K-theories and free inductive graded rings in abstract quadratic forms theories

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Abstract. We build on previous work on multirings ([17]) that provides generalizations of the available abstract quadratic forms theories (special groups and real semigroups) to the context of multirings ([10], [14]). Here we raise one step in this generalization, introducing the concept of pre-special hyperfields and expand a fundamental tool in quadratic forms theory to the more general multivalued setting: the K-theory. We introduce and develop the K-theory of hyperbolic hyperfields that generalize simultaneously Milnor's K-theory ([11]) and Special Groups K-theory, developed by Dickmann-Miraglia ([5]). We develop some properties of this generalized K-theory, that can be seen as a free inductive graded ring, a concept introduced in [2] in order to provide a solution of Marshall's Signature Conjecture.

1 Introduction

Concerning Abstract Theories of Quadratic forms (in particular special groups and real semigroups), the references [3], [6] and [7] are central. The theory of special groups deals simultaneously reduced and non-reduced theories but focuses

on rings with an "expressive amount" of invertible coefficients to quadratic forms and the theory of real semigroups consider general coefficients of a ring, but only addresses the reduced case. Both are first-order theory, thus they allow the use of model theoretic methods.

M. Marshall in [10] introduced an approach to (reduced) theory of quadratic forms through the concept of multiring (roughly, a ring with a multi valued sum) [1]: this seems more intuitive for an algebraist since it encompasses (generalizes, in fact) some techniques of ordinary Commutative Algebra. Moreover, the multirings encode copies of special groups and real semigroups (see [14]) and still allows the use of model-theoretic tools, since multirings (hyperrings) endowed with convenient notion of morphisms constitutes a category that is isomorphic to a category of appropriate first-order structures.

In the recent work [17]: (i) we have considered interesting pairs (A, T) where A is a multiring and $T \subseteq A$ is a certain multiplicative subset in such a way to obtain models of abstract theories of quadratic forms (special groups and real semigroups) via natural quotients - Marshall's quotient construction and (ii) we have used this new setting to motivate a "non reduced" expansion of the theory of real semigroups to deal the formally real case, isolating axioms over pairs involving multirings and a multiplicative subset with some properties.

The uses of K-theoretic (and Boolean) methods in abstract theories of quadratic forms has been proved a very successful method, see for instance, these two papers of Dickmann and Miraglia: [2] where they give an affirmative answer to Marshall's Conjecture, and [4], where they give an affirmative answer to Lam's Conjecture (previously both conjecture have kept open for almost three decades). These two central papers makes us take a deeper look at the theory of special groups (and hence, hyperbolic/pre-special hyperfields) by itself. This is not mere exercise in abstraction: from Marshall's and Lam's Conjecture many questions arise in the abstract and concrete context of quadratic forms.

In the present paper we provide some new steps towards the development of tools of algebraic theory of quadratic forms in this multiring setting: we have defined and explored K-theory and graded rings in the context of hyperfields that, in particular, provides a generalization and unification of Milnor's K-theory ([11])

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[1]The main terminology in the literature is "hyperring". Moreover, M. Marshall makes a distinction between "multiring" and "hyperrings" which is important in the context of quadratic forms. But throughout this entire work, we deal essentially with multifields/hyperfields and then, the main terminology here will be "hyperfield".

and special groups K-theory ([5]). We develop some properties of this generalized K-theory, that can be seen as a free inductive graded ring.

Outline: In Section 2 we: recall the basic notions of K-theory in quadratic forms theories (Milnor's K-theory, Special Groups K-theory); provide the basic definitions, constructions and results on multirings and hyperfields; introduce the concept of pre-special hyperfield. In the third section we introduce the notion of K-theory of hyperfields and establish its main properties and some technical results. In Section 4 we recognize the K-theory of special hyperfields as a free construction in the category of inductive graded rings ([3]) and prove that this concept, encompasses both Milnor's K-theory and Special Groups K-theory. We finish the work indicating some themes of research motivated by the present paper.

Throughout this entire paper, we adopt the following convention: let A be an arbitrary structure (field, graded ring and so on) and let $T \subseteq A$ be a subset such that the quotient A/T is defined. We usually denote the elements of A/T by $\overline{a} \in A/T$. This notation is overwhelming here (as you will see rapidly). So, we will omit the overline symbol, and an equation "$\overline{a} = \overline{b}$" that took place in A/T will be denoted simply by "$a = b$ in A/T". In other words, we will add "in A/T" or "in A" to make distinctions between some set and its quotients.

2 Preliminaries

This section contains, basically, the fundamental definitions and results included for the convenience of the reader such as multirings, hyperfields, special groups, Minor's K-theory of fields and Dickmann-Miraglia K-theory of (pre) special groups. For more details, consult [11], [3], [5], [10], [14] or [17].

2.1 Milnor's K-theory Here we get some definitions and results about Milnor's K-theory, as developed in [11].

Definition 2.1 (The Milnor's K-theory of a Field [11])**.** For a field F (of characteristic not 2), K_*F is the graded ring

$$K_*F = (K_0F, K_1F, K_2F, ...)$$

defined by the following rules: $K_0F := \mathbb{Z}$. K_1F is the multiplicative group $\dot{F}$ written additively. With this purpose, we fix the canonical "logarithm" isomorphism

$$l : \dot{F} \to K_1F,$$

where $l(ab) = l(a) + l(b)$. Then K_nF is defined to be the quotient of the tensor algebra

$$K_1F \otimes K_1F \otimes ... \otimes K_1F \text{ (n times)}$$

by the (homogeneous) ideal generated by all $l(a) \otimes l(1-a)$, with $a \neq 0, 1$. We also have the reduced K-theory graded ring $k_*F = (k_0F, k_1F, ..., k_nF, ...)$, which is defined by the rule $k_nF := K_nF/2K_nF$ for all $n \geqslant 0$.

Theorem 2.2 (Theorem 4.1 of [11]). *There is only one morphism*

$$s_n : k_nF \to I^nF/I^{n+1}F$$

which carries each product $l(a_1)...l(a_n)$ *in* $K_nF/2K_nF$ *to the product* $(\langle a_1 \rangle - \langle 1 \rangle)...(\langle a_n \rangle - \langle 1 \rangle)$ *modulo* $I^{n+1}F$.

These morphisms determines an epimorphism $s_* : k_*F \to W_g(F)$, where

$$W_g(F) = (WF/IF, IF/I^2F, ..., I^nF/I^{n+1}F, ...).$$

For a field F, let F_s be the a separable closure of F and $G_F = \text{Gal}(F_s)$. Then, the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \dot{F}_s \overset{2}{\longrightarrow} \dot{F}_s \longrightarrow 1$$

is taken to the following exact sequence

$$H^0(G_F, \dot{F}_s) \overset{2}{\longrightarrow} H^0(G_F, \dot{F}_s) \overset{\delta}{\longrightarrow} H^1(G_F, \{\pm 1\}) \longrightarrow H^1(G_F, \dot{F}_s)$$

of cohomology groups. Identifying the two first groups with $\dot{F}$, and $\{\pm 1\}$ with $\mathbb{Z}/2\mathbb{Z}$ and applying Hilbert's 90, we have

$$\dot{F} \overset{2}{\longrightarrow} \dot{F} \overset{\delta}{\longrightarrow} H^1(G_F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 .$$

The quotient $\dot{F}/\dot{F}^2$ is identified with k_1F.

Theorem 2.3 (Lemma 6.1 of [11]). *The isomorfism* $l(a) \mapsto \delta(a)$ *from* $K_1F/2K_1F$ *to* $H^1(G_F, \mathbb{Z}/2\mathbb{Z})$ *admits a unique extension to a graded ring morphism*

$$h_f : k_*F \to H^*(G_F, \mathbb{Z}/2\mathbb{Z}).$$

The *Milnor Conjecture* consists of the assertion that s and h are graded rings isomorfisms, which makes the fuctors $K_*F/2K_*F$, $W_g(F)$, $H^*(G, \mathbb{Z}/2\mathbb{Z})$ isomorphic.

2.2 Dickmann-Miraglia K-theory for special groups There are some generalizations of Milnor's K-theory. In the quadratic forms context, maybe the most significant one is the Dickmann-Miraglia K-theory of Special Groups. It is a main tool in the proof of Marshall's and Lam's Conjecture. In this section, we get some definitions and results from [3] and [5].

Firstly, we make a brief summary on special groups. Let A be a set and $\equiv$ a binary relation on $A \times A$. We extend $\equiv$ to a binary relation $\equiv_n$ on A^n, by induction on $n \geqslant 1$, as follows:

i) $\equiv_1$ is the diagonal relation $\Delta_A \subseteq A \times A$.

ii) $\equiv_2 = \equiv$.

iii) If $n \geq 3$, $\langle a_1, ..., a_n \rangle \equiv_n \langle b_1, ..., b_n \rangle$ if and only there are $x, y, z_3, ..., z_n \in A$ such that

$$\langle a_1, x \rangle \equiv \langle b_1, y \rangle, \ \langle a_2, ..., a_n \rangle \equiv_{n-1} \langle x, z_3, ..., z_n \rangle,$$
$$\text{and } \langle b_2, ..., b_n \rangle \equiv_{n-1} \langle y, z_3, ..., z_n \rangle.$$

Whenever clear from the context, we frequently abuse notation and indicate the afore-described extension $\equiv$ by the same symbol.

Definition 2.4 (Special Group, 1.2 of [3])**.** A **special group** is a tuple $(G, -1, \equiv)$, where G is a group of exponent 2, i.e, $g^2 = 1$ for all $g \in G$; -1 is a distinguished element of G, and $\equiv \subseteq G \times G \times G \times G$ is a relation (the special relation), satisfying the following axioms for all $a, b, c, d, x \in G$:

SG 0 $\equiv$ is an equivalence relation on G^2;

SG 1 $\langle a, b \rangle \equiv \langle b, a \rangle$;

SG 2 $\langle a, -a \rangle \equiv \langle 1, -1 \rangle$;

SG 3 $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow ab = cd$;

SG 4 $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow \langle a, -c \rangle \equiv \langle -b, d \rangle$;

SG 5 $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow \langle ga, gb \rangle \equiv \langle gc, gd \rangle$, for all $g \in G$.

SG 6 (3-transitivity) the extension of $\equiv$ for a binary relation on G^3 is a transitive relation.

A group of exponent 2, with a distinguished element -1, satisfying the axioms SG0-SG3 and SG5 is called a **proto special group**; a **pre special group** is a proto special group that also satisfy SG4. Thus a **special group** is a pre-special group that satisfies SG6 (or, equivalently, for each $n \geq 1$, $\equiv_n$ is an equivalence relation on G^n).

A n-**form** (or form of dimension $n \geqslant 1$) is an n-tuple of elements of a pre-SG G. An element $b \in G$ is **represented** on G by the form $\varphi = \langle a_1, ..., a_n \rangle$, in symbols $b \in D_G(\varphi)$, if there exists $b_2, ..., b_n \in G$ such that $\langle b, b_2, ..., b_n \rangle \equiv \varphi$.

A pre-special group (or special group) $(G, -1, \equiv)$ is:

- **formally real** if $-1 \notin \bigcup_{n \in \mathbb{N}} D_G(n\langle 1 \rangle)$²;
- **reduced** if it is formally real and, for each $a \in G$, $a \in D_G(\langle 1, 1 \rangle)$ iff $a = 1$.

Definition 2.5 (1.1 of [3]). A map $(G, \equiv_G, -1) \xrightarrow{f} (H, \equiv_H, -1)$ between pre-special groups is a **morphism of pre-special groups or PSG-morphism** if $f : G \to H$ is a homomorphism of groups, $f(-1) = -1$ and for all $a, b, c, d \in G$

$$\langle a, b \rangle \equiv_G \langle c, d \rangle \Rightarrow \langle f(a), f(b) \rangle \equiv_H \langle f(c), f(d) \rangle$$

A **morphism of special groups or SG-morphism** is a pSG-morphism between the corresponding pre-special groups. f will be an isomorphism if is bijective and f, f^{-1} are PSG-morphisms.

It can be verified that a special group G is formally real iff it admits some SG-morphism $f : G \to 2$. The category of special groups (respectively reduced special groups) and their morphisms will be denoted by $\mathcal{SG}$ (respectively $\mathcal{RSG}$).

