $g \notin C_c(X)$. Consequently, X is not a $c.c.c$-space.
(3) The space $X = \beta\mathbb{R} \setminus \mathbb{R}$ is a $c.c.c$-space but not $c.c$-space. Clearly, $X = (\beta\mathbb{R}^+ \setminus \mathbb{R}^+) \cup (\beta\mathbb{R}^- \setminus \mathbb{R}^-)$ in which $\beta\mathbb{R}^+ = \mathrm{cl}_{\beta\mathbb{R}}\mathbb{R}^+$ and $\beta\mathbb{R}^- = \mathrm{cl}_{\beta\mathbb{R}}\mathbb{R}^-$. Now we set $Y = \beta\mathbb{R}^+ \setminus \mathbb{R}^+$ and $Z = \beta\mathbb{R}^- \setminus \mathbb{R}^-$. The spaces Y and Z are connected and compact, see [12, 6.10], so they are disjoint closed sets. Let $f \in C_c(X)$ and suppose that $f|_Y = r$, $f|_Z = s$. The cases $r = s = 0$ and $r, s \neq 0$ are clear. Hence suppose that $r = 0$ and $s \neq 0$. Now, we set $g : X \longrightarrow \mathbb{R}$ with:

$$g(x) = \begin{cases} 1 & x \in Y \\ \\ 0 & x \in Z \end{cases}$$

So, $g \in C_c(X)$. Since $fg = 0$ and $\mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g) = Y \cap Z = \emptyset$, then X is a $c.c.c$-space. Also, X is an almost P-space which is not a P-space. Thus, X is not a $c.c$-space. Similarly, if $X = \beta\mathbb{R}^+ \setminus \mathbb{R}^+$, then X is a connected space, see [12, 6.10], which is not a P-space. Thus, X is a $c.c.c$-space which is not a $c.c$-space.

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