Categories and
General Algebraic Structures
with Applications

Volume 17, Number 1, July 2022, 101-116.
https://doi.org/10.52547/cgasa.17.1.101

WWW.CGASA.ir

Universal extensions of specialization semilattices

Paolo Lipparini

Abstract. A specialization semilattice is a join semilattice together with a coarser preorder $\sqsubseteq$ satisfying an appropriate compatibility condition. If X is a topological space, then $(\mathcal{P}(X), \cup, \sqsubseteq)$ is a specialization semilattice, where $x \sqsubseteq y$ if $x \subseteq Ky$, for $x, y \subseteq X$, and K is closure. Specialization semilattices and posets appear as auxiliary structures in many disparate scientific fields, even unrelated to topology.

In a former work we showed that every specialization semilattice can be embedded into the specialization semilattice associated to a topological space as above. Here we describe the universal embedding of a specialization semilattice into an additive closure semilattice.

1 Introduction

The idea of *closure* is pervasive in mathematics. First, the notion is used in the sense of *hull*, *generated by*, for example when we consider the subgroup generated by a given subset of some group. In a slightly different but related sense, closure is a fundamental notion in topology. In both cases, "closed" sets are preserved under arbitrary intersections; in the topological case the union of two closed sets is still

Keywords: Specialization semilattice, closure semilattice, closure space, universal extension.
Mathematics Subject Classification [2010]: 06A15, 54A05, 06A12.
Received: 10 March 2022, Accepted: 28 May 2022.
ISSN: Print 2345-5853, Online 2345-5861.

closed; in most "algebraic" examples, the union of an upward directed family of closed subsets is still closed.

The general notion of a *closure space* which can be abstracted from the above examples has been dealt with or foreshadowed by such mathematicians as Schröder, Dedekind, Cantor, Riesz, Hausdorff, Moore, Čech, Kuratowski, Sierpiński, Tarski, Birkhoff and Ore, as listed in Erné [3], with applications, among others, to ordered sets, lattice theory, logic, algebra, topology, computer science and connections with category theory.

In many cases it is not necessary to describe the actual closure, we just need to know whether some object is contained or not in the closure. Even in topology, one frequently needs to consider only the *adherence* relation $p \in Ky$, meaning that the element p belongs to the topological closure of the subset y. Arguing in terms of adherence provides a conceivably more intuitive approach to continuity: as well-known, a function f between topological spaces is continuous if and only if f preserves the adherence relation, namely, if and only if $p \in Ky$ implies $f(p) \in Kf(y)$.

Similarly, we can consider the *specialization* relation $x \sqsubseteq y$ defined by $x \subseteq Ky$, for x, y subsets of some topological space X. It is a natural generalization of the *specialization preorder* defined on points of a topological space [6, Ex. 3.17e], [5]. As in the case of adherence, a function f from X to some other space Y is continuous if and only if the direct image function $f^{\rightarrow}$ is a homomorphism from the structure $(\mathcal{P}(X), \cup, \sqsubseteq)$ to $(\mathcal{P}(Y), \cup, \sqsubseteq)$.

The above "algebraization" of topology is thus significantly different from the classical approach presented in [10], where the operation K of closure is taken into account. The notion of homomorphism in [10] does not correspond to the notion of continuity. In fact, a function f between two spaces is continuous if and only if $f^{\rightarrow}(Kx) \subseteq Kf^{\rightarrow}(x)$, for all subsets x. On the other hand, a homomorphism φ of closure algebras [10] is assumed to satisfy the stronger condition $\varphi(Kx) = K\varphi(x)$.

In [9] we characterized *specialization semilattices*, those structures which can be embedded into $(\mathcal{P}(X), \cup, \sqsubseteq)$ for some topological space X, and *specialization posets*, which can be embedded into $(\mathcal{P}(X), \subseteq, \sqsubseteq)$. See (S1) - (S3) below. While our main interest was algebraic and model-theoretical, we realized that such structures appear in many distinct and unrelated settings.

A typical example of a specialization to which no closure can be associated is *inclusion modulo finite*. If X is an infinite set and we let $x \sqsubseteq y$ if $x \setminus y$ is finite, for $x, y \subseteq X$, then $(\mathcal{P}(X), \cup, \sqsubseteq)$ is a specialization semilattice. Inclusion modulo finite plays important roles, among other, in set theory, topology and model theory [1, 11]. From a slightly different perspective, working modulo finite corresponds to taking the quotient modulo the ideal of finite sets on the standard Boolean algebra on $\mathcal{P}(X)$. From the present point of view, a similar construction can be used to generate specialization semilattices: if $\varphi : \mathbf{S} \to \mathbf{T}$ is a semilattice homomorphism and we set $a \sqsubseteq b$ in S when $\varphi(a) \leq \varphi(b)$ in $\mathbf{T}$, then $\mathbf{S}$ is endowed with the structure of a specialization semilattice. As we shall show elsewhere, every specialization semilattice can indeed be constructed this way. Specialization semilattices are substructures of topological spaces in the language with union and $\sqsubseteq$, but, in a sense, they are also semilattices together with a quotient (or a congruence).

Under different terminology, specialization appears in [4] in the context of complete lattices, with deep and important applications to algebraic logic. See Conditions (1) - (2) in [4, Subsection 3.1]. Specialization semilattices arise also naturally in the theory of *tolerance spaces* [12], with applications to image analysis and information systems [13].

Causal spaces have been introduced by Kronheimer and Penrose [7] in connection with abstract foundations of general relativity. Causal spaces can be axiomatized as two partial orders, one finer than the other, and satisfying a further coherence condition. In particular, they are specialization posets. As another example, if μ is a measure on some set S of subsets of X, then $a \sqsubseteq_\mu b$ defined by $\mu(a) \leq \mu(b)$, for $a, b \in S$, is a preorder which forms a specialization poset together with inclusion. If μ is 2-valued, then we get a specialization semilattice. Such structures have been widely studied in connection with foundations of probability. See [8] and references there.

A *closure poset* (*semilattice*) is a partially ordered set (join semilattice) together with an isotone, extensive and idempotent operator K. See Remark 2.1. If K satisfies $K(a \vee b) = Ka \vee Kb$ and $K0 = 0$ in a closure semilattice with minimum 0, then K satisfies the Kuratowski axioms for topological closure. Closure posets and semilattices have many applications; see [3, 14] for references. As in the case of topological spaces, setting $a \sqsubseteq b$ if $a \leq Kb$ induces the structure of a specialization poset (semilattice) and a large part of the theory of closure posets applies to this more general setting. See [9] for more details and further examples.

