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Algebraic models of cubical weak higher structures

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Abstract. In this article we recast some of the results developped in articles [19, 22] but in the setup of cubical geometry. Thus we define a monad on $\mathbb{CS}\mathrm{ets}$ whose algebras are models of cubical weak ∞-groupoids with connections. In addition, we define a monad on the category $\mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets}$ whose algebras are models of cubical weak ∞-functors, and a monad on the category $\mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets}$ whose algebras are models of cubical weak ∞-natural transformations.

1 Introduction and preliminaries

This article follows [21]; in it we explain how to build algebraic models of :

- cubical weak ∞-groupoids with connections (see 2.3)
- cubical weak ∞-functors (see 3.1)

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- cubical weak ∞-natural transformations (see 3.2)

In particular cubical weak $(\infty, 0)$-categories known as *cubical weak ∞-groupoids* are very important for us because although other models of cubical weak ∞-groupoids exist, they are defined in a non-algebraic way [7, 8, 10, 11, 13], i.e defined as kind of *cubical Kan complexes.*

We believe that our models of cubical weak ∞-groupoids should open new perspectives on the Grothendieck conjecture on homotopy types of spaces 2.3, which is stated in the globular setting, and is as follow:

Conjecture 1.1 (Grothendieck)**.** The category of (some) models of globular weak ∞-groupoids is equipped with a Quillen model structure such that its localisation is equivalent to the category of CW-spaces.

As [21], this article is mainly concerned with *cubical stretchings* and to the *cubical higher structures* they provide. Our main steps are as follows:

- We start to define cubical $(\infty, 0)$-sets, using as a main tool the *cubical reversors*. These are the cubical analogue of the globular $(\infty, 0)$-sets[1] which have been defined in [22]. More precisely they are built by using cubical analogue of minimal $(\infty, 0)$-structures in the sense of [22]. Then we define the categories of cubical reflexive $(\infty, 0)$-magmas and the category of cubical $(\infty, 0)$-groupoidal stretchings, which is the cubical analogue of the globular $(\infty, 0)$-categorical stretchings, as defined in [22]. The introduction of these new sketches inside cubical stretchings allows us to build a monad on the category of cubical sets whose algebras are our models of cubical weak ∞-groupoids. This monad is the cubical analogue of the monad on globular sets built in [22] and whose algebras are globular models of weak ∞-groupoids.

- In the sections 3.1 and 3.2 we extend globular weak ∞-functors and globular weak ∞-natural transformations to the cubical setting. In particular we shall see that the monad of cubical weak ∞-functors acts on the category $\mathbb{CS}$ets $\times$ $\mathbb{CS}$ets and the monad of cubical weak natural ∞-transformations acts on the category $\mathbb{CS}$ets $\times$ $\mathbb{CS}$ets $\times$ $\mathbb{CS}$ets $\times$ $\mathbb{CS}$ets. In these last sections some interesting internal 2-cubes appear in ∞-$\mathbb{CCAT}$ which are in fact all classical cubical strict 2-categories.

[1] Also called $(\infty, 0)$-graphs in [22]

> We finish our article by sketching the construction of an expected cocubical object of monads which should be a W-coalgebra[2], which leads to an operadic approach of the cubical weak ∞-category of cubical weak ∞-categories.

In this article the reader has to take care to not confuse (∞, m)-structures and (n, ∞)-structures: (∞, m)-structures as described in [22] are some kind of higher categories with invertible cells; (n, ∞)-structures as described in 3.1 and 3.2 lead to n-cells in the cubical weak ∞-category of cubical weak ∞-categories[3]; for example, $(0, \infty)$-structures in 3.1 refers to all higher structures surrounding cubical weak ∞-functors, and $(1, \infty)$-structures in 3.2 refers to all higher structures which surround cubical weak ∞-natural transformations. Forgetful functors are generically written with the letter U plus some variations of it: exponents on right side are used in 2.3, and left side are used in 3.1 and 3.2.

2 Cubical weak ∞-groupoids

The author doesn't know of any work on cubical weak ∞-categories with some kind of inverses involved. However such work exists in the strict case, see [27]. In low dimensions, simplicial methods have been used in [7, 8, 10, 11, 13] to study it. Some applications of it to homology have been considered in [9–11], and other applications in algebraic topology have also been carried out in [12]. As we said in the beginning of this article our first aim is to provide these higher cubical notions with the perspective to carry on applications of these cubical higher structures to homological algebra, algebraic topology and computer sciences.

The models of cubical weak ∞-groupoids that we are going to define are algebras for a monad $\mathbb{W}^0$ on the category $\mathbb{CS}$ets of cubical sets. This monad is built with adapted stretchings: the *cubical ∞-groupoidal stretchings* 2.3, which themselves are built with cubical strict ∞-groupoids with connections 2.2 which has been characterized by Lucas in [27], and with cubical reflexive $(\infty, 0)$-magmas 2.3. As in [21] these cubical ∞-groupoidal stretchings are tools which fill with cubical coherences cells their underlying cubical reflex-

[2] W is the monad of cubical weak ∞-categories built in [21] but seen as an $\mathbb{S}$-operad.

[3] We discuss some important progress for its existence in [25].

ive $(\infty,0)$-magmas, and such fills are controlled by their underlying cubical strict ∞-groupoids.

2.1 Cubical $(\infty,0)$-sets. Cubical $(\infty,0)$-sets underlie a new sketch (see diagrams below) which we use in 2.3 to define algebraic models of cubical weak ∞-groupoids.

Here we define cubical version of the formalism developed in [22] for globular $(\infty,0)$-sets. The formalism of this cubical world is similar to its globular analogue.

Consider a cubical set $\mathcal{C}=(C_n,s^n_{n-1,j},t^n_{n-1,j})_{1\leq j\leq n}$. If $n\geq 1$ and $1\leq j\leq n$, then a (n,j)-reversor on it is given by a map $C_n \xrightarrow{j^n_j} C_n$ such that the following two diagrams commute:

$$\begin{tikzcd} C_n \arrow[rr, "j^n_j"] \arrow[dr, "s^n_{n-1,j}"'] & & C_n \arrow[dl, "t^n_{n-1,j}"] \\ & C_{n-1} & \end{tikzcd} \qquad \begin{tikzcd} C_n \arrow[rr, "j^n_j"] \arrow[dr, "t^n_{n-1,j}"'] & & C_n \arrow[dl, "s^n_{n-1,j}"] \\ & C_{n-1} & \end{tikzcd}$$

If for each $n>0$ and for each $1\leq j\leq n$, there are such (n,j)-reversor j^n_j on $\mathcal{C}$, then we say that $\mathcal{C}$ is a cubical $(\infty,0)$-set. The family of maps $(j^n_j)_{n>0,1\leq j\leq n}$ for all $(n\in\mathbb{N}^*)$ is called an $(\infty,0)$-structure and in that case we shall say that $\mathcal{C}$ is equipped with the $(\infty,0)$-structure $(j^n_j)_{n>0,1\leq j\leq n}$. When we speak about such $(\infty,0)$-structure $(j^n_j)_{n>0,1\leq j\leq n}$ on $\mathcal{C}$, it means that it is for all integers $n\in\mathbb{N}^*$ such that C_n is non-empty. Seen as cubical $(\infty,0)$-set we denote it by:

$$\mathcal{C}=((C_n,s^n_{n-1,j},t^n_{n-1,j})_{1\leq j\leq n},(j^n_j)_{n>0,1\leq j\leq n})$$

and if

$$\mathcal{C}'=((C'_n,s'^n_{n-1,j},t'^n_{n-1,j})_{1\leq j\leq n},(j'^n_j)_{n>0,1\leq j\leq n})$$

is another $(\infty,0)$-set, then a morphism of $(\infty,0)$-sets:

$$\mathcal{C}\xrightarrow{f}\mathcal{C}'$$

is given by a morphism of cubical sets such that for each $n > 0$ and for each $1 \leq j \leq n$ we have the following commutative diagrams:

$$\begin{tikzcd} C_n \arrow[r, "j_j^n"] \arrow[d, "f_n"'] & C_n \arrow[d, "f_n"] \\ C'_n \arrow[r, "j'^n_j"'] & C'_n \end{tikzcd}$$

The category of cubical $(\infty,0)$-sets is denoted by $(\infty,0)$-$\mathbb{CS}$ets.

