Categories and
General Algebraic Structures
with Applications

Volume 16, Number 1, January 2022, 143-187.
https://doi.org/10.52547/cgasa.16.1.143

Algebraic models of cubical weak ∞-categories with connections

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Abstract. In this article we adapt some aspects of Penon's article [23] to cubical geometry. More precisely we define a monad on the category $\mathbb{C}\mathbb{S}$ets of cubical sets (without degeneracies) whose algebras are models of cubical weak ∞-categories with connections.

1 Introduction and preliminaries

In this article we explain how to construct algebraic models of *cubical weak ∞-categories with connections* (see Subsection 4.1).

A very important feature of cubical higher category theory is, for them, the flexible possibility of having models of higher structures built by mimicking simplicial methods for presheaves on the classical category Δ, to presheaves on the *reflexive cubical category* $\mathbb{C}_r$ of cubical sets with connections (see Subsection 2.2). On the other hand, as we shall see in this article,

Keywords: Cubical weak ∞-groupoids with connections, homology theory, homotopy theory, computer science.
Mathematics Subject Classification [2010]: 18B40,18C15, 18C20, 18G55, 20L99, 55U35, 55P15.
Received: 17 March 2021, Accepted: 11 June 2021.
ISSN: Print 2345-5853, Online 2345-5861.
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to also have models of higher structures built by mimicking the *stretching method* initiated for the globular setting in [23]) and carrying further in [12, 14].

For this last point it is important to notice that cubical strict ∞-categories (see Section 3) are very close in nature to their globular analogue: their first data are given by a countable family of sets $(C_n)_{n\in\mathbb{N}}$, equipped both with sources and targets of a kind, and partial operations, and two kinds of *reflexors* (that we call *the classical reflexions* and the *connections*) on each set C_n, subject to axioms. See also [5].

Cubical sets have richer structure than globular sets, as do simplicial sets, and this richness allows one to translate many definitions of simplicial higher category to cubical higher category (see [1]). Yet we shall see that cubical higher category theory has the algebraic flexibility of globular higher category theory, which is a feature difficult to find for simplicial higher category theory. These important aspects of cubical higher category theory allow one to see it as a bridge between simplicial higher category theory and globular higher category theory.

Finally it is important to notice that cubical strict higher structures already have applications and impacts in homology [2] and in algebraic topology [6]. The use of connections with the simplicial method can be found in [1].

Cubical Stretchings are the main tools of this article: the first *stretching* structure occurred in the work of Jacques Penon in 1999 [23] for globular shapes. *Globular stretchings* are the main tool to build algebraic models of weak ∞-categories in Penon's sense. Let us briefly recall how these structures weakened algebraic structures. Five ingredients build them:

- A category $\mathcal{G}$ of presheaves[1] which formalizes a given geometry[2], that is, where a notion of dimension is involved and where the objects of the category of elements of any of its presheaves are called cells.
- A magmatic higher structure based on $\mathcal{G}$ with chosen partial operations;
- A strict higher structure based on $\mathcal{G}$, which in fact is a magmatic higher structure with chosen equations build with its operations.

[1] Usually $\mathcal{G}$ is just equipped with the empty cone.
[2] Globular, cubical, multiple, etc.

- A morphism of magmas between a magmatic higher structure and a strict higher structure based on the geometry of $\mathcal{G}$[3]. A magma means a structure without equations.
- A family of partial operations, called *bracketings*, which builds coherences (or homotopy) between two cells of its underlying magma, for those couples of cells with the same dimension and with some other extra conditions.

With its bracketing operations, a stretching fills its underlying magma with coherence cells, and this filling is controlled[4] by its strict structure. Thus it equipped its underlying magma with a weakened version of its underlying strict higher structure. The category $\mathbb{S}\mathrm{tr}$ of such stretchings is all the time projectively sketchable and one of its sketch $\mathcal{E}$ contains the projective sketch $\mathcal{E}'$ of $\mathcal{G}$: $\mathcal{E}' \xhookrightarrow{i} \mathcal{E}$, thus the induced functor[5] $\mathbb{M}\mathrm{od}(\mathcal{E}) \xrightarrow{i^\star} \mathbb{M}\mathrm{od}(\mathcal{E}')$ is a right adjoint [9], and we get a monad $\mathbb{W}$ on $\mathcal{G} = \mathbb{M}\mathrm{od}(\mathcal{E}')$. $\mathbb{W}$-algebras are models of the weakened version of its corresponding strict higher structures. The free algebraic structures are the free stretchings on objects of $\mathcal{G}$, and they are our basic weak higher structure which weakened the strict higher structures involved. In [23] it is shown that free categorical stretchings on globular sets can be described by using basic method of logic, i.e constructing terms with an adapted language to the globular geometry which is defined with operations such as the $\circ^n_p$ and the bracketings $[-;-]$; it has no variables and has only constants built out with globular sets. Such a method can be apply to build thr free stretchings of this article, but it is quite technical and goes beyond our scope here. The category of strict higher structures of such stretchings is the category of algebras for a monad $\mathbb{S}$ on $\mathcal{G}$. The monad $\mathbb{W}$ is a kind of (or weakly equivalent to) cofibrant replacement of the monad $\mathbb{S}$. A true advantage of weakness with stretchings is it doesn't need the cartesianality of the monad $\mathbb{S}$ of its underlying strict higher structures. But it seems that the corresponding monad $\mathbb{W}$ could be an operad as is shown in [3]. Another good feature of the monad $\mathbb{W}$ is that it preserves α-filtered colimits where α is a cardinal bounding the size of

[3]Both in the same kind of higher structure as $\mathcal{G}$.

[4]It is similar to cofibrantly generated weak factorization system where the small object argument of Kan-Quillen shows that a given factorization is built by using a sequence controlled by a small set of maps.

[5]This induced functor $i^\star$ has no reason to be monadic.

all cones of both sketches $\mathcal{E}$ and $\mathcal{E}'$. This shows that the category $\mathbb{W}$-$\mathbb{A}$lg of $\mathbb{W}$-algebras is locally presentable.

This article is mainly concerned with *cubical stretchings* and the *cubical higher structures* they provide. It is devoted to several works for which the main steps are as follow:

- Our own terminology is chosen to be as close as possible to the notation of the globular environment, and in it we define the monad of cubical strict ∞-categories with connections on the category of cubical sets[6]
- We define the category of cubical categorical stretchings which is the cubical anologue of the category of globular categorical stretchings of [23]. The key ingredient is a cubical analogue of the globular contractions build in [23]. Then we give a monad $\mathbb{W}$ on the category of cubical sets whose algebras are our models of cubical weak ∞-categories with connections. This monad is the cubical analogue of the monad $\mathbb{P}^0_C$ in [23], whose $\mathbb{P}^0_C$-algebras are the globular weak ∞-categories of Penon.

