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Quantum determinants in ribbon category

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Abstract. The aim of this paper is to introduce an abstract notion of determinant which we call quantum determinant, verifying the properties of the classical one. We introduce $\mathcal{R}-$basis and $\mathcal{R}-$solution on rigid objects of a monoidal $Ab-$category, for a compatibility relation $\mathcal{R}$, such that we require the notion of duality introduced by Joyal and Street, the notion given by Yetter and Freyd and the classical one, then we show that $\mathcal{R}-$solutions over a semisimple ribbon $Ab-$category form as well a semisimple ribbon $Ab-$category. This allows us to define a concept of so-called quantum determinant in ribbon category. Moreover, we establish relations between these and the classical determinants. Some properties of the quantum determinants are exhibited.

1 Introduction

The theory of *monoidal* categories was studied and developed by many authors [1, 7], see also [8, 9]. In particular *duality* in such categories introduced by Joyal and Street [8], (see also [2, 11]) as well as the concept of *braiding* -as a weaker version of commutativity- which came along firstly with Joyal and Street [8]. The notion of *determinant* dates a long time as an essential tool in linear algebra. Since then,

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many versions and analogs were introduced and developed in the setting of square matrices of commutative entries as well as for non commutative entries (among widely used ones: q−determinants, Dieudonné determinant, quasideterminants...). It is well known that the notion of *trace* has been generalized to the context of categories (with tensor product and duality) [3–6, 13]. In particular, every *ribbon* category [10] or called *tortile tensor* category (in [15]), admits a canonical notion of (*quantum*) *trace* (well behaved: cyclicity and multiplicativity) and dimension [10], in the way that it generalizes the classical one of vector spaces in linear algebra. These traces are used to construct quantum invariants of links and 3−manifolds. Motivated by that, this paper introduces an abstract notion of determinant which we call "*quantum determinant*", verifying the properties of the classical one. The name quantum here is justified by the fact that this element uses the quantum trace; in fact it is nothing but the quantum trace of the endomorphism $f^n \Lambda_A^n$ (Proposition 5.1).
We begin with the introduction of a concept of $\mathcal{R}-$basis on an object V of a monoidal $Ab-$category C, for a (compatible) congruence relation $\mathcal{R}$, as a family of morphisms

$$d_V^i : V^* \otimes V \longrightarrow I \ , \ \ b_V^i : I \longrightarrow V \otimes V^* \ \ and \ \ \pi_V^i : V \longrightarrow V \ ; \ \forall i \in J$$

for a finite index set J, such as they verify some axioms. We prove that it coincides with the usual basis when we consider the category of finite dimensional vector spaces over a certain field. The existence of such $\mathcal{R}-$basis in the context of *semisimple ribbon* $Ab-$categories, is ensured. In fact, we show that semisimplicity gives rise to an $\mathcal{R}-$basis on every object of the category. Moreover, we define a notion of $\mathcal{R}-$*solution* on an object V as a quadruple $(V; d_V; b_V; \pi_V)$ obeing to some axioms.
Finally, we introduce the notion of quantum determinant in a semisimple ribbon $Ab-$category and show that its formula is independent of the choice of the $\mathcal{R}-$*solution* on the object. We prove that in fact, the quantum determinant of an endomorphism $f \in End_{C/\mathcal{R}}(V)$ coincides with the classical determinant of an associated square matrix M_f over the ground commutative ring $End_C(I)$ of C, denoted by $\mathbb{K}_C$ and that under some conditions, there is a bijective correspondence between the $\mathbb{K}_C-$algebras $End_C(V)$ and $M_n(\mathbb{K}_C)$ of square matrices over $\mathbb{K}_C$. In this case, $f \in End_{C/\mathcal{R}}(V)$ is an automorphism, if and only if, its quantum determinant is invertible in $\mathbb{K}_C$.

2 Preliminaries

Throughout this paper, $\mathbb{K}$ states for a base field with unit and C for a *strict* monoidal category $(C;\otimes;I)$ with unit object I.
We recall some notions from the theory of monoidal categories. For more details, we refer to [12] and [17].
A *monoidal category* $C = (C;\otimes;I;\alpha;l;r)$ consists of a category C, a bifunctor $\otimes : C \times C \longrightarrow C$, a unit object I and natural isomorphisms $\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, $l : I \otimes A \longrightarrow A$ and $r : A \otimes I \longrightarrow A$ called associativity constraint, left and right unitality constraints respectively such that the pentagon and triangle axioms hold.
C is called strict if all components α, l and r are identities.
A result due to Mac-Lane's (see [12]) coherence Theorem, asserts that any monoidal category is necessarily equivalent to a strict one.
C is called an $Ab-$category provided that the hom sets $Hom_C(U,V)$ are additive abelian groups and the composition map $Hom_C(U,V) \times Hom_C(V,W) \longrightarrow Hom_C(U,W)$, $(f,g) \mapsto g \circ f$ is bilinear.
A *braiding* (firstly introduced in [8]) for a monoidal category C consists of a family of natural isomorphisms

$$c_{V;W} : V \otimes W \longrightarrow W \otimes V$$

for all V and W in C, such that for any three objects U, V and W we have

$$c_{U;V\otimes W} = (id_V \otimes c_{U;W})(c_{U;V} \otimes id_W)$$

$$c_{U\otimes V;W} = (c_{U;W} \otimes id_V)(id_U \otimes c_{V;W}).$$

For a monoidal category C with a braiding c; a twist (see [7]) consists of a family of isomorphisms

$$\theta_V : V \longrightarrow V, \quad V \in Ob(C)$$

such that $\theta_I = id_I$ and for any two objects V and W of C we have

$$\theta_{X\otimes Y} = c_{Y;X}(\theta_Y \otimes \theta_X)c_{X;Y}$$

The naturality of the twist θ means that for any morphism $f : U \longrightarrow V$ of C, we have $\theta_V f = f\theta_U$.

Let $(C;\otimes;I)$ be a strict monoidal category with tensor product $\otimes$ and unit I. It is a monoidal category with *left duality* if for each object V of C there exists an object V^* and morphisms $b_V : I \longrightarrow V\otimes V^*$ and $d_V : V^*\otimes V \longrightarrow I$ in C such that

$$(id_V\otimes d_V)(b_V\otimes id_V) = id_V \;\; and \;\; (d_V\otimes id_{V^*})(id_{V^*}\otimes b_V) = id_{V^*}.$$

For any morphism $f : U \longrightarrow V$, we define its *dual* morphism $f^* : V^* \longrightarrow U^*$ by

$$f^* = (d_V\otimes id_{U^*})(id_{V^*}\otimes f\otimes id_{U^*})(id_{V^*}\otimes b_U)$$

and the morphism $\lambda_{U;V} : V^*\otimes U^* \longrightarrow (U\otimes V)^*$ (see [10, page 344], for more details) defined by

$$\lambda_{U;V} = (d_V\otimes id_{(U\otimes V)^*})(id_{V^*}\otimes d_U\otimes id_{V\otimes(U\otimes V)^*})(id_{V^*\otimes U^*}\otimes b_{U\otimes V}) \qquad (2.1)$$

is an isomorphism for any two objects U and V of C.
We say that duality is compatible with the braiding c and the twist θ if for any object V of C we have

$$\theta_{V^*} = (\theta_V)^*.$$

In this case, the double dual $V^{**} := (V^*)^*$ of an object V is canonically isomorphic to V.
A *ribbon category* is a monoidal category C equipped with a twist θ, a braiding c and a compatible duality $(*;b;d)$.
Let C be a ribbon category with unit I. For any endomorphism f of an object V of C, we define the *quantum trace* $Tr_q(f)$ of f as the element

$$Tr_q(f) = d_V c_{V;V^*}(\theta_V f\otimes id_{V^*})b_V \in \mathbb{K}_C$$

When C is the category $vect_{\mathbb{K}}$ of finite dimensional vector spaces over a field $\mathbb{K}$, this concept of quantum trace coincides with the usual one.
We collect in following Theorem the principal properties of the quantum trace.

Theorem 2.1. ([10]) *For any morphisms f and g in a ribbon category, we have*

(a) $Tr_q(fg) = Tr_q(gf)$;

(b) $Tr_q(f\otimes g) = Tr_q(f)Tr_q(g)$;

(c) $Tr_q(f^*) = Tr_q(f)$ *and*

(d) $Tr_q(k) = k$ *for any* $k\in\mathbb{K}_C$.

Proof. See [10] for proof, where string diagrams are used to simplify the proof and make the passages more obvious. □

For any object V of C, the *quantum dimension* $dim_q(V)$ of V is the element defined by $dim_q(V) = Tr_q(id_V)$ and we have

$$dim_q(V \otimes W) = dim_q(V)dim_q(W) \;\; and \;\; dim_q(V^*) = dim_q(V).$$

Let C be a ribbon category and V an object of C (for the following setup, we mainly follow the terminology adopted in [17]).
V is called *simple* provided that the map

$$\mathcal{K} : \mathbb{K}_C \longrightarrow End_C(V); \; k \longmapsto k \otimes id_V$$

is a bijection and $dim_q(V)$ is invertible in $\mathbb{K}_C$.
An object V of a ribbon $Ab-$category C is said to be *dominated* by n simple objects $\{V_i\}_i$ of C, if there exists a finite family of morphisms $\{\varepsilon^i_V : V \longrightarrow V_i \; ; \; \mu^i_V : V_i \longrightarrow V\}_{1\leq i\leq n}$ such that the endomorphism

$$\sum_i \mu^i_V \varepsilon^i_V - id_V$$

of V, is *negligible* as defined below.
The set of negligible morphisms between two objects U and V is denoted $Negl(U;V)$ and it is defined as

$$Negl(U;V) := \{f \in Hom_C(U,V) \mid \forall g \in Hom_C(V,U),\; Tr_q(fg) = 0\}.$$

Obviously $Negl(I;I) = \{0\}$.
We call a ribbon $Ab-$category *semisimple* provided that every object is dominated by a finite set of simple ones.
Recall from [12, page 52], that a relation $\mathcal{R}$, is a *congruence* on a category C if for any objects X and Y of C, $\mathcal{R}_{X,Y}$ is an equivalence relation on the hom set $Hom_C(X,Y)$ and for all $f, g : X \longrightarrow Y$ such that $f \; \mathcal{R}_{X,Y} \; g$, we have $(vfu)\mathcal{R}_{A,B}(vgu)$, for any morphisms $u : A \rightarrow X$ and $v : Y \rightarrow B$ of C.