Henceforth we were convinced that the notion of a specialization semilattice deserves an accurate study, both for its possible foundational relevance in connection with topology, and since the notion appears in many disparate fields.

The main result in [9] asserts that every specialization semilattice or poset can be embedded into a "topological" one. The extensions constructed in [9, Section 4] are not minimal and possibly neither canonical nor functorial. In search for a better-behaved extension, here we explicitly describe the universal embedding of a specialization semilattice into an additive closure semilattice. This is done in Section 3. In Section 4 we then show that the existence of such an embedding, as well as the existence of a multitude of other embeddings, follow from an abstract argument.

2 Preliminaries

A *specialization semilattice* [9, Definition 3.1] is a join semilattice endowed with a further preorder $\sqsubseteq$ which is coarser than the order $\leq$ induced by $\vee$ and satisfies the further compatibility relation (S3) below. In detail, a specialization semilattice $\mathbf{S}$ is a triple $(S, \vee, \sqsubseteq)$ such that $(S, \vee)$ is a semilattice and moreover

$$a \leq b \Rightarrow a \sqsubseteq b, \tag{S1}$$

$$a \sqsubseteq b \;\&\; b \sqsubseteq c \Rightarrow a \sqsubseteq c, \tag{S2}$$

$$a \sqsubseteq b \;\&\; a_1 \sqsubseteq b \Rightarrow a \vee a_1 \sqsubseteq b, \tag{S3}$$

for all elements $a, b, c, a_1 \in S$. Notice that from (S1) one gets

$$a \sqsubseteq a, \tag{S4}$$

for every a in S.

It can be shown [9, Remark 3.4(a)] that every specialization semilattice satisfies

$$a \sqsubseteq b \;\&\; a_1 \sqsubseteq b_1 \Rightarrow a \vee a_1 \sqsubseteq b \vee b_1. \tag{S7}$$

A *specialization poset* is a partially ordered set with a further preorder satisfying (S1) - (S2). Specialization posets occur naturally in many situations, but the theory of specialization semilattices is much cleaner and here we shall be mainly interested in the latter.

A *homomorphism* of specialization semilattices is a semilattice homomorphism η such that $a \sqsubseteq b$ implies $\eta(a) \sqsubseteq \eta(b)$. An *embedding* is an injective homomorphism satisfying the additional condition that $\eta(a) \sqsubseteq \eta(b)$ implies $a \sqsubseteq b$.

If $\mathbf{S}$ is a specialization semilattice, $a \in S$ and the set $S_a = \{b \in S \mid b \sqsubseteq a\}$ has a $\leq$-maximum, such a maximum shall be denoted by Ka and shall be called the *closure* of a. Notice that we require Ka to be the maximum of S_a, not just a supremum. Namely, we require $Ka \sqsubseteq a$.

In general, Ka need not exist in an arbitrary specialization semilattice: consider the example of inclusion modulo finite mentioned in the introduction. If Ka exists for every $a \in S$, then **S** shall be called a *principal* specialization semilattice.

Remark 2.1. (a) Principal specialization semilattices are in a one-to one correspondence with *closure semilattices*, that is, semilattices with a further operation K such that $a \leq Ka$, $KKa = Ka$, and $K(a \vee b) \geq Ka \vee Kb$.

If **C** is a closure semilattice, then setting $a \sqsubseteq b$ if $a \leq Kb$ makes **C** a specialization semilattice, and K turns out to be closure also in the sense of specialization semilattices. See [3, Section 3.1], in particular, [3, Proposition 3.9] for details.

(b) The clause $K(a \vee b) \geq Ka \vee Kb$ is equivalent to the condition that $c \geq a$ implies $Kc \geq Ka$. As a consequence, we get $K(a \vee b) \leq K(a \vee Kb)$ in closure semilattices. Moreover, $K(a \vee b) \geq Ka \geq a$, $K(a \vee b) \geq Kb$, so $K(a \vee b) \geq a \vee Kb$, hence $K(a \vee b) = KK(a \vee b) \geq K(a \vee Kb)$. In conclusion, as well-known, $K(a \vee b) = K(a \vee Kb)$ in every closure semilattice.

By the same argument, we could even prove $K(a \vee b) = K(Ka \vee Kb)$, but we shall not need this in what follows.

(c) If a and b are elements of some specialization semilattice and both Ka and Kb exist, then $Ka \leq Kb$ if and only if $a \sqsubseteq b$. Indeed, from $a \leq a$ and (S1) we get $a \sqsubseteq a$, thus $a \leq Ka$, by the definition of Ka. Hence if $Ka \leq Kb$, then $a \leq Kb$, which means $a \sqsubseteq b$, by the definition of Kb. Conversely, if $a \sqsubseteq b$, then from $Ka \sqsubseteq a$ and (S2) we get $Ka \sqsubseteq b$, which means $Ka \leq Kb$.

In particular, in a principal specialization semilattice, $Ka = Kb$ if and only if both $a \sqsubseteq b$ and $b \sqsubseteq a$.

If **S** and **T** are principal specialization semilattices, a *K-homomorphism* from **S** to **T** is a homomorphism η which preserves K, that is $\eta(Ka) = K\eta(a)$. Thus K-homomorphisms correspond to the natural notion of homomorphism for closure semilattices.

If η is a semilattice homomorphism between two principal specialization semilattices and η satisfies $\eta(Ka) = K\eta(a)$, then η is also a $\sqsubseteq$-homomorphism. Indeed, $a \sqsubseteq b$ is equivalent to $a \leq Kb$, hence $\eta(a) \leq \eta(Kb) = K\eta(b)$, which implies $\eta(a) \sqsubseteq \eta(b)$.

Notice that, even when **S** and **T** are principal, a specialization homomorphism need not be a K-homomorphism; see, for example, the second sentence in the following Remark 2.2. Of course, if either **S** or **T** fails to be principal, then it is not even possible to apply the notion of K-homomorphism. Whenever we speak of a homomorphism without further specifications, we always mean a homomorphism of specialization semilattices as introduced above, that is, we do not assume that homomorphisms are K-homomorphisms, unless specified otherwise.