In [12, 14, 23] some computations were described for globular higher structures born with globular stretchings. For example in [23] it is proved that, in dimension 2, the globular weak ∞-categories of Penon are bicategories. In Subsection 4.3 we gave a precise definition of the dimension for algebras of our models of cubical weak ∞-categories. Computations in dimension 2 lead to long computations and go beyond the scope of this article, but the reader interested in the dimension 2 can verify that our models of dimension 2 (without the structure of connections) are weak double categories in the sense of Verity [26]; however in Subsection 4.1 we exhibited an example of cubical coherence cell[7] in dimension 2.

Main proofs of this article use Sketch Theory initiated by Charles Ehresmann and his students, especially Christian Lair [8]. Two sketches are *Morita equivalent* if their category of $\mathbb{S}$ets-models are equivalent. Also when we write "sketch of a category $\mathcal{C}$", we mean that this category $\mathcal{C}$ is projectively sketchable, and that a sketch of it has been fixed up to Morita equivalence.

[6] Cubical sets in our terminology are the *precubical sets* of [6].

[7] In [16] we exhibited the same kind of 2-cell for the cubical operad of cubical weak ∞-categories. Globular stretchings and globular operads are quite close in nature (for example they share the same notion of contractions) and for computations [3].

2 Cubical sets

Cubical sets are set-valued functors on a specific small category $\mathbb{C}$ (see Section 2.1). This small category contains the combinatorics underlying the geometric idea of cubes and higher cubes. The following internal cocubical complex in $\mathbb{T}\mathrm{op}$, where $I = [0,1]$ is the usual interval in $\mathbb{R}$ and where I^n is the product of I with itself n-times [6], is an archetypal example of the shape modelled by $\mathbb{C}$:

$$I^0 \;\begin{array}{c} \xrightarrow{s^0_1} \\ \xrightarrow[t^0_1]{} \end{array}\; I^1 \;\begin{array}{c} \xrightarrow{s^2_{1,2}} \\ \xrightarrow{s^2_{1,1}} \\ \xrightarrow[t^2_{1,1}]{} \\ \xrightarrow[t^2_{1,2}]{} \end{array}\; I^2 \;\begin{array}{c} \xrightarrow{s^3_{2,3}} \\ \xrightarrow{s^3_{2,2}} \\ \xrightarrow{s^3_{2,1}} \\ \xrightarrow[t^3_{2,1}]{} \\ \xrightarrow[t^3_{2,2}]{} \\ \xrightarrow[t^3_{2,3}]{} \end{array}\; I^3 \cdots I^{n-1} \;\begin{array}{c} \xrightarrow{s^n_{n-1,1}} \\ \dashrightarrow \\ \xrightarrow{s^n_{n-1,i}} \\ \dashrightarrow \\ \xrightarrow{s^n_{n-1,n-1}} \\ \xrightarrow[t^n_{n-1,n-1}]{} \\ \dashrightarrow \\ \xrightarrow[t^n_{n-1,i}]{} \\ \dashrightarrow \\ \xrightarrow[t^n_{n-1,1}]{} \end{array}\; I^n \cdots$$

Here *sources* $I^{n-1} \xrightarrow{s^n_{n-1,j}} I^n$ for each $j \in \{1,..,n\}$ and *targets* $I^{n-1} \xrightarrow{t^n_{n-1,j}} I^n$ for each $j \in \{1,..,n\}$ such that for $1 \leq i < j \leq n$, follow the cocubical relations

(i) $s^n_{n-1,j} \circ s^{n-1}_{n-2,i} = s^n_{n-1,i} \circ s^{n-1}_{n-2,j-1}$,

(ii) $t^n_{n-1,j} \circ s^{n-1}_{n-2,i} = s^n_{n-1,i} \circ t^{n-1}_{n-2,j-1}$,

(iii) $s^n_{n-1,j} \circ t^{n-1}_{n-2,i} = t^n_{n-1,i} \circ s^{n-1}_{n-2,j-1}$,

(iv) $t^n_{n-1,j} \circ t^{n-1}_{n-2,i} = t^n_{n-1,i} \circ t^{n-1}_{n-2,j-1}$,

which are the dual relations described below in 2.1. This cocubical complex is used in [17] to build the functor of fundamental cubical weak ∞-groupoids for spaces[8]. More references on cubical sets can be found in [1, 10].

[8] It is a $\mathbb{B}^0_C$-coalgebra, where $\mathbb{B}^0_C$ is the cubical operad of cubical weak ∞-categories [16].

2.1 The cubical category Consider the small category $\mathbb{C}$ with integers $\underline{n} \in \mathbb{N}$ as objects. Generators for $\mathbb{C}$ are, for all $\underline{n} \in \mathbb{N}$ given by *sources* $\underline{n} \xrightarrow{s^n_{n-1,j}} \underline{n-1}$ for each $j \in \{1,..,n\}$, and by *targets* $\underline{n} \xrightarrow{t^n_{n-1,j}} \underline{n-1}$ for each $j \in \{1,..,n\}$ such that for $1 \leq i < j \leq n$ we have the following cubical relations

(i) $s^{n-1}_{n-2,i} \circ s^n_{n-1,j} = s^{n-1}_{n-2,j-1} \circ s^n_{n-1,i}$,

(ii) $s^{n-1}_{n-2,i} \circ t^n_{n-1,j} = t^{n-1}_{n-2,j-1} \circ s^n_{n-1,i}$,

(iii) $t^{n-1}_{n-2,i} \circ s^n_{n-1,j} = s^{n-1}_{n-2,j-1} \circ t^n_{n-1,i}$,

(iv) $t^{n-1}_{n-2,i} \circ t^n_{n-1,j} = t^{n-1}_{n-2,j-1} \circ t^n_{n-1,i}$

These generators plus these relations give the small category $\mathbb{C}$ called the *cubical category* whose objects can be represented schematically by the low dimensional diagram:

$$
\begin{tikzcd}[column sep=huge]
\cdots C_4
\arrow[r, shift left=14, "s^4_{3,4}"]
\arrow[r, shift left=10, "s^4_{3,3}"]
\arrow[r, shift left=6, "s^4_{3,2}"]
\arrow[r, shift left=2, "s^4_{3,1}"]
\arrow[r, shift right=2, "t^4_{3,1}"']
\arrow[r, shift right=6, "t^4_{3,2}"']
\arrow[r, shift right=10, "t^4_{3,3}"']
\arrow[r, shift right=14, "t^4_{3,4}"']
& C_3
\arrow[r, shift left=10, "s^3_{2,3}"]
\arrow[r, shift left=6, "s^3_{2,2}"]
\arrow[r, shift left=2, "s^3_{2,1}"]
\arrow[r, shift right=2, "t^3_{2,1}"']
\arrow[r, shift right=6, "t^3_{2,2}"']
\arrow[r, shift right=10, "t^3_{2,3}"']
& C_2
\arrow[r, shift left=6, "s^2_{1,2}"]
\arrow[r, shift left=2, "s^2_{1,1}"]
\arrow[r, shift right=2, "t^2_{1,1}"']
\arrow[r, shift right=6, "t^2_{1,2}"']
& C_1
\arrow[r, shift left=2, "s^1_0"]
\arrow[r, shift right=2, "t^1_0"']
& C_0
\end{tikzcd}
$$

and this category $\mathbb{C}$ gives also the sketch $\mathcal{E}_{\mathbb{S}}$ of cubical sets used especially in 3.2, 4.1, to produce the monads $\mathbb{S} = (S, \lambda, \mu)$, $\mathbb{W} = (W, \eta, \nu)$ on $\mathbb{C}\mathbb{S}$ets, whose algebras are respectively cubical strict ∞-categories and cubical weak ∞-categories.