3 Notion of $\mathcal{R}-$basis

Definition 3.1. We call compatibility relation on a monoidal category C, any congruence relation $\mathcal{R}$ on C verifying the following

(i) For any morphisms f and g between dualizable objects U and V of C, such that $f\ \mathcal{R}_{U,V}\ g$, we have $f^*\ \mathcal{R}_{V^*,U^*}\ g^*$.

(ii) For any objects U, V, A and B of C and any morphisms $f, g : U \longrightarrow V$ and $f', g' : A \longrightarrow B$ such that $f\ \mathcal{R}_{U,V}\ g$ and $f'\ \mathcal{R}_{A,B}\ g'$, then $(f \otimes f')\ \mathcal{R}_{U\otimes A,V\otimes B}\ (g \otimes g')$.

Lemma 3.2. *Let C be a ribbon $Ab-$category. The relation $\mathcal{R}$ defined on hom sets by*

$$\forall U, V \in Ob(C),\ \forall f, g : U \longrightarrow V;\ f\ \mathcal{R}_{U,V}\ g\ \Leftrightarrow\ f - g\ \in Negl(U, V) \tag{3.1}$$

is a compatibility relation on C.

Proof. $\mathcal{R}_{U,V}$ is clearly an equivalence relation on each hom set $Hom_C(U, V)$. The axioms of Definition 3.1 hold by the fact that the dual of a negligible morphism is negligible and the tensor product of negligible morphisms is again negligible [17]. □

Remark 3.3. (a) Let C be a ribbon $Ab-$category and U and V two objects of C. Then, $Negl(U; V)$ is an ideal in C and the class modulo $\mathcal{R}_{U,V}$ of the zero arrow is the ideal $Negl(U; V)$. Recall [16] that a set R_X of arrows to an object X of C, is called a right $X-$ideal if for all $f, g : B \longrightarrow X$ in R_X, for all $v : A \longrightarrow B$, $A, B \in Ob(C)$, one has $(f + g)v$ is in R_X. Left $X-$ideal is defined similarly. A set of arrows from an object X to an object Y of C is called an ideal if it is both a right $X-$ideal and a left $Y-$ideal.

(b) Let C be a ribbon $Ab-$category and $f \in Hom_C(X, Y)$. Consider the sets

$$R_X := \{g : A \longrightarrow X,\ fg \in Negl(A; Y)\}_{A\in Ob(C)};$$

$$R_Y := \{g : Y \longrightarrow B,\ gf \in Negl(X; B)\}_{B\in Ob(C)}.$$

Then, R_X is a right $X-$ideal and R_Y is a left $Y-$ideal.

(c) Let C be a ribbon $Ab-$category. A set of arrows R is an ideal in C, if and only if, the set

$$R^* := \{f^*,\ f \in R\}$$

is an ideal in the category $\overline{C}$ defined by $Ob(\overline{C}) := \{X^*, X \in Ob(C)\}$ and $Mor(\overline{C}) := \{f^*, f \in Mor(C)\}$.

From now on, $\mathcal{R}$ will denote always the above compatibility relation (3.1), whenever C is considered as a ribbon $Ab-$category.

Definition 3.4. Let C be a monoidal $Ab-$category, equipped with a compatibility relation $\mathcal{R}$ and V a dualizable object of C with duality structures $(V^*; d_V; b_V)$. An $n-\mathcal{R}-$basis $(V; d_V^i; b_V^i; \pi_V^i)_{1\leq i\leq n}$ on V, is a family of morphisms

$$d_V^i : V^* \otimes V \longrightarrow I,\ b_V^i : I \longrightarrow V \otimes V^* \ and\ \pi_V^i : V \longrightarrow V$$

such that for all $1 \leq i, j \leq n$, the following hold

$$(id_V \otimes d_V^i)(b_V^j \otimes id_V) = \pi_V^j \pi_V^i;$$

$$(d_V^i \otimes id_{V^*})(id_{V^*} \otimes b_V^j) = (\pi_V^j \pi_V^i)^*$$

$$and \quad \sum_{i=1}^{i=n} \pi_V^i = 1_V\ mod(\mathcal{R}_{V,V}).$$

Remark 3.5. Let V be a dualizable object of C with dual V^* and consider an $n-\mathcal{R}-$basis $(V; d_V^i; b_V^i; \pi_V^i)_{1\leq i\leq n}$ on V.

(a) We know that the dual object in a monoidal category is unique up to a unique isomorphism [18, page 23]. Let $V^\vee$ be another dual of V and $f : V^* \longrightarrow V^\vee$ be the unique isomorphism between the duals. Then, it is not difficult to verify that
$$(V; d_V^i(f \otimes id_V); (f^{-1} \otimes id_V)b_V^i; \pi_V^i)_{1\leq i\leq n}$$
is another $n-\mathcal{R}-$basis on V.

(b) Every sub family of an $n-\mathcal{R}-$basis is again an $m-\mathcal{R}-$basis with $m \leq n$.

Note that this notion generalizes the standard notion of basis for vector spaces over a field $\mathbb{K}$.

Example 3.6. Every object V of the category $(vect_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ of finite dimensional vector spaces over a field $\mathbb{K}$, admits an $n-\mathcal{R}-$basis where n is its dimension and $\mathcal{R}$ is any compatibility relation.

Proof. Consider the family:

$$d^l_V: \begin{array}{ccc} V^*\otimes V & \longrightarrow & \mathbb{K} \\ e^j\otimes e_i & \longmapsto & \delta_{lj}\delta_{li} \end{array}, \quad b^l_V: \begin{array}{ccc} \mathbb{K} & \longrightarrow & V\otimes V^* \\ 1 & \longmapsto & e_l\otimes e^l \end{array} \quad and \quad \pi^l_V: \begin{array}{ccc} V & \longrightarrow & V \\ e_i & \longmapsto & \delta_{li}e_i \end{array}$$

for all $1 \leq l \leq n$, where $\{e_i\}_{1\leq i\leq n}$ and $\{e^i\}_{1\leq i\leq n}$ are respectively a basis and its dual basis of V and its dual V^*. Then, the family $(d^l_V; b^l_V; \pi^l_V)_{1\leq l\leq n}$ defines an $n-\mathcal{R}-$basis on V. □

Example 3.7. Let C be a monoidal $Ab-$category equipped with a compatibility relation $\mathcal{R}$ and V a dualizable object of C with duality structures denoted $(V^*; d_V; b_V)$. Then, $(V; d_V; b_V; 1_V)$ is a $1-\mathcal{R}-$basis on V.

Proposition 3.8. *Any semisimple ribbon $Ab-$category C admits an $\mathcal{R}-$basis on each of its objects.*

Proof. An object V of a semisimple ribbon $Ab-$category is dominated by a finite family $(V_i)_{1\leq i\leq n}$ of simple objects of C, i.e, there exists a family of morphisms $\{\varepsilon^i_V : V \longrightarrow V_i\ ;\ \mu^i_V : V_i \longrightarrow V\}_{i=1}^{i=n}$ such that $\sum_i \mu^i_V\varepsilon^i_V - id_V$ is a negligible endomorphism of V. Then

$$(V; d_{V_i}((\mu^i_V)^*\otimes\varepsilon^i_V); (\mu^i_V\otimes(\varepsilon^i_V)^*)b_{V_i}; \mu^i_V\varepsilon^i_V)$$

is an $n-\mathcal{R}-$basis on V. In fact, the following hold

$$(1_V\otimes d_{V_i}((\mu^i_V)^*\otimes\varepsilon^i_V))((\mu^j_V\otimes(\varepsilon^j_V)^*)b_{V_j}\otimes 1_V) = \mu^j_V\varepsilon^j_V\mu^i_V\varepsilon^i_V\ ;$$

$$(d_{V_i}((\mu^i_V)^*\otimes\varepsilon^i_V)\otimes 1_{V^*})(1_{V^*}\otimes(\mu^j_V\otimes(\varepsilon^j_V)^*)b_{V_j}) = (\mu^j_V\varepsilon^j_V\mu^i_V\varepsilon^i_V)^*$$

$$and \quad \sum_i \mu^i_V\varepsilon^i_V = id_V \quad mod(\mathcal{R}_{V,V}).$$

□

Definition 3.9. Let C be a monoidal $Ab-$category, equipped with a compatibility relation $\mathcal{R}$ and let V be a dualizable object of C. Denote by r_V, the minimum cardinal, as explained below, of $\mathcal{R}-$bases $(V; d^i_V; b^i_V; \pi^i_V)_{1\leq i\leq n}$ on V, $n\in\mathbb{N}$; for which, $\pi^i_V \neq 1_V$, for all i, $1\leq i\leq n$.
$r_V = n$ is a minimum cardinal in the sense that:

(i) There exists an $\mathcal{R}-$basis $(V; d_V^i; b_V^i; \pi_V^i)_{1\leq i\leq n}$ on V such that there exist no morhisms d_V^{n+1}, b_V^{n+1} and π_V^{n+1} suth that

$$(V; d_V^i; b_V^i; \pi_V^i)_{1\leq i\leq n} \cup (V; d_V^{n+1}; b_V^{n+1}; \pi_V^{n+1})$$

is an $(n+1)-\mathcal{R}-$basis on V.

(ii) There is no $m-\mathcal{R}-$basis on V verifying (i) such that $m < n$.