Remark 2.1. A cubical set $\mathcal{C} = (C_n, s^n_{n-1,j}, t^n_{n-1,j})_{1\leq j\leq n}$ can be equipped with other operations giving the inverse for degeneracies which come from connections. These operations are called *connectors* in [20]. For each integer $n \in \mathbb{N}^*$ such that C_n is non-empty, we define connectors on $\mathcal{C}$ as maps:

$$C_n \xrightarrow{j_j^{n,\gamma}} C_n$$

such that we have the following commutative diagrams:

$$\begin{tikzcd} C_n \arrow[rr, "j_j^{n,\gamma}"] \arrow[dr, "s^n_{n-1,j}"'] & & C_n \arrow[dl, "s^n_{n-1,j+1}"] \\ & C_{n-1} & \end{tikzcd} \qquad \begin{tikzcd} C_n \arrow[rr, "j_j^{n,\gamma}"] \arrow[dr, "t^n_{n-1,j}"'] & & C_n \arrow[dl, "t^n_{n-1,j+1}"] \\ & C_{n-1} & \end{tikzcd}$$

where $j \in \{1, \cdots, n-1\}$. This provides on $\mathcal{C}$ another kind of structure of $(\infty,0)$-set, which could be used to define cubical inverses related to connections. But we prefer to avoid such structure, though very interesting, because our scope is first to define cubical weak ∞-groupoids with connections, which use (through adapted stretchings, see 2.3) the characterization in [27] of cubical strict ∞-groupoids with connections, just by using reversors as above. Such cubical set $\mathcal{C}$ can be also equipped with reversors and connectors, which is still another kind of structure of $(\infty,0)$-set that deserves more investigations in the future. It is also important to notice that the different kinds of inverses for strict cubical ∞-categories have been invested by Lucas in [27] but without our level of structures.

Remark 2.2. This structural approach of inverses is much more powerful than the simplicial methods because with it we are able to build any kind of *reversible higher structure*. For example in our framework it is a simple exercise to build some exotic one which could be difficult to build with simplicial method.

2.2 Cubical strict ∞-groupoids In this section we use the characterization of Lucas [27] for cubical strict ∞-groupoids with connections. Thus cubical strict ∞-groupoids are just cubical strict ∞-categories with connections such that all n-cells for $n > 0$ are $\circ^n_j$-isomorphisms for $1 \leq j \leq n$. The studies of cubical strict structures using inverses has been done in [7, 8, 10, 11, 13, 27], especially in the aim to generalize many known results involving low dimensional groupoids and algebraic topology. For example, a generalization of the notion of cubical strict fundamental groupoids to higher dimensions has been undertaken in [9, 10] in order to obtain higher version of Van Kampen type Theorem.

In this article we use cubical strict ∞-groupoids as an underlying part of the structure of the *cubical* $(\infty,0)$*-groupoidal stretchings* (see 2.3) which are the adapted stretchings to weakened cubical strict ∞-groupoids. Thus they are an important step in our approach of cubical weak ∞-groupoids.

Consider a cubical strict ∞-category $\mathcal{C}$ as defined in [21]. We say that it is a cubical strict ∞-groupoid if its underlying cubical set is equipped with an $(\infty,0)$-structure $(j^n_j)_{n>0,1\leq j\leq n}$ satisfying the following identities: $\forall j,n$ such that $1 \leq j \leq n$ and $0 < n$,

$$\forall \alpha \in C_n,\ \alpha \circ^n_j j^n_j(\alpha) = 1^{n-1}_{n,j}(t^n_{n-1,j}(\alpha)) \text{ and } j^n_j(\alpha) \circ^n_j \alpha = 1^{n-1}_{n,j}(s^n_{n-1,j}(\alpha))$$

Proposition 2.3. A cubical strict ∞-groupoids $\mathcal{C}$ as above has a unique underlying $(\infty,0)$-set.

A cubical strict ∞-functor preserve (k,j)-reversors. Thus morphisms between cubical strict ∞-groupoids are just cubical strict ∞-functors. The category of cubical strict ∞-groupoids is denoted ∞-$\mathbb{CG}$rp. Also we have the following proposition

Proposition 2.4. The evident forgetful functor:

$$\infty\text{-}\mathbb{CG}\mathrm{rp} \xrightarrow{\ U\ } \mathbb{CS}\mathrm{ets}$$

is right adjoint and monadic.

Proof. Write $\mathcal{E}_{C_0}$ for the sketch of the cubical strict ∞-groupoids with connections. It is based on the sketch $\mathcal{E}_C$ of the cubical strict ∞-categories with connections, plus diagrams of the $(\infty,0)$-structure with the easy diagrammatical form of the axioms just above. But the sketch $\mathcal{E}_C$ was described in [21]. Thus $\mathcal{E}_{C_0}$ is projectively sketchable and it contains the sketch $\mathcal{E}_S$ (which is also projective) of cubical sets. Thus we just apply the Foltz theorem [17] for its right adjointness. The fact that it is monadic is straightforward. $\square$

The monad of cubical strict ∞-groupoids on cubical sets is denoted:

$$\mathbb{S}^0 = (S^0, \lambda^0, \mu^0)$$

Here λ^0 is the unit map of $\mathbb{S}^0$:

$$1_{\mathbb{CS}\mathrm{ets}} \xrightarrow{\quad \lambda^0 \quad} S^0$$

and μ^0 is the multiplication of $\mathbb{S}^0$:

$$(S^0)^2 \xrightarrow{\quad \mu^0 \quad} S^0$$

2.3 The category of cubical weak ∞-groupoids

A cubical reflexive $(\infty,0)$-magma is an object of ∞-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$ such that its underlying cubical set is equipped with an $(\infty,0)$-structure. Morphisms between cubical reflexive $(\infty,0)$-magmas are those of ∞-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$ which are also morphisms of $(\infty,0)$-$\mathbb{CS}ets$, that is, they preserve the underlying $(\infty,0)$-structures. The category of cubical reflexive $(\infty,0)$-magmas is denoted $(\infty,0)$-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$.

The category $(\infty,0)$-$\mathbb{CE}\mathrm{tG}$ of cubical ∞-groupoidal stretchings has as objects quintuples:

$$\mathbb{E} = (M, C, \pi, ([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n\}}, ([-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n\};\gamma\in\{-,+\}})$$

where M is a cubical reflexive $(\infty,0)$-magma, C is a cubical strict ∞-groupoid, π is a morphism in $(\infty,0)$-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$:

$$M \xrightarrow{\quad \pi \quad} C$$

and

$$([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,...,n\}},([-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,...,n\};\gamma\in\{-,+\}}$$

are the cubical bracketing structures which have already been defined in [21].