Definition 2.1. The category of cubical sets $\mathbb{C}\mathbb{S}$ets is the category $[\mathbb{C}; \mathbb{S}\text{ets}]$ of set-valued functors on $\mathbb{C}$. The terminal cubical set is denoted 1.

Occasionally a cubical set shall be denoted with the notation

$$\mathcal{C} = (C_n, s^n_{n-1,j}, t^n_{n-1,j})_{1 \leq j \leq n,\ n \in \mathbb{N}}$$

in case we want to point out its underlying structures.

2.2 Reflexive cubical sets Reflexivity for cubical sets are of two sorts: one is "classical" in the sense that they are very similar to their globular analogue; thus we shall use the notation $(1^n_{n+1,j})_{n\in\mathbb{N},j\in\{1,..,n\}}$ to denote these maps $C(n) \xrightarrow{1^n_{n+1,j}} C(n+1)$ which formally behave like globular reflexivity ([14]); the others are called *connections* and are given by maps $C(n) \xrightarrow{\Gamma} C(n+1)$ where the notation using the Greek letter "*Gamma*" seems to be the usual notation. However we do prefer to use instead the notation $C(n) \xrightarrow{1^{n,\gamma}_{n+1,j}} C(n+1)$ $(\gamma \in \{+,-\})$ in order to point out the reflexive nature of connections.

Consider the cubical category $\mathbb{C}$. For all $n \in \mathbb{N}$ we add generators $\underline{n-1} \xrightarrow{1^{n-1}_{n,j}} \underline{n}$ to it for each $j \in \{1,..,n\}$ subject to the relations:

(i) $1^n_{n+1,i} \circ 1^{n-1}_{n,j} = 1^n_{n+1,j+1} \circ 1^{n-1}_{n,i}$ if $1 \leq i \leq j \leq n$;

(ii) $s^n_{n-1,i} \circ 1^{n-1}_{n,j} = 1^{n-2}_{n-1,j-1} \circ s^{n-1}_{n-2,i}$ if $1 \leq i < j \leq n$;

(iii) $s^n_{n-1,i} \circ 1^{n-1}_{n,j} = 1^{n-2}_{n-1,j} \circ s^{n-1}_{n-2,i-1}$ if $1 \leq j < i \leq n$;

(iv) $s^n_{n-1,i} \circ 1^{n-1}_{n,j} = id(\underline{n-1})$ if $i = j$.

and

(i) $1^n_{n+1,i} \circ 1^{n-1}_{n,j} = 1^n_{n+1,j+1} \circ 1^{n-1}_{n,i}$ if $1 \leq i \leq j \leq n$;

(ii) $t^n_{n-1,i} \circ 1^{n-1}_{n,j} = 1^{n-2}_{n-1,j-1} \circ t^{n-1}_{n-2,i}$ if $1 \leq i < j \leq n$;

(iii) $t^n_{n-1,i} \circ 1^{n-1}_{n,j} = 1^{n-2}_{n-1,j} \circ t^{n-1}_{n-2,i-1}$ if $1 \leq j < i \leq n$;

(iv) $t^n_{n-1,i} \circ 1^{n-1}_{n,j} = id(\underline{n-1})$ if $i = j$.

These generators and relations give the small category $\mathbb{C}_{\text{sr}}$ called the *semireflexive cubical category* where a quick look at its underlying semireflexive structure is given by the following diagram:

$$\begin{tikzcd} C_0 \arrow[r, "1^0_1"] & C_1 \arrow[r, shift left, "1^1_{2,1}"] \arrow[r, shift right, "1^1_{2,2}"'] & C_2 \arrow[r, shift left=2, "1^2_{3,2}"] \arrow[r, "1^2_{3,1}"] \arrow[r, shift right=2, "1^2_{3,3}"'] & C_3 \arrow[r, shift left=3, "1^3_{4,3}"] \arrow[r, shift left=1, "1^3_{4,1}"] \arrow[r, shift right=1, "1^3_{4,2}"] \arrow[r, shift right=3, "1^3_{4,4}"'] & C_4 \cdots \end{tikzcd}$$

Definition 2.2. The category of semireflexive cubical sets $\mathbb{C}_{\mathrm{sr}}\mathbb{S}\mathrm{ets}$ is the functor category $[\mathbb{C}_{\mathrm{sr}};\mathbb{S}\mathrm{ets}]$. The terminal semireflexive cubical set is denoted 1_{sr}.

Consider the semireflexive cubical category $\mathbb{C}_{\mathrm{sr}}$. For all integers $n \geq 1$ we add generators $\underline{n-1} \xrightarrow{1^{n-1,\gamma}_{n,j}} \underline{n}$ for each $j \in \{1,..,n-1\}$ subject to the relations:

(i) for $1 \leq j < i \leq n$, $1^{n,\gamma}_{n+1,i} \circ 1^{n-1,\gamma}_{n,j} = 1^{n,\gamma}_{n+1,j+1} \circ 1^{n-1,\gamma}_{n,i}$;

(ii) for $1 \leq i \leq n-1$, $1^{n,\gamma}_{n+1,i} \circ 1^{n-1,\gamma}_{n,i} = 1^{n,\gamma}_{n+1,i+1} \circ 1^{n-1,\gamma}_{n,i}$;

(iii) for $1 \leq i,j \leq n$,
$$\begin{cases} 1^{n,\gamma}_{n+1,i} \circ 1^{n-1}_{n,j} & = 1^{n}_{n+1,j+1} \circ 1^{n-1,\gamma}_{n,i} \quad \text{if} \quad 1 \leq i < j \leq n \\ & = 1^{n}_{n+1,j} \circ 1^{n-1,\gamma}_{n,i-1} \quad \text{if} \quad 1 \leq j < i \leq n \end{cases};$$

(iv) for $1 \leq j \leq n$, $1^{n,\gamma}_{n+1,j} \circ 1^{n-1}_{n,j} = 1^{n}_{n+1,j} \circ 1^{n-1}_{n,j}$;

(v) for $1 \leq i,j \leq n$,
$$\begin{cases} s^{n}_{n-1,i} \circ 1^{n-1,\gamma}_{n,j} & = 1^{n-2,\gamma}_{n-1,j-1} \circ s^{n-1}_{n-2,i} \quad \text{if} \quad 1 \leq i < j \leq n-1 \\ & = 1^{n-2,\gamma}_{n-1,j} \circ s^{n-1}_{n-2,i-1} \quad \text{if} \quad 2 \leq j+1 < i \leq n \end{cases};$$
and
$$\begin{cases} t^{n}_{n-1,i} \circ 1^{n-1,\gamma}_{n,j} & = 1^{n-2,\gamma}_{n-1,j-1} \circ t^{n-1}_{n-2,i} \quad \text{if} \quad 1 \leq i < j \leq n-1 \\ & = 1^{n-2,\gamma}_{n-1,j} \circ t^{n-1}_{n-2,i-1} \quad \text{if} \quad 2 \leq j+1 < i \leq n \end{cases};$$