Note that the condition on π_V^i is just to avoid the trivial case when C is rigid, where, for any object V of C, $r_V = 1$ by Example 3.7.

The following lemmata will be useful in claiming forthcoming results on the integer r_V introduced in the very definition.

Lemma 3.10. *Let C be a semisimple ribbon $Ab-$category and A and B be isomorphic objects in C. Then*

(a) *A is dominated by n simple objects, if and only if B is;*

(b) $(A; d_A^i; b_A^i; \pi_A^i)_{i=1}^{i=n}$ *is an $\mathcal{R}-$basis on A, if and only if*

$$(B; d_A^i(f^* \otimes g); (f \otimes g^*)b_A^i; f\pi_A^i g)_{i=1}^{i=n}$$

is an $\mathcal{R}-$basis on B.

Proof. (a) Let $f \in Hom_C(A; B)$ be an isomorphism with inverse g.
Assume that A is dominated by $(V_i; \varepsilon_A^i; \mu_A^i)_{i=1}^{i=n}$. Let $\varepsilon_B^i = \varepsilon_A^i g$ and $\mu_B^i = f\mu_A^i$. Then, B is dominated by $(V_i; \varepsilon_B^i; \mu_B^i)_{i=1}^{i=n}$.

Inversely, if B is dominated by $(V_i; \varepsilon_B^i; \mu_B^i)_{i=1}^{i=n}$, one easily checks that A is dominated by $(V_i; \varepsilon_B^i f; g\mu_B^i)_{i=1}^{i=n}$.

(b) Let $(A; d_A^i; b_A^i; \pi_A^i)_{i=1}^{i=n}$ be an $\mathcal{R}-$basis on A. Then

$$(B; d_A^i(f^* \otimes g); (f \otimes g^*)b_A^i; f\pi_A^i g)_{i=1}^{i=n}$$

is an $\mathcal{R}-$basis on B. In fact, we have to prove the following three identities:

$$(id_B \otimes d_A^j(f^* \otimes g))((f \otimes g^*)b_A^i \otimes id_B) = (f\pi_A^j g)(f\pi_A^i g),$$

$$(d_A^j(f^* \otimes g) \otimes id_{B^*})(id_{B^*} \otimes (f \otimes g^*)b_A^i) = (f\pi_A^j g)^*(f\pi_A^i g)^*$$

$$and \quad \sum_{i=1}^{i=n} f\pi_A^i g = 1_B \; mod(\mathcal{R}_{B,B}).$$

We have

$$\begin{aligned}(1_B \otimes d_A^j(f^*\otimes g))((f \otimes g^*)b_A^i \otimes 1_B)\\ &= (1_B \otimes d_A^j)(1_B \otimes f^* \otimes g)(f \otimes g^* \otimes 1_B)(b_A^i \otimes 1_B)\\ &= (1_B \otimes d_A^j)(f \otimes 1_{A^*} \otimes g)(b_A^i \otimes 1_B)\\ &= f(1_A \otimes d_A^j)(b_A^i \otimes 1_A)g\\ &= (f\pi_A^j g)(f\pi_A^i g)\end{aligned}$$

and

$$\begin{aligned}(d_A^j(f^* \otimes g) \otimes id_{B^*})(id_{B^*} \otimes (f \otimes g^*)b_A^i)\\ &= (d_A^j \otimes id_{B^*})(f^* \otimes g \otimes id_{B^*})(id_{B^*} \otimes f \otimes g^*)(id_{B^*} \otimes b_A^i)\\ &= g^*(d_A^j \otimes id_{A^*})(id_{A^*} \otimes b_A^i)f^*\\ &= (f\pi_A^j g)^*(f\pi_A^i g)^*.\end{aligned}$$

The third identity is obvious.
Inversely, if B admits an $\mathcal{R}-$basis on it, then by the same previous procedure interchanging the roles of f and g, we get an $\mathcal{R}-$basis on A. □

Lemma 3.11. *Let C be a semisimple ribbon $Ab-$category and A an object of C. Then*

(a) *A is dominated by n simple objects, if and only if A^* is;*

(b) *$(A; d_A^i; b_A^i; \pi_A^i)_{i=1}^{i=n}$ is an $\mathcal{R}-$basis on A, if and only if*

$$(A^*; (d_A^i)_*; (b_A^i)_*; (\pi_A^i)_*)_{i=1}^{i=n}$$

is an $\mathcal{R}-$basis on A^*.

Proof. (a) Assume that A is dominated by $(V_i; \varepsilon_A^i; \mu_A^i)_{i=1}^{i=n}$. Then A^* is dominated by $(V_i^*; (\mu_A^i)^*; (\varepsilon_A^i)^*)_{i=1}^{i=n}$.
Inversely, this holds due to the fact that $A^{**} \simeq A$ is verified in light of the compatible duality of C.

(b) Let $(A; d^i_A; b^i_A; \pi^i_A)_{i=1}^{i=n}$ be an $\mathcal{R}-$basis on A. Then $(A^*; (d^i_A)_*; (b^i_A)_*; (\pi^i_A)_*)_{i=1}^{i=n}$ is an $\mathcal{R}-$basis on A^*, where

$$(d^i_A)_* := d_I(b^i_A)^*\lambda_{A;A^*}\ ; \quad (b^i_A)_* := \lambda^{-1}_{A;A^*}(d^i_A)^*b_I \quad and \quad (\pi^i_A)_* := (\pi^i_A)^*$$

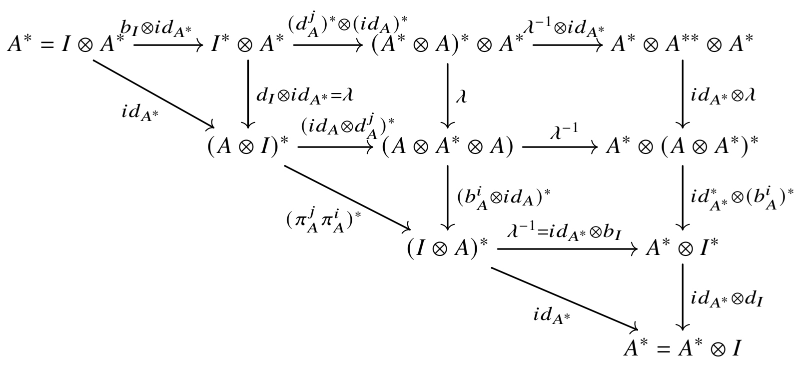
for all $1 \le i \le n$.
Along with the proof and the rest of the paper, by λ we mean $\lambda_{A;A^*}$ and by λ^{-1} we mean $\lambda^{-1}_{A;A^*}$ to reduce notations (where λ is defined as in (2.1)).
In fact, we prove the three identities:

$$(id_{A^*} \otimes d_I(b^i_A)^*\lambda)(\lambda^{-1}(d^j_A)^*b_I \otimes id_{A^*}) = (\pi^j_A)^*(\pi^i_A)^*,$$

$$(d_I(b^i_A)^*\lambda \otimes id_{(A^*)^*})(id_{(A^*)^*} \otimes \lambda^{-1}(d^j_A)^*b_I) = ((\pi^j_A)^*)^*((\pi^i_A)^*)^*$$

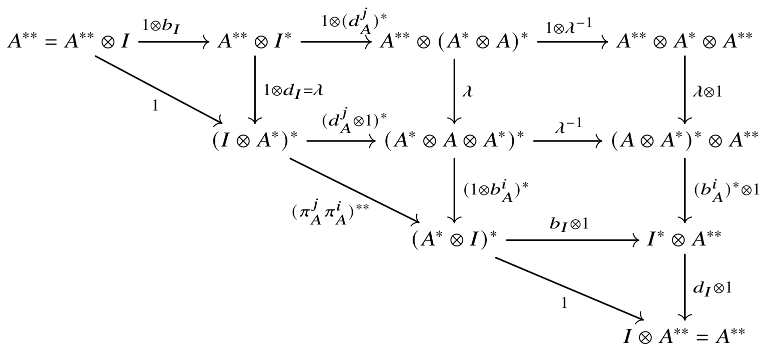
$$and \quad \sum_{i=1}^{i=n}(\pi^i_A)^* = 1_{A^*}\ mod(\mathcal{R}_{A^*,A^*}).$$

The first equality is justified by the following commutative diagram:



The two middle squares are commutative by the naturality of λ (where, $\lambda_{A;I} = (d_I \otimes id_{A^*})(id_{I^*} \otimes d_A \otimes id_{A^*})(id_{A^*} \otimes b_A) = (d_I \otimes id_{A^*})(d_A \otimes id_{A^*})(id_{A^*} \otimes b_A) = d_I \otimes id_{A^*}$ and the same thing for $\lambda^{-1}_{A;I}$).

With similar arguments one can prove the second identity which is justified by the following commutative diagram:



The third is obvious.

Inversely, an $\mathcal{R}-$basis on A^* gives similarly an $\mathcal{R}-$basis on $A^{**}(\simeq A)$. □

Proposition 3.12. *Let C be a semisimple ribbon $Ab-$category and A and B be isomorphic objects in C. Assume that A (or B) is dominated by a finite set of simple objects. Then A, A^*, B and B^* admit $\mathcal{R}-$bases on them and we have*

(a) $r_B = r_A$;

(b) $r_{A^*} = r_A$.

Proof. A (resp. B) being dominated by simple objects ensures by using Lemma 3.10, the existence of $\mathcal{R}-$bases on A, A^*, B and B^*.

(a) Using Lemma 3.10, (ii), we obtain an $r_A - \mathcal{R}-$basis on B which is minimal (among the cardinals of the other $\mathcal{R}-$bases on B) and vice versa. Hence, $r_B = r_A$.

(b) Identically to the above, using this time Lemma 3.11, (ii).

□

Definition 3.13. Let C be a semisimple ribbon $Ab-$category and V an object of C. We call quantum rank of V denoted by $ran_q(V)$, the nonnegative integer defined as

$$ran_q(V) = min(n)$$

where n runs over all finite cardinals of dominating families $(V_i; \varepsilon_V^i; \mu_V^i)_{i=1}^{i=n}$ of simple objects of V.