A principal specialization semilattice (or a closure semilattice) is *additive* if $K(a \vee b) = Ka \vee Kb$.

Remark 2.2. If X is a topological space with topological closure K, then $(\mathcal{P}, \cup, K)$ is an additive closure semilattice, thus $(\mathcal{P}, \cup, \sqsubseteq)$ is a principal additive specialization semilattice, by Remark 2.1(a).

It can be checked that topological continuity corresponds to the notion of homomorphism between the associated specialization semilattices [9, Proposition 2.4]; on the other hand, the notion of K-homomorphism is stronger, and corresponds to the notion of a closed continuous map.

All the above comments apply to *closure spaces*, which are like topological spaces, except that the union of two closed subsets is not assumed to be closed, equivalently, closure is not assumed to satisfy $K(a \cup b) \subseteq Ka \cup Kb$. In a closure space the closure of the empty set is not assumed to be the empty set, either. Closure spaces occur naturally in algebra; for example, if **G** is a group, then $\mathcal{P}(G)$ becomes a closure space if subgroups are considered as the closed subsets of G. See [3] for more examples and details. Of course, in the case of a closure space, the associated specialization semilattice as above is still principal, but not necessarily additive.

Further details about the above notions can be found in [9].

A *specialization semilattice with* 0 is a specialization semilattice with a constant 0 which is a neutral element with respect to the semilattice operation, thus a minimal element in the induced order, and furthermore satisfies

$$a \sqsubseteq 0 \Rightarrow a = 0. \tag{S0}$$

A homomorphism η of specialization semilattices with 0 is required to satisfy $\eta(0) = 0$. When some risk of ambiguity might occur, we shall explicitly mention that the homomorphism is 0-*preserving*.

Remark 2.3. We shall generally assume that specialization semilattices have a 0, but this assumption is only for simplicity. In fact, if **S** is an arbitrary specialization semilattice, then by adding a new $\vee$-neutral element 0 and setting $0 \sqsubseteq a$, for every

$a \in S \cup \{0\}$, and $a \not\sqsubseteq 0$, for every $a \in S$, we get a specialization semilattice with 0. Conversely, if **S** is a specialization semilattice with 0, then $S \setminus \{0\}$ has naturally the structure of a specialization semilattice.

3 Universal extensions

Given any specialization semilattice **S**, we now construct a "universal" principal additive extension $\widetilde{\mathbf{S}}$ of **S**.

Definition 3.1. Suppose that **S** is a specialization semilattice.

On the product $S \times S$ define an equivalence relation $\sim$ by

(*) $(a, b) \sim (c, d)$ if and only if, in **S**, $b \sqsubseteq d$, $d \sqsubseteq b$ and there are $a_1, c_1 \in S$ such that $a_1 \sqsubseteq b$, $c_1 \sqsubseteq d$ and $a \leq c \vee c_1$, $c \leq a \vee a_1$.

We shall check in Lemma 3.3(i) below that $\sim$ is actually an equivalence relation. Let $\widetilde{S} = (S \times S)/\sim$.

Define $K : \widetilde{S} \to \widetilde{S}$ by $K[a, b] = [a, a \vee b]$, where $[x, y]$ is the $\sim$ class of the pair (x, y). In Lemma 3.3(ii)(iii) we shall prove that K is well-defined and that $\widetilde{S}$ naturally inherits a semilattice operation $\vee$ from the semilattice product $\mathbf{S} \times \mathbf{S}$.

Define $\sqsubseteq$ on $\widetilde{S}$ by $[a, b] \sqsubseteq [c, d]$ if $[a, b] \leq K[c, d]$, where $\leq$ is the order induced by $\vee$ and let $\widetilde{\mathbf{S}} = (\widetilde{S}, \vee, \sqsubseteq)$, $\widetilde{\mathbf{S}}' = (\widetilde{S}, \vee, K)$.

If **S** is a specialization semilattice with 0, define $\upsilon_{\mathbf{S}} : S \to \widetilde{S}$ by $\upsilon_{\mathbf{S}}(a) = [a, 0]$.

We intuitively think of $[a, b]$ as $a \vee Kb$, where Kb is the "new" closure we need to introduce; in particular, $[a, 0]$ corresponds to a and $[0, b]$ corresponds to a new element Kb.

Theorem 3.2. *Suppose that* **S** *is a specialization semilattice with* 0*. Let* $\widetilde{\mathbf{S}}$ *and* $\upsilon_{\mathbf{S}}$ *be as in Definition 3.1. Then the following statements hold.*

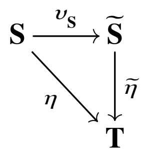
(1) $\widetilde{\mathbf{S}}$ *is a principal additive specialization semilattice with* 0.

(2) $\upsilon_{\mathbf{S}}$ *is a* 0*-preserving embedding of* **S** *into* $\widetilde{\mathbf{S}}$.

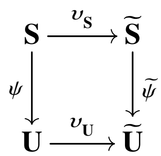
(3) *The pair* $(\widetilde{\mathbf{S}}, \upsilon_{\mathbf{S}})$ *has the following universal property.*

For every principal additive specialization semilattice **T** *and every homomorphism* $\eta : \mathbf{S} \to \mathbf{T}$*, there is a unique* K*-homomorphism* $\widetilde{\eta} : \widetilde{\mathbf{S}} \to \mathbf{T}$ *such that*

$\eta = \upsilon_{\mathbf{S}} \circ \widetilde{\eta}$.



(4) *If* $\mathbf{U}$ *is another specialization semilattice with* 0 *and* $\psi : \mathbf{S} \to \mathbf{U}$ *is a* 0-*preserving homomorphism, then there is a unique* K-*homomorphism* $\widetilde{\psi} : \widetilde{\mathbf{S}} \to \widetilde{\mathbf{U}}$ *making the following diagram commute.*



We first need to check that Definition 3.1 is correct. This is the content of the following lemma.

Lemma 3.3. *Assume the notation and the definitions in* 3.1.

(i) *The relation* $\sim$ *on* $S \times S$ *is an equivalence relation.*

(ii) *The operation* K *is well-defined on the* $\sim$-*equivalence classes.*

(iii) *The relation* $\sim$ *is a semilattice congruence on the semilattice product* $\mathbf{S}\times\mathbf{S}$, *hence* $\widetilde{\mathbf{S}}$ *inherits a semilattice structure from* $\mathbf{S} \times \mathbf{S}$.