A morphism of cubical groupoidal stretchings:

$$\mathbb{E} \xrightarrow{(m,c)} \mathbb{E}'$$

is given by commutative squares in $(\infty,0)$-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$:

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "\pi"'] & M' \arrow[d, "\pi'"] \\ C \arrow[r, "c"'] & C' \end{tikzcd}$$

such that for all $n \in \mathbb{N}$, and for all $(\alpha,\beta) \in \widetilde{M_n}$,

$$m_{n+1}([\alpha,\beta]^n_{n+1,j}) = [m_n(\alpha),m_n(\beta)]^n_{n+1,j} \;\; (j \in \{1,...,n+1\})$$

and

$$m_{n+1}([\alpha,\beta]^{n,\gamma}_{n+1,j}) = [m_n(\alpha),m_n(\beta)]^{n,\gamma}_{n+1,j} \;\; (j \in \{1,...,n\}, \gamma \in \{-,+\})$$

The category of cubical groupoidal stretchings is denoted $(\infty,0)$-$\mathbb{CE}$tG.

Consider the forgetful functor:

$$(\infty,0)\text{-}\mathbb{CE}\mathrm{tG} \xrightarrow{U^0} \mathbb{CS}\mathrm{ets}$$

defined on objects by:

$$(M,C,\pi,([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,...,n\}},([-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,...,n\};\gamma\in\{-,+\}}) \longmapsto M$$

This functor is right adjoint which produces a monad $\mathbb{W}^0 = (W^0,\eta^0,\nu^0)$ on the category of cubical sets. Its right adjointness comes from the fact that the category $(\infty,0)$-$\mathbb{CE}$tG is projectively sketchable and contains the sketch $\mathcal{E}_S$. Main parts of this sketch have been already described in [21].

Definition 2.5. Cubical weak ∞-groupoids are algebras for the monad $\mathbb{W}^0 = (W^0, \eta^0, \nu^0)$ defined above on the category $\mathbb{C}\mathbb{S}$ets of cubical sets.

Thus the category of our models of cubical weak ∞-groupoids is denoted $\mathbb{W}^0$-$\mathbb{A}$lg.

If it is evident to see that we have "an embedding" of ∞-$\mathbb{C}\mathbb{G}$rp in ∞-$\mathbb{C}\mathbb{C}$at, this is also true in the weak case, but it is a bit more subtle; it comes from the forgetful functor:

$$(\infty, 0)\text{-}\mathbb{C}\mathbb{E}\text{tG} \xrightarrow{\quad U \quad} \infty\text{-}\mathbb{C}\mathbb{E}\text{tC}$$

which forgets the underlying $(\infty, 0)$-structures (see [21] for the description of the category ∞-$\mathbb{C}\mathbb{E}$tC). We also have the following morphism of the category $\mathbb{A}$dj of adjunctions:

$$\begin{tikzcd}
(\infty, 0)\text{-}\mathbb{C}\mathbb{E}\text{tG} \arrow[r, "V"] \arrow[d, shift left, "U^0"] & \infty\text{-}\mathbb{C}\mathbb{E}\text{tC} \arrow[d, shift left, "U"] \\
\mathbb{C}\mathbb{S}\text{ets} \arrow[r, "id"] \arrow[u, shift left, "F^0"] \arrow[u, phantom, "\dashv"] & \mathbb{C}\mathbb{S}\text{ets} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

because $U \circ V = U^0$ (see [19]), thus it produces a morphism V^* in the category $\mathbb{M}$nd of monads:

$$(\mathbb{C}\mathbb{S}\text{ets}, \mathbb{W}) \xrightarrow{\quad V^* \quad} (\mathbb{C}\mathbb{S}\text{ets}, \mathbb{W}^0)$$

and passing to algebras, gives the following functor $\mathbb{A}\text{lg}(V^*)$ which is the expected "embedding":

$$\mathbb{W}^0\text{-}\mathbb{A}\text{lg} \xrightarrow{\quad \mathbb{A}\text{lg}(V^*) \quad} \mathbb{W}\text{-}\mathbb{A}\text{lg}$$

In [1] it is proved that the category of cubical strict ∞-categories with connections is equivalent to the category of globular strict ∞-categories. In [27] it is proved that the category of cubical strict (∞, m)-categories with connections is equivalent to the category of globular strict (∞, m)-categories (which were first defined in [22]) and in particular he proved that the category of cubical strict ∞-groupoids with connections is equivalent to the category of globular strict ∞-groupoids.

Our models of cubical weak ∞-groupoids with connections are the direct analogue of globular weak ∞-groupoids (models of it are built in [4, 22, 30]). Thus the conjecture of Batanin in [3] which says that his globular weak ∞-categories are "equivalent" to Penon globular weak ∞-categories, plus the recent result of Bourke [5] which resolves the conjecture of Ara [2] which says that Batanin's globular weak ∞-categories are equivalent to Grothendieck weak ∞-categories, leads us to put formulate the following hypothesis:

Conjecture 2.6 (1)**.** The category $\mathbb{W}^0$-$\mathbb{A}$lg of cubical weak ∞-groupoids with connections, is equipped with a Quillen model structure, and such a model structure is Quillen equivalent to the category of globular weak ∞-groupoids also equipped with a genuine Quillen model structure

Conjecture 2.7 (2)**.** The category $\mathbb{W}^0$-$\mathbb{A}$lg of cubical weak ∞-groupoids with connections, is equiped with a Quillen model structure, the one of the first conjecture, such that its localization is equivalent to the category of CW-spaces.

This second conjecture is inspired by the result in [29] which says that the category $\mathbb{C}_r$ defined in [21] is a *test category*, that is the category of presheaves on $\mathbb{C}_r$ is equipped with a Quillen model structure (the *Cisinski one*, see [29]), such that its localization is equivalent to the category of CW-spaces.

These hypotheses together must solve the *Grothendieck conjecture on homotopy types*:

Conjecture 2.8 (Grothendieck)**.** The category of globular weak ∞-groupoids is equipped with a Quillen model structure such that its localization is equivalent to the category of CW-spaces.

3 Steps toward the cubical weak ∞-category of cubical weak ∞-categories.

In this last section we are going to show how cubical stretchings give algebraic models of cubical weak ∞-functors and cubical weak ∞-natural transformations. We can go further as it was done for the globular paradigm in [19], but just with this level 2 of cubical algebras we obtain some interesting canonical 2-cubes in 3.2.1, 3.2.2 and 3.2.3. In the end of this section 3.2.3 we draw the cocubical object in some category of monads that we hope to describe with more precision in the future and which contains all cubical analogue of globular algebras of [19]. This cocubical object[4] should be an important step to obtain the expected cubical weak ∞-category of cubical weak ∞-categories, and this lead to accurate formulation of cubical Grothendieck ∞-topos, cubical ∞-stacks, etc. More details on such a cubical higher structure can be found in [25].

3.1 The category of cubical weak ∞-functors In Theoretical Physics basic data of an *Extended Topological Quantum Field Theory* (*ETQFT*) [16, 28] is given by a weak monoidal ∞-functor between a higher monoidal category of cobordisms and some kind of higher categorification of the monoidal category of Hilbert spaces. For example in [28] the *simplicial geometry* is used to define ETQFT, while in [16] they use instead the *multiple geometry* of Charles Ehresmann [15]. In this section we are going to define algebraic model of cubical weak ∞-functors with the hope that in the future it can be used for accurate algebraic models of cubical ETQFT.