Proposition 3.14. *Let C be a semisimple ribbon $Ab-$category and A and B be isomorphic objects in C. Then*

(a) $ran_q(V) = 1$ *for every simple object* V *of* C*;*

(b) $ran_q(B) = ran_q(A)$*;*

(c) $ran_q(A^*) = ran_q(A)$.

Proof. Straightforward from Lemma 3.10 (i) and Lemma 3.11 (i). □

4 Categorification of bilinear forms

Definition 4.1. Let C be a monoidal $Ab-$category equipped with a compatibility relation $\mathcal{R}$ and V a dualizable object of C. An $\mathcal{R}-$solution on V, is a quadruple $(V; d_V; b_V; \pi_V)$, such that:

$$\begin{aligned} (1_V \otimes d_V)(b_V \otimes 1_V) &= \pi_V^2; \\ (d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) &= (\pi_V^2)^*; \\ d_V(1_{V^*} \otimes \pi_V) &= d_V(\pi_V^* \otimes 1_V); \\ (\pi_V \otimes 1_{V^*})b_V &= (1_V \otimes \pi_V^*)b_V; \\ \pi_V &= 1_V \; mod(\mathcal{R}_{V,V}). \end{aligned}$$

Example 4.2. Let V be an object of the category $(vect_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ of finite dimensional vector spaces over a field $\mathbb{K}$ and $\mathcal{R}$ any compatibility relation. Then, V admits an $\mathcal{R}-$solution on it.

Proof. $(V; d_V; b_V; 1_V)$ is an $\mathcal{R}-$solution on V, where:

$$\begin{array}{lccc} d_V: & V^* \otimes V & \longrightarrow & \mathbb{K} \\ & e^j \otimes e_i & \longmapsto & \delta_{ij} \end{array} \quad and \quad \begin{array}{lccc} b_V: & \mathbb{K} & \longrightarrow & V \otimes V^* \\ & 1 & \longmapsto & \sum_i e_i \otimes e^i \end{array}$$

such that $\{e_i\}_i$ and $\{e^i\}_i$ are respectively a basis and its dual basis of V and its dual V^*. □

Example 4.3. The $1 - \mathcal{R}-$basis in Example 3.7 of the previous section 3, is an $\mathcal{R}-$solution on V.

Proposition 4.4. *Let C be a monoidal $Ab-$category, $\mathcal{R}$ a compatibility relation on C and V a dualizable object of C. Then, for every morphism $\pi_V : V \longrightarrow V$, such that $\pi_V^2 = \pi_V$ and $\pi_V = 1_V \; mod(\mathcal{R}_{V,V})$, the quadruple*

$$(V; d_V(\pi_V^* \otimes \pi_V); (\pi_V \otimes \pi_V^*)b_V; \pi_V)$$

is an $\mathcal{R}-$solution on V, where d_V and b_V are duality structures on V.

Proof. Straightforward. □

Example 4.5. Let C be a semisimple ribbon $Ab-$category and V an object of C dominated by $(V_i;\varepsilon_i;\mu_i)_{i=1}^{i=n}$. Assume that $\varepsilon_i\mu_j = \delta_{i,j}$, for all i, j; $1 \leq i, j \leq n$. Then

$$(V; d_V(T_V^* \otimes T_V); (T_V \otimes T_V^*)b_V; T_V)$$

is an $\mathcal{R}-$solution on V, where $T_V = \sum_i \mu_i\varepsilon_i$ and d_V and b_V are duality structures on V.

Proof. In fact, $\varepsilon_i\mu_j = \delta_{i,j}$, $1 \leq \forall i, j \leq n \Rightarrow T_V^2 = \sum_i\sum_j \mu_i\varepsilon_i\mu_j\varepsilon_j = T_V$. Hence, applying Proposition 4.4, the result holds. □

Proposition 4.6. *Let C be a monoidal $Ab-$category equipped with a compatibility relation $\mathcal{R}$ and $f : A \longrightarrow B$ be an isomorphism between dualizable objects in C. Then, the following are equivalent*

(a) $(A; d_A; b_A; \pi_A)$ *is an $\mathcal{R}-$solution on A;*

(b) $(B; d_A(f^* \otimes f^{-1}); (f \otimes (f^{-1})^*)b_A; f\pi_A f^{-1})$ *is an $\mathcal{R}-$solution on B.*

Proof. (a)⇒(b): Let $(A; d_A; b_A; \pi_A)$ be an $\mathcal{R}-$solution on A. We have to prove

$$(1_B \otimes d_A(f^* \otimes f^{-1}))((f \otimes (f^{-1})^*)b_A \otimes 1_B) = f\pi_A^2 f^{-1} \ ;$$

$$(d_A(f^* \otimes f^{-1}) \otimes 1_{B^*})(1_{B^*} \otimes (f \otimes (f^{-1})^*)b_A) = (f\pi_A^2 f^{-1})^* \ ;$$

$$d_B(f^* \otimes \pi_A f^{-1}) = d_B((f\pi_A)^* \otimes f^{-1}) \ ;$$

$$(\pi_A f^{-1} \otimes f^*)b_B = (f^{-1} \otimes (f\pi_A)^*)b_B$$

$$and \quad \pi_B = 1_B \ mod(\mathcal{R}_{B,B}).$$

The proof of the first and second identities is similar to the one of Lemma 3.10 (ii).
For the third one, we have

$$\begin{aligned} d_B(f^* \otimes \pi_A f^{-1}) &= d_B(1_{A^*} \otimes \pi_A)(f^* \otimes f^{-1}) \\ &= d_B(\pi_A^* \otimes 1_A)(f^* \otimes f^{-1}) \\ &= d_B((f\pi_A)^* \otimes f^{-1}) \end{aligned}$$

and similarely for the fourth one.
For the fifth, we have:

$$\pi_A = 1_A \ mod(\mathcal{R}_{A,A}) \Rightarrow \pi_B := f\pi_A f^{-1} = 1_B \ mod(\mathcal{R}_{B,B}).$$

(b)⇒(a): Let $d_A : A^* \otimes A \longrightarrow I$, $b_A : I \longrightarrow A \otimes A^*$ and $\pi_A : A \longrightarrow A$ be morphisms such that

$$(B; d_A(f^* \otimes f^{-1}); (f \otimes (f^{-1})^*)b_A; f\pi_A f^{-1})$$

is an $\mathcal{R}-$solution on B. Then

$$(A; d_A(f^* \otimes f^{-1})((f^{-1})^* \otimes f); (f^{-1} \otimes f^*)(f \otimes (f^{-1})^*)b_A; f^{-1}f\pi_A f^{-1}f)$$

is an $\mathcal{R}-$solution on A (by the first sense), i.e: $(A; d_A; b_A; \pi_A)$ is an $\mathcal{R}-$solution on A. □

Proposition 4.7. *Let C be a ribbon $Ab-$category equipped with a compatibility relation $\mathcal{R}$ and $A \in Ob(C)$ endowed with an $\mathcal{R}-$solution $(A; d_A; b_A; \pi_A)$ on it. Define the morphisms*

$$(d_A)_* := d_I b_A^* \lambda_{A;A^*}; \quad (b_A)_* := \lambda_{A;A^*}^{-1} d_A^* b_I \quad \textit{and} \quad (\pi_A)_* := \pi_A^*.$$

Then, $(A^*; (d_A)_*; (b_A)_*; (\pi_A)_*)$ is an $\mathcal{R}-$solution on A^*.

Proof. We have to prove the five identities

$$\begin{aligned} (1_{A^*} \otimes d_I b_A^* \lambda)(\lambda^{-1} d_A^* b_I \otimes 1_{A^*}) &= (\pi_A^2)^* \\ (d_I b_A^* \lambda \otimes 1_{V^*})(1_{(A^*)^*} \otimes \lambda^{-1} d_A^* b_I) &= (\pi_A^2)^* \\ d_I b_A^* \lambda(1_{(A^*)^*} \otimes (\pi_A)_*) &= d_I b_A^* \lambda(\pi_A^{**} \otimes 1_{A^*}) \\ (\pi_{A^*} \otimes 1_{(A^*)^*})\lambda^{-1} d_A^* b_I &= (1_{A^*} \otimes \pi_A^{**})\lambda^{-1} d_A^* b_I \\ \pi_A^* &= 1_{A^*} \ mod(\mathcal{R}_{A^*,A^*}). \end{aligned}$$

The proof of the first and second is exactly similar to the proof given in Lemma 3.12 (ii).
For the third one, we have

$$\begin{aligned} d_I b_A^* \lambda((\pi_{A^*})^* \otimes 1_{A^*}) &= d_I b_A^* \lambda \lambda^{-1} (1_A \otimes \pi_A^*)^* \lambda \\ &= d_I [(1_A \otimes \pi_A^*) b_A]^* \lambda \\ &= d_I [(\pi_A \otimes 1_{A^*}) b_A]^* \lambda \\ &= d_I b_A^* \lambda \lambda^{-1} (\pi_A \otimes 1_{A^*})^* \lambda \\ &= d_I b_A^* \lambda (1_{(A^*)^*} \otimes \pi_A^*). \end{aligned}$$

The fourth: similar to the third.
The fifth identity is straightforward. □

In order to study the properties of $\mathcal{R}-$solutions, we introduce the tensor product of bilinear forms in C.