(iv) *If* $\mathbf{S}$ *is a specialization semilattice with* 0, *then* K *satisfies*

$$K[a, b] = [0, a \vee b] = [a, a \vee b].$$

Proof. (i) The relation $\sim$ is symmetric, since its definition is symmetric. It is reflexive because of (S4), since if $a = c$ and $b = d$, then we can take $a_1 = c_1 = b$ in (*). We now check transitivity. Let $(a, b) \sim (c, d)$ be witnessed by elements a_1, c_1 as in (*). Let $(c, d) \sim (e, f)$ be witnessed by c'_1, e'_1, thus $d \sqsubseteq f$, $f \sqsubseteq d$, $c'_1 \sqsubseteq d$, $e'_1 \sqsubseteq f$ and $c \leq e \vee e'_1$, $e \leq c \vee c'_1$. Then $b \sqsubseteq f$, by $b \sqsubseteq d$, $d \sqsubseteq f$ and (S2). Symmetrically $f \sqsubseteq b$. From $a \leq c \vee c_1$ and $c \leq e \vee e'_1$ we get $a \leq e \vee e'_1 \vee c_1$. Moreover, $c_1 \sqsubseteq d \sqsubseteq f$, hence $c_1 \sqsubseteq f$ by (S2), thus $e'_1 \vee c_1 \sqsubseteq f$, by $e'_1 \sqsubseteq f$ and (S3). Symmetrically, $e \leq a \vee a_1 \vee c'_1$ and $a_1 \vee c'_1 \sqsubseteq b$. This means that the elements $a''_1 = a_1 \vee c'_1$ and $e''_1 = e'_1 \vee c_1$ witness $(a, b) \sim (e, f)$.

(ii) We have to show that if $(a, b) \sim (c, d)$, then $(a, a \vee b) \sim (c, c \vee d)$. Suppose that $(a, b) \sim (c, d)$ is witnessed by a_1 and c_1, as given by (*) in Definition 3.1.

From $a \leq c \vee c_1$ and $c_1 \sqsubseteq d$ we get $a \sqsubseteq c \vee c_1 \sqsubseteq c \vee d$, by (S1), (S4) and (S7), hence $a \sqsubseteq c \vee d$, by (S2). Since $b \sqsubseteq d \leq c \vee d$, hence $b \sqsubseteq c \vee d$, by (S1) and (S2), then we also have $a \vee b \sqsubseteq c \vee d$, by (S3). Symmetrically, $c \vee d \sqsubseteq a \vee b$. The remaining conditions in Clause (*) in Definition 3.1 are verified by using the same a_1 and c_1. Indeed, from $a_1 \sqsubseteq b$ and $b \leq a \vee b$, hence $b \sqsubseteq a \vee b$ by (S1), we get $a_1 \sqsubseteq a \vee b$, by (S2). Similarly, $c_1 \sqsubseteq c \vee d$. The last two conditions in (*) hold since these conditions do not involve b and d, and a and c have remained unchanged.

Hence $(a, a \vee b) \sim (c, c \vee d)$. This means that K is well-defined.

(iii) We have to show that if $(a, b) \sim (c, d)$, then $(a, b) \vee (e, f) \sim (c, d) \vee (e, f)$, that is, $(a \vee e, b \vee f) \sim (c \vee e, d \vee f)$. Since $(a, b) \sim (c, d)$, then $b \sqsubseteq d$, hence $b \vee f \sqsubseteq d \vee f$ follows from (S4) and (S7). Symmetrically, $d \vee f \sqsubseteq b \vee f$. Again by $(a, b) \sim (c, d)$, there is $c_1 \sqsubseteq d$ such that $a \leq c \vee c_1$. Then $c_1 \sqsubseteq d \vee f$ by (S2) (since $d \sqsubseteq d \vee f$ by (S1)); moreover, $a \vee e \leq c \vee e \vee c_1$. Performing the symmetrical argument, we get $a_1 \sqsubseteq b \vee f$ and $c \vee e \leq a \vee e \vee a_1$. This means that the same elements c_1 and a_1 witnessing $(a, b) \sim (c, d)$ also witness $(a \vee e, b \vee f) \sim (c \vee e, d \vee f)$. We have shown that $\sim$ is a semilattice congruence.

(iv) Since we have shown that K is well defined on the equivalence classes, in order to get the equation in (iv) it is enough to check that if $\mathbf{S}$ has a 0, then $(a, a \vee b) \sim (0, a \vee b)$. The condition $a \vee b \sqsubseteq a \vee b$ is from (S4). Then take $c_1 = a \vee b$ and $a_1 = 0$ in order to witness (*). □

Proof of Theorem 3.2. Definition 3.1 is justified by Lemma 3.3. In order to prove Clause (1) in the theorem it is easier to deal with the structure $\widetilde{\mathbf{S}}'$ from Definition 3.1.

Claim. $\widetilde{\mathbf{S}}' = (\widetilde{S}, \vee, K)$ is an additive closure semilattice.

We have shown in Lemma 3.3(iii) that $(\widetilde{S}, \vee)$ is a semilattice; it remains to check that K is an additive closure. Indeed, by the definition of K,

$$\begin{aligned}
&[a, b] \leq [a, a \vee b] = K[a, b],\\
&KK[a, b] = K[a, a \vee b] = [a, a \vee a \vee b] = K[a, b], \text{ and}\\
&K([a, b] \vee [c, d]) = K[a \vee c, b \vee d] = [a \vee c, a \vee b \vee c \vee d]\\
&\qquad\qquad = [a, a \vee b] \vee [c, c \vee d] = K[a, b] \vee K[c, d].
\end{aligned}$$

Having proved the claim, Clause (1) in the theorem follows from Remark 2.1(a). The 0 of $\widetilde{\mathbf{S}}$ is the class $[0, 0]$, since $(0, 0)$ is a neutral element for $\mathbf{S} \times \mathbf{S}$, hence $[0, 0]$ is neutral for the quotient $\widetilde{\mathbf{S}} = \mathbf{S}/\sim$. Moreover, $[a, b] \sqsubseteq [0, 0]$ means

$[a, b] \leq K[0, 0] = [0, 0]$ and this implies $[a, b] = [0, 0]$, since $[0, 0]$ is the minimum of $\widetilde{\mathbf{S}}$.