In [32] Jacques Penon proposed algebraic models of globular weak ∞-functors which were extended to all kind of globular weak higher transformations in [19]. The methods used in [19, 32] consists in the use of different kinds of *stretchings* in order to weaken different kinds of strict higher structures. For example in [32] Penon has built a category of stretchings [5] (called the category of $(0,\infty)$-*categorical stretchings* in [19]) which were adapted to weakened strict ∞-functors. And in [19] the author used the category of (n,∞)-*categorical stretchings* to weaken all kinds of globular strict n-transformations for all $n \geq 2$ (strict natural ∞-transformations correspond to $n = 2$ and strict ∞-modifications correspond to $n = 3$, etc.). As we are going to see, our models of cubical weak ∞-functors are built with a similar technology: we are going to define *cubical functorial stretchings* which con-

[4]This cocubical object should be a W-coalgebra, where W means the cubical operad which algebras are cubical weak ∞-categories with connections. See [25].

[5]$(0,\infty)$-categorical stretchings must not be confused with $(\infty,0)$-categorical stretchings used in 2.3 to weaken cubical strict $(\infty,0)$-categories and which were used in [22] to weaken globular strict $(\infty,0)$-categories.

tain all the "information"[6] on the structure behind cubical weak ∞-functors. This structure produces a monad on the category $\mathbb{C}\mathbb{S}\mathrm{ets}\times\mathbb{C}\mathbb{S}\mathrm{ets}$ whose algebras are our models of cubical weak ∞-functors. In 3.2 we shall investigate similar constructions for models of cubical weak ∞-natural transformations.

Cubical strict ∞-functors have been defined in [21]. A morphism between two cubical strict ∞-functors $C \xrightarrow{F} D$ and $C' \xrightarrow{F'} D'$ is given by a commutative 2-cube in $\mathbb{C}\mathbb{C}\mathrm{AT}$:

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "F"'] & C' \arrow[d, "F'"] \\ D \arrow[r, "d"'] & D' \end{tikzcd}$$

The category of cubical strict ∞-functors is denoted ∞-$\mathbb{C}\mathbb{F}\mathrm{unct}$.

3.1.1 The category of cubical $(0,\infty)$-magmas

A cubical $(0,\infty)$-magma is given by a morphism $M_0 \xrightarrow{F_M} M_1$ of $\mathbb{C}\mathbb{S}\mathrm{ets}$ such that M_0 and M_1 are objects of ∞-$\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}}$ (defined in [21]). This object is denoted (M_0, F_M, M_1).

A morphism between $(0,\infty)$-magmas:

$$(M_0, F_M, M_1) \xrightarrow{\quad m \quad} (M'_0, F'_M, M'_1)$$

is given by two morphisms of ∞-$\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}}$:

$$M_0 \xrightarrow{m_0} M'_0 \ , \ M_1 \xrightarrow{m_1} M'_1$$

such that the following diagram commutes in $\mathbb{C}\mathbb{S}\mathrm{ets}$:

[6] here we put these brackets, because these structural informations must be thought of as up to some weak equivalences, following the idea of *models* developed in any higher category theory

$$\begin{tikzcd} M_0 \arrow[r, "F_M"] \arrow[d, "m_0"'] & M_1 \arrow[d, "m_1"] \\ M'_0 \arrow[r, "F'_M"'] & M'_1 \end{tikzcd}$$

The category of cubical $(0,\infty)$-magmas is denoted by $(0,\infty)$-$\mathbb{CM}\mathrm{ag_r}$

3.1.2 The category of cubical $(0,\infty)$-categorical stretchings

A $(0,\infty)$-stretching is given by a quadruple:

$$\mathbb{E} = (\mathbb{E}_0, \mathbb{E}_1, F_M, F_C)$$

such that $\mathbb{E}_0$, $\mathbb{E}_1$ are cubical categorical stretchings (defined in [21]) given by:

$$\mathbb{E}_0 = (M_0, C_0, \pi_0, ({}^0[-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n\}}, ({}^0[-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n\};\gamma\in\{-,+\}})$$

and

$$\mathbb{E}_1 = (M_1, C_1, \pi_1, ({}^1[-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n\}}, ({}^1[-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n\};\gamma\in\{-,+\}})$$

where (M_0, F_M, M_1) is an object of $(0,\infty)$-$\mathbb{CM}\mathrm{ag_r}$ and $C_0 \xrightarrow{F_C} C_1$ is a strict cubical ∞-functor, such that the following square is commutative in $\mathbb{CS}\mathrm{ets}$:

$$\begin{tikzcd} M_0 \arrow[r, "F_M"] \arrow[d, "\pi_0"'] & M_1 \arrow[d, "\pi_1"] \\ C_0 \arrow[r, "F_C"'] & C_1 \end{tikzcd}$$

A morphism of $(0,\infty)$-stretchings:

$$\mathbb{E} = (\mathbb{E}_0, \mathbb{E}_1, F_M, F_C) \longrightarrow \mathbb{E}' = (\mathbb{E}'_0, \mathbb{E}'_1, F'_M, F'_C)$$

is given by the following commutative diagram in $\mathbb{CS}$ets:

$$\begin{tikzcd}
& M'_0 \arrow[rr, "F'_M"] \arrow[dd, "\pi'_0"] & & M'_1 \arrow[dd, "\pi'_1"] \\
M_0 \arrow[ur, "m_0"] \arrow[rr, "F_M"] \arrow[dd, "\pi_0"] & & M_1 \arrow[ur, "m_1"] \arrow[dd, "\pi_1"] & \\
& C'_0 \arrow[rr, "F'_C"] & & C'_1 \\
C_0 \arrow[ur, "c_0"] \arrow[rr, "F_C"] & & C_1 \arrow[ur, "c_1"] &
\end{tikzcd}$$

such that (m_0, m_1) is a morphism of $(0,\infty)$-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$, (m_0, c_0) and (m_1, c_1) are morphisms of ∞-$\mathbb{CE}$t$\mathbb{C}$. The category of $(0,\infty)$-stretchings is denoted $(0,\infty)$-$\mathbb{CE}$t$\mathbb{C}$.

Consider the forgetful functor:

$$(0,\infty)\text{-}\mathbb{CE}\mathrm{t}\mathbb{C} \xrightarrow{\ {}^0U\ } \mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets}$$

defined on objects by:

$$\mathbb{E} = (\mathbb{E}_0, \mathbb{E}_1, F_M, F_C) \longmapsto (M_0, M_1)$$

It is not difficult to show that the category $(0,\infty)$-$\mathbb{CE}$t$\mathbb{C}$ is projectively sketchable and that its sketch contains the projective sketch of $\mathbb{CS}$ets $\times$ $\mathbb{CS}$ets. Thus this functor has a left adjoint ${}^0F$ which produces a monad $\mathbb{T}^0 = (T^0, \lambda^0, \mu^0)$ on the category $\mathbb{CS}$ets $\times$ $\mathbb{CS}$ets.

Definition 3.1. Cubical weak ∞-functors are algebras for the monad $\mathbb{T}^0$ above.

Thus a cubical weak ∞-functor is given by a quadruple (C_0, C_1, v_0, v_1) such that if we note $T^0(C_0, C_1) = (T^0_0(C_0, C_1), T^0_1(C_0, C_1)$ then we get its underling morphisms of $\mathbb{CS}$ets:

$$T_0^0(C_0,C_1) \xrightarrow{\quad v_0 \quad} C_0 \qquad T_1^0(C_0,C_1) \xrightarrow{\quad v_1 \quad} C_1$$

and these morphisms of $\mathbb{CS}\mathrm{ets}$ put on (C_0, C_1) a structure of cubical weak ∞-functor $C_0 \xrightarrow{\quad F \quad} C_1$, defined by:

$$F = v_1 \circ F_M \circ \lambda_0^0(C_0,C_1)$$

with:

$$\begin{tikzcd}
C_0 \arrow[d, "\lambda_0^0(C_0,C_1)"'] & \\
T_0^0(C_0,C_1) \arrow[r, dotted, "F"] \arrow[d, "v_0"'] & T_1^0(C_0,C_1) \arrow[d, "v_1"] \\
C_0 \arrow[r, dotted, "v_1 \circ F_M \circ \lambda_0^0"'] & C_1
\end{tikzcd}$$

3.2 The category of cubical weak ∞-natural transformations

Now we describe a monad on the category

$$(\mathbb{CS}\mathrm{ets})^4 = \mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets} \times \mathbb{CS}\mathrm{ets}$$

whose algebras are our models of cubical weak ∞-natural transformations. In [19] we defined globular ∞-natural transformations by using the structure given by an adapted category of globular stretchings, namely the category of $(1,\infty)$-stretchings. Here we use similar technology by defining first the category of *cubical* $(1,\infty)$-*stretchings* which contains the underlying structure needed to weaken cubical strict ∞-natural transformations. This last monad $\mathbb{T}^1$ (see below) gives some 2-cubes in 3.2.3 and a first flavor of an expected cocubical object of operads described in the end of this article.