(vi) for $1 \leq j \leq n-1$, $s^{n}_{n-1,j} \circ 1^{n-1,-}_{n,j} = s^{n}_{n-1,j+1} \circ 1^{n-1,-}_{n,j} = 1_{\underline{n-1}}$ and $t^{n}_{n-1,j} \circ 1^{n-1,+}_{n,j} = t^{n}_{n-1,j+1} \circ 1^{n-1,+}_{n,j} = 1_{\underline{n-1}}$;

(vii) for $1 \leq j \leq n-1$, $s^{n}_{n-1,j} \circ 1^{n-1,+}_{n,j} = s^{n}_{n-1,j+1} \circ 1^{n-1,+}_{n,j} = 1^{n-2}_{n-1,j} \circ s^{n-1}_{n-2,j}$;

(viii) for $1 \leq j \leq n-1$, $t^{n}_{n-1,j} \circ 1^{n-1,-}_{n,j} = t^{n}_{n-1,j+1} \circ 1^{n-1,-}_{n,j} = 1^{n-2}_{n-1,j} \circ t^{n-1}_{n-2,j}$.

These generators and relations give the small category $\mathbb{C}_r$ called the *reflexive cubical category* and in it, connections have the following shape:

$$
\begin{tikzcd}[column sep=huge]
C_1 \arrow[r, shift left=1ex, "1^{1,-}_{2,1}"] \arrow[r, shift right=1ex, "1^{1,+}_{2,1}"'] & C_2 \arrow[r, shift left=3ex, "1^{2,-}_{3,2}"] \arrow[r, shift left=1ex, "1^{2,-}_{3,1}"] \arrow[r, shift right=1ex, "1^{2,+}_{3,1}"'] \arrow[r, shift right=3ex, "1^{2,+}_{3,2}"'] & C_3 \arrow[r, shift left=5ex, "1^{3,-}_{4,3}"] \arrow[r, shift left=3ex, "1^{3,-}_{4,2}"] \arrow[r, shift left=1ex, "1^{3,-}_{4,1}"] \arrow[r, shift right=1ex, "1^{3,+}_{4,1}"'] \arrow[r, shift right=3ex, "1^{3,+}_{4,2}"'] \arrow[r, shift right=5ex, "1^{3,+}_{4,3}"'] & C_3 \arrow[r, shift left=7ex, "1^{4,-}_{5,4}"] \arrow[r, shift left=5ex, "1^{4,-}_{5,3}"] \arrow[r, shift left=3ex, "1^{4,-}_{5,2}"] \arrow[r, shift left=1ex, "1^{4,-}_{5,1}"] \arrow[r, shift right=1ex, "1^{4,+}_{5,1}"'] \arrow[r, shift right=3ex, "1^{4,+}_{5,2}"'] \arrow[r, shift right=5ex, "1^{4,+}_{5,3}"'] \arrow[r, shift right=7ex, "1^{4,+}_{5,4}"'] & C_4 \cdots
\end{tikzcd}
$$

Definition 2.3. The category of reflexive cubical sets $\mathbb{C}_{\mathrm{r}}\mathbb{S}\mathrm{ets}$ is the functor category $[\mathbb{C}_{\mathrm{r}}; \mathbb{S}\mathrm{ets}]$. The terminal reflexive cubical set is denoted 1_{r}

It is important to note that this small category $\mathbb{C}_{\mathrm{r}}$ is a *strict test category* [22]; that is, the category $\mathbb{C}_{\mathrm{r}}\mathbb{S}\mathrm{ets}$ of reflexive cubical sets can be equipped with a Quillen model structure which is Quillen equivalent to the category of spaces equipped with its classical Quillen model structure [11].

3 The category of strict cubical ∞-categories

Cubical strict ∞-categories have been studied in [5, 6]. In [5] the authors proved that the category of cubical strict ∞-categories with cubical strict ∞-functors as morphisms is equivalent to the category of globular strict ∞-categories with globular strict ∞-functors as morphisms.

These higher structures are very important for us because they are used in 4.1 to control the cubical coherence cells that we need to add in reflexive cubical ∞-magmas in order to be our basic cubical weak ∞-categories: in fact they allow the construction of the free cubical weak ∞-categories on each cubical sets. The primary structure behind cubical strict ∞-categories are cubical ∞-magmas that we are now going to define.

Consider a cubical reflexive set

$$(C, (1^n_{n+1,j})_{n\in\mathbb{N}, j\in[\![1,n+1]\!]}, (1^{n,\gamma}_{n+1,j})_{n\geq 1, j\in[\![1,n]\!]})$$

equipped with partial operations $(\circ^n_j)_{n\geq 1, j\in[\![1,n]\!]}$ where if $a, b \in C(n)$ then $a \circ^n_j b$ is defined for $j \in \{1, ..., n\}$ if $s^n_j(b) = t^n_j(a)$. We also require these operations to follow the following axioms of position:

(i) For $1 \leq j \leq n$, we have:

$$s^n_{n-1,j}(a \circ^n_j b) = s^n_{n-1,j}(a) \text{ and } t^n_{n-1,j}(a \circ^n_j b) = t^n_{n-1,j}(a),$$

(ii) $s^n_{n-1,i}(a \circ^n_j b) = \begin{cases} s^n_{n-1,i}(a) \circ^{n-1}_{j-1} s^n_{n-1,i}(b) \text{ if } 1 \leq i < j \leq n \\ s^n_{n-1,i}(a) \circ^{n-1}_{j} s^n_{n-1,i}(b) \text{ if } 1 \leq j < i \leq n \end{cases}$

(iii) $t^n_{n-1,i}(a \circ^n_j b) = \begin{cases} t^n_{n-1,i}(a) \circ^{n-1}_{j-1} t^n_{n-1,i}(b) \text{ if } 1 \leq i < j \leq n \\ t^n_{n-1,i}(a) \circ^{n-1}_{j} t^n_{n-1,i}(b) \text{ if } 1 \leq j < i \leq n \end{cases}$

Definition 3.1. Cubical ∞-magmas are cubical sets equipped with partial operations as above. A morphism between two cubical ∞-magmas is a morphism of their underlying cubical sets which respects the partial operations $(\circ^n_j)_{n\geq 1, j\in [\![1,n]\!]}$.The category of cubical ∞-magmas is denoted ∞-$\mathbb{CM}$ag.

The following sketch $\mathcal{E}_M$ of *axioms of position* as above shall be used in 3.2 to justify the existence of the monad on $\mathbb{CS}$ets of cubical strict ∞-categories. It is important to notice that the sketch just below has only *one generation* which means that diagrams and cones involved in it are not built with previous data of other diagrams and cones. The terminology used in [21] is *sketch with one floor*[9]. See also [8].