Now we prove (2). We have $\upsilon_{\mathrm{S}}(a \vee b) = [a \vee b, 0] = [a, 0] \vee [b, 0] = \upsilon_{\mathrm{S}}(a) \vee \upsilon_{\mathrm{S}}(b)$, hence υ_{S} is a semilattice homomorphism. Moreover, υ_{S} is injective, since $\upsilon_{\mathrm{S}}(a) = \upsilon_{\mathrm{S}}(c)$ means $(a, 0) \sim (c, 0)$ and this happens only if $a \leq c$ and $c \leq a$, that is, $a = c$. Indeed, if $b = d = 0$ and $a_1 \sqsubseteq b$, $c_1 \sqsubseteq d$ in Clause (*) in Definition 3.1, then $a_1 = c_1 = 0$ by (S0).

Furthermore, if $a \sqsubseteq b$ in $\mathbf{S}$, then $a \vee b \sqsubseteq b$, by (S4) and (S3). We then get $(a \vee b, b) \sim (b, b)$, by (S4) and taking $c_1 = a \vee b$ and $a_1 = b$ in (*) from Definition 3.1. Hence $[a, 0] \leq [a \vee b, b] = [b, b] = K[b, 0]$, that is, $\upsilon_{\mathrm{S}}(a) \sqsubseteq \upsilon_{\mathrm{S}}(b)$, according to the definition of $\sqsubseteq$ on $\widetilde{S}$ in Definition 3.1. This shows that υ_{S} is a $\sqsubseteq$-homomorphism.

In fact, υ_{S} is an embedding, since from $\upsilon_{\mathrm{S}}(a) \sqsubseteq \upsilon_{\mathrm{S}}(b)$, that is, $[a, 0] \leq K[b, 0] = [b, b]$, we get $[a \vee b, b] = [a, 0] \vee [b, b] = [b, b]$, that is, $(a \vee b, b) \sim (b, b)$, hence, according to (*) in Definition 3.1, $a \vee b \leq b \vee c_1$, for some $c_1 \sqsubseteq b$. From $a \leq a \vee b \leq b \vee c_1$ and (S1) we get $a \sqsubseteq b \vee c_1$. From $c_1 \sqsubseteq b$, (S4) and (S3) we get $b \vee c_1 \sqsubseteq b$, hence $a \sqsubseteq b$ by (S2). This shows that υ_{S} is an embedding.

Since $\upsilon_{\mathrm{S}}(0) = [0, 0]$, then υ_{S} is 0-preserving.

We now deal with (3). If $\eta : \mathbf{S} \to \mathbf{T}$ is a homomorphism and there exists $\widetilde{\eta}$ such that $\eta = \upsilon_{\mathrm{S}} \circ \widetilde{\eta}$, then $\widetilde{\eta}([a, 0]) = \widetilde{\eta}(\upsilon_{\mathrm{S}}(a)) = \eta(a)$, for every $a \in S$. If furthermore $\widetilde{\eta}$ is a K-homomorphism, then $\widetilde{\eta}([0, b]) =^{3.3} \widetilde{\eta}(K[b, 0]) = K\widetilde{\eta}([b, 0]) = K\eta(b)$, by the equation in Lemma 3.3(iv). Since $\widetilde{\eta}$ is supposed to be a lattice homomorphism, it follows that $\widetilde{\eta}([a, b]) = \widetilde{\eta}([a, 0]) \vee \widetilde{\eta}([0, b]) = \eta(a) \vee K\eta(b)$, hence if $\widetilde{\eta}$ exists it is unique. The above considerations make sense since $\mathbf{T}$ is assumed to be principal, so that $K\eta(b)$ exists.

It is then enough to show that the above condition $\widetilde{\eta}([a, b]) = \eta(a) \vee K\eta(b)$ actually determines a K-homomorphism $\widetilde{\eta}$ from $\widetilde{\mathbf{S}}$ to $\mathbf{T}$.

First, we need to check that if $(a, b) \sim (c, d)$, then $\eta(a) \vee K\eta(b) = \eta(c) \vee K\eta(d)$, so that $\widetilde{\eta}$ is well-defined. In fact, suppose that $(a, b) \sim (c, d)$ is given by (*) in 3.1. From $b \sqsubseteq d$ and $d \sqsubseteq b$, we get $\eta(b) \sqsubseteq \eta(d)$ and $\eta(d) \sqsubseteq \eta(b)$, since η is a homomorphism, so that $K\eta(b) = K\eta(d)$, by the final sentence in Remark 2.1(c). Moreover, if $c_1 \sqsubseteq d$, then $\eta(c_1) \sqsubseteq \eta(d)$, so that $\eta(c_1) \leq K\eta(d)$, by the definition of K. Since in addition $a \leq c \vee c_1$, then $\eta(a) \leq \eta(c \vee c_1) = \eta(c) \vee \eta(c_1) \leq \eta(c) \vee K\eta(d)$, since η is a homomorphism, so that $\eta(a) \vee K\eta(b) \leq \eta(c) \vee K\eta(d)$, since we have already shown that $K\eta(b) = K\eta(d)$. Symmetrically, $\eta(c) \vee K\eta(d) \leq \eta(a) \vee K\eta(b)$, hence $\eta(c) \vee K\eta(d) = \eta(a) \vee K\eta(b)$. This means that $\widetilde{\eta}$ is well-defined.

We now check that $\widetilde{\eta}$ is a semilattice homomorphism. Indeed,

$$\begin{aligned}\widetilde{\eta}([a,b]) \vee \widetilde{\eta}([c,d]) &= \eta(a) \vee K\eta(b) \vee \eta(c) \vee K\eta(d)\\ &= \eta(a) \vee \eta(c) \vee K\eta(b) \vee K\eta(d)\\ &=^{\mathrm{A}} \eta(a \vee c) \vee K(\eta(b) \vee \eta(d))\\ &= \eta(a \vee c) \vee K\eta(b \vee d) = \widetilde{\eta}([a \vee c, b \vee d]),\end{aligned}$$

where we have used the definition of $\widetilde{\eta}$, the assumption that η is a semilattice homomorphism and in the identity marked with the superscript A we have used the assumption that $\mathbf{T}$ is additive.