3.2.1 The category of cubical strict ∞-natural transformations

Cubical strict natural transformations were introduced in [18]. Here we give the evident strict and higher version of it. A cubical strict ∞-natural transformation is given by a 2-cube in ∞-$\mathbb{CCAT}$:

$$\begin{tikzcd} C_{0,0} \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\tau \Downarrow"] & C_{1,0} \arrow[d, "G"] \\ C_{0,1} \arrow[r, "K"'] & C_{1,1} \end{tikzcd}$$

whose 0-cells correspond to four cubical strict ∞-categories $C_{0,0}$, $C_{0,1}$, $C_{1,0}$, $C_{1,1}$, whose 1-cells correspond to four cubical strict ∞-functors F, G, H, K, and whose only 2-cell τ corresponds, for all 0-cells a in $C_{0,0}$, to a 1-cell:

$$G(F(a)) \xrightarrow{\tau(a)} K(H(a))$$

such that for all 1-cells $a \xrightarrow{f} b$ of $C_{0,0}$ we have the following commutative diagram:

$$\begin{tikzcd} G(F(a)) \arrow[r, "\tau(a)"] \arrow[d, "G(F(f))"'] & K(H(a)) \arrow[d, "K(H(f))"] \\ G(F(b)) \arrow[r, "\tau(b)"'] & K(H(b)) \end{tikzcd}$$

A morphism between two cubical strict ∞-natural transformations τ and τ' is given by a 3-cube in ∞-$\mathbb{CCAT}$:

$$
\begin{tikzcd}
& C'_{0,0} \arrow[rr, "F'"] \arrow[dd, "H'"] \arrow[dd, phantom, "\tau \Downarrow", shift right=4ex] & & C'_{1,0} \arrow[dd, "G'"] \\
C_{0,0} \arrow[ur, "c_{0,0}"] \arrow[rr, "F"] \arrow[dd, "H"'] & & C_{1,0} \arrow[ur, "c_{1,0}"] \arrow[dd, "G"] \arrow[r, phantom, "\tau' \Downarrow"] & \\
& C'_{0,1} \arrow[rr, "K'"] & & C'_{1,1} \\
C_{0,1} \arrow[ur, "c_{0,1}"] \arrow[rr, "K"'] & & C_{1,1} \arrow[ur, "c_{1,1}"'] &
\end{tikzcd}
$$

such that $c_{1,0}F = F'c_{0,0}$, $c_{1,1}G = G'c_{1,0}$, $c_{0,1}H = H'c_{0,0}$ and $c_{1,1}K = K'c_{0,1}$. The category of cubical strict ∞-natural transformations is denoted $(1,\infty)$-$\mathbb{CT}$rans, and we obtain an internal 2-cube in $\mathbb{C}$AT:

$$
\begin{tikzcd}
(1,\infty)\text{-}\mathbb{CT}\mathrm{rans} \arrow[r, shift left=4.5, "\sigma^2_{1,1}"] \arrow[r, shift left=1.5, "\sigma^2_{1,2}"] \arrow[r, shift right=1.5, "\tau^2_{1,1}"'] \arrow[r, shift right=4.5, "\tau^2_{1,2}"'] & \infty\text{-}\mathbb{CF}\mathrm{unct} \arrow[r, shift left=1.5, "\sigma^1_0"] \arrow[r, shift right=1.5, "\tau^1_0"'] & \mathbb{CC}\mathrm{AT}
\end{tikzcd}
$$

Proposition 3.2. The internal 2-cube of $\mathbb{C}$AT just above can be structured in a strict cubical 2-category

Proof. Consider the following object $\tau \in (1,\infty)$-$\mathbb{CT}$rans:

$$
\begin{tikzcd}
C_{0,0} \arrow[r, "F"] \arrow[d, "H"'] & C_{1,0} \arrow[d, "G"] \\
C_{0,1} \arrow[r, "K"'] \arrow[ur, phantom, "\tau \Downarrow"] & C_{1,1}
\end{tikzcd}
$$

such that $\sigma^2_{1,1}(\tau) = F$, $\sigma^2_{1,2}(\tau) = H$, $\tau^2_{1,1}(\tau) = K$ and $\tau^2_{1,2}(\tau) = G$, and such that σ^1_0 and τ^1_0 are clearly defined.

Definition of the classical reflexivity:

$$\begin{tikzcd}
(1,\infty)\text{-}\mathbb{C}\mathbb{T}\text{rans} & \infty\text{-}\mathbb{C}\mathbb{F}\text{unct} \arrow[l, shift right, "1^1_{2,1}"'] \arrow[l, shift left, "1^1_{2,2}"] & \mathbb{C}\mathbb{C}\text{AT} \arrow[l, "1^0_1"']
\end{tikzcd}$$

$1^1_{2,1}(F)$ is given by:

$$\begin{tikzcd}
C_{0,0} \arrow[r, "F"] \arrow[d, "1_{C_{0,0}}"'] & C_{1,0} \arrow[d, "1_{C_{1,0}}"] \\
C_{0,0} \arrow[r, "F"'] \arrow[ur, phantom, "1^1_{2,1}(F)\Downarrow"] & C_{1,0}
\end{tikzcd}$$

and is such that $1^1_{2,1}(F)(a) = 1^0_1(F(a))$ for all 0-cells $a \in C_{0,0}(0)$, and also $1^1_{2,2}(F)$ is given by:

$$\begin{tikzcd}
C_{0,0} \arrow[r, "1_{C_{0,0}}"] \arrow[d, "F"'] & C_{0,0} \arrow[d, "F"] \\
C_{1,0} \arrow[r, "1_{C_{1,0}}"'] \arrow[ur, phantom, "1^1_{2,2}(F)\Downarrow"] & C_{1,0}
\end{tikzcd}$$

and is such that $1^1_{2,2}(F)(a) = 1^0_1(F(a))$ for all 0-cells $a \in C_{0,0}(0)$.