Definition 2.6 (The Dickmann-Miraglia K-theory [5]). For each special group G (written multiplicatively) we associate a graded ring

$$k_* G = (k_0 G, k_1 G, ..., k_n G, ...)$$

as follows: $k_0 G := \mathbb{F}_2$ and $k_1 G := G$ written additively. With this purpose, we fix the canonical "logarithm" isomorphism $\lambda : G \to k_1 G$ by $\lambda(ab) = \lambda(a) + \lambda(b)$. Observe that $\lambda(1)$ is the zero of $k_1 G$ and $k_1 G$ has exponent 2, i.e, $\lambda(a) = -\lambda(a)$ for all $a \in G$. In the sequel, we define $k_* G$ by the

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²Here the notation $n\langle 1 \rangle$ means the form $\langle a_1, ..., a_n \rangle$ where $a_j = 1$ for all $j = 1, ..., n$. In other words, $n\langle 1 \rangle$ is the form $\langle 1, ..., 1 \rangle$ with n entries equal to 1.

quotient of the $\mathbb{F}_2$-graded algebra

$$(\mathbb{F}_2, k_1G, k_1G \otimes_{\mathbb{F}_2} k_1G, k_1G \otimes_{\mathbb{F}_2} k_1G \otimes_{\mathbb{F}_2} k_1G, ...)$$

by the (graded) ideal generated by $\{\lambda(a) \otimes \lambda(ab), a \in D_G(1, b)\}$. In other words, for each $n \geqslant 2$,

$$k_nG := T^n(k_1G)/Q^n(G),$$

where

$$T^n(k_1G) := k_1G \otimes_{\mathbb{F}_2} k_1G \otimes_{\mathbb{F}_2} ... \otimes_{\mathbb{F}_2} k_1G$$

and $Q^n(G)$ is the subgroup generated by all expressions of type $\lambda(a_1) \otimes \lambda(a_2) \otimes ... \otimes \lambda(a_n)$ such that for some i with $1 \leqslant i < n$, there exist $b \in G$ such that $a_i \in D_G(1, b)$ and $a_i = a_{i+1}b$, which in symbols, means

$$Q^n(G) := \langle\{\lambda(a_1) \otimes \lambda(a_2) \otimes ... \otimes \lambda(a_n) : \text{ exists } 1 \leqslant i < n \text{ and } b \in G$$
$$\text{such that } a_i = a_{i+1}b \text{ and } a_i \in D_G(1, b)\}\rangle.$$

Since $\lambda(a) + \lambda(a) = 0$ for all $a \in k_1G$, it follow that $\eta + \eta = 0$ for all $\eta \in k_nG$, so this is a group of exponent 2.

Moreover, we will denote "$\lambda(a_1) \otimes \lambda(a_2) \otimes ... \otimes \lambda(a_n)$" simply by "$\lambda(a_1)\lambda(a_2)...\lambda(a_n)$". Clearly, $k_*(G)$ is a graded ring, and in particular, a ring, so that we are able to multiply elements $\eta \in k_nG$, $\tau \in k_mG$. Whenever we want to do this, we will denote $\eta \cdot \tau$, in order to avoid confusion with the simplifications described above.

Finally, since we only take tensorial products with parameters in $\mathbb{F}_2$, we abbreviate "$A \otimes_{\mathbb{F}_2} B$" simply by "$A \otimes B$". In this way, $T^n(k_1G)$ we will be denoted simply by

$$T^n(k_1G) = k_1G \otimes k_1G \otimes ... \otimes k_1G.$$

Next, we have a result that approximates Dickmann-Miraglia's K-theory with the Milnor's reduced K-theory:

Proposition 2.7 (2.1 [5])**.** *Let G be a special group, $x, y, a_1, ..., a_n \in G$ and σ be a permutation on n elements.*

(a) *In k_2G, $\lambda(a)^2 = \lambda(a)\lambda(-1)$. Hence in k_mG, $\lambda(a)^m = \lambda(a)\lambda(-1)^{m-1}$, $m \geqslant 2$;*

(b) *In k_2G, $\lambda(a)\lambda(-a) = \lambda(a)^2 = 0$;*

(c) *In $k_n G$, $\lambda(a_1)\lambda(a_2)...\lambda(a_n) = \lambda(a_{\sigma 1})\lambda(a_{\sigma 2})...\lambda(a_{\sigma n})$;*

(d) *For $n \geqslant 1$ and $\xi \in k_n G$, $\xi^2 = \lambda(-1)^n \xi$;*

(e) *If G is a reduced special group, then $x \in D_G(1, y)$ and the equality $\lambda(y)\lambda(a_1)...\lambda(a_n) = 0$ implies*

$$\lambda(x)\lambda(a_1)\lambda(a_2)...\lambda(a_n) = 0.$$

Definition 2.8 (2.4 [5])**.**

a) A reduced special group is [MC] if for all $n \leqslant 1$ and all forms φ over G,

For all $\sigma \in X_G$, if $\sigma(\varphi) \equiv 0 \bmod 2^n$ then $\varphi \in I^n G$.

b) A reduced special group is [SMC] if for all $n \geqslant 1$, the multiplication by $\lambda(-1)$ is an injection of $k_n G$ in $k_{n+1} G$.

***Fact* 2.9:** We summarize some properties of the Dickmann-Miraglia K-theory below:

i) An inductive system of special groups

$$\mathcal{G} = (G_i; \{f_{ij} : i \leqslant j \in I\}),$$

provides an inductive system of graded ring, which nodes are $k_* G_i$ and morphisms are

$$(f_{ij})_* : k_* G_i \rightarrow k_* G_j, \text{ for } i \leqslant j \text{ in } I.$$

ii) (4.5 of [5]) Let $\mathcal{G} = (G_i; \{f_{ij} : i, j \in I, i \leqslant j\})$ an inductive system of special groups over a directed poset I and $(G; \{f_i : i \in I\}) = \varinjlim \mathcal{G}$. Then

$$k_* G \cong \varinjlim_{i \in I} k_* G_i.$$

iii) (4.6, 5.1, 5,7 and 6.8 of [5]) The inductive limit, finite products, SG-sum and extension of SMC groups is a SMC group.

iv) (5.1 of [5]) Let $G_1, ..., G_m$ be special groups and $\prod_{i=1}^{m} G_i$. Then there exists a graded isomorphism

$$\gamma : k_* P \rightarrow \bigoplus_{i=1}^{m} k_* G_i.$$

2.3 Multifields/Hyperfields Roughly speaking, a multiring is a "ring" with a multivalued addition, a notion introduced in the 1950s by Krasner's works. The notion of multiring was joined to the quadratic forms tools by the hands of M. Marshall in the last decade ([10]). We gather the basic information about multirings/hyperfields and expand some details that we use in the context of K-theories. For more detailed calculations involving multirings/hyperfields and quadratic forms we indicate to the reader the reference [14] (or even [8] and [17]). Of course, multi-structures is an entire subject of research (which escapes from the "quadratic context"), and in this sense, we indicate the references [12], [18], [1].

Definition 2.10. A multigroup is a quadruple $(G, *, r, 1)$, where G is a non-empty set, $* : G \times G \to \mathcal{P}(G) \setminus \{\emptyset\}$ and $r : G \to G$ are functions, and 1 is an element of G satisfying:

   i) If $z \in x * y$ then $x \in z * r(y)$ and $y \in r(x) * z$.

   ii) $y \in 1 * x$ if and only if $x = y$.

   iii) With the convention $x * (y * z) = \bigcup_{w \in y*z} x * w$ and $(x * y) * z = \bigcup_{t \in x*y} t * z$,

$$x * (y * z) = (x * y) * z \text{ for all } x, y, z \in G.$$

     A multigroup is said to be **commutative** if

   iv) $x * y = y * x$ for all $x, y \in G$.

Observe that by (i) and (ii), $1 * x = x * 1 = \{x\}$ for all $x \in G$. When $a * b = \{x\}$ be a unitary set, we just write $a * b = x$.

Definition 2.11 (Adapted from Definition 2.1 in [10])**.** A multiring is a sextuple $(R, +, \cdot, -, 0, 1)$ where R is a non-empty set, $+ : R \times R \to \mathcal{P}(R) \setminus \{\emptyset\}$, $\cdot : R \times R \to R$ and $- : R \to R$ are functions, 0 and 1 are elements of R satisfying:

   i) $(R, +, -, 0)$ is a commutative multigroup;

   ii) $(R, \cdot, 1)$ is a commutative monoid;

   iii) $a.0 = 0$ for all $a \in R$;

   iv) If $c \in a + b$, then $c.d \in a.d + b.d$. Or equivalently, $(a + b).d \subseteq a.d + b.d$.

Note that if $a \in R$, then $0 = 0.a \in (1 + (-1)).a \subseteq 1.a + (-1).a$, thus $(-1).a = -a$.

R is said to be an hyperring if for $a, b, c \in R$, $a(b + c) = ab + ac$.

A multring (respectively, a hyperring) R is said to be a multidomain (hyperdomain) if it has no zero divisors. A multring R will be a multifield if every non-zero element of R has multiplicative inverse; note that hyperfields and multifields coincide. We will use "hyperfield" since this is the prevailing terminology.

Definition 2.12. Let A and B multirings. A map $f : A \to B$ is a morphism if for all $a, b, c \in A$:

i) $c \in a + b \Rightarrow f(c) \in f(a) + f(b)$;

ii) $f(-a) = -f(a)$;

iii) $f(0) = 0$;

iv) $f(ab) = f(a)f(b)$;

v) $f(1) = 1$.

If A and B are multirings, a morphism $f : A \to B$ is a *strong morphism* if for all $a, b, c \in A$, if $f(c) \in f(a) + f(b)$, then there are $a', b', c' \in A$ with $f(a') = f(a), f(b') = f(b), f(c') = f(c)$ such that $c' \in a' + b'$. In the quadratic context, there is a more detailed analysis in Example 2.10 of [14].

Example 2.13.

(a) Suppose that $(G, +, 0)$ is an abelian group. Defining $a + b = \{a + b\}$ and $r(g) = -g$, we have that $(G, +, r, 0)$ is an abelian multigroup. In this way, every ring, domain and field is a multring, multidomain and hyperfield, respectively.

(b) $Q_2 = \{-1, 0, 1\}$ is hyperfield with the usual product (in $\mathbb{Z}$) and the multivalued sum defined by relations

$$\begin{cases} 0 + x = x + 0 = x, \text{ for every } x \in Q_2 \\ 1 + 1 = 1, (-1) + (-1) = -1 \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\} \end{cases}$$

(c) Let $K = \{0, 1\}$ with the usual product and the sum defined by relations $x + 0 = 0 + x = x$, $x \in K$ and $1 + 1 = \{0, 1\}$. This is a hyperfield called Krasner's hyperfield [9].

Now, another example that generalizes $Q_2 = \{-1, 0, 1\}$. Since this is a new one, we will provide the entire verification that it is a multiring:

Example 2.14 (Kaleidoscope, Example 2.7 in [14]). Let $n \in \mathbb{N}$ and define

$$X_n = \{-n, ..., 0, ..., n\} \subseteq \mathbb{Z}.$$

We define the n-**kaleidoscope multiring** by $(X_n, +, \cdot, -, 0, 1)$, where $- : X_n \to X_n$ is restriction of the opposite map in $\mathbb{Z}$, $+ : X_n \times X_n \to \mathcal{P}(X_n) \setminus \{\emptyset\}$ is given by the rules:

$$a + b = \begin{cases} \{a\}, & \text{if } b \neq -a \text{ and } |b| \leqslant |a| \\ \{b\}, & \text{if } b \neq -a \text{ and } |a| \leqslant |b| \\ \{-a, ..., 0, ..., a\} & \text{if } b = -a \end{cases},$$

and $\cdot : X_n \times X_n \to X_n$ is given by the rules:

$$a \cdot b = \begin{cases} \operatorname{sgn}(ab) \max\{|a|, |b|\} & \text{if } a, b \neq 0 \\ 0 & \text{if } a = 0 \text{ or } b = 0 \end{cases}.$$

With the above rules we have that $(X_n, +, \cdot, -, 0, 1)$ is a multiring.

Now, another example that generalizes $K = \{0, 1\}$.

Example 2.15 (H-hyperfield, Example 2.8 in [14]). Let $p \geqslant 1$ be a prime integer and $H_p := \{0, 1, ..., p - 1\} \subseteq \mathbb{N}$. Now, define the binary multioperation and operation in H_p as follows:

$$a + b = \begin{cases} H_p & \text{if } a = b, \, a, b \neq 0 \\ \{a, b\} & \text{if } a \neq b, \, a, b \neq 0 \\ \{a\} & \text{if } b = 0 \\ \{b\} & \text{if } a = 0 \end{cases}$$

$$a \cdot b = k \text{ where } 0 \leqslant k < p \text{ and } k \equiv ab \text{ mod p.}$$

$(H_p, +, \cdot, -, 0, 1)$ is a hyperfield such that for all $a \in H_p$, $-a = a$. In fact, these H_p are a kind of generalization of K, in the sense that $H_2 = K$.

There are many natural constructions on the category of multrings as: products, directed inductive limits, quotients by an ideal, localizations by multiplicative subsets and quotients by ideals. Now, we present some constructions that will be used further. For the first one, we need to restrict our category:

Definition 2.16 (Definition 3.1 of [17])**.** An **hyperbolic multiring** is a multiring R such that $1 - 1 = R$. The category of hyperbolic multirings and hyperbolic hyperfields will be denoted by $\mathcal{HMR}$ and $\mathcal{HMF}$ respectively.

Let F_1 and F_2 be two hyperbolic hyperfields. We define a new hyperbolic hyperfield $(F_1 \times_h F_2, +, -, \cdot, (0,0), (1,1))$ by the following: the adjacent set of this structure is
$$F_1 \times_h F_2 := (\dot{F}_1 \times \dot{F}_2) \cup \{(0,0)\}.$$
For $(a,b), (c,d) \in F_1 \times_h F_2$ we define

$$-(a,b) = (-a,-b),$$
$$(a,b) \cdot (c,d) = (a \cdot c, b \cdot d),$$
$$(a,b) + (c,d) = \{(e,f) \in F_1 \times F_2 : e \in a + c \text{ and } f \in b + d\} \cap (F_1 \times_h F_2).$$
$$\tag{2.1}$$

In other words, $(a,b) + (c,d)$ is defined in order to avoid elements of $F_1 \times F_2$ of type $(x,0), (0,y), x, y \neq 0$.