- For $1 \leq i < j \leq n$ we consider the following two cones:

$$\begin{tikzcd} M_n \underset{M_{n-1,j}}{\times} M_n \arrow[r, "\pi^n_{1,j}"] \arrow[d, "\pi^n_{0,j}"'] & M_n \arrow[d, "s^n_{n-1,j}"] \\ M_n \arrow[r, "t^n_{n-1,j}"'] & M_{n-1} \end{tikzcd} \qquad \begin{tikzcd} M_{n-1} \underset{M_{n-2,j-1}}{\times} M_{n-1} \arrow[r, "\pi^{n-1}_{1,j-1}"] \arrow[d, "\pi^{n-1}_{0,j-1}"'] & M_{n-1} \arrow[d, "s^{n-1}_{n-2,j-1}"] \\ M_{n-1} \arrow[r, "t^{n-1}_{n-2,j-1}"'] & M_{n-2} \end{tikzcd}$$

and the following commutative diagram (definition of $s^n_{n-1,i} \underset{j,j-1}{\times} s^n_{n-1,i}$):

[9] In French we say "*esquisse à un étage*".

$$\begin{tikzcd}
M_n \underset{M_{n-1,j}}{\times} M_n \arrow[rr, "\pi^n_{1,j}"] \arrow[dr, dotted, "s^n_{n-1,i} \underset{j,j-1}{\times} s^n_{n-1,i}"] \arrow[dd, "\pi^n_{0,j}"'] & & M_n \arrow[dr, "s^n_{n-1,i}"] & \\
& M_{n-1} \underset{M_{n-2,j-1}}{\times} M_{n-1} \arrow[rr, "\pi^{n-1}_{1,j-1}"] \arrow[dd, "\pi^{n-1}_{0,j-1}"'] & & M_{n-1} \arrow[dd, "s^{n-1}_{n-2,j-1}"] \\
M_n \arrow[dr, "s^n_{n-1,i}"'] & & & \\
& M_{n-1} \arrow[rr, "t^{n-1}_{n-2,j-1}"'] & & M_{n-2}
\end{tikzcd}$$

which gives the following commutative diagram:

$$\begin{tikzcd}
M_n \underset{M_{n-1,j}}{\times} M_n \arrow[r, "s^n_{n-1,i} \underset{j,j-1}{\times} s^n_{n-1,i}"] \arrow[d, "\star^n_j"'] & M_{n-1} \underset{M_{n-2,j-1}}{\times} M_{n-1} \arrow[d, "\star^{n-1}_{j-1}"] \\
M_n \arrow[r, "s^n_{n-1,i}"'] & M_{n-1}
\end{tikzcd}$$

- For $1 \leq j < i \leq n$ we consider the following two cones:

$$\begin{tikzcd}
M_n \underset{M_{n-1,j}}{\times} M_n \arrow[r, "\pi^n_{1,j}"] \arrow[d, "\pi^n_{0,j}"'] & M_n \arrow[d, "s^n_{n-1,j}"] \\
M_n \arrow[r, "t^n_{n-1,j}"'] & M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{n-1} \underset{M_{n-2,j}}{\times} M_{n-1} \arrow[r, "\pi^{n-1}_{1,j}"] \arrow[d, "\pi^{n-1}_{0,j}"'] & M_{n-1} \arrow[d, "s^{n-1}_{n-2,j}"] \\
M_{n-1} \arrow[r, "t^{n-1}_{n-2,j}"'] & M_{n-2}
\end{tikzcd}$$

and the following commutative diagram (definition of $s^n_{n-1,i} \underset{j,j}{\times} s^n_{n-1,i}$)

$$
\begin{tikzcd}
M_n \underset{M_{n-1,j}}{\times} M_n \arrow[rr, "\pi^n_{1,j}"] \arrow[dr, dotted, "s^n_{n-1,i} \underset{j,j}{\times} s^n_{n-1,i}"] \arrow[dd, "\pi^n_{0,j}"'] & & M_n \arrow[dr, "s^n_{n-1,i}"] & \\
& M_{n-1} \underset{M_{n-2,j}}{\times} M_{n-1} \arrow[rr, "\pi^{n-1}_{1,j}"] \arrow[dd, "\pi^{n-1}_{0,j}"'] & & M_{n-1} \arrow[dd, "s^{n-1}_{n-2,j}"] \\
M_n \arrow[dr, "s^n_{n-1,i}"'] & & & \\
& M_{n-1} \arrow[rr, "t^{n-1}_{n-2,j}"'] & & M_{n-2}
\end{tikzcd}
$$

The previous data give the following commutative diagram of axioms:

$$
\begin{tikzcd}
M_n \underset{M_{n-1}}{\times} M_n \arrow[r, "s^n_{n-1,i} \underset{j,j}{\times} s^n_{n-1,i}"] \arrow[d, "\star^n_j"'] & M_{n-1} \underset{M_{n-2}}{\times} M_{n-1} \arrow[d, "\star^{n-1}_j"] \\
M_n \arrow[r, "s^n_{n-1,i}"'] & M_{n-1}
\end{tikzcd}
$$

and for $1 \leq j \leq n$ we have the following commutative diagram of axioms which completes the description of $\mathcal{E}_M$:

$$
\begin{tikzcd}
M_n \underset{M_{n-1}}{\times} M_n \arrow[r, "\pi_1"] \arrow[d, "\star^n_j"'] & M_n \arrow[d, "s^n_{n-1,j}"] \\
M_n \arrow[r, "s^n_{n-1,j}"'] & M_{n-1}
\end{tikzcd}
$$

Definition 3.2. Cubical reflexive ∞-magmas are cubical reflexive sets equipped with a structure of ∞-magmas. A morphism between two cubical reflexive ∞-magmas is a morphism of their underlying cubical reflexive sets which respects the partial operations $(\circ^n_j)_{n\geq 1, j\in [\![1,n]\!]}$. The category of cubical reflexive ∞-magmas is denoted ∞-$\mathbb{CM}\mathrm{ag}_\mathrm{r}$

3.1 Strict cubical ∞-categories Strict cubical ∞-categories are cubical reflexive ∞-magmas such that the partial operations are associative and also satisfy the following axioms:

(i) The interchange laws:

$$(a \circ_i^n b) \circ_j^n (c \circ_i^n d) = (a \circ_j^n c) \circ_i^n (b \circ_j^n d)$$

whenever both sides are defined

(ii) $1^n_{n+1,i}(a \circ^n_j b) = 1^n_{n+1,i}(a) \circ^{n+1}_{j+1} 1^n_{n+1,i}(b)$ if $1 \leq i \leq j \leq n$

$1^n_{n+1,i}(a \circ^n_j b) = 1^n_{n+1,i}(a) \circ^{n+1}_{j} 1^n_{n+1,i}(b)$ if $1 \leq j < i \leq n+1$