Definition of the connections:

$$\begin{tikzcd}
(1,\infty)\text{-}\mathbb{C}\mathbb{T}\text{rans} & \infty\text{-}\mathbb{C}\mathbb{F}\text{unct} \arrow[l, shift right, "1^{1,+}_{2,1}"'] \arrow[l, shift left, "1^{1,-}_{2,2}"]
\end{tikzcd}$$

$1^{1,-}_{2,1}(F)$ is given by:

$$\begin{tikzcd}
C_{0,0} \arrow[r, "F"] \arrow[d, "F"'] \arrow[dr, phantom, "1^{1,-}_{2,1}(F) \Downarrow"] & C_{1,0} \arrow[d, "1_{C_{1,0}}"] \\
C_{1,0} \arrow[r, "1_{C_{1,0}}"'] & C_{1,0}
\end{tikzcd}$$

and is such that $1^{1,-}_{2,1}(F)(a) = 1^0_1(F(a))$ for all 0-cells $a \in C_{0,0}(0)$,

and $1^{1,+}_{2,1}(F)$ is given by:

$$\begin{tikzcd}
C_{0,0} \arrow[r, "1_{C_{0,0}}"] \arrow[d, "1_{C_{0,0}}"'] \arrow[dr, phantom, "1^{1,+}_{2,1}(F) \Downarrow"] & C_{0,0} \arrow[d, "F"] \\
C_{0,0} \arrow[r, "F"'] & C_{1,0}
\end{tikzcd}$$

and is such that $1^{1,+}_{2,1}(F)(a) = 1^0_1(F(a))$ for all 0-cells $a \in C_{0,0}(0)$.

The following shape of 2-cells:

$$\begin{tikzcd}
C_{0,0} \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\tau \Downarrow"] & C_{1,0} \arrow[d, "G"] \\
C_{0,1} \arrow[r, "K"'] \arrow[d, "H'"'] \arrow[dr, phantom, "\rho \Downarrow"] & C_{1,1} \arrow[d, "G'"] \\
C_{0,2} \arrow[r, "K'"'] & C_{1,2}
\end{tikzcd}$$

allows to define the composition $\rho \circ^2_{1,1} \tau$:

$$\begin{tikzcd}
C_{0,0} \arrow[r, "F"] \arrow[d, "H' \circ H"'] \arrow[dr, phantom, "\rho \circ^2_{1,1} \tau \Downarrow"] & C_{1,0} \arrow[d, "G' \circ G"] \\
C_{0,2} \arrow[r, "K"'] & C_{1,2}
\end{tikzcd}$$

by the formula:

$$(\rho \circ^2_{1,1} \tau)(a) = \rho(H(a)) \circ G'(\tau(a))$$

and the following shape of 2-cells:

$$\begin{tikzcd} C_{0,0} \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\tau \Downarrow"] & C_{1,0} \arrow[r, "F'"] \arrow[d, "G"] \arrow[dr, phantom, "\rho \Downarrow"] & C_{2,0} \arrow[d, "G'"] \\ C_{0,1} \arrow[r, "K"'] & C_{1,1} \arrow[r, "K'"'] & C_{2,1} \end{tikzcd}$$

allows to define the composition $\rho \circ^2_{1,2} \tau$:

$$\begin{tikzcd} C_{0,0} \arrow[r, "F' \circ F"] \arrow[d, "H"'] \arrow[dr, phantom, "\rho \circ^2_{1,2} \tau \Downarrow"] & C_{2,0} \arrow[d, "G"] \\ C_{0,1} \arrow[r, "K' \circ K"'] & C_{2,1} \end{tikzcd}$$

by the formula:

$$(\rho \circ^2_{1,2} \tau)(a) = K'(\tau(a)) \circ \rho(F(a)).$$

The proof that these datas put a structure of cubical strict 2-categories on the internal 2-cube of the proposition is left to the reader.

□

3.2.2 The category of cubical $(1, \infty)$-magmas

A cubical $(1, \infty)$-magma is an object with shape:

$$\begin{tikzcd} M_{0,0} \arrow[rr, "F_M"] \arrow[d, "H_M"'] & \arrow[d, phantom, "\tau_M \Downarrow"] & M_{1,0} \arrow[d, "G_M"] \\ M_{0,1} \arrow[rr, "K_M"'] & & M_{1,1} \end{tikzcd}$$

such that:

$$(M_{0,0}, F_M, M_{1,0}), (M_{1,0}, G_M, M_{1,1}), (M_{0,0}, H_M, M_{0,1}) \text{ and } (M_{0,1}, K_M, M_{1,1})$$

are objects of $(0, \infty)$-$\mathbb{CM}\mathrm{ag}_{\mathrm{r}}$, and such that τ_M is a map:

$$M_{0,0}(0) \xrightarrow{\tau_M} M_{1,1}(1)$$

which sends each 0-cells a of $M_{0,0}$ to an 1-cell $\tau_M(a) \in M_{1,1}(1)$ such that:

$$s_0^1(\tau_M(a)) = G_M(F_M(a)) \text{ and } t_0^1(\tau_M(a)) = K_M(H_M(a))$$

We prefer to avoid heavy notations and shall denote usually just by τ_M such object of a category $(1, \infty)$-$\mathbb{CM}\mathrm{ag}_{\mathrm{r}}$, where we have to think this greek letter τ as the variable usually used for natural transformations and the subscript M in it just means "Magmatic".

Given τ_M and τ'_M two objects of $(1, \infty)$-$\mathbb{CM}\mathrm{ag}_{\mathrm{r}}$, a morphism between them is given by a commutative diagram in ∞-$\mathbb{CS}\mathrm{ets}$:

$$\begin{tikzcd}
& M'_{0,0} \arrow[rr, "F'_M"] \arrow[dd, "H'_M" near end] & & M'_{1,0} \arrow[dd, "G'_M"] \\
M_{0,0} \arrow[ur, "m_{0,0}"] \arrow[rr, "F_M" near start] \arrow[dd, "H_M"'] & & M_{1,0} \arrow[ur, "m_{1,0}"] \arrow[dd, "G_M" near start] \arrow[r, phantom, "\tau'_M \Downarrow"] & \\
& M'_{0,1} \arrow[rr, "K'_M" near start] \arrow[l, phantom, "\tau_M \Downarrow"] & & M'_{1,1} \\
M_{0,1} \arrow[ur, "m_{0,1}"] \arrow[rr, "K_M"'] & & M_{1,1} \arrow[ur, "m_{1,1}"] &
\end{tikzcd}$$

such that $(m_{0,0}, m_{1,0})$, $(m_{1,0}, m_{1,1})$, $(m_{0,0}, m_{0,1})$, $(m_{0,1}, m_{1,1})$ are morphisms of $(0,\infty)$-$\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}}$. It is important to note that commutativity of this diagram means also the equality $m_{1,1} \circ \tau_M = \tau'_M \circ m_{0,0}$.

We obtain an internal 2-cube in $\mathbb{C}\mathbb{A}\mathbb{T}$:

$$\begin{tikzcd}[column sep=large]
(1,\infty)\text{-}\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}} \arrow[r, shift left=4.5, "\sigma^2_{1,1}"] \arrow[r, shift left=1.5, "\sigma^2_{1,2}"] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5, "\tau^2_{1,1}", "\tau^2_{1,2}"'] & (0,\infty)\text{-}\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}} \arrow[r, shift left=1.5, "\sigma^1_0"] \arrow[r, shift right=1.5, "\tau^1_0"'] & \infty\text{-}\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}}
\end{tikzcd}$$

Proposition 3.3. The internal 2-cube of $\mathbb{C}\mathbb{A}\mathbb{T}$ just above can be structured in a cubical reflexive 2-magma

Proof. The proof is easy and basic datas have been already defined in 3.2. $\square$