Theorem 2.17 (Product of Hyperbolic Hyperfields)**.** *Let F_1, F_2 be hyperbolic hyperfields and $F_1 \times_h F_2$ as above. Then $F_1 \times_h F_2$ is a hyperbolic hyperfield and satisfy the Universal Property of product for F_1 and F_2.*

Proof. We will verify the conditions of definition 2.11 (in a very similar manner as in Theorem 3.3 of [14]). Note that by the definition of multivalued sum in $F_1 \times_h F_2$ we have for all $a, c \in F_1$ and all $b, d \in F_2$ that $(a,b) + (c,d) \neq \emptyset$ and $(a,c) - (a,c) = F_1 \times_h F_2$ if $a, c \neq 0$.

  i) In order to prove that $(F_1 \times_h F_2, +, -, (0,0))$ is a multigroup we follow the steps below.

  (a) We will prove that if $(e,f) \in (a,b) + (c,d)$, then $(a,b) \in (c,d) + (-e,-f)$ and $(c,d) \in (-a,-b) + (e,f)$.

  If $(a,b) = (0,0)$ (or $(c,d) = (0,0)$) or $(a,b) = (-c,-d)$, then $(e,f) \in (a,b) + (c,d)$ means $(e,f) = (c,d)$ or $e \in a - a, f \in b - b$. In both

cases we get $(a, b) \in (c, d) + (-e, -f)$ and $(c, d) \in (-a, -b) + (e, f)$. Now suppose $a, b, c, d \neq 0$ with $a \neq -c$, $b \neq -d$. Let $(e, f) \in (a, b) + (c, d)$. Then $(e, f) \in F_1 \times_h F_2$ with $e \in a + c$ and $f \in c + d$. Moreover $a \in c - e$ and $b \in d - f$ with $(a, b) \in F_1 \times_h F_2$, implying $(a, b) \in (c, d) + (-e, -f)$. We prove $(c, d) \in (-a, -b) + (e, f)$ by the very same argument.

(b) Commutativity and $((a, b) \in (c, d) + (0, 0)) \Leftrightarrow ((a = b) \wedge (c = d))$ are direct consequence of the definition of multivaluated sum.

(c) Now we prove the associativity, that is,

$$[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)].$$

In fact (see the remarks after Lemma 2.4 of [14]), it is enough to show

$$[(a, b) + (c, d)] + (e, f) \subseteq (a, b) + [(c, d) + (e, f)].$$

If $(a, b) = 0$ or $(c, d) = 0$ or $(e, f) = 0$ we are done. Now let $a, b, c, d, e, f \neq 0$ and $(v, w) \in [(a, b) + (c, d)] + (e, f)$. If $(c, d) = -(e, f)$, we have

$$(a, b) + [(c, d) + (e, f)] = (a, b) + F_1 \times_h F_2 = F_1 \times_h F_2 \supseteq [(a, b) + (c, d)] + (e, f).$$

If $-(e, f) \in (a, b) + (c, d)$ then $-(a, b) \in (c, d) + (e, f)$ and we have

$$[(a, b) + (c, d)] + (e, f) = F_1 \times_h F_2 = (a, b) + [(c, d) + (e, f)]$$

Now suppose $a, b, c, d, e, f \neq 0$, $(c, d) \neq -(e, f)$, $-(e, f) \notin (a, b) + (c, d)$. Let $(x, y) \in [(a, b) + (c, d)] + (e, f)$. Then there exists $(v, w) \in F_1 \times_h F_2$ such that $(v, w) \in (a, b) + (c, d)$ and $(x, y) \in (v, w) + (e, f)$. This imply $(v \in a + c) \wedge (x \in v + e)$ and $(w \in b + y) \wedge (y \in d + f)$, so there exists $p \in F_1, q \in F_2$ such that $(v \in a + p) \wedge (p \in c + e)$ and $(w \in b + q) \wedge (q \in d + f)$. If $p, q = 0$ or $p, q \neq 0$ we have $(p, q) \in F_1 \times_h F_2$, which imply $(v, w) \in [(a, b) + (c, d)] + (e, f)$. If $p = 0$ and $q \neq 0$ (the case $q = 0$ and $p \neq 0$ is analogous), then $v = a$ and $c = -e$. Since $a, c \neq 0$ and F_1 is hyperbolic, we have $a - a = c - c = F_1$. Then $(v \in a - a) \wedge (-a \in c - c)$ and $(w \in b + q) \wedge (q \in d + f)$, with $(-a, q) \in F_1 \times_h F_2$ and again, we get $(v, w) \in [(a, b) + (c, d)] + (e, f)$.

ii) Since $(F_1 \times_h F_2, \cdot, (1, 1))$ is an abelian group, we conclude that it is a commutative monoid. Beyond this, every nonzero element of $F_1 \times_h F_2$ has an inverse.

iii) $(a, b) \cdot (0, 0) = (0, 0)$ for all $(a, b) \in F_1 \times_h F_2$ is direct from definition.

iv) For the distributive property, let $(a, b), (c, d), (e, f) \in F_1 \times_h F_2$ and consider $(x, y) \in (e, f)[(a, b) + (c, d)]$. We need to prove that
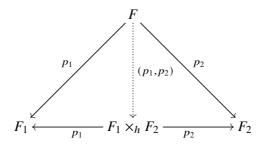
$$(x, y) \in (e, f) \cdot (a, b) + (e, f) \cdot (c, d). \tag{*}$$

It is the case if $(a, b) = (0, 0)$, or $(c, d) = (0, 0)$ or $(e, f) = (0, 0)$. Moreover (*) also holds if $a, b, c, d, e, f \neq 0$ and $(c, d) = -(a, b)$.

Now suppose $a, b, c, d, e, f \neq 0$ and $(c, d) \neq -(a, b)$. Then we have $(x, y) = (ev, fw)$ for some $(v, w) \in (a, b) + (c, d)$. Since $(e, f) \in F_1 \times_h F_2$, $e = 0$ iff $f = 0$, which imply $(ev, fw) \in F_1 \times_h F_2$, with $ev \in ea + ec$ and $fw \in fb + fd$. Therefore $(x, y) = (ev, fw) \in (e, f) \cdot (a, b) + (e, f) \cdot (c, d)$, as desired.

Then $(F_1 \times_h F_2, +, -, \cdot, (0, 0), (1, 1))$ is a hyperbolic hyperfield. Moreover we have projections $\pi_1 : F_1 \times_h F_2 \to F_1$, $\pi_2 : F_1 \times_h F_2 \to F_2$ given respectively by the rules $\pi_1(x, y) = x$, $\pi_2(x, y) = y$.

Finally, suppose that F is another hyperfield with morphisms $p_1 : F \to F_1$ and $p_2 : F \to F_2$. Consider $(p_1, p_2) : F \to F_1 \times_h F_2$ given by the rule $(p_1, p_2)(x) = (p_1(x), p_2(x))$. It is immediate that (p_1, p_2) is the unique morphism making the following diagram commute

so $F_1 \times_h F_2$ is the product in the category of hyperbolic hyperfields, completing the proof. $\qquad \square$

In order to avoid confusion and mistakes, we denote the binary product in $\mathcal{HMF}$ by $F_1 \times_h F_2$. For hyperfields $\{F_i\}_{i \in I}$, we denote the product of this family by

$$\prod_{i \in I}^{h} F_i,$$

with underlying set defined by

$$\prod_{i \in I}^{h} F_i := \left(\prod_{i \in I} \dot{F}_i \right) \cup \{(0_i)_{i \in I}\}$$

and operations defined by rules similar to the ones defined in 2.1. If $I = \{1, ...n\}$, we denote

$$\prod_{\substack{i \in I \\ [h]}}^{h} F_i = \prod_{i=1}^{n} F_i.$$

Example 2.18. Note that if F_1 (or F_2) is not hyperbolic, then $F_1 \times_h F_2$ is not a hyperfield. Let F_1 be a field (considered as a hyperfield), for example $F_1 = \mathbb{R}$ and F_2 be another hyperfield. Then if $a, b \in F_2$, we have $1 - 1 = \{0\}$, so $(1, a) + (-1, b) = \{0\} \times (a - b)$, and

$$[\{0\} \times (a - b)] \cap (F_1 \times_h F_2) = \emptyset.$$

Proposition 2.19 (Example 2.6 in [10])**.** *Fix a multiring A and a multiplicative subset S of A such that $1 \in S$. Define an equivalence relation $\sim$ on A by $a \sim b$ if and only if $as = bt$ for some $s, t \in S$. Denote by $\overline{a}$ the equivalence class of a and set $A/_m S = \{\overline{a} : a \in A\}$. Then, we define in agreement with Marshall's notation, $\overline{a} + \overline{b} = \{\overline{c} : cv \in as + bt, \text{ for some } s, t, v \in S\}$, $-\overline{a} = \overline{-a}$, and $\overline{a}\overline{b} = \overline{ab}$.*

Then $A/_m S$ are multirings. Moreover, if A is a hyperring, the same holds for $A/_m S$. The canonical projection $\pi : A \to A/_m S$ is a morphism.

Proposition 2.20 (2.19 in [14])**.** *Let A, B be a multiring and $S \subseteq A$ a multiplicative subset of A. Then for every morphism $f : A \to B$ such that $f[S] = \{1\}$, there exist a unique morphism $\tilde{f} : A/_m S \to B$ such that the following diagram commute:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \pi\ } & A/_m S \\
 & \searrow{\scriptstyle f} & \downarrow{\scriptstyle !\tilde{f}} \\
 & & B
\end{array}
$$

where $\pi : A \to A/_m S$ is the canonical projection $\pi(a) = \overline{a}$.

Proposition 2.21 (3.13 of [14])**.** *Let $(G, \equiv, -1)$ be a special group and define $M(G) = G \cup \{0\}$ where $0 := \{G\}$[3]. Then $(M(G), +, -, \cdot, 0, 1)$ is a hyperfield, where*

---

[3]Here, the choice of the zero element was ad hoc. Indeed, we can define $0 := \{x\}$ for any $x \notin G$.

- $a \cdot b = \begin{cases} 0 \ \textit{if } a = 0 \textit{ or } b = 0 \\ a \cdot b \ \textit{otherwise} \end{cases}$

- $-(a) = (-1) \cdot a$

- $a + b = \begin{cases} \{b\} \ \textit{if } a = 0 \\ \{a\} \ \textit{if } b = 0 \\ M(G) \ \textit{if } a = -b, \textit{ and } a \neq 0 \\ D_G(a,b) \ \textit{otherwise} \end{cases}$

Corollary 2.22 (3.14 of [14]). *The correspondence $G \mapsto M(G)$ extends to a faithful functor $M : \mathcal{SG} \to MField$.*

Definition 2.23 (3.15-3.19 of [14]). A hyperfield F is a **special multifield** if there exist a special group G such that $F = M(G)$. The category of special multifields will be denoted by $\mathcal{SMF}$.

Definition 2.24 (Definition 3.2 of [17]). A **Dickmann-Miraglia multiring (or DM-multiring for short)** [4] is a pair (R, T) such that R is a multiring, $T \subseteq R$ is a multiplicative subset of $R \setminus \{0\}$, and (R, T) satisfies the following properties:

DM0 $R/_m T$ is hyperbolic.

DM1 If $\overline{a} \neq 0$ in $R/_m T$, then $\overline{a}^2 = \overline{1}$ in $R/_m T$. In other words, for all $a \in R \setminus \{0\}$, there are $r, s \in T$ such that $ar = s$.

DM2 For all $a \in R$, $(\overline{1} - \overline{a})(\overline{1} - \overline{a}) \subseteq (\overline{1} - \overline{a})$ in $R/_m T$.

DM3 For all $a, b, x, y, z \in R \setminus \{0\}$, if

$$\begin{cases} \overline{a} \in \overline{x} + \overline{b} \\ \overline{b} \in \overline{y} + \overline{z} \end{cases} \quad \text{in } R/_m T,$$

then exist $\overline{v} \in \overline{x} + \overline{z}$ such that $\overline{a} \in \overline{y} + \overline{v}$ and $\overline{vb} \in \overline{xy} + \overline{az}$ in $R/_m T$.

If R is a ring, we just say that (R, T) is a DM-ring, or R is a DM-ring. A Dickmann-Miraglia hyperfield (or DM-hyperfield) F is a hyperfield such that $(F, \{1\})$ is a DM-multiring (satisfies DM0-DM3). In other words, F is a DM-hyperfield if F is hyperbolic and for all $a, b, v, x, y, z \in F^*$,

---

[4] The name "Dickmann-Miraglia" is given in honor to professors Maximo Dickmann and Francisco Miraglia, the creators of the special group theory.

i) $a^2 = 1$.

ii) $(1 - a)(1 - a) \subseteq (1 - a)$.

iii) If $\begin{cases} a \in x + b \\ b \in y + z \end{cases}$ then there exists $v \in x + z$ such that $a \in y + v$ and
$vb \in xy + az$.

Theorem 2.25 (Theorem 3.4 of [17]). *Let (R, T) be a DM-multiring and denote*

$$Sm(R, T) = (R/_m T).$$

Then $Sm(R)$ is a special hyperfield (thus $Sm(R, T)^\times$ is a special group).

Theorem 2.26 (Theorem 3.9 of [17]). *Let F be a hyperfield satisfying DM0-DM2. Then F satisfies DM3 if and only if satisfies SMF4. In other words, F is a DM-hyperfield if and only if it is a special hyperfield.*

In this sense, we define the following category:

Definition 2.27. A **pre-special hyperfield** is a hyperfield satisfying DM0, DM1 and DM2. In other words, a pre-special hyperfield is a hyperbolic hyperfield F such that for all $a \in \dot{F}$, $a^2 = 1$ and $(1 - a)(1 - a) \subseteq 1 - a$.

The category of pre-special hyperfields will be denoted by $PSMF$.