(iii) $1^{n,\gamma}_{n+1,i}(a \circ^n_j b) = 1^{n,\gamma}_{n+1,i}(a) \circ^{n+1}_{j+1} 1^{n,\gamma}_{n+1,i}(b)$ if $1 \leq i < j \leq n$

$1^{n,\gamma}_{n+1,i}(a \circ^n_j b) = 1^{n,\gamma}_{n+1,i}(a) \circ^{n+1}_{j} 1^{n,\gamma}_{n+1,i}(b)$ if $1 \leq j < i \leq n$

(iv) First transport laws: for $1 \leq j \leq n$

$$1^{n,+}_{n+1,j}(a \circ^n_j b) = \begin{bmatrix} 1^{n,+}_{n+1,j}(a) & 1^n_{n+1,j}(a) \\ 1^n_{n+1,j+1}(a) & 1^{n,+}_{n+1,j}(b) \end{bmatrix}$$

(v) Second transport laws: for $1 \leq j \leq n$

$$1^{n,-}_{n+1,j}(a \circ^n_j b) = \begin{bmatrix} 1^{n,-}_{n+1,j}(a) & 1^n_{n+1,j+1}(b) \\ 1^n_{n+1,j}(b) & 1^{n,-}_{n+1,j}(b) \end{bmatrix}$$

(vi) for $1 \leq j \leq n$, $1^{n,+}_{n+1,i}(x) \circ^{n+1}_i 1^{n,-}_{n+1,i}(x) = 1^n_{n+1,i+1}(x)$ and $1^{n,+}_{n+1,i}(x) \circ^{n+1}_{i+1} 1^{n,-}_{n+1,i}(x) = 1^n_{n+1,i}(x)$

The category ∞-$\mathbb{C}$CAT of strict cubical ∞-categories is the full subcategory of ∞-$\mathbb{C}\mathbb{M}\mathrm{ag}_\mathrm{r}$ spanned by strict cubical ∞-categories. A morphism in ∞-$\mathbb{C}$CAT is called a *strict cubical ∞-functor*.

3.2 The monad of cubical strict ∞-categories In this section we describe cubical strict ∞-categories as algebras for a monad $\mathbb{S}$ on $\mathbb{C}\mathbb{S}$ets. We intend it to be an ingredient in the comparison of globular strict ∞-categories with cubical strict ∞-categories. Also we conjecture that $\mathbb{S}$ is cartesian.

Consider the forgetful functor:

$$\infty\text{-}\mathbb{C}\mathbb{C}\mathrm{AT} \xrightarrow{\quad U \quad} \mathbb{C}\mathbb{S}\mathrm{ets}$$

which associates to any strict cubical ∞-category its underlying cubical set and which associates to any strict cubical ∞-functor its underlying morphism of cubical sets.

Proposition 3.3. The functor U is right adjoint and monadic.

Its left adjoint is denoted F. The proof is based on exhibiting a good morphism of projective sketches and it is also based on two results of Foltz [9] and Lair [21]: the non-trivial part is to exhibit the projective sketch $\mathcal{E}_{\mathrm{C}}$ of the category ∞-$\mathbb{C}\mathbb{C}$AT and we shall easily see that we get an inclusion of projective sketches: $\mathcal{E}_{\mathrm{S}} \xhookrightarrow{\quad i \quad} \mathcal{E}_{\mathrm{C}}$ where $\mathcal{E}_{\mathrm{S}}$ is the sketch of cubical sets 2.1. Now we have the commutative diagram

$$\begin{tikzcd} Mod(\mathcal{E}_{\mathrm{C}}) \arrow[r, "Mod(i)"] \arrow[d, "iso"'] & Mod(\mathcal{E}_{\mathrm{C}}) \arrow[d, "iso"] \\ \infty\text{-}\mathbb{C}\mathbb{C}\mathrm{AT} \arrow[r, "U"] & \mathbb{C}\mathbb{S}\mathrm{ets} \end{tikzcd}$$

when passing to models in $\mathbb{S}$ets. It shows that i induces the forgetful functor U and that U is right adjoint thanks to the *sheafification theorem* of Foltz [9].

Remark 3.4. This result of Foltz is called the *sheafification theorem*, because it generalizes the construction of the associated sheaf on a presheaf for a given site.

Following the terminology of [21] we say that the functor U is *projectively sketchable*. Also we shall easily see that each distinguished cone of $\mathcal{E}_{\mathrm{C}}$ has a base which factorizes i and each object of $\mathcal{E}_{\mathrm{C}}$ which is not in the image of i is the vertex of at least one distinguished cone of $\mathcal{E}_{\mathrm{C}}$. Thus by the theorem of Lair in [21] about monadicity, it follows that U is monadic.

Proof. The proof is very similar to those in [23]: actually we are going to see that the category ∞-$\mathbb{C}\mathbb{C}$AT is projectively sketchable. Let us denote by $\mathcal{E}_{\mathrm{C}}$ the sketch of ∞-$\mathbb{C}\mathbb{C}$AT. The description of $\mathcal{E}_{\mathrm{C}}$ started with the description of $\mathcal{E}_{\mathrm{M}}$ in 3. We carry on with it by describing the sketch behind the interchange laws, which shall complete the main parts of $\mathcal{E}_{\mathrm{C}}$[10].

- We start with three limit cones:

$$\begin{tikzcd}
Z_n \underset{Z_{n-1,i}}{\times} Z_n \arrow[r, "\rho^n_{1,i}"] \arrow[d, "\rho^n_{0,i}"'] & Z_n \arrow[d, "s^n_{n-1,i}"] \\
Z_n \arrow[r, "t^n_{n-1,i}"'] & Z_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_n \underset{Z_{n-1,j}}{\times} Z_n \arrow[r, "\rho^n_{1,j}"] \arrow[d, "\rho^n_{0,j}"'] & Z_n \arrow[d, "s^n_{n-1,j}"] \\
Z_n \arrow[r, "t^n_{n-1,j}"'] & Z_{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & E_{ijn} \arrow[dlll, "\pi^n_{00}"'] \arrow[dl, "\pi^n_{10}"] \arrow[dr, "\pi^n_{01}"] \arrow[drrr, "\pi^n_{11}"] & & & \\
Z_n \arrow[dr, "t^n_{n-1,i}"'] & & Z_n \arrow[dl, "s^n_{n-1,i}"'] \arrow[dr, "s^n_{n-1,j}"] & & Z_n \arrow[dl, "t^n_{n-1,j}"'] \arrow[dr, "t^n_{n-1,i}"] & & Z_n \arrow[dl, "s^n_{n-1,i}"] \\
 & Z_{n-1} & & Z_{n-1} & & Z_{n-1} &
\end{tikzcd}$$