Finally, $\widetilde{\eta}$ is a K-homomorphism, since

$$\begin{aligned}\widetilde{\eta}(K[a,b]) = \widetilde{\eta}([a, a \vee b]) = \eta(a) \vee K\eta(a \vee b) =^{\diamond} K\eta(a \vee b) =\\ K(\eta(a) \vee \eta(b)) =^{2.1} K(\eta(a) \vee K\eta(b)) = K\widetilde{\eta}([a,b]),\end{aligned}$$

where we have used the definitions of K and $\widetilde{\eta}$, the assumption that η is a homomorphism of specialization semilattices with 0 and Remark 2.1(b). The equation marked with $\diamond$ follows from $\eta(a) \leq \eta(a \vee b) \leq K\eta(a \vee b)$.

Notice that we do not need to assume that $\mathbf{T}$ has a 0, in order to get (3). However, if $\mathbf{T}$ has a 0 and η is 0-preserving, then $\widetilde{\eta}$ is 0-preserving, too, since we have proved that υ_{S} is 0-preserving and that the diagram in (3) commutes.

Having proved Clause (3) in the theorem, we now prove Clause (4). Suppose that $\mathbf{U}$ is a specialization semilattice with 0 and $\psi : \mathbf{S} \to \mathbf{U}$ is a 0-preserving homomorphism. Then $\eta = \psi \circ \upsilon_{\mathrm{U}}$ is a 0-preserving homomorphism from $\mathbf{S}$ to $\widetilde{\mathbf{U}}$, by clauses (1) and (2) applied to $\mathbf{U}$. Applying Clause (3) to η and $\mathbf{T} = \widetilde{\mathbf{U}}$, and by the last comment in the proof of (3), we get that there is a unique 0-preserving K-homomorphism $\widetilde{\eta}$ from $\widetilde{\mathbf{S}}$ to $\widetilde{\mathbf{U}}$ such that $\eta = \upsilon_{\mathrm{S}} \circ \widetilde{\eta}$. Letting $\widetilde{\psi} = \widetilde{\eta}$, then the diagram in (4) commutes, since we have taken $\eta = \psi \circ \upsilon_{\mathrm{U}}$. Conversely, if $\widetilde{\psi}$ is a K-homomorphism which makes the diagram in (4) commute, then necessarily $\widetilde{\psi} = \widetilde{\eta}$, because of the unicity of $\widetilde{\eta}$ in (3). □

Notice that υ_{S} as given by Theorem 3.2(2) does not necessarily preserve existing closures in $\mathbf{S}$: just consider the case in which $\mathbf{S}$ is principal but not additive, then closures necessarily are modified, since $\widetilde{\mathbf{S}}$ turns out to be additive.

Moreover, it is necessary to ask that $\widetilde{\eta}$ is a K-homomorphism in Theorem 3.2(3); it is not enough to assume that $\widetilde{\eta}$ is just a homomorphism. Indeed, let $\mathbf{S} = \mathbb{N}$ with max as join and with $n \sqsubseteq m$, for all $m, n > 0$. Then $\widetilde{\mathbf{S}}$ is isomorphic to $\mathbf{S} \cup \{\infty\}$, where $Ka = \infty$, for every $a \in S \cup \{\infty\}$, $a \neq 0$. Let $T = \{0, 1, 2\}$ with $2 \sqsubseteq 1$ and with the standard interpretation otherwise. Let $\eta : \mathbf{S} \to \mathbf{T}$ with $\eta(0) = 0$ and $\eta(n) = 1$ otherwise. Then the only K-homomorphism extending η must send ∞ to $2 = K(1)$. However, if we set $\eta^*(\infty) = 1$, we still get a (not K-) homomorphism from $\widetilde{\mathbf{S}}$ to $\mathbf{T}$ extending η.

Remark 3.4. For simplicity, we have stated and proved Theorem 3.2 for specialization semilattices with 0, but the theorem holds for arbitrary specialization semilattices.

If $\mathbf{S}_1$ does not have a 0, first apply the theorem to $\mathbf{S} = \mathbf{S}_1 \cup \{0\}$ as constructed in Remark 2.3 and then restrict to $\mathbf{S}_1$ and $\widetilde{\mathbf{S}} \setminus \{0\}$. Notice that $\upsilon_{\mathbf{S}}$ sends 0 to 0.

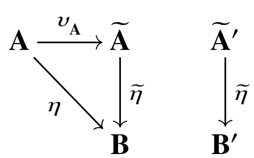
In order to prove (3), if $\eta_1 : \mathbf{S}_1 \to \mathbf{T}_1$, add a new 0 to $\mathbf{T}_1$, getting some specialization semilattice $\mathbf{T}$. Extend η_1 to some homomorphism $\eta : \mathbf{S} \to \mathbf{T}$ by setting $\eta(0) = 0$. Having obtained (3) in the extended situation, it follows that (3) holds for the original η_1, $\mathbf{S}_1$ and $\mathbf{T}_1$.

4 More general universal extensions

In the present section we assume that the reader is familiar with some basic notions of model theory [2]. The following lemma about the existence of universal objects is a folklore argument. A *subreduct* is a substructure of some reduct.

In the next lemma $\mathcal{L} \subseteq \mathcal{L}'$ are two languages, $\mathcal{K}'$ is a class of models for $\mathcal{L}'$ and $\mathcal{K}$ is the class of all subreducts in the language $\mathcal{L}$ of members of $\mathcal{K}'$. We adopt the convention that models in $\mathcal{K}'$ are denoted by $\mathbf{A}'$, $\mathbf{B}'$, ... and $\mathbf{A}$, $\mathbf{B}$, ... are the corresponding $\mathcal{L}$-reducts.