Example 2.28 (An hyperfield satisfying DM3 but not DM2). Let H_3 as in Example 2.15. We have that H_3 is hyperbolic and has exponent 2 but does not satisfies $DM2$:

$$1 + 2 = \{1, 2\} \text{ and } (1 + 2)(1 + 2) = 1 + 2 + 2 + 1 = (1 + 1) + (2 + 2) = H_3.$$

Moreover, by a case analysis we have that H_3 satisfies DM3, and we prove it by a case analysis: we have 27 possible choices for the triple (x, y, z) with $x, y, z \in H_3$.

(i) Let (x, y, z) such that $z = 0$. Then $b \in y + z$ means $b = y$ and $v \in x + z$ means $v = x$. Let $a \in x + b$ with $b \in y + z$. Then $a \in b + x = y + v$ and

$$vb = xy \in xy + az.$$

So we cover the cases

$$(x, y, z) \in \{(0, 0, 0), (0, 1, 0), (0, 2, 0), (1, 0, 0), (1, 1, 0), (1, 2, 0), (2, 0, 0),$$
$$(2, 1, 0), (2, 2, 0)\}.$$

(ii) Let (x, y, z) such that $x = 0$. Then $v \in x + z$ means $v = z$. Let $a \in x + b = 0 + b = \{b\}$ with $b \in y + z$. Then $a \in b + z = y + z$, $az = bz$ and

$$vb = bz \in xy + bz = xy + az.$$

So we cover the cases

$$(x, y, z) \in \{(0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2)\}.$$

(iii) Let (x, y, z) such that $y = 0$ and $x, z \neq 0$. Then $\{1, 2\} \subseteq x + z$, $b \in y + z$ means $b = z$ and $a \in y + v$ means $a = v$. Let $a \in x + b = x + z$. Then $vb = az$ and

$$vb = az \in 0 + az = xy + az.$$

So we cover the cases

$$(x, y, z) \in \{(1, 0, 1), (1, 0, 2), (2, 0, 1), (2, 0, 2)\}.$$

(iv) Let (x, y, z) such that $x, y, z \neq 0$ and $x + z = H_3$. Let $a \in x + b$ with $b \in y + z$. If $a = 0$ then $b = x$. Taking $v = y \in H_3 = x + z$ we get $a \in y + y = y + v$ and

$$vb = xy \in xy + 0 = xy + az.$$

If $b = 0$ then $z = y$ $a = x$ (so $az = xy$). Taking $v = y \in H_3 = x + z$ we get $a \in y + y = y + v$ and

$$vb = 0 \in xy + xy = xy + az.$$

If $a, b \neq 0$, then there exist $v \in H_3 = x + z$ such that $vb = xy$. Then $a \in \{1, 2\} \subseteq y + v$ (because $x, y, z, v \neq 0$) and

$$vb = xy \in xy + az.$$

So we cover the cases

$$(x, y, z) \in \{(1, 1, 1), (1, 2, 1), (2, 1, 2), (2, 2, 2)\}.$$

(v) Let (x, y, z) such that $x, y, z \neq 0$ and $y + z = H_3$. Let $a \in x + b$ with $b \in y + z$. If $a = 0$ then $b = x$. Taking $v = y \in \{1, 2\} \subseteq x + z$ we get $a \in y + y = y + v$ and

$$vb = xy \in xy + 0 = xy + az.$$

If $b = 0$ then $z = y$ $a = x$ (so $az = xy$). Taking $v = y \in \{1, 2\} \subseteq x + z$ we get $a \in y + y = y + v$ and

$$vb = 0 \in xy + xy = xy + az.$$

If $a, b \neq 0$, then there exist $v \in \{1, 2\} \subseteq x + z$ such that $vb = xy$. Then $a \in \{1, 2\} \subseteq y + v$ (because $x, y, z, v \neq 0$) and

$$vb = xy \in xy + az.$$

So we cover the cases

$$(x, y, z) \in \{(1, 1, 2), (1, 2, 2), (2, 1, 1), (2, 2, 1)\}.$$

Example 2.28 is a big surprise in the sense that, in the context of special groups, this example exhibits a group that it is not pre-special but satisfies SG6.

3 The K-theory for Multifields/Hyperfields

In this section we introduce the notion of K-theory of a hyperfield essentially repeating the construction in 2.1 replacing the word "field" by "hyperfield" and explore some of this basic properties. In particular, Theorem 3.8 is an extension of a result [19], that gives us some evidence, that apart from the obvious resemblance, more technical aspects of this new theory can be developed (but with other proofs) in multistructure setting in parallel with classical K-theory.

Definition 3.1 (The K-theory of a Hyperfield)**.** For a hyperfield F, K_*F is the graded ring

$$K_*F = (K_0F, K_1F, K_2F, ...)$$

defined by the following rules: $K_0F := \mathbb{Z}$. K_1F is the multiplicative group $\dot{F}$ written additively. With this purpose, we fix the canonical "logarithm" isomorphism

$$\rho : \dot{F} \to K_1F,$$

where $\rho(ab) = \rho(a) + \rho(b)$. Then K_nF is defined to be the quotient of the tensor algebra

$$K_1F \otimes K_1F \otimes ... \otimes K_1F \ (n \text{ times})$$

by the (homogeneous) ideal generated by all $\rho(a) \otimes \rho(b)$, with $a \neq 0, 1$ and $b \in 1 - a$.

In other words, for each $n \geqslant 2$,

$$K_n F := T^n(K_1 F)/Q^n(K_1(F)),$$

where

$$T^n(K_1 F) := K_1 F \otimes_{\mathbb{Z}} K_1 F \otimes_{\mathbb{Z}} ... \otimes_{\mathbb{Z}} K_1 F$$

and $Q^n(K_1(F))$ is the subgroup generated by all expressions of type $\rho(a_1) \otimes \rho(a_2) \otimes ... \otimes \rho(a_n)$ such that $a_i \in 1 - a_j$ for some i, j with $1 \leqslant i, j \leqslant n$.

To avoid carrying the overline symbol, we will adopt all the conventions used in Dickmann-Miraglia's K-theory (as explained in above definition 2.6). Just as it happens with the previous K-theories, a generic element $\eta \in K_n F$ has the pattern

$$\eta = \rho(a_1) \otimes \rho(a_2) \otimes ... \otimes \rho(a_n)$$

for some $a_1, ..., a_n \in \dot{F}$, with $a_i \in 1 - a_j$ for some $1 \leqslant i < j \leqslant n$. Note that if F is a field, then "$b \in 1 - a$" just means $b = 1 - a$, and the hyperfield and Milnor's K-theory for F coincide.

The very first task, is to extend the basic properties valid in Milnor's and Dickmann-Miraglia's K-theory to ours. Here we already need to restrict our attention to hyperbolic hyperfields:

Lemma 3.2 (Basic Properties I). *Let F be an hyperbolic hyperfield. Then*

(a) $\rho(1) = 0$.

(b) *For all $a \in \dot{F}$, $\rho(a)\rho(-a) = 0$ in $K_2 F$.*

(c) *For all $a, b \in \dot{F}$, $\rho(a)\rho(b) = -\rho(a)\rho(b)$ in $K_2 F$.*

(d) *For every $a_1, ..., a_n \in \dot{F}$ and every permutation $\sigma \in S_n$,*

$$\rho(a_1)...\rho(a_i)...\rho(a_n) = sgn(\sigma)\rho(a_1)...\rho(a_n) \text{ in } K_n F.$$

(e) *For every $\xi \in K_m F$ and $\eta \in K_n F$, $\eta\xi = (-1)^{mn}\xi\eta$ in $K_{m+n}F$.*

(f) *For all $a \in \dot{F}$, $\rho(a)^2 = \rho(a)\rho(-1)$.*

Proof.

(a) Is an immediate consequence of the fact that ρ is an isomorphism.

(b) Since F hiperbolic, $1 - 1 = F$. Then $-a^{-1} \in 1 - 1$ for all $a \in \dot{F}$, and hence, $-1 \in -1 + a^{-1}$. Multiplying this by $-a$, we get $a \in 1 - a$. By definition, this imply $\rho(a)\rho(-a) = 0$.

(c) By item (b), $\rho(ab)\rho(-ab) = 0$ in K_2F. But

$$\rho(ab)\rho(-ab) = \rho(a)\rho((-a)b) + \rho(b)\rho((-b)a)$$
$$= \rho(a)\rho(-a) + \rho(a)\rho(b) + \rho(b)\rho(-b) + \rho(b)\rho(a)$$
$$= \rho(a)\rho(b) + \rho(b)\rho(a).$$

From $\rho(a)\rho(b) + \rho(b)\rho(a) = \rho(ab)\rho(-ab) = 0$, we get the desired result $\rho(a)\rho(b) = -\rho(a)\rho(b)$ in K_2F.

(d) This is a consequence of item (c) and an inductive argument.

(e) This is a consequence of item (d) and an inductive argument, using the fact that an element in K_nF has pattern

$$\eta = \rho(a_1) \otimes \rho(a_2) \otimes ... \otimes \rho(a_n)$$

for some $a_1, ..., a_n \in \dot{F}$, with $a_i \in 1 - a_j$ for some $1 \leqslant i < j \leqslant n$.

(f) Follow from the fact that F is hyperbolic i.e, for all $a \in \dot{F}$, $a \in 1 - 1$.

$\square$

An element $a \in \dot{F}$ induces a morphism of graded rings of degree 1, $\omega^a = \{\omega_n^a\}_{n \geqslant 1} : K_*F \to K_*F$, where $\omega_n^a : K_nF \to K_{n+1}F$ is the multiplication by $\lambda(-a)$. When $a = -1$, we write

$$\omega = \{\omega_n\}_{n \geqslant 1} = \{\omega_n^{-1}\}_{n \geqslant 1} = \omega^{-1}.$$

Proposition 3.3 (Adapted from 3.3 of [5]). *Let F, K be hyperbolic hyperfields and $\varphi : F \to L$ be a morphism. Then φ induces a morphism of graded rings*
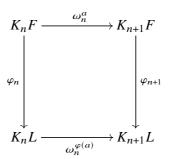
$$\varphi_* = \{\varphi_n : n \geqslant 0\} : K_*F \to K_*L,$$

where $\varphi_0 = Id_{\mathbb{Z}}$ and for all $n \geqslant 1$, φ_n is given by the following rule on generators
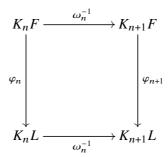
$$\varphi_n(\rho(a_1)...\rho(a_n)) = \rho(\varphi(a_1))...\rho(\varphi(a_n)).$$

Moreover if φ is surjective then φ_* is also surjective, and if $\psi : L \to M$ is another morphism then

(a) $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ *and* $Id_* = Id$.

(b) *For all* $a \in \dot{F}$ *the following diagram commute:*

$$
\begin{array}{ccc}
K_n F & \xrightarrow{\ \omega_n^a\ } & K_{n+1} F \\
\Big\downarrow{\scriptstyle \varphi_n} & & \Big\downarrow{\scriptstyle \varphi_{n+1}} \\
K_n L & \xrightarrow[\ \omega_n^{\varphi(a)}\ ]{} & K_{n+1} L
\end{array}
$$

(c) *If* $\varphi(1) = 1$ *then for all* $n \geqslant 1$ *the following diagram commute:*

$$
\begin{array}{ccc}
K_n F & \xrightarrow{\ \omega_n^{-1}\ } & K_{n+1} F \\
\Big\downarrow{\scriptstyle \varphi_n} & & \Big\downarrow{\scriptstyle \varphi_{n+1}} \\
K_n L & \xrightarrow[\ \omega_n^{-1}\ ]{} & K_{n+1} L
\end{array}
$$

Proof. Firstly, note that φ extends to a function $\varphi_1 : K_1 F \to K_1 L$ given by the rule

$$\varphi_1(\rho(a)) = \rho(\varphi(a)).$$

Certainly φ_1 is a morphism because

$$\varphi_1(0) = \varphi_1(\rho(1)) = \rho(\varphi(1)) = \rho(1) = 0,$$

and for all $\rho(a), \rho(b) \in K_1 F$,

$$\varphi_1(\rho(a) + \rho(b)) = \varphi_1(\rho(ab)) = \rho(\varphi(ab)) = \rho(\varphi(a)\varphi(b)) = \rho(\varphi(a)) + \rho(\varphi(b)).$$

Proceeding inductively, for all $n \geqslant 1$ we extend φ to a function $\varphi_n : \prod_{i=1}^{n} K_1 F \to K_n L$ given by the rule

$$\varphi(\rho(a_1), ..., \rho(a_n)) := \varphi_1(\rho(a_1))...\varphi_1(\rho(a_n)) = \rho(\varphi(a_1))...\rho(\varphi(a_n)).$$

Then if $i = 1, ..., n$ and $b_i \in k_1 F$ we have

$$\varphi_n(\rho(a_1), ..., \rho(a_i) + \rho(b_i), ..., \rho(a_n)) = \varphi_n(\rho(a_1), ..., \rho(a_i b_i), ..., \rho(a_n)) =$$
$$\rho(\varphi(a_1))...\rho(\varphi(a_i b_i)...\rho(\varphi(a_n)) = \rho(\varphi(a_1))...\rho(\varphi(a_i)\varphi(b_i))...\rho(\varphi(a_n)) =$$
$$\rho(\varphi(a_1)...[\rho(\varphi(a_i) + \varphi(b_i))]...\rho(\varphi(a_n)) =$$
$$\rho(\varphi(a_1))...\rho(\varphi(a_i))...\rho(\varphi(a_n)) + \rho(\varphi(a_1))...\rho(\varphi(b_i))...\rho(\varphi(a_n)) =$$
$$\varphi_n(\rho(a_1), ..., \rho(a_i), ..., \rho(a_n)) + \varphi_n(\rho(a_1), ..., \rho(b_i), ..., \rho(a_n)),$$

then for each n, $\varphi_n : \prod_{i=1}^n K_1 F \to K_n L$ is multilinear and by the universal property of tensor product there is an unique morphism

$$\tilde{\varphi}_n : \bigotimes_{j=1}^n K_1 F \to K_n L$$

extending φ_n. By construction (and using the fact that φ is a morphism), $\mathrm{Ker}(\tilde{\varphi}_n) = Q^n(K_1 F)$, which provides an unique morphism

$$\overline{\varphi}_n : T^n(K_1 F)/Q^n(K_1(F) \to K_n L$$

such that $\tilde{\varphi}_n = \overline{\varphi}_n \circ \pi_n$, where π_n is the canonical projection $T^n(K_1 F)$ in $Q^n(k_1 F)$. Then taking $\varphi_0 = Id_{\mathbb{Z}}$, we get a morhism $\varphi_* : K_* F \to K_* L$, given by $\varphi_* = \{\overline{\varphi}_n : n \geqslant 0\}$.