Proposition 4.8. *Let C be a monoidal $Ab-$category equipped with a compatibility relation $\mathcal{R}$ and let $(A; d_A; b_A; \alpha)$ and $(B; d_B; b_B; \beta)$ be $\mathcal{R}-$solutions on two dualizable objects A and B of C. Then,*

$$(A \otimes B; d_A \otimes_- d_B; b_A \otimes_+ b_B; \alpha \otimes \beta)$$

is an $\mathcal{R}-$solution on $A \otimes B$; where the tensor products $\otimes_-$ of d_A, d_B and $\otimes_+$ of b_A, b_B; are defined as

$$d_A \otimes_- d_B := d_B(1_{B^*} \otimes d_A \otimes 1_B)(\lambda_{A;B}^{-1} \otimes 1_A \otimes 1_B);$$

$$b_A \otimes_+ b_B := (1_A \otimes 1_B \otimes \lambda_{A;B})(1_A \otimes b_B \otimes 1_{A^*}) b_A.$$

Proof. The domains and codomains of the defined tensor products are as follows:

$$d_A \otimes_- d_B : (A \otimes B)^* \otimes A \otimes B \longrightarrow B^* \otimes A^* \otimes A \otimes B \to B^* \otimes B \to I$$

and

$$b_A \otimes_+ b_B : I \longrightarrow A \otimes A^* \longrightarrow A \otimes B \otimes B^* \otimes A^* \longrightarrow A \otimes B \otimes (A \otimes B)^*.$$

Let's prove the first identity:
$[1_{A\otimes B} \otimes d_B(1_{B^*} \otimes d_A \otimes 1_B)(\lambda^{-1} \otimes 1_A \otimes 1_B)][(1_A \otimes 1_B \otimes \lambda)(1_A \otimes b_B \otimes 1_{A^*}) b_A \otimes$

$1_{A\otimes B}] = (\alpha \otimes \beta)^2$.
We have:
$[1_A \otimes 1_B \otimes d_B(1_{B^*} \otimes d_A \otimes 1_B)(\lambda^{-1} \otimes 1_A \otimes 1_B)][(1_A \otimes 1_B \otimes \lambda)(1_A \otimes b_B \otimes 1_{A^*})b_A \otimes 1_A \otimes 1_B]$
$= [1_A \otimes 1_B \otimes d_B(1_{B^*} \otimes d_A \otimes 1_B)][1_A \otimes 1_B \otimes (\lambda^{-1} \otimes 1_A \otimes 1_B)][(1_A \otimes 1_B \otimes \lambda) \otimes 1_A \otimes 1_B][(1_A \otimes b_B \otimes 1_{A^*})b_A \otimes 1_A \otimes 1_B]$
$= [1_A \otimes 1_B \otimes d_B(1_{B^*} \otimes d_A \otimes 1_B)][(1_A \otimes b_B \otimes 1_{A^*})b_A \otimes 1_A \otimes 1_B]$
$= [1_A \otimes 1_B \otimes d_B][1_A \otimes b_B \otimes 1_B][1_A \otimes d_A \otimes 1_B][b_A \otimes 1_A \otimes 1_B]$
$= [1_A \otimes \beta^2][1_A \otimes d_A \otimes 1_B][b_A \otimes 1_A \otimes 1_B]$
$= \alpha^2 \otimes \beta^2$
$= (\alpha \otimes \beta)^2$.
The proof of the other identities is done similarly. □

Corollary 4.9. *Let C be a monoidal $Ab-$category equipped with a compatibility relation $\mathcal{R}$ and let $(V_i; d^i_V; b^i_V; \pi^i_V)$ be $1-\mathcal{R}-$bases on dualizable objects V_i of C, for any i, $1 \le i \le n$, $n \ge 2$. Then*

$$(V_1 \otimes ... \otimes V_n; d^1_V \otimes_- (d^1_V \otimes_- (... \otimes_- d^n_V)...); b^1_V \otimes_+ (b^1_V \otimes_+ (... \otimes_+ b^n_V)...); \pi^1_V \otimes ... \otimes \pi^n_V)$$

is a $1-\mathcal{R}-$basis on $V_1 \otimes ... \otimes V_n$.

Proof. By induction on n, using Proposition 4.8 and remarking that in fact, an $\mathcal{R}-$solution on an object is in particular a $1-\mathcal{R}-$basis on it. □

The following definition serves to establish a forthcoming result.

Definition 4.10. Let C be a monoidal $Ab-$category; V an object of C and $(V; d_V; b_V; 1_V)$ a particular solution of the triangular system

$$(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V \ ;$$

$$(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*}.$$

Let $(V; D_V; B_V; 1_V)$ be another solution of the same system. Then, for any automorphism $f : V \longrightarrow V$. Define the morphisms

$$f^{*^1} = (D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes b_V);$$

$$f^{*^2} = (d_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes B_V);$$

and

$$f^{-1}.b_V := (f \otimes (f^{-1})^{*^2})b_V : I \longrightarrow V \otimes V^*;$$

$$d_V.f := d_V(f^{*^1} \otimes f^{-1}) : V^* \otimes V \longrightarrow I.$$

Proposition 4.11. *Let* $(V; d_V; b_V; 1_V)$ *be a particular solution of the triangular system*

$$(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V \ ;$$

$$(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*}.$$

Then, all solutions are given by

$$(V; d_V.f; f^{-1}.b_V; 1_V), \quad f \in Aut_C(V).$$

Proof. Let $(V; d_V; b_V; 1_V)$ be a particular solution and $f \in Aut_C(V)$. Then we get $(V; d_V.f; f^{-1}.b_V; 1_V)$ is a solution of the above system for any other solution $(V; D_V; B_V; 1_V)$ (including the fixed particular one). In fact, we have
$(1_V \otimes d_V(f^{*^1} \otimes f^{-1}))((f \otimes (f^{-1})^{*^2})b_V \otimes 1_V)$
$= (1_V \otimes d_V)(1 \otimes D_V \otimes 1_{V^*} \otimes 1_V)(1_V \otimes 1_{V^*} \otimes f \otimes 1_{V^*} \otimes f^{-1})(1_V \otimes 1_{V^*} \otimes b_V \otimes 1_V)(1_V \otimes d_V \otimes 1_{V^*} \otimes 1_V)(f \otimes 1_{V^*} \otimes f^{-1} \otimes 1_{V^*} \otimes 1_V)(1_V \otimes 1_{V^*} \otimes B_V \otimes 1_V)(b_V \otimes 1_V)$
$= (1_V \otimes D_V)(1_V \otimes 1_{V^*} \otimes f)(1_V \otimes 1_{V^*} \otimes 1_V \otimes d_V)(1_V \otimes 1_{V^*} \otimes b_V \otimes 1_V)(1_V \otimes 1_{V^*} \otimes f^{-1})(f \otimes 1_{V^*} \otimes 1_V)(1_V \otimes d_V \otimes 1_{V^*} \otimes 1_V)(b_V \otimes 1_V \otimes 1_{V^*} \otimes 1_V)(f^{-1} \otimes 1_{V^*} \otimes 1_V)(B_V \otimes 1_V)$
$= (1_V \otimes D_V)(1_V \otimes 1_{V^*} \otimes f)(f^{-1} \otimes 1_{V^*} \otimes 1_V)(B_V \otimes 1_V)$
$= 1_V.$

And

$(d_V(f^{*^1} \otimes f^{-1}) \otimes 1_{V^*})(1_{V^*} \otimes (f \otimes (f^{-1})^{*^2})b_V)$
$= (d_V \otimes 1_{V^*})(D_V \otimes 1_{V^*} \otimes 1_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes b_V \otimes 1_V \otimes 1_{V^*})(1_{V^*} \otimes 1_V \otimes d_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes 1_V \otimes 1_{V^*} \otimes B_V)(1_{V^*} \otimes b_V)$
$= (D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes 1_V \otimes d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V \otimes 1_V \otimes 1_{V^*})(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes 1_V \otimes d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V \otimes 1_V \otimes 1_{V^*})(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes B_V)$
$= (D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes B_V)$
$= 1_{V^*}.$

Now let $(V; D_V; B_V; 1_V)$ be a solution of the triangular system and let

$$f = (1_V \otimes d_V)(B_V \otimes 1_V) \ \ (resp.\ f = (1_V \otimes D_V)(b_V \otimes 1_V)).$$

Then, f is invertible and its inverse is

$$f^{-1} = (1_V \otimes D_V)(b_V \otimes 1_V) \ \ (resp.\ f^{-1} = (1_V \otimes d_V)(B_V \otimes 1_V))$$

and we have

$$\begin{aligned} d_V.f &= d_V(f^{*^1} \otimes f^{-1}) \\ &= d_V((D_V \otimes 1_{V^*})(1_{V^*} \otimes f \otimes 1_{V^*})(1_{V^*} \otimes b_V) \otimes f^{-1}) \\ &= D_V(1_{V^*} \otimes f)(1_{V^*} \otimes 1_V \otimes d_V)(1_{V^*} \otimes b_V \otimes 1_V)(1_{V^*} \otimes f^{-1}) \\ &= D_V \end{aligned}$$

and

$$\begin{aligned} f^{-1}.b_V &= (f \otimes (f^{-1})^{*^2})b_V \\ &= (f \otimes (d_V \otimes 1_{V^*})(1_{V^*} \otimes f^{-1} \otimes 1_{V^*})(1_{V^*} \otimes B_V))b_V \\ &= (f \otimes 1_{V^*})(1_V \otimes d_V \otimes 1_{V^*})(b_V \otimes 1_V \otimes 1_{V^*})(f^{-1} \otimes 1_{V^*})B_V \\ &= B_V. \end{aligned}$$

□

In general, we have the following.

Proposition 4.12. *Let $(V; d_V; b_V; 1_V)$ be a particular solution of the triangular system*

$$(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V;$$

$$(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*}.$$

Denote by $Sol_C(V)$ the set of all solutions of the above system on V and consider the map $\varphi : Aut_C(V) \longrightarrow Sol_C(V), f \longmapsto (V; d_V.f; f^{-1}.b_V; 1_V)$. Then, φ is surjective but not injective.