3.2.3 The category of cubical $(1,\infty)$-categorical stretchings

A cubical $(1,\infty)$-categorical stretching is given by a commutative diagram in ∞-$\mathbb{C}\mathbb{S}$ets:

$$\begin{tikzcd}
& M_{0,1} \arrow[rr, "K_M"] \arrow[dd, "\pi_{0,1}"] \arrow[r, phantom, "\tau_M \Rightarrow"] & & M_{1,1} \arrow[dd, "\pi_{1,1}"] \\
M_{0,0} \arrow[ur, "H_M"] \arrow[rr, "F_M"] \arrow[dd, "\pi_{0,0}"'] & & M_{1,0} \arrow[ur, "G_M"] \arrow[dd, "\pi_{1,0}"] & \\
& C_{0,1} \arrow[rr, "K_C"] \arrow[r, phantom, "\tau_C \Rightarrow"] & & C_{1,1} \\
C_{0,0} \arrow[ur, "H_C"] \arrow[rr, "F_C"'] & & C_{1,0} \arrow[ur, "G_C"'] &
\end{tikzcd}$$

such that $(\pi_{0,0}, F_M, F_C, \pi_{1,0}), (\pi_{1,0}, G_M, G_C, \pi_{1,1}), (\pi_{0,0}, H_M, H_C, \pi_{0,1})$ and $(\pi_{0,1}, K_M, K_C, \pi_{1,1})$ are objects of $(0,\infty)$-$\mathbb{CE}$tC, and also τ_M is an object of $(1,\infty)$-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$ and τ_C is an object of $(1,\infty)$-$\mathbb{CT}$rans. It is important to note that commutativity of this diagram means also that the equality $\pi_{1,1} \circ \tau_M = \tau_C \circ \pi_{0,0}$ holds. Such cubical $(1,\infty)$-categorical stretching can be denoted (τ_M, τ_C). Given an other cubical $(1,\infty)$-categorical stretching (τ'_M, τ'_C):

$$\begin{tikzcd}
& M'_{0,1} \arrow[rr, "K'_M"] \arrow[dd, "\pi'_{0,1}"] \arrow[r, phantom, "\tau'_M \Rightarrow"] & & M'_{1,1} \arrow[dd, "\pi'_{1,1}"] \\
M'_{0,0} \arrow[ur, "H'_M"] \arrow[rr, "F'_M"] \arrow[dd, "\pi'_{0,0}"'] & & M'_{1,0} \arrow[ur, "G'_M"] \arrow[dd, "\pi'_{1,0}"] & \\
& C'_{0,1} \arrow[rr, "K'_C"] \arrow[r, phantom, "\tau'_C \Rightarrow"] & & C'_{1,1} \\
C'_{0,0} \arrow[ur, "H'_C"] \arrow[rr, "F'_C"'] & & C'_{1,0} \arrow[ur, "G'_C"'] &
\end{tikzcd}$$

a morphism $(\tau_M, \tau_C) \longrightarrow (\tau'_M, \tau'_C)$ of such cubical $(1,\infty)$-categorical stretchings is given by:

- a morphism of $(1,\infty)$-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$ underlied by $(m_{0,0}, m_{1,0}, m_{0,1}, m_{1,1})$, a morphism of $(1,\infty)$-$\mathbb{C}\mathbb{T}$rans underlied by $(c_{0,0}, c_{1,0}, c_{0,1}, c_{1,1})$:

$$\begin{tikzcd}
 & M'_{0,0} \arrow[rr, "F'_M"] \arrow[dd, "H'_M"] \arrow[ddrr, phantom, "\tau'_M \Downarrow"] & & M'_{1,0} \arrow[dd, "G'_M"] \\
M_{0,0} \arrow[ur, "m_{0,0}"] \arrow[rr, "F_M"] \arrow[dd, "H_M"'] \arrow[ddrr, phantom, "\tau_M \Downarrow"] & & M_{1,0} \arrow[ur, "m_{1,0}"] \arrow[dd, "G_M"] & \\
 & M'_{0,1} \arrow[rr, "K'_M"] & & M'_{1,1} \\
M_{0,1} \arrow[ur, "m_{0,1}"] \arrow[rr, "K_M"'] & & M_{1,1} \arrow[ur, "m_{1,1}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & C'_{0,0} \arrow[rr, "F'_C"] \arrow[dd, "H'_C"] \arrow[ddrr, phantom, "\tau'_C \Downarrow"] & & C'_{1,0} \arrow[dd, "G'_C"] \\
C_{0,0} \arrow[ur, "c_{0,0}"] \arrow[rr, "F_C"] \arrow[dd, "H_C"'] \arrow[ddrr, phantom, "\tau_C \Downarrow"] & & C_{1,0} \arrow[ur, "c_{1,0}"] \arrow[dd, "G_C"] & \\
 & C'_{0,1} \arrow[rr, "K'_C"] & & C'_{1,1} \\
C_{0,1} \arrow[ur, "c_{0,1}"] \arrow[rr, "K_C"'] & & C_{1,1} \arrow[ur, "c_{1,1}"] &
\end{tikzcd}$$

- the following morphisms:

$((m_{0,0}, c_{0,0}), (m_{1,0}, c_{1,0})), ((m_{1,0}, c_{1,0}), (m_{1,1}, c_{1,1})), ((m_{0,0}, c_{0,0}), (m_{0,1}, c_{0,1}))$

and $((m_{0,1}, c_{0,1}), (m_{1,1}, c_{1,1}))$, of $(0,\infty)$-$\mathbb{C}\mathbb{E}$tC:

$$\begin{tikzcd}
& M'_{0,0} \arrow[rr, "F'_M"] \arrow[dd, "\pi'_{0,0}"] & & M'_{1,0} \arrow[dd, "\pi'_{1,0}"] \\
M_{0,0} \arrow[ur, "m_{0,0}"] \arrow[rr, "F_M"] \arrow[dd, "\pi_{0,0}"'] & & M_{1,0} \arrow[ur, "m_{1,0}"] \arrow[dd, "\pi_{1,0}"] & \\
& C'_{0,0} \arrow[rr, "F'_C"] & & C'_{1,0} \\
C_{0,0} \arrow[ur, "c_{0,0}"] \arrow[rr, "F_C"'] & & C_{1,0} \arrow[ur, "c_{1,0}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& M'_{0,0} \arrow[rr, "H'_M"] \arrow[dd, "\pi'_{0,0}"] & & M'_{0,1} \arrow[dd, "\pi'_{0,1}"] \\
M_{0,0} \arrow[ur, "m_{0,0}"] \arrow[rr, "H_M"] \arrow[dd, "\pi_{0,0}"'] & & M_{0,1} \arrow[ur, "m_{0,1}"] \arrow[dd, "\pi_{0,1}"] & \\
& C'_{0,0} \arrow[rr, "H'_C"] & & C'_{0,1} \\
C_{0,0} \arrow[ur, "c_{0,0}"] \arrow[rr, "H_C"'] & & C_{0,1} \arrow[ur, "c_{0,1}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& M'_{1,0} \arrow[rr, "G'_M"] \arrow[dd, "\pi'_{1,0}"] & & M'_{1,1} \arrow[dd, "\pi'_{1,1}"] \\
M_{1,0} \arrow[ur, "m_{1,0}"] \arrow[rr, "G_M"] \arrow[dd, "\pi_{1,0}"'] & & M_{1,1} \arrow[ur, "m_{1,1}"] \arrow[dd, "\pi_{1,1}"] & \\
& C'_{1,0} \arrow[rr, "G'_C"] & & C'_{1,1} \\
C_{1,0} \arrow[ur, "c_{1,0}"] \arrow[rr, "G_C"'] & & C_{1,1} \arrow[ur, "c_{1,1}"] &
\end{tikzcd}$$

$$
\begin{tikzcd}
& M'_{0,1} \arrow[rr, "K'_M"] \arrow[dd, "\pi'_{0,1}"] & & M'_{1,1} \arrow[dd, "\pi'_{1,1}"] \\
M_{0,1} \arrow[ur, "m_{0,1}"] \arrow[rr, "K_M"] \arrow[dd, "\pi_{0,1}"'] & & M_{1,1} \arrow[ur, "m_{1,1}"] \arrow[dd, "\pi_{1,1}"] & \\
& C'_{0,1} \arrow[rr, "K'_C"] & & C'_{1,1} \\
C_{0,1} \arrow[ur, "c_{0,1}"] \arrow[rr, "K_C"'] & & C_{1,1} \arrow[ur, "c_{1,1}"] &
\end{tikzcd}
$$

We denote $(1, \infty)$-$\mathbb{C}\mathbb{E}$tC the category of cubical $(1, \infty)$-categorical stretchings.