Lemma 4.1. *Under the above assumptions, if $\mathcal{K}'$ is closed under isomorphism, substructures and products, then, for every $\mathbf{A} \in \mathcal{K}$, there are $\widetilde{\mathbf{A}}' \in \mathcal{K}'$ and an $\mathcal{L}$-embedding $\upsilon_{\mathbf{A}} : \mathbf{A} \to \widetilde{\mathbf{A}}$ such that, for every $\mathbf{B}' \in \mathcal{K}'$ and $\mathcal{L}$-homomorphism $\eta : \mathbf{A} \to \mathbf{B}$, there is a unique $\mathcal{L}'$-homomorphism $\widetilde{\eta} : \widetilde{\mathbf{A}}' \to \mathbf{B}'$ such that $\eta = \upsilon_{\mathbf{A}} \circ \widetilde{\eta}$.*



The structure $\widetilde{\mathbf{A}}'$ is unique up to isomorphism over $\upsilon_{\mathbf{A}}(A)$. As a consequence, if $\mathbf{E} \in \mathcal{K}$ and $\psi : \mathbf{A} \to \mathbf{E}$ is an $\mathcal{L}$-homomorphism, then ψ lifts to an $\mathcal{L}'$-homomorphism $\widetilde{\psi} : \widetilde{\mathbf{A}}' \to \widetilde{\mathbf{E}}'$ making the following diagram commute.

$$
\begin{array}{ccccc}
\mathbf{A} & \xrightarrow{\upsilon_{\mathbf{A}}} & \widetilde{\mathbf{A}} & \qquad & \widetilde{\mathbf{A}}' \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\widetilde{\psi}} & & \downarrow{\scriptstyle\widetilde{\psi}} \\
\mathbf{E} & \xrightarrow{\upsilon_{\mathbf{E}}} & \widetilde{\mathbf{E}} & & \widetilde{\mathbf{E}}'
\end{array}
$$

Proof. The proof is a standard construction of free objects. Since $\mathbf{A} \in \mathcal{K}$, then $\mathbf{A}$ is a subreduct of some $\mathbf{C}' \in \mathcal{K}'$. Since $\mathcal{K}'$ is closed under substructures, we can choose $\mathbf{C}'$ in such a way that $\mathbf{C}'$ is generated by A in the language $\mathcal{L}'$. Consider the class of all $\mathbf{C}' \in \mathcal{K}'$ such that there is a homomorphism ξ from $\mathbf{A}$ to $\mathbf{C}$ and $\mathbf{C}'$ is generated by $\xi(A)$ in the language $\mathcal{L}'$; by the preceding sentence this class is nonempty. If $\xi, \mathbf{C}'$ and $\xi_1, \mathbf{C}'_1$ are as above, let us call the pairs $(\xi, \mathbf{C}')$ and $(\xi_1, \mathbf{C}'_1)$ *equivalent* if there is an isomorphism $\psi : \mathbf{C}'_1 \to \mathbf{C}'$ such that $\xi = \xi_1 \circ \psi$, namely, $\mathbf{C}'$ and $\mathbf{C}'_1$ are isomorphic over the image of A. Let $(\mathbf{C}'_i, \xi_i)_{i \in I}$ be a family of representatives for each equivalence class. Since each $\mathbf{C}'$ as above is generated by $\xi(A)$ in the language $\mathcal{L}'$, we have $|C'| \leq \sup\{\omega, |A|, |\mathcal{L}'|\}$, hence any representative can be taken over some fixed set of cardinality $\sup\{\omega, |A|, |\mathcal{L}'|\}$; in conclusion, there is a set—not a proper class—of such representatives.

Let $\mathbf{D}' = \prod_{i \in I} \mathbf{C}'_i$, thus $\mathbf{D}' \in \mathcal{K}'$, since $\mathcal{K}'$ is closed under products. Let $\widetilde{\mathbf{A}}'$ be the substructure of $\mathbf{D}'$ $\mathcal{L}'$-generated by the sequences $(\xi_i(a))_{i \in I}$, for a varying in A. Since $\mathcal{K}'$ is closed under substructures, then $\widetilde{\mathbf{A}}' \in \mathcal{K}'$. Moreover, the function which assigns to $a \in A$ the sequence $(\xi_i(a))_{i \in I}$ is an $\mathcal{L}$-embedding $\upsilon_{\mathbf{A}}$ from $\mathbf{A}$ to $\widetilde{\mathbf{A}}$; $\upsilon_{\mathbf{A}}$ is an embedding because of the first sentence in the proof. Notice that, for every $i \in I$, the projection π_i from $\mathbf{D}'$ to $\mathbf{C}'_i$ induces a homomorphism $\zeta_i : \widetilde{\mathbf{A}}' \to \mathbf{C}'_i$ such that $\xi_i = \upsilon_{\mathbf{A}} \circ \zeta_i$.

If $\mathbf{B}' \in \mathcal{K}'$ and $\eta : \mathbf{A} \to \mathbf{B}$ is a homomorphism, let $\mathbf{B}'_1$ be the $\mathcal{L}'$-substructure of $\mathbf{B}'$ generated by $\eta(A)$, let ι be the inclusion embedding from $\mathbf{B}'_1$ to $\mathbf{B}'$ and let η_1 be the function induced by η from $\mathbf{A}$ to $\mathbf{B}_1$, that is, $\eta = \eta_1 \circ \iota$. By the choice of the $\mathbf{C}'_i$s, $\mathbf{B}'_1$ is isomorphic to $\mathbf{C}'_i$, for some $i \in I$, through an isomorphism ψ such that $\eta_1 = \xi_i \circ \psi$, hence $\eta = \eta_1 \circ \iota = \xi_i \circ \psi \circ \iota = \upsilon_{\mathbf{A}} \circ \zeta_i \circ \psi \circ \iota$. It follows that $\widetilde{\eta} = \zeta_i \circ \psi \circ \iota$ is the desired homomorphism.

Since, by construction, $\widetilde{\mathbf{A}}'$ is $\mathcal{L}'$-generated by $\upsilon_{\mathbf{A}}(A)$, then every element of $\widetilde{\mathbf{A}}'$ has the form $t(\upsilon_{\mathbf{A}}(a_1), \dots, \upsilon_{\mathbf{A}}(a_n))$, for some natural number n, some term t of $\mathcal{L}'$ and elements $a_1, \dots, a_n$ of A. The requests that $\widetilde{\eta}$ be an $\mathcal{L}'$-homomorphism and that $\eta = \upsilon_{\mathbf{A}} \circ \widetilde{\eta}$ imply that

$$\begin{aligned}\widetilde{\eta}(t(\upsilon_{\mathbf{A}}(a_1), \dots, \upsilon_{\mathbf{A}}(a_n))) &= t(\widetilde{\eta}(\upsilon_{\mathbf{A}}(a_1)), \dots, \widetilde{\eta}(\upsilon_{\mathbf{A}}(a_n))) \\ &= t(\eta(a_1), \dots, \eta(a_n)),\end{aligned}$$

hence $\widetilde{\eta}$ is uniquely determined.