Proof. Immediate from Proposition 4.11. □

Definition 4.13. Let C be a semisimple ribbon $Ab-$category.
$\mathcal{R}-$solutions over objects of C form a category which is denoted by $Fin(C)$; the unit object is given by $\overline{I} = (I; d_I; b_I; 1_I)$, where d_I and b_I are duality structures on I.
A morphism

$$f : (A; d_A; b_A; \pi_A) \longrightarrow (B; d_B; b_B; \pi_B)$$

of $Fin(C)$, where A and B are two objects of C; consists of a morphism $f : A \longrightarrow B$ in C, such that

$$f.d_A = d_B.f \quad and \quad f.b_A = b_B.f$$

where, $f.d_A := d_A(f^* \otimes 1_A)$, $d_B.f := d_B(1_{B^*} \otimes f)$, $f.b_A := (f \otimes 1_{A^*})b_A$ and $b_B.f := (1_B \otimes f^*)b_B$ (notations here are independent from those of Definition 4.10).

Lemma 4.14. *Let C be a semisimple ribbon $Ab-$category and*

$$f : (A; d_A; b_A; \pi_A) \longrightarrow (B; d_B; b_B; \pi_B)$$

a morphism in $Fin(C)$. Then, the dual morphism f^* of f defined a morphism

$$f^* : (A^*; (d_A)_*; (b_A)_*; (\pi_A)_*) \longrightarrow (B^*; (d_B)_*; (b_B)_*; (\pi_B)_*)$$

in $Fin(C)$.

Proof. We have to prove the following

$$(d_B)_*(f^{**} \otimes 1_{B*}) = (d_A)_*(1_{A^{**}} \otimes f^*);$$

$$(1_{A^*} \otimes f^{**})(b_A)_* = (f^* \otimes 1_{B^{**}})(b_B)_*.$$

For the first identity, we have

$$\begin{aligned}
(d_B)_*(f^{**} \otimes 1_{B^*}) &= d_I b_B^* \lambda\lambda^{-1}(1_B \otimes f^*)^* \lambda \\
&= d_I[(1_B \otimes f^*)b_B]^* \lambda \\
&= d_I[(f \otimes 1_{A^*})b_A]^* \\
&= d_I b_A^* (f \otimes 1_{A^*})^* \lambda \\
&= d_I b_A^* \lambda\lambda^{-1}(f \otimes 1_{A^*})^* \lambda \\
&= (d_A)_*(1_{A^{**}} \otimes f^*).
\end{aligned}$$

The third passage is due to the axioms of f being a morphism in $Fin(C)$.
Similarly for the second identity using the other axioms of f as a morphism in $Fin(C)$. □

Proposition 4.15. *Let C be a semisimple ribbon $Ab-$category. Then, $Fin(C)$ is also a semisimple ribbon $Ab-$category.*

Proof. The category $Fin(C)$ may be provided with canonical tensor product, duality and braiding (inherited from those of C), which makes it a braided monoidal category with duality.
The tensor product of a couple of $\mathcal{R}-$solutions $(A; d_A; b_A; \pi_A)$ and $(B; d_B; b_B; \pi_B)$ is given by

$$(A; d_A; b_A; \pi_A) \otimes (B; d_B; b_B; \pi_B) = (A \otimes B; d_A \otimes_- d_B; b_A \otimes_+ b_B; \pi_A \otimes \pi_B).$$

The category $Fin(C)$ is provided with canonical duality as follows: to each object $(A; d_A; b_A; \pi_A)$, there are associated an object

$$(A; d_A; b_A; \pi_A)^* := (A^*; (d_A)_*; (b_A)_*; (\pi_A)_*)$$

and morphisms

$$\overline{b_A} := b_{(A;d_A;b_A;\pi_A)} : \overline{I} \longrightarrow (A; d_A; b_A; \pi_A) \otimes (A^*; (d_A)_*; (b_A)_*; (\pi_A)_*)$$

and

$$\overline{d_A} := d_{(A;d_A;b_A;\pi_A)} : (A^*; (d_A)_*; (b_A)_*; (\pi_A)_*) \otimes (A; d_A; b_A; \pi_A) \longrightarrow \overline{I}$$

given by b_A and d_A respectively, such that the identities hold

$$(1 \otimes \overline{d_A})(\overline{b_A} \otimes 1) = 1$$

$$(\overline{d_A} \otimes 1)(1 \otimes \overline{b_A}) = 1$$

The dual f^* of an arbitrary morphism

$$f : (A; X_A; Y_A; \alpha) \longrightarrow (B; Z_B; T_B; \beta)$$

is well defined by Lemma 4.14, and it is given by the formula

$$f^* = (Z_B \otimes 1_{A^*})(1_{B^*} \otimes f \otimes 1_{A^*})(1_{B^*} \otimes Y_A).$$

It is easy to deduce that for any objects $(A;X_A;Y_A;\alpha)$ and $(B;Z_B;T_B;\beta)$ of $Fin(C)$, there is a natural family of isomorphisms between

$$(B;Z_B;T_B;\beta)^* \otimes (A;X_A;Y_A;\alpha)^*$$

and

$$((A;X_A;Y_A;\alpha) \otimes (B;Z_B;T_B;\beta))^*$$

defined as

$$(\alpha \otimes \beta)^*(Z_B \otimes 1_{(A\otimes B)^*})(1_{B^*} \otimes X_A \otimes 1_B \otimes 1_{(A\otimes B)^*})(1_{B^*} \otimes 1_{A^*} \otimes Y_A \otimes_+ T_B)(\beta^* \otimes \alpha^*).$$

We provide $Fin(C)$ with the braiding induced from C.
$Fin(C)$ is twisted as follows: the twist $\theta_{(A;d_A;b_A;\pi_A)}$ on an object $(A;d_A;b_A;\pi_A)$, consists of the twist θ_A. In fact, $\theta_A.d_A = d_A.\theta_A$ and $\theta_A.b_A = b_A.\theta_A$ by the naturality of θ.
Consequently, $(*;\overline{b_A};\overline{d_A})$ is a compatible duality in $Fin(C)$. Hence, the later is a ribbon category.
For semisimplicity, it is easy to verify that every object $(A;d_A;b_A;\pi_A)$ of $Fin(C)$ is dominated by $\{(V_i;d_{V_i};b_{V_i};1_{V_i});\varepsilon_i;\mu_i\}_{i=1}^{i=n}$, where A is dominated by $(V_i;\varepsilon_i;\mu_i)_{i=1}^{i=n}$. □

5 The concept of a determinant

In all the sequel, we write $Tr(f)$ instead of $Tr_q(f)$ to reduce indices as well as for dimension and we identify V^n with $V^{\otimes^n}$ and $f^{\otimes^n}$ with f^n, for all $V \in Ob(C)$; $f \in End_C(V)$.

Let C be a *semisimple ribbon* $Ab-$category and A an object of C of *rank* n dominated by simple objects $(V_i)_{1\leq i\leq n}$ with domination morphisms denoted $\{\varepsilon_i : V \longrightarrow V_i \ ; \ \mu_i : V_i \longrightarrow V\}_i$. Let $[1;n] \cap \mathbb{N} = I_1 \cup I_2 \cup ... \cup I_m$ be a partition of $[1;n] \cap \mathbb{N}$ into isomorphic classes. Denote $card(I_j) = n_j$ for all $1 \leq j \leq m$; W_j a representative of the isomorphic objects indixed by indices in I_j and C^{w_j} the identity endomorphism of $W_j^{n_j}$.
We define the endomorphism Λ_A^n of A^n as:

$$\Lambda_A^n = \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)Tr(C^{w_{n_1}})^{-1}D_\sigma^1 \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)Tr(C^{w_{n_m}})^{-1}D_\sigma^m$$

where D^j_σ is the endomorphism of A^{n_j} defined by

$$D^j_\sigma = \mu_{j_1}\varepsilon_{\sigma(j_1)} \otimes .... \otimes \mu_{j_{n_j}}\varepsilon_{\sigma(j_{n_j})} \tag{5.1}$$

with $I_j = [j_1, j_{n_j}] \cap \mathbb{N}$ and σ a permutation of $\mathfrak{S}_{n_j}$.
If $n = 1$, we consider $\Lambda^1_A = Tr(id_A)^{-1} id_A$.

Proposition 5.1. *Let $(A; X_A; Y_A; 1_A)$ be a particular solution on A and $f \in End_C(A)$. Then, the quantum determinant, $det^C_n(f)$, of f defined by*

$$det^C_n(f) = X_A^{\otimes^n_-}(f^{n^*} \otimes \Lambda^n_A \theta_{A^n}) c_{A^n,(A^n)^*} Y_A^{\otimes^n_+}$$

is independent of the choice of the solution on A.

Proof. $(A; X_A; Y_A; 1_A)$ is a paricular solution of the triangular system as in Proposition 4.11. If $(A; \overline{X}_A; \overline{Y}_A; 1_A)$ is another particular solution, then $(A^n; X_A^{\otimes^n_-}; Y_A^{\otimes^n_+}; 1)$ and $(A^n; \overline{X}_A^{\otimes^n_-}; \overline{Y}_A^{\otimes^n_+}; 1)$ are solutions on A^n by Proposition 4.8. Using now Proposition 4.11, we obtain

$$\overline{X}_A^{\otimes^n_-} = X_A^{\otimes^n_-}.h := X_A^{\otimes^n_-}(h^{*^1} \otimes h^{-1}) \ ; \ \ \overline{Y}_A^{\otimes^n_+} = Y_A^{\otimes^n_+}.h^{-1} := (h \otimes (h^{-1})^{*^2}) Y_A^{\otimes^n_+}$$

where h is the automorphism $(1 \otimes X_A^{\otimes^n_-})(\overline{Y}_A^{\otimes^n_+} \otimes 1)$ of A^n. Hence, we have

$$\begin{aligned}
&\overline{X}_A^{\otimes^n_-}(f^{n^*} \otimes \Lambda^n_A \theta_{A^n}) c_{A^n,(A^n)^*} \overline{Y}_A^{\otimes^n_+} \\
&\quad = X_A^{\otimes^n_-}(h^* \otimes h^{-1})(f^{n^*} \otimes \Lambda^n_A \theta_{A^n}) c_{A^n,(A^n)^*}(h \otimes (h^{-1})^*) Y_A^{\otimes^n_+} \\
&\quad = \overline{X}_A^{\otimes^n_-}(1 \otimes h)(1 \otimes 1 \otimes h)(1 \otimes Y_A^{\otimes^n_+} \otimes 1)(1 \otimes h^{-1})(1 \otimes f^n \Lambda^n_A \theta_{A^n}) \\
&\quad c_{A^n,(A^n)^*}(h \otimes 1)(1 \otimes X_A^{\otimes^n_-} \otimes 1)(Y_A^{\otimes^n_+} \otimes 1 \otimes 1)(h^{-1} \otimes 1)\overline{Y}_A^{\otimes^n_+} \\
&\quad = \overline{X}_A^{\otimes^n_-}(1 \otimes h)(1 \otimes h^{-1})(1 \otimes f^n \Lambda^n_A \theta_{A^n}) c_{A^n,(A^n)^*}(h \otimes 1)(h^{-1} \otimes 1)\overline{Y}_A^{\otimes^n_+} \\
&\quad = X_A^{\otimes^n_-}(f^{n^*} \otimes \Lambda^n_A \theta_{A^n}) c_{A^n,(A^n)^*} Y_A^{\otimes^n_+}.
\end{aligned}$$