- Then we consider the following commutative diagrams:

$$\begin{tikzcd}
E_{ijn} \arrow[dr, dotted, "p^n_{1000}"] \arrow[drr, "\pi^n_{10}"] \arrow[ddr, "\pi^n_{00}"'] & & \\
 & Z_n \underset{Z_{n-1,i}}{\times} Z_n \arrow[r, "\rho^n_{1,i}"] \arrow[d, "\rho^n_{0,i}"'] & Z_n \arrow[d, "s^n_{n-1,i}"] \\
 & Z_n \arrow[r, "t^n_{n-1,i}"'] & Z_n
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{ijn} \arrow[dr, dotted, "p^n_{1101}"] \arrow[drr, "\pi^n_{11}"] \arrow[ddr, "\pi^n_{01}"'] & & \\
 & Z_n \underset{Z_{n-1,i}}{\times} Z_n \arrow[r, "\rho^n_{1,i}"] \arrow[d, "\rho^n_{0,i}"'] & Z_n \arrow[d, "s^n_{n-1,i}"] \\
 & Z_n \arrow[r, "t^n_{n-1,i}"'] & Z_n
\end{tikzcd}$$

[10]Other parts of $\mathcal{E}_{\mathrm{C}}$ are straightforward.

$$
\begin{tikzcd}
E_{ijn} \arrow[dr, dotted, "p^n_{1011}"] \arrow[drr, "\pi^n_{10}"] \arrow[ddr, "\pi^n_{11}"'] & & \\
& Z_n \underset{Z_{n-1,j}}{\times} Z_n \arrow[r, "\rho^n_{1,j}"] \arrow[d, "\rho^n_{0,j}"'] & Z_n \arrow[d, "s^n_{n-1,j}"] \\
& Z_n \arrow[r, "t^n_{n-1,j}"'] & Z_n
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{ijn} \arrow[dr, dotted, "p^n_{0001}"] \arrow[drr, "\pi^n_{00}"] \arrow[ddr, "\pi^n_{01}"'] & & \\
& Z_n \underset{Z_{n-1,j}}{\times} Z_n \arrow[r, "\rho^n_{1,j}"] \arrow[d, "\rho^n_{0,j}"'] & Z_n \arrow[d, "s^n_{n-1,j}"] \\
& Z_n \arrow[r, "t^n_{n-1,j}"'] & Z_n
\end{tikzcd}
$$

- We consider then the following two commutative diagrams:

$$
\begin{tikzcd}
& & E_{ijn} \arrow[dll, "p^n_{1000}"'] \arrow[drr, "p^n_{1101}"] \arrow[d, dotted, "c^n_1"] & & \\
Z_n \underset{Z_{n-1,i}}{\times} Z_n \arrow[dd, "\star^n_{n-1,i}"'] & & Z_n \underset{Z_{n-1,j}}{\times} Z_n \arrow[ddll, "\rho^n_{1,j}"'] \arrow[ddrr, "\rho^n_{0,j}"] & & Z_n \underset{Z_{n-1,i}}{\times} Z_n \arrow[dd, "\star^n_{n-1,i}"] \\
& & & & \\
Z_n \arrow[drr, "s^n_{n-1,j}"'] & & & & Z_n \arrow[dll, "t^n_{n-1,j}"] \\
& & Z_{n-1} & &
\end{tikzcd}
$$

$$\begin{tikzcd}
 & & E_{ijn} \arrow[dll, "p^n_{1011}"'] \arrow[drr, "p^n_{0001}"] \arrow[dd, dotted, "c^n_2"] & & \\
Z_n \underset{Z_{n-1,j}}{\times} Z_n \arrow[dd, "\star^n_{n-1,j}"'] & & & & Z_n \underset{Z_{n-1,j}}{\times} Z_n \arrow[dd, "\star^n_{n-1,j}"] \\
 & & Z_n \underset{Z_{n-1,i}}{\times} Z_n \arrow[dll, "\rho^n_{1,i}"'] \arrow[drr, "\rho^n_{0,i}"] & & \\
Z_n \arrow[drr, "s^n_{n-1,i}"'] & & & & Z_n \arrow[dll, "t^n_{n-1,i}"] \\
 & & Z_{n-1} & &
\end{tikzcd}$$

- Finally we consider the following commutative diagram, which diagrammatically formalizes the interchange laws:

$$\begin{tikzcd}
 & E_{ijn} \arrow[dl, "c^n_1"'] \arrow[dr, "c^n_2"] & \\
Z_n \underset{Z_{n-1}}{\times} Z_n \arrow[dr, "\star^n_{n-1,j}"'] & & Z_n \underset{Z_{n-1}}{\times} Z_n \arrow[dl, "\star^n_{n-1,i}"] \\
 & Z_n &
\end{tikzcd}$$

□

The monad of strict cubical ∞-categories with connections defined on the category of cubical sets is denoted $\mathbb{S} = (S, \lambda, \mu)$. Here λ is the unit map of $\mathbb{S}$:

$$1_{\mathbb{C}\mathbb{S}\mathrm{ets}} \xrightarrow{\quad\lambda\quad} S$$

and μ is the multiplication of $\mathbb{S}$:

$$S^2 \xrightarrow{\quad\mu\quad} S$$

We have the following important result proved in [18]

Theorem 3.5. *The monad $\mathbb{S} = (S, \lambda, \mu)$ is cartesian.*

4 The category of cubical weak ∞-categories

In this section we exhibit the first algebraic models of cubical weak ∞-categories. Thus these models are algebras for a specific monad that we describe below, acting on the category $\mathbb{C}\mathbb{S}$ets of cubical sets 2.1. We shall propose in [16] other algebraic models of cubical weak ∞-categories defined as algebras for a specific cubical operad, and these operadical models are possible up to the conjecture 3.5 just above. However the algebraic models of this article do not need the conjecture 3.5, and this is one advantage of the *Weakened by Stretchings* initiated by Jacques Penon in [23] which does not require cartesianess of monads. However, because in [3] Michael Batanin proved that there is a morphism of operads $\mathbb{B}^0_C \longrightarrow \mathbb{P}^0_C$ where $\mathbb{B}^0_C$-algebras are weak ∞-categories of Batanin and $\mathbb{P}^0_C$-algebras are weak ∞-categories of Penon, we suspect the same phenomenon between the operad which underlies the monad described just below and the operad of cubical weak ∞-categories described in [16].

4.1 The category of cubical categorical stretchings We have defined the category ∞-$\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}}$ of cubical reflexive ∞-magmas in 3. Objects of this category plus cubical strict ∞-categories, allow us to define *cubical categorical stretchings* (see below), which are objects of the category ∞-$\mathbb{C}\mathbb{E}\mathrm{t}\mathbb{C}$. This category is the key ingredient for weakened cubical strict ∞-categories as done in [23] for the globular paradigm. Our cubical weak ∞-categories are algebraic in the sense that they are algebras (4.2) for a monad on $\mathbb{C}\mathbb{S}$ets which is built by using the category of cubical categorical stretchings. Our way to build the category ∞-$\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}}$ allows us to weakened the whole structure of cubical strict ∞-categories. As we shall see, the central notion of *cubical contractions* (see below) are more subtle than globular contractions of [23]: in particular they involve an inductive definition on the dimension n of the n-cells ($n \in \mathbb{N}$).