To prove the last statement, just take $\eta = \psi \circ \upsilon_{\mathbf{E}}$, $\mathbf{B} = \widetilde{\mathbf{E}}$ and argue as in the proof of clause (4) in Theorem 3.2. □

In particular, Lemma 4.1 applies when $\mathcal{K}'$ is the class of the models of some universal Horn first-order theory T' in the language $\mathcal{L}'$.

Lemma 4.1, together with the above comment, can be applied in all the situations described below.

(C1) $\mathcal{L}'$ is the language of Boolean algebras plus a binary relation symbol $\sqsubseteq$ and a unary operation symbol K. T' is the theory of *closure algebras*, that is, T' contains the axioms for Boolean algebras plus axioms saying that $K0 = 0$ and K is extensive, idempotent and additive and let us add to T' an axiom defining $\sqsubseteq$, namely, $a \sqsubseteq b \Leftrightarrow a \leq Kb$.

Finally, $\mathcal{L} = \{\vee, \sqsubseteq\}$.

(C2) $\mathcal{L}'$ is the language of closure semilattices plus a binary relation symbol $\sqsubseteq$. T' is the theory of closure semilattices plus axioms defining $\sqsubseteq$, as above, $\mathcal{L} = \{\vee, \sqsubseteq\}$.

(C3) As in (C1), but K is only assumed to be extensive, idempotent and isotone.

(C4) As in (C2), plus the assumption that K is additive.

(C5) As in (C2), plus the assumption that K satisfies $a \vee Kb = K(a \vee b)$.

(C6) $\mathcal{L}'$ is the language of closure posets plus a binary relation symbol $\sqsubseteq$. T' is the theory of closure posets plus axioms defining $\sqsubseteq$. Let $\mathcal{L} = \{\leq, \sqsubseteq\}$.

(C7) We can allow $\mathcal{L} = \{\leq, \sqsubseteq\}$ also in all cases (C1)-(C5), adding the symbol $\leq$ to $\mathcal{L}'$, with its definition $a \leq b \Leftrightarrow a \vee b = b$.

Recall that $\mathcal{K}$ is the class of all $\mathcal{L}$-subreducts of models of $\mathcal{K}'$. In cases (C1) - (C5) the class $\mathcal{K}$ turns out to be the class of all specialization semilattices, since we have proved in [9, Theorem 4.10] that every specialization semilattice can be embedded into the specialization semilattice associated to some topological space X. In particular, this provides an embedding into the specialization closure algebra $(P(X), \cap, \cup, \complement, \emptyset, X, K, \sqsubseteq)$; for cases (C2) - (C4) it is then sufficient to consider an appropriate reduct.

For case (C5), it follows from the proof of [9, Theorem 4.8] that every specialization semilattice can be extended to some principal specialization semilattice satisfying $a \vee Kb = K(a \vee b)$. In fact, for case (C5) the construction in the proof [9, Theorem 4.8] provides an explicit description for the universal object whose existence follows from Lemma 4.1. Notice also that Theorem 3.2 here provides a description for the universal object corresponding to (C4).

In cases (C6) and (C7) the class K is the class of specialization posets, since we have shown in [9, Proposition 4.15] that every specialization poset can be embedded into the order-reduct of some specialization semilattice. Then use the arguments for (C1) - (C5).

It is an open problem to provide an explicit description of the structure $\widetilde{\mathbf{A}}'$ given by Lemma 4.1 in cases (C1) - (C3) and (C6) - (C7).

Acknowledgement

We thank the anonymous referee for many useful comments which helped to improve the paper.

Work performed under the auspices of G.N.S.A.G.A.

The author acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

References

[1] Blass, A., *Combinatorial cardinal characteristics of the continuum*, in Foreman, M., and Kanamori, A. (eds.), “Handbook of Set Theory”, Springer, Dordrecht, 2010, 395-489.

[2] Chang, C.C. and Keisler, H.J., “Model theory”, Studies in Logic and the Foundations of Mathematics **73**, North-Holland Publishing Co., American Elsevier Publishing Co., Inc., 1973, third expanded edition, 1990.

[3] Erné, M., *Closure*, in Mynard, F., and Pearl E. (eds), “Beyond topology”, Contemp. Math. **486**, Amer. Math. Soc., Providence, RI, 2009, 163-238.

[4] Galatos, N. and Tsinakis, C., *Equivalence of consequence relations: an order-theoretic and categorical perspective*, J. Symbolic Logic **74** (2009), 780-810.

[5] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., and Scott, D.S., "Continuous Lattices and Domains", Encyclopedia of Mathematics and its Applications **93**, Cambridge University Press, 2003.

[6] Hartshorne, R., "Algebraic Geometry", Graduate Texts in Mathematics **52**, Springer-Verlag, 1977.

[7] Kronheimer, E.H. and Penrose, R., *On the structure of causal spaces*, Proc. Cambridge Philos. Soc. **63** (1967), 481-501.

[8] Lehrer, E., *On a representation of a relation by a measure*, J. Math. Econom. **20** (1991), 107-118.

[9] Lipparini, P., *A model theory of topology*, arXiv:2201.00335v1 (2022), 1-30.

[10] McKinsey, J.C.C. and Tarski, A., *The algebra of topology*, Ann. of Math. **45** (1944), 141-191.

[11] Montalbán, A. and Nies, A., *Borel structures: a brief survey*, in Greenberg, N., Hamkins, J.D., Hirschfeldt, D., and Miller, R., *Effective mathematics of the uncountable*, Lect. Notes Log. **41**, Assoc. Symbol. Logic, La Jolla, CA, 2013, 124-134.

[12] Peters, J. and Naimpally, S., *Applications of near sets*, Notices Amer. Math. Soc. **59** (2012), 536-542.

[13] Peters, J. F. and Wasilewski, P., *Tolerance spaces: Origins, theoretical aspects and applications*, Inform. Sci. **195** (2012), 211-225.

[14] Ranzato, F., *Closures on CPOs form complete lattices*, Inform. and Comput. **152** (1999), 236-249.

Paolo Lipparini *Dipartimento di Matematica, Viale della Ricerca Scientifica Non Chiusa, Università di Roma "Tor Vergata", I-00133 ROME ITALY.*

Email: lipparin@axp.mat.uniroma2.it