Consider the forgetful functor:

$$
(1, \infty)\text{-}\mathbb{C}\mathbb{E}\mathrm{tC} \xrightarrow{\ {}^1U\ } (\mathbb{C}\mathbb{S}\mathrm{ets})^4
$$

defined on objects by:

$$
(\tau_M, \tau_C) \longmapsto (M_{0,0}, M_{1,0}, M_{0,1}, M_{1,1})
$$

This functor has a left adjoint[7]: ${}^1F$, which produces a monad $\mathbb{T}^1 = (T^1, \lambda^1, \mu^1)$ on the category $(\mathbb{C}\mathbb{S}\mathrm{ets})^4$.

Definition 3.4. Cubical weak natural ∞-transformations are algebras for the monad $\mathbb{T}^1$ above.

Thus we obtain a 2-cube in the category $\mathbb{A}$dj of pairs of adjunctions defined in [19]:

[7] We carry on the use of Foltz theorem: indeed we are in the situation where the category $(1, \infty)$-$\mathbb{C}\mathbb{E}$tC is projectively sketchable and contains the projective sketch of the category $(\mathbb{C}\mathbb{S}\mathrm{ets})^4$.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
(1,\infty)\text{-}\mathbb{C}\mathbb{E}t\mathbb{C} \arrow[r, shift left=4.5, "\sigma^2_{1,1}"] \arrow[r, shift left=1.5, "\sigma^2_{1,2}"] \arrow[r, shift right=1.5, "\tau^2_{1,1}"'] \arrow[r, shift right=4.5, "\tau^2_{1,2}"'] \arrow[d, shift left, "{}^1U"] & (0,\infty)\text{-}\mathbb{C}\mathbb{E}t\mathbb{C} \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\tau^1_0"'] \arrow[d, shift left, "{}^0U"] & \infty\text{-}\mathbb{C}\mathbb{E}t\mathbb{C} \arrow[d, shift left, "U"] \\
(\mathbb{C}\mathbb{S}ets)^4 \arrow[r, shift left=4.5, "\sigma^2_{1,1}"] \arrow[r, shift left=1.5, "\sigma^2_{1,2}"] \arrow[r, shift right=1.5, "\tau^2_{1,1}"'] \arrow[r, shift right=4.5, "\tau^2_{1,2}"'] \arrow[u, shift left, "{}^1F"] \arrow[u, phantom, "\dashv"] & (\mathbb{C}\mathbb{S}ets)^2 \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\tau^1_0"'] \arrow[u, shift left, "{}^0F"] \arrow[u, phantom, "\dashv"] & \mathbb{C}\mathbb{S}ets \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

which allow to obtain a 2-cocube in the category $\mathbb{M}$nd of categories equipped with monads defined in [19]:

$$
\begin{tikzcd}[column sep=huge]
((\mathbb{C}\mathbb{S}ets)^4, \mathbb{T}^1) & ((\mathbb{C}\mathbb{S}ets)^2, \mathbb{T}^0) \arrow[l, shift right=4.5, "\sigma^2_{1,1}"'] \arrow[l, shift right=1.5, "\sigma^2_{1,2}"'] \arrow[l, shift left=1.5, "\tau^2_{1,1}"] \arrow[l, shift left=4.5, "\tau^2_{1,2}"] & (\mathbb{C}\mathbb{S}ets, \mathbb{W}) \arrow[l, shift right, "\sigma^1_0"'] \arrow[l, shift left, "\tau^1_0"]
\end{tikzcd}
$$

And finally it gives the following 2-cube in $\mathbb{C}$AT:

$$
\begin{tikzcd}[column sep=huge]
\mathbb{T}^1\text{-}\mathbb{A}lg \arrow[r, shift left=4.5, "\sigma^2_{1,1}"] \arrow[r, shift left=1.5, "\sigma^2_{1,2}"] \arrow[r, shift right=1.5, "\tau^2_{1,1}"'] \arrow[r, shift right=4.5, "\tau^2_{1,2}"'] & \mathbb{T}^0\text{-}\mathbb{A}lg \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\tau^1_0"'] & \mathbb{W}\text{-}\mathbb{A}lg
\end{tikzcd}
$$

Proposition 3.5. The internal 2-cube of $\mathbb{C}$AT just above can be structured in a cubical weak 2-category with connections.

Proof. The details of the proof are quite long but not difficult. For example basic datas of such structure are similar to those built in 3.2. $\square$

We finish this article by drawing the cocubical shape of monads for all cubical weak higher transformations that we hope to describe in a future work. For clarity we change the denotation of the monads $\mathbb{W}$, $\mathbb{T}^0$ and $\mathbb{T}^1$ described in this article with: $\mathbb{W}^0 := \mathbb{W}$, $\mathbb{W}^1 := \mathbb{T}^0$ and $\mathbb{W}^2 := \mathbb{T}^1$:

$$\mathbb{W}^0 \begin{array}{c} \xrightarrow{s_1^0} \\ \xrightarrow[t_1^0]{} \end{array} \mathbb{W}^1 \begin{array}{c} \xrightarrow{s_{1,2}^2} \\ \xrightarrow{s_{1,1}^2} \\ \xrightarrow[t_{1,1}^2]{} \\ \xrightarrow[t_{1,2}^2]{} \end{array} \mathbb{W}^2 \begin{array}{c} \xrightarrow{s_{2,3}^3} \\ \xrightarrow{s_{2,2}^3} \\ \xrightarrow{s_{2,1}^3} \\ \xrightarrow[t_{2,1}^3]{} \\ \xrightarrow[t_{2,2}^3]{} \\ \xrightarrow[t_{2,3}^3]{} \end{array} \mathbb{W}^3 \ldots \mathbb{W}^{n-1} \begin{array}{c} \xrightarrow{s_{n-1,1}^n} \\ \dashrightarrow \\ \xrightarrow{s_{n-1,i}^n} \\ \dashrightarrow \\ \xrightarrow{s_{n-1,n-1}^n} \\ \xrightarrow[t_{n-1,n-1}^n]{} \\ \dashrightarrow \\ \xrightarrow[t_{n-1,i}^n]{} \\ \dashrightarrow \\ \xrightarrow[t_{n-1,1}^n]{} \end{array} \mathbb{W}^n \ldots$$

For example $\mathbb{W}^3$-algebras are cubical weak ∞-modifications[8]. This cocubical object of monads should be a cocubical object of operads [25] and we believe that it is in fact a $\mathbb{W}^0$-coalgebra or at least a $\mathbb{B}_C^0$-coalgebra in the sense of [24] where $\mathbb{B}_C^0$-algebras are the operadic models of cubical weak ∞-categories described in [24]. If this cocubical object is a $\mathbb{W}^0$-coalgebra then it means that the cubical weak ∞-category with connections of cubical weak ∞-categories with connections exists, by using this technology of cubical stretchings. Thus this result opens the perspective of an accurate approach of the cubical weak ∞-topos of Grothendieck with connections and to the cubical weak ∞-stacks with connections.

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[8]Its globular analogue has been described in [19].

[9]Arithmétique et Géométrie Algébrique, LMO, Paris-Saclay.

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