For items (a) and (b), it is enough to note that these properties holds for $\tilde{\varphi}_n$, $n \geqslant 0$, and after the application of projection, we get the validity for $\overline{\varphi}_n = \pi_n \circ \tilde{\varphi}_n$.

Item (c) follows by the same argument of itens (a) and (b), noting that $\varphi(1) = 1$ imply $\varphi(-1) = -1$. By abuse of notation, we denote

$$\varphi_* = \{\overline{\varphi}_n : n \geqslant 0\} = \{\varphi_n : n \geqslant 0\}.$$

$\square$

We also have the reduced K-theory graded ring $k_* F = (k_0 F, k_1 F, ..., k_n F, ...)$ in the hyperfield context, which is defined by the rule $k_n F := K_n F/2K_n F$ for all $n \geqslant 0$. Of course for all $n \geqslant 0$ we have an epimorphism $q : K_n F \to k_n F$ simply denoted by $q(a) := [a]$, $a \in K_n F$. It is immediate that $k_n F$ is additively generated by

$$\{[\rho(a_1)]..[\rho(a_n)] : a_1, ..., a_n \in \dot{F}\}.$$

We simply denote such a generator by $\tilde{\rho}(a_1)...\tilde{\rho}(a_n)$ or even $\rho(a_1)...\rho(a_n)$ whenever the context allows it.

We also have some basic properties of the reduced K-theory, which proof is just a translation of 2.1 of [5]:

Lemma 3.4 (Adapted from 2.1 [5]). *Let F be a hyperbolic hyperfield, $x, y, a_1, ..., a_n \in \dot{F}$ and σ be a permutation on n elements.*

(a) *In k_2F, $\rho(a)^2 = \rho(a)\rho(-1)$. Hence in k_mF, $\rho(a)^m = \rho(a)\rho(-1)^{m-1}$, $m \geqslant 2$;*

(b) *In k_2F, $\rho(a)\rho(-a) = \rho(a)^2 = 0$;*

(c) *In k_nF, $\rho(a_1)\rho(a_2)...\rho(a_n) = \rho(a_{\sigma 1})\rho(a_{\sigma 2})...\rho(a_{\sigma n})$;*

(d) *For $n \geqslant 1$ and $\xi \in k_nF$, $\xi^2 = \rho(-1)^n\xi$;*

(e) *If F is a real reduced hyperfield, then $x \in 1 + y$ and $\rho(y)\rho(a_1)...\rho(a_n) = 0$ implies*

$$\rho(x)\rho(a_1)\rho(a_2)...\rho(a_n) = 0.$$

Moreover the results in Proposition 3.3 continue to hold if we took $\varphi_* = \{\varphi_n : n \geqslant 0\} : k_*F \to k_*L$.

Proposition 3.5. *Let F be a hyperfield and $T \subseteq F$ be a multiplicative subset such that $F \subseteq T$. Then*

$$K(F/_mT^*) \cong k(F/_mT^*).$$

Proof. Since $F^2 \subseteq T$, for all $a \in F/_mT^*$ we have

$$0 = \rho(a^2) = \rho(a) + \rho(a).$$

Then $2K(F/_mT^*) = 0$ and we get $K(F/_mT^*) \cong k(F/_mT^*)$. $\qquad\square$

Theorem 3.6. *Let F be a hyperbolic hyperfield and $T \subseteq F$ be a multiplicative subset such that $F \subseteq T$. Then there is a surjective morphism*

$$k(F) \to k(F/_mT^*).$$

Moreover,

$$k(F) \cong K(F/_m\dot{F}^2) \cong k(F/_m\dot{F}^2).$$

Berfore we prove it, we need a Lemma:

Lemma 3.7. *Let F be a hyperfield and $n \geqslant 1$. Then*

$$K_n(F) = \{ \sum_{j=1}^{p} \rho(a_{j1})...\rho(a_{jn}) : \text{for all } j \text{ there is an index } k \text{ such that}$$

$$a_{jk} = b_i^2, \; b_i \in \dot{F} \},$$

$$2K_n(F) = \{ \sum_{j=1}^{p} \rho(a_{j1})...\rho(a_{jn}) : \text{for all } j \text{ there is an index } k \text{ such that}$$

$$a_{jk} = b_i^2, \; b_i \in \dot{F} \}$$

Proof. Let $\eta \in 2K_n F$. Then

$$\eta = \left(\sum_{j=1}^{p} \rho(a_{j1})...\rho(a_{jn}) \right) + \left(\sum_{j=1}^{p} \rho(a_{j1})...\rho(a_{jn}) \right), \; d_{ij} \in \dot{F}.$$

By induction, we only need to consider the case $p = 1$, so

$$\rho(a_1)...\rho(a_n) + \rho(a_1)...\rho(a_n) = \rho(a_1^2)\rho(a_2)...\rho(a_n).$$
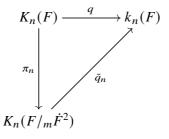
and we get $\subseteq$. The reverse inclusion follow by the same calculation. $\square$

Proof of Theorem 3.6. Let $\pi : F \to F/_m T^*$ denote the canonical projection. By Proposition 3.3 there is a morphism $\pi_* : K(F) \to K(F/_m T^*)$. Since π is surjective, π_* is surjective.

Now, let $\pi : F \to F/_m \dot{F}^2$ and $q : K(F) \to k(F)$ the canonical projections. Denote elements in $F/_m \dot{F}^2$ by $[a] \in F/_m \dot{F}^2$, $a \in F$ and elements in $k_n(F)$ by $\tilde{\rho}(a_1)...\tilde{\rho}(a_n)$. For all $n \geqslant 1$ we have an induced morphism $\tilde{q}_n : K_n(F/_m \dot{F}^2) \to k_n(F)$ given by the rule

$$\tilde{q}_n(\rho([a_1])...\rho([a_n])) := \tilde{\rho}(a_1)...\tilde{\rho}(a_n).$$

This morphism $\tilde{\pi}_n$ makes the following diagram commute

$$
\begin{array}{ccc}
K_n(F) & \xrightarrow{\;\;q\;\;} & k_n(F) \\
\downarrow{\scriptstyle \pi_n} & \nearrow{\scriptstyle \tilde{q}_n} & \\
K_n(F/_m \dot{F}^2) & &
\end{array}
$$

and then, $\tilde{q}_n$ is surjective. Finally, if $\tilde{q}_n(\rho([a_1])...\rho([a_n])) = 0$, then we have $\tilde{\rho}(a_1)...\tilde{\rho}(a_n) = 0$, and hence $\rho(a_1)...\rho(a_n) \in 2K_n(F)$. By Lemma 3.7

$$\rho(a_1)...\rho(a_n) = \sum_{j=1}^{p} \rho(d_{j1})...\rho(d_{jn}), \, d_{ij} \in \dot{F}$$

and for all i there is an index k such that $a_{ik} = b_i^2$, $b_i \in \dot{F}$. Therefore

$$\pi_n(\rho(a_1)...\rho(a_n)) = \pi_n\left(\sum_{j=1}^{p} \rho(d_{j1})...\rho(d_{jn})\right)$$

$$= \sum_{j=1}^{p} \pi_n(\rho(d_{j1})...\rho(d_{jn})) = \sum_{j=1}^{p} \rho([d_{j1}])...\rho([d_{jn}])$$

$$= \sum_{j=1}^{p} [d_{j1}])...\rho([1])...\rho([d_{jn}]) = 0.$$

Then $\text{Ker}(\tilde{q}_n) = [0]$, proving that $\tilde{q}_n$ is injective. Then $\tilde{q}_n$ is an isomorphism, and composing all the isomorphisms obtained here we get

$$k(F) \cong K(F/_m\dot{F}^2) \cong k(F/_m\dot{F}^2).$$

$\square$

The Theorem 3.8 below generalizes Proposition 5.10 of [19]: this constitutes a fundamental technical step to build profinite (Galois) groups associated to a pre-special hyperfield in [15].

Lets establish some notation: for $n \geqslant 0$ we denote

$$P(n) = \mathcal{P}(\{0, ..., n-1\}) \setminus \{\emptyset\}$$

and for $0 \leqslant i \leqslant n-1$, denote

$$P(n, i) = \{X \in P(n) : i \in X\}.$$

For a be a pre-special hyperfield F and $\{a_0, ..., a_{n-1}\} \subseteq F^* \, \mathbb{F}_2$-linearly independent, if $S \in P(n)$ we denote

$$a_S := a_0^{\varepsilon_0}...a_{n-1}^{\varepsilon_{n-1}},$$

where $\varepsilon_0 \in \{0, 1\}$ for all $i = 0, .., n - 1$ and $\varepsilon_i = 1$ if and only if $i \in S$.

Remember that by the very definition of $k_n(F)$,

$$k_n(F) := [k_1(G) \otimes k_1(G)]/M,$$

where M is the subgroup of $k_1(G) \otimes k_1(G)$ generated by

$$\{\rho(a)\rho(b) : a \in D_G(1, b), a\}.$$

Theorem 3.8. *Let F be a pre-special hyperfield and $\{a_0, ..., a_{n-1}\} \subseteq F^* \; \mathbb{F}_2$-linearly independent. The following conditions are equivalent:*

(i) *There exists $\{b_0, ..., b_{n-1}\} \subseteq F^*$ such that*

$$\sum_{k<n} \rho(a_k)\rho(b_k) = 0 \text{ in } k_2(F).$$

(ii) *There exist subsets $\{c_0, ..., c_{m-1}\}, \{d_0, ..., d_{n-1}\}$ of F^* with $m \geqslant n$ such that*

    (a) *$\{c_0, ..., c_{m-1}\}$ is linearly independent and $c_i = a_i$ for all $i < n$;*

    (b) *$d_i = b_i$ for all $i < n$ and $d_i = 1$ for $i = n, ..., m - 1$.*

    (c) *For all $x \in C := [c_0, ..., c_{m-1}]$, there is some $r_x \in 1 + x \setminus \{0\}$ such that for each $i < m$*

$$d_i = \prod_{x \in C_i} r_x$$

where

$$C_i = \left\{ \prod_{k<m} c_k^{\varepsilon_k} : \varepsilon_k \in \{0, 1\} \text{ and } \varepsilon_i = 1 \right\}.$$

In other words, C_i is "counting" all products $c_0^{r_0}...c_i^1...c_{m-1}^{r_{m-1}}$. Since for all $x \in C := [c_0, ..., c_{m-1}]$ there exist $S \in P(m)$ such that

$$x = \prod_{i \in S} c_i := c_S.$$

Denoting r_x by r_S we can rewrite

$$d_i = \prod_{x \in C_i} r_x = \prod_{S \in P(m)} r_S.$$

Proof of Theorem 3.8. (i) $\Rightarrow$ (ii). Let

$$\sum_{k<n} \rho(a_k)\rho(b_k) = 0 \text{ in } k_2(F).$$

Then there exist $u_0, ..., u_{p-1}, v_0, ..., v_{p-1} \in F^*$ such that $v_i \in 1+u_i$ for $i = 0, ..., p-1$ and

$$\sum_{k<n} \rho(a_k)\rho(b_k) = \sum_{k<n} \rho(a_k)\rho(b_k) \text{ in } k_1(F) \otimes k_1(F).$$

Enlarge the set $\{a_0, ..., a_{n-1}\}$ to a base for $[\{a_0, ..., a_{n-1}, u_0, ..., u_{p-1}\}]$, namely $\{c_0, ..., c_{m-1}\}$ with $c_i = a_i$ for all $i < n$. For all $x \in C := [c_0, ..., c_{m-1}]$ there exist $S \in P(m)$ such that

$$x = \prod_{i \in S} c_i := c_S.$$

Moreover, since $\{c_0, ..., c_{m-1}\}$ is a basis, for each $i = 0, ..., p-1$ there is only one $S_i \in P(m)$ such that

$$u_i = c_{S_i}.$$

For each $S \in P(m)$, set

$$r_S := \prod_{\substack{those\ j\ with \\ S_j=S}} v_j.$$

If no $S_j = S$, set $r_S = 1$. Note that if there is an index j with $S = S_j$, this index must be unique (because the expression $u_i = c_{S_i}$ is unique). Then by construction

$r_S \in 1 + c_S \setminus \{0\}$ and in $k_2(F)$ we get

$$\sum_{k<m} \rho(a_k)\rho(b_k) = \sum_{k<n} \rho(c_k)\rho(d_k) = \sum_{k<p} \rho(u_k)\rho(v_k)$$

$$= \sum_{S \in P(m)} \rho \left(\prod_{\substack{those\ j\ with \\ S_j=S}} c_j \right) \rho(v_j)$$

$$= \sum_{S \in P(m)} \sum_{\substack{those\ j\ with \\ S_j=S}} \rho(c_j)\rho(v_j)$$

$$= \sum_{S \in P(m)} \rho(c_S)\rho \left(\prod_{\substack{those\ j\ with \\ S_j=S}} v_S \right)$$

$$= \sum_{S \in P(m)} \rho(c_S)\rho(r_S) = \sum_{S \in P(m)} \sum_{k \in S} \rho(c_k)\rho(r_S)$$

$$= \sum_{k<m} \rho(c_k)\rho \left(\prod_{S \in P(n)} r_S \right).$$

Since $\{c_0, ..., c_{m-1}\}$ is a basis, it follows that

$$d_i = \prod_{S \in P(n)} r_S$$

as desired.