□

Theorem 5.2. *Let C be a semisimple ribbon $Ab-$category, A an object of C of rank n dominated by a family $(V_i; \varepsilon_i; \mu_i)_{1 \le i \le n}$ of simple objects and $f \in End_C(A)$. Then, the quantum determinant $det^C_n(f)$ of f, verifies the following*

(a) $det^C_n(f) \in \mathbb{K}_C$;

(b) $det_1^C(1_V) = 1_I$ *where* V *is a simple object;*

(c) *Assume that* $\varepsilon_i\mu_j = \delta_{i;j}$ *for all* $1 \le i; j \le n$. *Then,* $det_n^C(1_A) = 1_I$;

(d) $det_n^C(q \otimes f) = q^n det_n^C(f)$ *for all* $q \in U(\mathbb{K}_C)$;

(e) $det_n^C(f^*) = det_n^C(f)$.

Proof. (a) By definition.

(b) Straightforward.

(c) $det_n^C(1_A)$
$= X_A^{\otimes_-^n}(1_{(A^n)^*} \otimes \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}(Y_A^{\otimes_+^n})$
$= Tr(\Lambda_A^n)$
$= Tr(\sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)Tr(C_\sigma^{w_{n_1}})^{-1}D_\sigma^1 \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)Tr(C^{w_{n_m}})^{-1}D_\sigma^m)$
$= \varepsilon(1_{\mathfrak{S}_{n_1}})Tr(C^{w_{n_1}})^{-1}Tr(D^1_{1_{\mathfrak{S}_{n_1}}}) ...\varepsilon(1_{\mathfrak{S}_{n_m}})Tr(C^{w_{n_m}})^{-1}Tr(D^m_{1_{\mathfrak{S}_{n_m}}})$
$+ \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)Tr(C^{w_{n_1}})^{-1}Tr(D_\sigma^1) ... \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)Tr(C^{w_{n_m}})^{-1}Tr(D_\sigma^m)$
(where in the second term of the summand, at least one of $\sigma \in \mathfrak{S}_{n_i}$ is non identity for some $1 \le i \le m$)
$= 1_I + 0$
$= 1_I$.

(d)

$$\begin{aligned} det_n^C(q \otimes f) &= X_A^{\otimes_-^n}((q \otimes f)^{n^*} \otimes \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}(Y_A^{\otimes_+^n}) \\ &= X_A^{\otimes_-^n}(1_{(A^n)^*} \otimes q^n f^n \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}(Y_A^{\otimes_+^n}) \\ &= q^n X_A^{\otimes_-^n}(1_{(A^n)^*} \otimes f^n \Lambda_A^n \theta_{A^n})c_{A^n,(A^n)^*}(Y_A^{\otimes_+^n}) \\ &= q^n det_n^C(f). \end{aligned}$$

(e) V^* is dominated by $(V_i^*)_{1\le i\le n}$ with $\overline{\mu}_i = \varepsilon_i^*$ and $\overline{\varepsilon}_i = \mu_i^*$. Then:

$$\Lambda_{A^*}^n = \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)Tr(C^{w_{n_1}})^{-1}D_\sigma^1 \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)Tr(C^{w_{n_m}})^{-1}D_\sigma^m$$

where

$$D_\sigma^j = \overline{\mu}_{j_1}\overline{\varepsilon}_{\sigma(j_1)} \otimes .... \otimes \overline{\mu}_{j_{n_j}}\overline{\varepsilon}_{\sigma(j_{n_j})}$$

as in (5.1) and we have

$$
\begin{aligned}
& det_n^C(f^*) \\
&= Tr((f^*)^n \Lambda^n_{A^*}) \\
&= Tr\Big(\sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{G}}}} \varepsilon(\sigma) Tr(C^{w^*_{n_1}})^{-1} (f^*)^{n_1} D^1_\sigma \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{m}}}} \varepsilon(\sigma) Tr(C^{w^*_{n_m}})^{-1} \\
& (f^*)^{n_m} D^m_\sigma\Big) \\
&= Tr\Big(\sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{G}}}} \varepsilon(\sigma) Tr((C^{w_{n_1}})^*)^{-1} (f^*)^{n_1} (\overline{\mu}_{1_1}\overline{\varepsilon}_{\sigma(1_1)} \otimes ... \otimes \overline{\mu}_{1_{n_1}}\overline{\varepsilon}_{\sigma(1_{n_1})})\Big) \\
& ... Tr\Big(\sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{m}}}} \varepsilon(\sigma) Tr((C^{w_{n_m}})^*)^{-1} (f^*)^{n_m} (\overline{\mu}_{m_1}\overline{\varepsilon}_{\sigma(m_1)} \otimes ... \otimes \overline{\mu}_{m_{n_m}}\overline{\varepsilon}_{\sigma(m_{n_m})})\Big) \\
&= \sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{G}}}} \varepsilon(\sigma) Tr(C^{w_{n_1}})^{-1} Tr(f^*\overline{\mu}_{1_1}\overline{\varepsilon}_{\sigma(1_1)}) ... Tr(f^*\overline{\mu}_{1_{n_1}}\overline{\varepsilon}_{\sigma(1_{n_1})}) \\
& ... \sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{m}}}} \varepsilon(\sigma) Tr(C^{w_{n_m}})^{-1} Tr(f^*\overline{\mu}_{m_1}\overline{\varepsilon}_{\sigma(m_1)}) ... Tr(f^*\overline{\mu}_{m_{n_m}}\overline{\varepsilon}_{\sigma(m_{n_m})}) \\
&= \sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{G}}}} \varepsilon(\sigma) Tr(C^{w_{n_1}})^{-1} Tr((\mu_{\sigma(1_1)}\varepsilon_{1_1} f)^*) ... Tr((\mu_{\sigma(1_{n_1})}\varepsilon_{1_{n_1}} f)^*) \\
& ... \sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{m}}}} \varepsilon(\sigma) Tr(C^{w_{n_m}})^{-1} Tr((\mu_{\sigma(m_1)}\varepsilon_{m_1} f)^*) ... Tr((\mu_{\sigma(m_{n_m})}\varepsilon_{m_{n_m}} f)^*) \\
&= \sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{G}}}} \varepsilon(\sigma) Tr(C^{w_{n_1}})^{-1} Tr(\varepsilon_{1_1} f \mu_{\sigma(1_1)}) ... Tr(\varepsilon_{1_{n_1}} f \mu_{\sigma(1_{n_1})}) \\
& ... \sum_{\sigma\in\mathfrak{S}_{\mathfrak{n}_{\mathfrak{m}}}} \varepsilon(\sigma) Tr(C^{w_{n_m}})^{-1} Tr(\varepsilon_{m_1} f \mu_{\sigma(m_1)}) ... Tr(\varepsilon_{m_{n_m}} f \mu_{\sigma(m_{n_m})}) \\
&= det_n^C(f).
\end{aligned}
$$

□

Theorem 5.3. *Let C be a semisimple ribbon $Ab-$category, A an object of C of rank n dominated by a family $(V_i;\varepsilon_i;\mu_i)_{1\le i\le n}$ of simple objects and $f \in End_{C/\mathcal{R}}(A)$. To f, we associate the matrix $M_f^C = (a^f_{i,j})_{1\le i,j\le n}$, where*

$$
a^f_{i,j} = \begin{cases} Tr(\varepsilon_i f \mu_j) dim(V_i)^{-1} & if\ V_i \simeq V_j; \\ 0 & else. \end{cases}
$$

Then

1. *$det_n^C(f) = det(M_f^C)$;*
2. *The map $End_{C/\mathcal{R}}(A) \longrightarrow \Bbbk_C$, $f \mapsto det_n^C(f)$ is muliplicative, i.e $det_n^C(fg) = det_n^C(f) det_n^C(g)$; $\forall g \in End_{C/\mathcal{R}}(A)$;*
3. *The map $\psi : End_{C/\mathcal{R}}(A) \longrightarrow M_n(\Bbbk_C)$, $f \longmapsto M_f^C$ is a monomorphism of $\Bbbk_C-$algebras;*
4. *Assume that $V_i \simeq V$, for all $1 \le i \le n$, then $Tr(f) = dim(V) Tr(M_f^C)$.*

Proof. (1) M_f^C is a block diagonal matrix: $M_f^C = diag(M_1, ..., M_m)$ where $M_j = (Tr(\varepsilon_l f \mu_k) dim(V_l)^{-1})_{j_1\le l,k\le j_{n_j}}$ $\forall\ 1 \le j \le m$. Then, we have