(ii)$\Rightarrow$(i). Under the hypotheses of (ii) we get

$$\sum_{k<n} \rho(a_k)\rho(b_k) = \sum_{k<m} \rho(c_k)\rho(d_k) = \sum_{k<m} \rho(c_k)\rho \left(\prod_{S \in P(n)} r_S \right)$$

$$= \sum_{k<m} \sum_{S \in P(m)} \rho(c_k)\rho(r_S) = \sum_{S \in P(m)} \sum_{k<m} \rho(c_i)\rho(r_S)$$

$$= \sum_{S \in P(m)} \rho(c_S)\rho(r_S) = 0.$$

$\square$

4   Inductive graded rings and interchanging of K-theories

After the three K-theories defined in the above sections, it is desirable (or, at least, suggestive) to build of an abstract environment that encapsulates all them, and of course, provide an axiomatic approach to guide new extensions of the concept of K-theory in the context of the algebraic and abstract theories of quadratic forms. The inductive graded rings fit well to this purpose[5].

Definition 4.1 (Inductive Graded Rings First Version (Definition 9.7 of [3]))**.** An **inductive graded ring** (or **Igr** for short) is a structure $R = ((R_n)_{n \geqslant 0}, (h_n)_{n \geqslant 0}, *_{nm})$ where

  i) $R_0 \cong \mathbb{F}_2$.

 ii) R_n is a group of exponent 2 with a distinguished element $\top_n$.

iii) $h_n : R_n \to R_{n+1}$ is a group homomorphism such that $h_n(\top_n) = \top_{n+1}$.

 iv) For all $n \geqslant 0$, $h_n = *_{1n}(\top_1, \_)$.

  v) The ring

$$R = \bigoplus_{n \geqslant 0} R_n$$

     is a commutative graded ring.

 vi) For $0 \leqslant s \leqslant t$ define

$$h_s^t = \begin{cases} Id_{R_s} & \text{if } s = t \\ h_{t-1} \circ \ldots \circ h_{s+1} \circ h_s & \text{if } s < t. \end{cases}$$

     Then if $p \geqslant n$ and $q \geqslant m$, for all $x \in R_n$ and $y \in R_m$,

$$h_n^p(x) * h_m^q(y) = h_{n+m}^{p+q}(x * y).$$

A **morphism** between Igr's R and S is a pair $f = (f, (f_n)_{n \geqslant 0})$ where $f_n : R_n \to S_n$ is a morphism of pointed groups and

$$f = \bigoplus_{n \geqslant 0} f_n : R \to S$$

is a morphism of commutative rings with unity. The category of inductive graded rings (in first version) and their morphisms will be denoted by Igr.

---

[5]A categorical and systematic development of inductive graded rings in connection with "quadratic multirings" is carried out in [16].

A first consequence of these definitions is that: if

$$f : ((R_n)_{n \geqslant 0}, (h_n)_{n \geqslant 0}, *_{nm}) \to ((S_n)_{n \geqslant 0}, (l_n)_{n \geqslant 0}, *_{nm})$$

is a morphism of Igr's then $f_{n+1} \circ h_n = l_n \circ f_n$.

$$
\begin{array}{ccccccccccc}
R_0 & \xrightarrow{h_0} & R_1 & \xrightarrow{h_1} & R_2 & \xrightarrow{h_2} & \dots & \xrightarrow{h_{n-1}} & R_n & \xrightarrow{h_n} & R_{n+1} & \xrightarrow{h_{n+1}} & \dots \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & & & \downarrow f_n & & \downarrow f_{n+1} & & \\
S_0 & \xrightarrow{l_0} & S_1 & \xrightarrow{l_1} & S_2 & \xrightarrow{l_2} & \dots & \xrightarrow{l_{n-1}} & S_n & \xrightarrow{l_n} & S_{n+1} & \xrightarrow{l_{n+1}} & \dots
\end{array}
$$

In fact, $f_0 : R_0 \to S_0$ is an isomorphism, so $f_1 \circ h_0 = l_0 \circ f_0$. If $n \geqslant 1$, for all $a_n \in R_n$ holds

$$f_{n+1} \circ h_n(a_n) = f_{n+1} \circ (*_{1n}(\top_1, a_n)) = f_1(\top_1) *_{1n} f_n(a_n)$$
$$= \top_1 *_{1n} f_n(a_n) = l_n(f_n(a_n)) = l_n \circ f_n(a_n).$$

Example 4.2.

(a) Let F be a field of characteristic not 2. The main actors here are WF, the Witt ring of F and IF, the fundamental ideal of WF. Is well know that $I^n F$, the n-th power of IF is additively generated by n-fold Pfister forms over F. Now, let $R_0 = WF/IF \cong \mathbb{F}_2$ and $R_n = I^n F/I^{n+1} F$. Finally, let $h_n = \_ \otimes \langle 1, 1 \rangle$. With these prescriptions we have an inductive graded ring R associated to F.

(b) The previous example still works if we change the Witt ring of a field F for the Witt ring of a (formally real) special group G.

Definition 4.3. We denote Igr_{fin} the full subcategory of Igr such that

$$\mathrm{Obj}(\mathrm{Igr}_{fin}) = \{R \in \mathrm{Obj}(\mathrm{Igr}) : |R_n| < \omega \text{ for all } n \geqslant 1\}.$$

Of course,

$$\left\{ R \in \mathrm{Obj}(\mathrm{Igr}) : \left| \bigoplus_{n \geqslant 1} R_n \right| < \omega \right\} \neq \mathrm{Obj}(\mathrm{Igr}_{fin}),$$

for example, in 4.2(a), $\mathbb{F}_2[x] \in \mathrm{Obj}(\mathrm{Igr}_{fin})$ and $\mathbb{F}_2[x]$ is not finite.

We finally this section with an use of Igr's to interchanging the three K-theory notions presented before in a functorial fashion. Lets first, look more carefully at theorem 4.4. We make the following distinctions between K-theories:

$$K^{mil} \text{ will denote the Milnor's K-theory,}$$

$$K^{dm} \text{ will denote the Dickmann-Miraglia's K-theory,}$$

$$K^{mult} \text{ will denote the K-theory of Hyperfields.}$$

We will deal with some different categories, so in order to ease the exposition lets establish some dictionary:

1. An object in Igr will be denoted simply by $R = (R_n, h_n, *_{nm})$ (being implicit the identation $n \in \mathbb{N}$ or $n \geqslant 0$). We even denote simply $R = (R_n, h, *)$ if the context allows it.

2. The distinguished elements $\top_n \in R_n$ will be denoted simply by $\top$, whenever the context allows it, and we do the same for the symbols $+_n, 0_n$.

3. An IGR-morphism between R and S will be simply denoted by $f : R \rightarrow S$, with the immediate convention that $f = (f_n)_{n \geqslant 0}$, with $f_n : R_n \rightarrow S_n$ being a morphism of pointed $\mathbb{F}_2$-modules.

4. Let C be a category. We use the notation "$A, B \in C$" as a synonym of "A, B be objects of C".

5. Let C be a category, A, B be objects of C. We use the notations "$f \in C(A, B)$ and $f \in \mathrm{Hom}_C(A, B)$" as a synonyms of "$f : A \rightarrow B$ is a morphism in C".

6. We will define various functors below. In order to facilitate the exposition, we define a functor $F : C \rightarrow \mathcal{D}$ by the sentence "for $f : A \rightarrow B$, we define $F(A) := ...$ and $F(f) := ...$"; being implicit that A and B are objects and $f : A \rightarrow B$ is a morphism in category C.

Of course, we need the following theorem:

Theorem 4.4.   (a) *Let F be a field. Then $k_*^{mil} F$ (the reduced Milnor K-theory) is an inductive graded ring.*

  (b) *Let G be a special group. Then $k_*^{dm} G$ (the Dickmann-Miraglia K-theory of G) is an inductive graded ring.*

  (c) *Let F be a hyperfield. Then $k_*^{mult} F$ (our reduced K-theory) is an inductive graded ring.*

Proof. Item (a) is the content of Lemma 9.11 in [3], and item (b) is the content of Lemma 9.12 in [3]. We prove item (c) and itens (a) and (b) will proceed by the same argument.

Let $k_*^{mult} F = (k_0 F, k_1 F, ..., k_n F, ...)$ be the reduced K-theory of a hyperfield F. Let $\top_0 = 1$ and for each $n \geqslant 1$, we set $\top_n = l(-1)^n$ as the distinguished element of $m - n$. For each $n \geqslant 0$, let

$$\theta_n : \prod_{j=1}^{n} K_1^{mult} F \to \otimes_{j=1}^{n+1} K_{n+1}^{mult} F$$

given by the rule

$$\theta_n(\rho(a_1), ..., \rho(a_n)) := \rho(-1)\rho(a_1)...\rho(a_n).$$

We have for each $i \in \{1, ..., n\}$ and each $a_1, ..., a_n, b_i \in F^*$ that

$$\theta_n(\rho(a_1), ...\rho(a_i) + \rho(b_i), ..., \rho(a_n)) = \theta_n(\rho(a_1), ...\rho(a_i b_i), ..., \rho(a_n)) :=$$
$$\rho(-1)\rho(a_1)...\rho(a_i b_i)...\rho(a_n) = \rho(-1)\rho(a_1)...[\rho(a_i) + \rho(b_i)]...\rho(a_n) =$$
$$\rho(-1)\rho(a_1)...\rho(a_i)...\rho(a_n) + \rho(-1)\rho(a_1)...\rho(b_i)...\rho(a_n) =$$
$$\theta_n(\rho(a_1), ..., \rho(a_i), ..., \rho(a_n)) + \theta_n(\rho(a_1)...\rho(b_i)...\rho(a_n)),$$

then θ_n is multilinear. By the universal property of tensor product, we have a group homomorphism $\tilde{\theta}_n : K_n^{mult} F \to K_{n+1}^{mult} F$ given by the rule[6]

$$\tilde{\theta}_n(\rho(a_1)...\rho(a_n)) = \rho(-1)\rho(a_1)...\rho(a_n).$$

In order to make distinctions between reduced and non-reduced K-theories, we denote an element in $k_n^{mult} F := K_n^{mult} F / 2K_n^{mult} F$ by $\tilde{\rho}(a_1)...\tilde{\rho}(a_n)$. Lets also denote the canonical projection by $\pi_n : K_n^{mult} F \to k_n^{mult} F$. We define $\omega_n : k_n^{mult} F \to k_{n+1}^{mult} F$ by the following rule (on generators): for $a_1, ..., a_n \in \dot{F}$,

$$\omega_n(\tilde{\rho}(a_1)...\tilde{\rho}(a_n)) = \tilde{\rho}(-1)\tilde{\rho}(a_1)...\tilde{\rho}(a_n).$$

In fact, if $\rho(a_1)...\rho(b_n) - \rho(b_1)...\rho(b_n) \in 2K_n^{mult} F$ then

$$\rho(-1)\rho(a_1)...\rho(b_n) - \rho(-1)\rho(b_1)...\rho(b_n)$$
$$= \rho(-1)[\rho(a_1)...\rho(b_n) - \rho(b_1)...\rho(b_n)] \in 2K_{n+1}^{mult} F,$$

---

[6]Remember that we are using the simplificated notation for elements in $K_n^{mult} F$ (and all other K-theories), which is $\rho(a_1)...\rho(a_n) := \rho(a_1) \otimes ... \otimes \rho(a_n)$.

which proves that ω_n is in fact a group homomorphism making the following diagram commute

$$
\begin{array}{ccc}
K_n^{mult} F & \xrightarrow{\tilde{\theta}_n} & K_{n+1}^{mult} F \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} \\
k_n^{mult} F & \xrightarrow{\omega_n} & k_{n+1}^{mult} F
\end{array}
$$

With these rules, we already have the properties (i)-(iv) of Definition 4.1 holding in $k_* F$, remaining only property (v). Note that $\omega_s^t = \tilde{\rho}(-1)^{t-s}$ for $0 \leqslant s < t$.

Now let $m, n, p, q \in \mathbb{N}$, $p \geqslant n, q \geqslant m$ and consider $x \in k_n^{mult} F$ and $y \in k_m^{mult} F$. Note that $\omega_n^p(x) = \tilde{\rho}(-1)^{p-n} \cdot x$ and $\omega_m^q(y) = \tilde{\rho}(-1)^{q-m} \cdot y$. Then

$$
\begin{aligned}
\omega_n^p(x) \cdot \omega_m^q(y) &= (\tilde{\rho}(-1)^{p-n} \cdot x)(\tilde{\rho}(-1)^{q-m} \cdot y) \\
&= \tilde{\rho}(-1)^{p-n+q-m} \cdot (x \cdot y) = \omega_{n-m}^{p+q}(x),
\end{aligned}
$$

completing the proof. □

Using this Theorem (in addition with the argument of Lemma 3.3 in [5]) we obtain the following.

Corollary 4.5. *Let F be a field. We have a functor and $k : Field_2 \to Igr$ induced by K-theory and Milnor's reduced K-theory.*

Corollary 4.6. *Let F be a hyperfield.   Then,   we   have   a   functor $k^{mult} : MField_2 \to Igr$ induced by our reduced K-theory.*

Theorem 4.7 (Theorem 2.5 in [4]). *Let F be a field. Then the functor $G : Field_2 \to SG$ provides a functor $k_*^{dm} : Field_2 \to Igr$ (the special group K-theory functor) given on the objects by $k_*^{dm}(F) : k_*^{dm}(G(F))$ and on the morphisms $f : F \to K$ by $k_*^{dm}(f) := G(f)_*$ (in the sense of Lemma 3.3 of [5]). Moreover, this functor commutes with the functors G and k, i.e, for all $F \in Field$, $k_*^{dm}(G(F)) \cong k_*(F)$.*

Theorem 4.8. *Let G be a special group. The equivalence of categories $M : SG \to SMF$ induces a functor $k_*^{mult} : SG \to Igr$ given on the objects by $k_*^{mult}(G) := k_*^{mult}(M(G))$, and given on the morphisms $f : G \to H$ by $k_*^{mult}(f) := k_*^{mult}(M(f))$. Moreover, this functor commutes with M and k^{dm}, i.e, for all $G \in SG$, $k_*^{mult}(M(G)) \cong k_*^{dm}(G)$.*

Proof. The only part requiring proof is that for all $G \in SG$, $k_*^{mult}(M(G))$ is isomorphic to $k_*^{dm}(G)$. The very first observation is that: since G is an exponent 2 group, the reduced and non-reduced K^{mult}-theory of $M(G)$ coincide.

Following the argument of Theorem 2.5 in [4], it is enough to show the following two statements:

(i) For all $a, b \in G$, if $b \in 1 - a$ in $M(G)$ then $\lambda(b)\lambda(a) = 0$;

(ii) For all $a, b \in G$, if $b \in D_G(1, a)$ then $\rho(b)\rho(a) = 0$.

For (i), if $b \in 1 - a$ in $M(G)$ then $b \in D_G(1, -a)$ and then, $\lambda(b)\lambda(-a) = 0$. Hence

$$\lambda(b)^2 = \lambda(b)\lambda(-a) = \lambda(b)\lambda(a) + \lambda(b)\lambda(-1).$$

Since $\lambda(b)\lambda(-1) = \lambda(b)^2$, we get $\lambda(b)\lambda(a) = 0$.

For (ii) we just use the same argument: if $b \in D_G(1, a)$ then $b \in 1 + a$ in $M(G)$ and then, $\rho(b)\rho(-a) = 0$. Hence

$$\rho(b)^2 = \rho(b)\rho(-a) = \rho(b)\rho(a) + \rho(b)\rho(-1).$$

Since $\rho(b)\rho(-1) = \rho(b)^2$, we get $\rho(b)\rho(a) = 0$. $\qquad\square$

Combining Theorems 4.4, 4.7, 4.8, 3.6, and Corollaries 4.5 and 4.6, we obtain the following Theorem that unificate in some sense all three K-theories:

Theorem 4.9 (Interchanging K-theories Formulas)**.** *Let $F \in Field_2$. Then*

$$k^{mil}(F) \cong k^{dm}(G(F)) \cong k^{mult}(M(G(F))).$$

If F is formally real and T is a preordering of F, then

$$k^{dm}(G_T(F)) \cong k^{mult}(M(G_T(F))).$$

Moreover, since $M(G(F)) \cong F/_m\dot{F}^2$ and $M(G_T(F)) \cong F/_m T^*$, we get

$$k^{mil}(F) \cong k^{dm}(G(F)) \cong k^{mult}(F/_m\dot{F}^2) \text{ and}$$
$$k^{dm}(G_T(F)) \cong k^{mult}(F/_m T^*).$$

Corollary 4.10. *Let F be a field. Then*

$$k^{mil}(F) \cong k^{mult}(F/_m\dot{F}^2).$$

Proof. Using the previous Corollary, we already have

$$k^{mil}(F) \cong k^{dm}(G(F)) \cong k^{mult}(M(G(F))).$$

Now, is enough to observe that $M(G(F)) \cong F/_m\dot{F}^2$. □

Combining Theorem 4.9, Corollary 4.10 and Theorem 3.6 we get the following Corollaries.

Corollary 4.11. *Let F be a formally real field and T be a preordering. Then we have a surjective map*

$$k^{mil}(F) \rightarrow k^{mult}(F/_mT^*).$$

Corollary 4.12. *Let G be a pre-special group and $H \subseteq G$ be a subgroup of G. Let $M(G)$ be the pre-special multifield associated to G and let $M(H) = H \cup \{0\} \subseteq M(G)$. Then*

$$G/H \cong M(G)/_mM(H)^*.$$

Moreover, $M(H) \subseteq M(G)$ is a preordering if and only if H is saturated.

Corollary 4.13. *Let G be a special group and H be a saturated subgroup. Then we have a surjective map*

$$k^{dm}(G) \rightarrow k^{mult}(G/_mH) \cong k^{dm}(G/H).$$

We close this paper "embedding" the three K-theories considered until now into the framework of inductive graded rings, in such a way that this "embedding" provides a sort of free objects in Igr. Lets firstly define functors that formalize the construction of K-theory.

Definition 4.14. We define the **graded subring generated by the level 1** functor

$$\mathbb{1} : \text{Igr} \rightarrow \text{Igr}$$

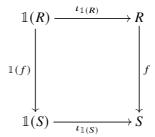as follow: for an object $R = (R_n, h_n, *_{nm})$,

   i) $\mathbb{1}(R)_0 := R_0 \cong \mathbb{F}_2$,

   ii) $\mathbb{1}(R)_1 := R_1$,

iii) for $n \geqslant 2$,

$$\mathbb{1}(R)_n := \{x \in R_n : x = \sum_{j=1}^{r} a_{1j} *_{11} \ldots *_{11} a_{nj},$$

with $a_{ij} \in R_1$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$ for some $r \geqslant 1\}$.

Of course, $\mathbb{1}(R)$ provides an inclusion $\iota_{\mathbb{1}(R)} : \mathbb{1}(R) \to R$ in the obvious way.

On the morphisms, for $f \in \mathrm{Igr}(R, S)$, we define $\mathbb{1}(f) \in \mathrm{Igr}(\mathbb{1}(R), \mathbb{1}(S))$ by the restriction $\mathbb{1}(f) = f \restriction_{\mathbb{1}(R)}$. In other words, $\mathbb{1}(f)$ is the only Igr-morphisms that makes the following diagram commute:

$$
\begin{array}{ccc}
\mathbb{1}(R) & \xrightarrow{\iota_{\mathbb{1}(R)}} & R \\
\downarrow{\scriptstyle \mathbb{1}(f)} & & \downarrow{\scriptstyle f} \\
\mathbb{1}(S) & \xrightarrow{\iota_{\mathbb{1}(S)}} & S
\end{array}
$$

We denote $\mathrm{Igr}_{\mathbb{1}}$ the full subcategory of Igr such that

$$\mathrm{Obj}(\mathrm{Igr}_{\mathbb{1}}) = \{R \in \mathrm{Igr} : \iota_{\mathbb{1}(R)} : \mathbb{1}(R) \to R \text{ is an isomorphism}\}.$$

Definition 4.15. We define the **quotient graded ring functor**

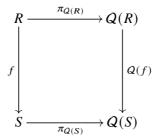$$Q : \mathrm{Igr} \to \mathrm{Igr}$$

as follow: for an object $R = (R_n, h_n, *_{nm})$, $Q(R) := R/Q$, where $Q = (Q_n)_{n \geqslant 0}$ is the ideal generated by $\{(\top + a) *_{11} a \in R_2 : a \in R_1\}$. More explicit,

i) $Q_0 := \{0\} \subseteq R_0$,

ii) $Q_1 := \{0\} \subseteq R_1$,

iii) for $n \geqslant 2$, $Q_n \subseteq R_n$ is the pointed $\mathbb{F}_2$-submodule generated by

$$\{x \in R_n : x = y_l *_{l1} (\top + a_1) *_{11} a_1 *_{1r} z_r,$$

with $a_1 \in R_1$, $y_l \in R_l$, $z_r \in R_r$, $1 \leqslant r, l \leqslant n - 2$, $l + r = n - 2\}$.

Of course, $Q(R)$ provides a projection $\pi_{Q(R)} : R \to Q(R)$ in the obvious way.

On the morphisms, for $f \in \mathrm{Igr}(R, S)$, we define $Q(f) \in \mathrm{Igr}(Q(R), Q(S))$ by the only Igr-morphisms that makes the following diagram commute:

$$
\begin{array}{ccc}
R & \xrightarrow{\;\pi_{Q(R)}\;} & Q(R) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Q(f)} \\
S & \xrightarrow[\;\pi_{Q(S)}\;]{} & Q(S)
\end{array}
$$

We denote Igr_q the full subcategory of Igr such that

$$\mathrm{Obj}(\mathrm{Igr}_q) = \{R \in \mathrm{Igr} : \pi_Q : R \to Q(R) \text{ is an isomorphism}\}.$$

Definition 4.16. We denote by Igr_+ the full subcategory of Igr such that

$$\mathrm{Obj}(\mathrm{Igr}_+) = \mathrm{Obj}(\mathrm{Igr}_{\mathbb{1}}) \cap \mathrm{Obj}(\mathrm{Igr}_q).$$

We denote by $j_+ : \mathrm{Igr}_+ \to \mathrm{Igr}$ the inclusion functor.

By the very definition of the K-theory of hyperfields (with the notations in Theorem 3.3) we define the following functor.

Definition 4.17 (K-theories Functors). With the notations of Theorem 3.3 we have a functors $k : \mathcal{HMF} \to \mathrm{Igr}_+$, $k : \mathcal{PSMF} \to \mathrm{Igr}_+$ induced by the reduced K-theory for hyperfields.

Now, let $R \in \mathrm{Igr}$. We define a hyperfield $(\Gamma(R), +, -\cdot, 0, 1)$ by the following: firstly, fix an isomorphism $e_R : (R_1, +_1, 0_1, \top_1) \to (G(R), \cdot, 1, -1)$. This isomorphism makes, for example, an element $a *_{11} (\top + b) \in R_2, a, b \in R_1$ take the form $(e_R^{-1}(x)) *_{11} (e_R^{-1}((-1) \cdot y)) \in R_2, x, y \in \Gamma(R)$. By an abuse of notation, we simply write $x *_{11} (-y) \in R_2, x, y \in \Gamma(R)$. In this sense, an element in Q_2 (see Definition 4.15) has the form $x *_{11} (-x), x \in \Gamma(R)$, and we can extend this terminology for all $Q_n, n \geqslant 2$.

Now, let $\Gamma(R) := G(R) \cup \{0\}$ and for $a, b \in \Gamma(R)$ we define

$$-a := (-1) \cdot a,$$
$$a \cdot 0 = 0 \cdot a := 0,$$
$$a + 0 = 0 + a = \{a\},$$
$$a + (-a) = \Gamma(R),$$

for $a, b \neq 0, a \neq -b$ define
$$a + b := \{c \in \Gamma(R) : \text{there exists } d \in G(R) \text{ such that}$$
$$a \cdot b = c \cdot d \text{ and } a *_{11} b = c *_{11} d \in R_2\}. \tag{4.1}$$

Proposition 4.18. *With the above rules, $(\Gamma(R), +, -\cdot, 0, 1)$ is a pre-special hyperfield.*

Proof. The proof is similar to Theorem 2.17: we will verify the conditions of Definition 2.11. Note that by the definition of multivalued sum once we proof that $\Gamma(R)$ is an hyperfield, it will be hyperbolic.

i) In order to prove that $(\Gamma(R), +, -\cdot, 0, 1)$ is a multigroup we follow the steps below.

   (a) Commutativity and $(a \in b + 0) \Leftrightarrow (a = b)$ are direct consequence of the definition of multivaluated sum and the fact that $a *_{11} b = b *_{11} a$.

   (b) We will prove that if $c \in a + b$, then $a \in b - c$ and $b \in -c + a$.
   If $a = 0$ (or $b = 0$) or $a = -b$, then $c \in a + b$ means $c = a$ or $c \in a - a$. In both cases we get $a \in b - c$ and $b \in -c + a$.
   Now suppose $a, b \neq 0$ with $a \neq -b$. Let $c \in a + b$. Then $a \cdot b = c \cdot d$ and $a *_{11} b = c *_{11} d \in R_2$ for some $d \in G(R)$. Then $b \cdot (-c) = a \cdot (-d)$ and

   $$a *_{11} b = c *_{11} d \Rightarrow$$
   $$a *_{11} b *_{11} [(-b) *_{11} (-d)] = c *_{11} d *_{11} [(-b) *_{11} (-d)]$$
   $$\Rightarrow a *_{11} (-d) = b *_{11} (-c),$$

   proving that $a \in b - c$. Similarly we prove that $b \in -c + a$.