$det(M_f^C) = det(M_1)\ ...\ det(M_m)$
$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)(dim(V_1)^{-1})^{n_1} Tr(\varepsilon_{1_1} f\mu_{\sigma(1_1)})...Tr(\varepsilon_{1_{n_1}} f\mu_{\sigma(1_{n_1})})...$
$\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)(dim(V_m)^{-1})^{n_m} Tr(\varepsilon_{m_1} f\mu_{\sigma(m_1)})...Tr(\varepsilon_{m_{n_m}} f\mu_{\sigma(m_{n_m})})$
$= \sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)Tr(C^{w_{n_1}})^{-1} Tr(f^{n_1}D^1_\sigma)... \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)Tr(C^{w_{n_m}})^{-1}$
$Tr(f^{n_m}D^m_\sigma)$
$= Tr(f^{n_1}(\sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)Tr(C^{w_{n_1}})^{-1}D^1_\sigma)\ ...\ Tr(f^{n_m}(\sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)$
$Tr(C^{w_{n_m}})^{-1}D^m_\sigma))$
$= Tr(f^n(\sum_{\sigma\in\mathfrak{S}_{n_1}} \varepsilon(\sigma)Tr(C^{w_{n_1}})^{-1}D^1_\sigma \otimes ... \otimes \sum_{\sigma\in\mathfrak{S}_{n_m}} \varepsilon(\sigma)Tr(C^{w_{n_m}})^{-1}D^m_\sigma))$
$= det^C_n(f).$

(2) We have

$$
\begin{aligned}
(a^{fg}_{i,j})_{i,j} &= (Tr(\varepsilon_i\ f1_A g\ \mu_j)dim(V)^{-1})_{i,j} \\
&= (Tr(\varepsilon_i\ f\ \sum_{l=1}^{l=n}\mu_l\varepsilon_l\ g\ \mu_j)dim(V)^{-1})_{i,j} \quad (Negl(I,I)=\{0\}) \\
&= (\sum_{l=1}^{l=n} Tr(\varepsilon_i f\mu_l\ \varepsilon_l g\mu_j)dim(V)^{-1})_{i,j} \\
&= (\sum_{l=1}^{l=n} Tr(\ (k_{i,l}\otimes 1_V)\ \varepsilon_l g\mu_j)dim(V)^{-1})_{i,j} \\
&= (\sum_{l=1}^{l=n} Tr(k_{i,l}\otimes \varepsilon_l g\mu_j)dim(V)^{-1})_{i,j} \\
&= (\sum_{l=1}^{l=n} Tr(k_{i,l})Tr(\varepsilon_l g\mu_j)dim(V)^{-1})_{i,j} \\
&= (\sum_{l=1}^{l=n} Tr(k_{i,l}\otimes 1_V)dim(V)^{-1}Tr(\varepsilon_l g\mu_j)dim(V)^{-1})_{i,j} \\
&= (\sum_{l=1}^{l=n} a^f_{i,l}a^g_{l,j})_{i,j}.
\end{aligned}
$$

where $k_{i,l}$ is unique in $\mathbb{K}_C$ because $\varepsilon_i f \mu_l$ is an endomorphism of a simple object V. Then

$$det_n^C(f)det_n^C(g) = det(M_f^C)det(M_g^C) = det(M_f^C M_g^C) = det(M_{fg}^C) = det_n^C(fg).$$

(3)

(i) ψ is a morphism of $\mathbb{K}_C-$algebras. In fact, linearity is obtained by the fact that for any objects V and W of C, the group $Hom_C(V;W)$ acquires the structure of a $\mathbb{K}_C-$module with bilinear composition of morphisms. Furthermore, we have $\psi(fg) = \psi(f)\psi(g)$ by Theorem 5.3 (2).

(ii) Let $f \in End_{C/\mathcal{R}}(A)$ such that $\psi(f) = 0$. Then, $Tr(\varepsilon_i f \mu_j) = 0$ for all $1 \leq i, j \leq n$; but $\varepsilon_i f \mu_j$ is a morphism of a simple object, then, $\varepsilon_i f \mu_j = k_{i,j} \otimes 1_V$ for some unique $k_{i,j} \in \mathbb{K}_C$. Hence, for all $1 \leq i, j \leq n$ we have $k_{i,j} = 0$, because V is simple. Thus $\mu_i \varepsilon_i f \mu_j \varepsilon_j = 0$, and so $\sum_{i,j} \mu_i \varepsilon_i f \mu_j \varepsilon_j = 0$ (composition with μ_i in left and ε_j in right, then entering summand). Therefore, $f = 0 \ \ mod(\mathcal{R}_{A,A})$. Thus, ψ is injective.

(4)

$$\begin{aligned}
a_{i,i}^f &= Tr(\varepsilon_i f \mu_i)dim(V)^{-1} \ ; \quad 1 \leq \forall i \leq n \\
&\Leftrightarrow \ Tr(M_f^C) = \sum_{i=1}^{n} Tr(\varepsilon_i f \mu_i)dim(V)^{-1} \\
&\Leftrightarrow \ dim(V)Tr(M_f^C) = \sum_{i=1}^{n} Tr(\mu_i \varepsilon_i f) \\
&\Leftrightarrow \ dim(V)Tr(M_f^C) = Tr((\sum_{i=1}^{n} \mu_i \varepsilon_i)f) \\
&\Leftrightarrow \ dim(V)Tr(M_f^C) = Tr(f).
\end{aligned}$$

□

Remark 5.4. From the above Theorem 5.3 (2) and under the same hypotheses; naturality of the quantum determinant is then trivial, i.e:

$$\forall f \in End_{C/\mathcal{R}}(A), \ \ det_n^C(g^{-1}fg) = det_n^C(f), \ \ \forall g \in Aut_{C/\mathcal{R}}(A).$$

Remark 5.5. We can construct in some cases dominating families of simple objects verifying $\varepsilon_i\mu_j = \delta_{i;j}$, for all i, j, $1 \le i, j \le n$. In fact, let C be a semisimple ribbon $Ab-$category enriched over finite dimensional vector spaces over a field $\mathtt{K}$ (i.e, for any objects V and W of C, $Hom_C(V, W)$ is a finite dimensional $\mathtt{K}-$vector space) and let A be an object of C and V a simple one. The $\mathtt{K}-$vector space $Hom_C(V; A)$ is dualizable and its dual is $Hom_C(A; V)$; consider a basis $(\mu_i)_{i=1}^{i=n}$ of $Hom_C(V; A)$ (where n is its dimension over $\mathtt{K}$) and its dual basis $(\varepsilon_i)_{i=1}^{i=n}$ of $Hom_C(A; V)$. Then, A is dominated by $(V; \varepsilon_i; \mu_i)_{1\le i\le n}$, $ran(A) = n$ and moreover, we have $\varepsilon_i\mu_j = \delta_{i;j}$, for all i, j, $1 \le i, j \le n$.

Corollary 5.6. *Under the same hypotheses of Theorem 5.3. Assume moreover that $\varepsilon_i\mu_j = \delta_{i;j}$, for all i, j, $1 \le i, j \le n$. Then*

(a) *The map $\psi : End_{C/\mathcal{R}}(A) \longrightarrow M_n(\mathtt{K}_C)$; $f \longmapsto M_f^C$ is an isomorphism of $\mathtt{K}_C-$algebras.*

(b) *f is invertible in $C/\mathcal{R}$, if and only if $det_n^C(f)$ is invertible in $\mathtt{K}_C$.*

Proof. (a) By Theorem 5.3 (3); we are just still have to show that ψ is surjective. Let $M = (a_{i,j})_{1\le i,j\le n}$ and $f = \sum_{i,j} a_{i,j}\mu_i\varepsilon_j$. Then $Tr(\varepsilon_{i_0} f \mu_{j_0}) dim(V)^{-1} = a_{i_0,j_0}$, for all i_0, j_0, $1 \le i_0, j_0 \le n$, and so $\psi(f) = M$.

(b) Assume that f is invertible in $C/\mathcal{R}$. Then:

$$\begin{aligned}
1_I &= det_n^C(1_A) \\
&= det_n^C(ff^{-1}) \\
&= det_n^C(f) det_n^C(f^{-1}) \\
&= det_n^C(f^{-1}) det_n^C(f).
\end{aligned}$$

Hence, $(det_n^C(f))^{-1} = det_n^C(f^{-1})$.
Inversely, if $det_n^C(f)$ is invertible, then M_f^C is invertible, so there exists $N \in M_n(\mathtt{K}_C)$ such that $M_f^C N = N M_f^C = I_n$, but $N = \psi(g)$ for some unique $g \in End_{C/\mathcal{R}}(A)$. Hence

$$\psi(1_A) = I_n = M_f^C N = \psi(f)\psi(g) = \psi(fg)$$

and similarly $\psi(1_A) = \psi(gf)$, then $fg = gf = 1_A \ \ mod(\mathcal{R}_{A,A})$. □

Example 5.7. Let $C = (Proj(R); \otimes_R; R)$ be the category of finitely genarated and projective modules over a commutative ring R. This is a modular category with simple objects isomorphic to R. Let V be a free finitely generated and projective $R-$module with basis $(x_i)_{i=1}^{i=n}$. By Corollary 5.6, $ran_q(V) = n$ and $det_n^C(f)$, $f \in End_C(V)$, coincides with its classical determinant, $i.e$ of a representative matrix of f.

Example 5.8. This is due to Reshetikhin and Turaev [14]. It deals with the semisimple ribbon $Ab-$category (in fact modular [14]) associated to the Hopf algebra $\overline{U}_q$, i.e, the finite dimensional quotient of the Hopf algebra $U_q(Sl_2(\mathbb{C}))$ for q a root of unity. Moreover, a general principe is given in [14] to construct modular categories upon categories of modules over quantum groups at roots of unity. The objects of C are finite dimensional $\overline{U}_q-$modules and the simple objects are highest weight modules $\{V_\lambda\}_\lambda$ (see [10, 14], for more details). Hence, the quantum determinant of an endomorphism f of an $\overline{U}_q-$module is computed via the associated square matrix of f, by Theorem 5.3 (1).

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