   (c) Now we prove the associativity, that is,

   $$(a + b) + c = a + (b + c).$$

In fact (see the remarks after Lemma 2.4 of [14]), it is enough to show

$$(a + b) + c \subseteq a + (b + c).$$

If $0 \in \{a, b, c\}$ we are done. Now let $a, b, c \neq 0$ and $x \in (a + b) + c$. If $b = -c$ we have

$$a + (b + c) = a + \Gamma(R) = \Gamma(R) \supseteq (a + b) + c.$$

If $-c \in a + b$, then $-a \in b + c$ and we have

$$(a + b) + c = \Gamma(R) = a + (b + c).$$

Now suppose $a, b, c \neq 0$, $c \neq -b$, $-c \notin a + b$. Let $p \in (a + b) + c$. Then $p \in x + c$ for some $x \in a + b$, and there exists $q, y \in G(R)$ such that

$$p \cdot q = x \cdot c \text{ and } p *_{11} q = x *_{11} c$$
$$x \cdot y = a \cdot b \text{ and } x *_{11} y = a *_{11} b.$$

Now let $z, w \in G(R)$ defined by $z = xca$, $w = abx$. We have $p \cdot q = x \cdot c = x \cdot c \cdot (a^2) = a \cdot z$ and $z \cdot w = (xca) \cdot (abx) = b \cdot c$. Moreover

$$p *_{11} q = x *_{11} c = x *_{11} c *_{11} (a^2) = x *_{11} c *_{11} a *_{11} a$$
$$= [x *_{11} c *_{11} a] *_{11} a = (xca) *_{11} a = z *_{11} a$$

and

$$z *_{11} w = (xca) *_{11} (abx)$$
$$= (cxa) *_{11} (axb) = c *_{11} (xa) *_{11} (ax) *_{11} b = c *_{11} b.$$

Then $p \in a + z$ with $z \in c + b = b + c$, and hence $p \in a + (b + c)$.

ii) Since $(G(R), \cdot, 1)$ is an abelian group, we conclude that $(\Gamma(R), \cdot, 1)$ is a commutative monoid. Beyond this, every nonzero element $a \in \Gamma(R)$ is such that $a^2 = 1$.

iii) $a \cdot 0 = 0$ for all $a \in \Gamma(R)$ is direct from definition.

iv) For the distributive property, let $a, b, d \in \Gamma(R)$ and consider $x \in d(a + b)$. We need to prove that

$$x \in d \cdot a + d \cdot b. \tag{*}$$

It is the case if $0 \in \{a, b, d\}$ or if $b = -a$.

Now suppose $a, b, d \neq 0$ with $b \neq -a$. Then there exist $y \in G(R)$ such that $x = dy$ and $y \in a + b$. Moreover, there exist some $z \in G(R)$ such that $y \cdot z = a \cdot b$ and $y *_{11} z = a *_{11} b$. Therefore $(dy) \cdot (dz) = (da) \cdot (db)$ and $(dy) *_{11} (dz) = (da) *_{11} (db)$, and $x = dy \in d \cdot a + d \cdot b$.

Then $(\Gamma(R), +, -, \cdot, 0, 1)$ is a hyperbolic hyperfield.

Finally, let $a \in \Gamma(R)$ and $x, y \in 1 - a$. If $a = 0$ or $a = 1$ then we automatically have $x \cdot y \in 1 - a$, so let $a \neq 0$ and $a \neq 1$. Then $x, y \in G(R)$ and there exist $p, q \in \Gamma(R)$ such that

$$x \cdot p = 1 \cdot a \text{ and } x *_{11} p = 1 *_{11} a$$
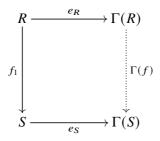$$y \cdot q = 1 \cdot a \text{ and } y *_{11} q = 1 *_{11} a.$$

Then $(xy) \cdot (pqa) = 1 \cdot a$ and

$$(xy) *_{11} (pqa) = x *_{11} y *_{11} p *_{11} q *_{11} a = [x *_{11} p] *_{11} a *_{11} [y *_{11} q]$$
$$= [1 *_{11} a] *_{11} a *_{11} [1 *_{11} a] = 1 *_{11} a,$$

then $xy \in 1 - a$, proving that $(1 - a)(1 - a) \subseteq (1 - a)$. $\qquad\square$

Definition 4.19. With the notations of Proposition 4.18 we have a functor $\Gamma :$ $\text{Igr}_+ \to \text{PSMF}$ defined by the following rules: for $R \in \text{Igr}_+$, $\Gamma(R)$ is the special hyperfield obtained in Proposition 4.18 and for $f \in \text{Igr}_+(R, S)$, $\Gamma(f) : \Gamma(R) \to \Gamma(S)$ is the unique morphism such that the following diagram commute

$$
\begin{array}{ccc}
R & \xrightarrow{\; e_R \;} & \Gamma(R) \\
{\scriptstyle f_1}\Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma(f)} \\
S & \xrightarrow[\; e_S \;]{} & \Gamma(S)
\end{array}
$$

In other words, for $x \in R$ we have

$$\Gamma(f)(x) = (e_S \circ f_1 \circ e_R^{-1})(x) = e_S(f_1(e_R^{-1}(x))).$$

Theorem 4.20. *The functor $k : \mathcal{PSMF} \to Igr_+$ is the left adjoint of $\Gamma : Igr_+ \to \mathcal{PSMF}$. The unity of the adjoint is the natural transformation $\phi : 1_{\mathcal{PSMF}} \to \Gamma \circ k$ defined for $F \in \mathcal{PSMF}$ by $\phi_F = e_{k(F)} \circ \rho_F$.*

Proof. We show that for all $f \in \mathcal{PSMF}(F, \Gamma(R))$ there is an unique $f^\sharp : Igr_+(k(F), R)$ such that $\Gamma(f^\sharp) \circ \phi_F = f$. Note that $\phi_F = e_{k(F)} \circ \rho_F$ is an isomorphism (because $e_{k(F)}$ and ρ_F are isomorphisms).

Let $f_0^\sharp : 1_{\mathbb{F}_2} : \mathbb{F}_2 \to \mathbb{F}_2$ and $f_1^\sharp := e_R^{-1} \circ f \circ (\phi_F)^{-1} \circ e_{k(F)} : k_1(F) \to R_1$. For $n \geqslant 2$, define $h_n : \prod_{i=1}^n k_1(F) \to R_n$ by the rule

$$h_n(\rho(a_1), ..., \rho(a_n)) := e_R^{-1}(f(a_1)) * ... * e_R^{-1}(f(a_n)).$$

We have that h_n is multilinear and by the Universal Property of tensor products we have an induced morphism $\bigotimes_{i=1}^n k_n(F) \to R_n$ defined on the generators by

$$h_n(\rho(a_1) \otimes ... \otimes \rho(a_n)) := e_R^{-1}(f(a_1)) * ... * e_R^{-1}(f(a_n).$$

Now let $\eta \in Q_n(F)$. Suppose without loss of generalities that $\eta = \rho(a_1) \otimes ... \otimes \rho(a_n)$ with $a_1 \in 1 - a_2$. Then $f(a_1) \in 1 - f(a_2)$ which imply $e_R^{-1}(f(a_1)) \in 1 - e_R^{-1}(f(a_2))$. Since $R_n \in Igr_+$,

$$h_n(\eta) := h_n(\rho(a_1) \otimes ... \otimes \rho(a_n)) = e_R^{-1}(f(a_1)) * ... * e_R^{-1}(f(a_n)) = 0 \in R_n.$$

Then h_n factors through Q_n, and we have an induced morphism $\overline{h}_n : k_n(F) \to R_n$. We set $f_n^\sharp := \overline{h}_n$. In other words, $f_n^\sharp$ is defined on the generators by

$$f_n^\sharp(\rho(a_1)...\rho(a_n)) := e_R^{-1}(f(a_1)) * ... * e_R^{-1}(f(a_n)).$$

Finally, we have

$$\begin{aligned}
\Gamma(f^\sharp) \circ \phi_F &= [e_R \circ (f_1^\sharp) \circ e_{k(F)}^{-1}] \circ [e_{k(F)} \circ \rho_F] = e_R \circ (f_1^\sharp) \circ \rho_F \\
&= e_R \circ [e_R^{-1} \circ f \circ (\phi_F)^{-1} \circ e_{k(F)}] \circ \rho_F \\
&= f \circ (\phi_F)^{-1} \circ [e_{k(F)} \circ \rho_F] \\
&= f \circ (\phi_F)^{-1} \circ \phi_F = f.
\end{aligned}$$

For the uniqueness, let $u, v \in \text{Igr}_+(k(F), R)$ such that $\Gamma(u) \circ \phi_F$ is equal to $\Gamma(v) \circ \phi_F$. Since ϕ_F is an isomorphism we have $u_1 = v_1$ and since $k(F) \in \text{Igr}_+$ we have $u = v$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5 Final remarks and future works

In [17] we emphasize that DM-multirings and DP-multirings provide a new way to look at the abstract theories of quadratic forms (for example, for special groups we obtained an easy way to describe the axiom SG6 in the theory of special groups).

In [13] is constructed a von Neumann hull functor from multiring category and that, when restricted to semi-real rings, it commutes with real semigroup functor. This allows us to obtain some quadratic forms properties of a semi-real ring by looking to its von Neumann regular hull. In fact, the Witt ring of a real semigroups is canonically isomorphic to the Witt ring of its Von Neumann regular hull and this connection allows the full description of the graded Witt ring of a real semigroup.

With these two observations in mind, it is natural consider possible expansions of "Milnor's triangle" of graded rings – where K-theory is interpolating graded Witt ring and graded Cohomology ring – to the multiring setting. In more details, below are some further roads to follow:

1. Extension of the K-theory framework to more general multirings (for example, to VN-multirings) with quadratic flavour.

2. Compare graded K-theory with graded Witt ring for VN-real semigroups as in the field case (Milnor [11]) and special groups (Dickmann-Miraglia [3]).

3. The definition and analysis of the structure of Witt ring (and graded Witt ring) of more general quadratic structures (not only obtained from special groups and real semigroups): this subject have already appeared in section 4, in connection with [8].

4. Still in the hyperfield case, investigate the extension of the concept of Galois group to hyperfields, comparing the Galois cohomology ring and analyse the existence of some canonical arrow from K-theory to this cohomology ring, in an attempt to recover the Milnor's Conjecture available in the classic algebraic quadratic forms context ([15], [16]).

Acknowledgement

References

[1] Ameri, R. Eyvazi, M., and Hoskova-Mayerova, S.. *Superring of polynomials over a hyperring*, Mathematics, 7(10) (2019), 902.

[2] Dickmann, M. and Miraglia, F., *On quadratic forms whose total signature is zero mod 2^n: Solution to a problem of M. Marshall*, *Invent. Math.* 133(2) (1998), 243-278.

[3] Dickmann, M. and Miraglia, F., "Special Groups: Boolean-theoretic Methods in the Theory of Quadratic Forms", Mem. Amer. Math. Soc. 145(689), American Mathematical Society, 2000.

[4] Dickmann, M. and Miraglia, F., *Lam's conjecture*, Algebra Colloq. 10 (2003), 149-176.

[5] Dickmann, M. and Miraglia, F., *Algebraic k-theory of special groups*, J. Pure Appl. Algebra, 204(1) (2006), 195-234.

[6] Dickmann, M. and Miraglia, F., "Faithfully Quadratic Rings", Mem. Amer. Math. Soc. 238(1128), American Mathematical Society, 2015.

[7] Dickmann, M. and Petrovich, A., *Real semigroups and abstract real spectra*, I., *Algebraic and arithmetic theory of quadratic forms*, Contemp. Math. 344 (2004), 99-120.

[8] Gladki, P. and Worytkiewicz, K., *Witt rings of quadratically presentable fields*, Categ. General Alg. Struct. Appl. 12(1) (2020), 1-23.

[9] Jun, J., *Algebraic geometry over hyperrings*, Adv. Math. 323 (2018), 142-192.

[10] Marshall, M., *Real reduced multirings and multifields*, J. Pure Appl. Algebra 205(2) (2006), 452-468.

[11] Milnor, J., *Algebraic K-theory and quadratic forms*, *Invent. Math.* 9(4) (1970), 318-344.

[12] Pelea, C. and Purdea, I., *Multialgebras, universal algebras and identities*, J. Austral. Math. Soc. 81(1) (2006), 121-140.

[13] Ribeiro, H.R.d.O. and Mariano, H.L., *von Neumann regular hyperrings and applications to real reduced multirings*, arXiv:2101.06527, 2021.

[14] Ribeiro, H.R.d.O., Roberto, K.M.d.A., and Mariano, H.L., *Functorial relationship between multirings and the various abstract theories of quadratic forms*, São Paulo J. Math. Sci., 2020, `https://doi.org/10.1007/s40863-020-00185-1`.

[15] Roberto, K.M.d.A. and Mariano, H.L., *Galois groups of special hyperfields*, preprint, 2021.

[16] Roberto, K.M.d.A., and Mariano, H.L., *Quadratic multirings and graded rings*, preprint, 2021.

[17] Roberto, K.M.d.A., Ribeiro, H.R.d.O., and Mariano, H.L., *Quadratic structures associated to (multi) rings*, Categ. General Alg. Struct. Appl., 2021.

[18] Viro, O., *Hyperfields for tropical geometry I., hyperfields and dequantization*, arXiv:1006.3034, 2010.

[19] Wadsworth, A., *Merkurjev's elementary proof of Merkurjev's theorem*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Parts I, II, *Contemp. Math.* 55 (1983), 741-776.

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