The category ∞-$\mathbb{C}\mathbb{E}$tC of cubical categorical stretchings has as objects quintuples:

$$\mathbb{E} = (M, C, \pi, ([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n+1\}}, ([-;-]^{n,\gamma}_{n+1,j})_{n\geq 1;j\in\{1,\dots,n\};\gamma\in\{-,+\}})$$

where M is a reflexive cubical ∞-magma, C is a cubical strict ∞-category, π is a morphism in ∞-$\mathbb{C}\mathbb{M}\mathrm{ag}_{\mathrm{r}}$:

$$M \xrightarrow{\quad\pi\quad} C$$

and

$$([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n+1\}}, ([-;-]^{n,\gamma}_{n+1,j})_{n\geq 1;j\in\{1,\dots,n\};\gamma\in\{-,+\}}$$

are extra structures called *cubical bracketings*, and which are the cubical analogue of the key structure of the Penon approach to weaken the axioms of strict ∞-categories; it is for us the key structure to weaken the axioms of cubical strict ∞-categories. More precisely: for all integer $n \geq 1$, consider the following subsets of $M_n \times M_n$:

- $\underline{M}_n = \{(\alpha, \beta) \in M_n \times M_n : \pi_n(\alpha) = \pi_n(\beta)\}$
- $\underline{M}_{n,j} = \{(\alpha, \beta) \in M_n \times M_n : s^n_{n-1,j}(\alpha) = s^n_{n-1,j}(\beta), t^n_{n-1,j}(\alpha) = t^n_{n-1,j}(\beta)$ and $\pi_n(\alpha) = \pi_n(\beta)\}$; if $n = 1$ then such set is denoted $\underline{M}_{1,0}$.

and for $n = 0$ we put: $\underline{M}_0 = \{(\alpha, \beta) \in M_0 \times M_0 : \alpha = \beta\}$
These extra structures are given by the operations:

$$([-;-]^n_{n+1,j} : \underline{M}_n \longrightarrow M_{n+1})_{n\in\mathbb{N};j\in\{1,\dots,n+1\}}$$

which are defined inductively, and are such that:

- If $1 \leq i < j \leq n+1$, then:
$$s^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [s^n_{n-1,i}(\alpha), s^n_{n-1,i}(\beta)]^{n-1}_{n,j-1},$$
and
$$t^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [t^n_{n-1,i}(\alpha), t^n_{n-1,i}(\beta)]^{n-1}_{n,j-1}$$
- If $1 \leq j < i \leq n+1$ then:
$$s^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [s^n_{n-1,i-1}(\alpha), s^n_{n-1,i-1}(\beta)]^{n-1}_{n,j},$$
and
$$t^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [t^n_{n-1,i-1}(\alpha), t^n_{n-1,i-1}(\beta)]^{n-1}_{n,j}$$
- If $1 \leq i = j \leq n+1$ then:
$$s^{n+1}_{n,j}([\alpha,\beta]^n_{n+1,j}) = \alpha \text{ and } t^{n+1}_{n,j}([\alpha,\beta]^n_{n+1,j}) = \beta$$

and also:

- $\pi_{n+1}([\alpha,\beta]^n_{n+1,j}) = 1^n_{n+1,j}(\pi_n(\alpha)) = 1^n_{n+1,j}(\pi_n(\beta))$
- $\forall \alpha \in M_n, [\alpha,\alpha]^n_{n+1,j} = 1^n_{n+1,j}(\alpha)$.

We use these operations $([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n+1\}}$ to define the other operations:

$$([-;-]^{n,-}_{n+1,j} : \underline{M}_{n,j} \longrightarrow M_{n+1})_{n\geq 1;j\in\{1,\ldots,n\}}$$

and

$$([-;-]^{n,+}_{n+1,j} : \underline{M}_{n,j} \longrightarrow M_{n+1})_{n\geq 1;j\in\{1,\ldots,n\}}$$

which are defined inductively, and are such that:

- if $1 \leq i < j \leq n$ then $s^{n+1}_{n,i}([\alpha;\beta]^{n,-}_{n+1,j}) = [s^n_{n-1,i}(\alpha); s^n_{n-1,i}(\beta)]^{n-1,-}_{n,j-1}$ and $t^{n+1}_{n,i}([\alpha;\beta]^{n,-}_{n+1,j}) = [t^n_{n-1,i}(\alpha); t^n_{n-1,i}(\beta)]^{n-1,-}_{n,j-1}$
- if $2 \leq j+1 < i \leq n+1$ then
$$s^{n+1}_{n,i}([\alpha;\beta]^{n,-}_{n+1,j}) = [s^n_{n-1,i-1}(\alpha); s^n_{n-1,i-1}(\beta)]^{n-1,-}_{n,j}$$
and $t^{n+1}_{n,i}([\alpha;\beta]^{n,-}_{n+1,j}) = [t^n_{n-1,i-1}(\alpha); t^n_{n-1,i-1}(\beta)]^{n-1,-}_{n,j}$

- if $1 \leq i = j \leq n$ then $s^{n+1}_{n,j}([\alpha;\beta]^{n,-}_{n+1,j}) = \alpha$ and $s^{n+1}_{n,j+1}([\alpha;\beta]^{n,-}_{n+1,j}) = \beta$, and $t^{n+1}_{n,j}([\alpha;\beta]^{n,-}_{n+1,j}) = t^{n+1}_{n,j+1}([\alpha;\beta]^{n,-}_{n+1,j}) = [t^{n}_{n-1,j}(\alpha); t^{n}_{n-1,j}(\beta)]^{n-1}_{n,j}$

and

- if $1 \leq i < j \leq n$ then $s^{n+1}_{n,i}([\alpha;\beta]^{n,+}_{n+1,j}) = [s^{n}_{n-1,i}(\alpha); s^{n}_{n-1,i}(\beta)]^{n-1,+}_{n,j-1}$ and $t^{n+1}_{n,i}([\alpha;\beta]^{n,+}_{n+1,j}) = [t^{n}_{n-1,i}(\alpha); t^{n}_{n-1,i}(\beta)]^{n-1,+}_{n,j-1}$
- if $2 \leq j+1 < i \leq n+1$ then

$$s^{n+1}_{n,i}([\alpha;\beta]^{n,+}_{n+1,j}) = [s^{n}_{n-1,i-1}(\alpha); s^{n}_{n-1,i-1}(\beta)]^{n-1,+}_{n,j}$$

  and $t^{n+1}_{n,i}([\alpha;\beta]^{n,+}_{n+1,j}) = [t^{n}_{n-1,i-1}(\alpha); t^{n}_{n-1,i-1}(\beta)]^{n-1,+}_{n,j}$
- if $1 \leq i = j \leq n$ then $s^{n+1}_{n,j}([\alpha;\beta]^{n,+}_{n+1,j}) = s^{n+1}_{n,j+1}([\alpha;\beta]^{n,+}_{n+1,j}) = [s^{n}_{n-1,j}(\alpha); s^{n}_{n-1,j}(\beta)]^{n-1}_{n,j}$ and $t^{n+1}_{n,j}([\alpha;\beta]^{n,+}_{n+1,j}) = \alpha$ and $t^{n+1}_{n,j+1}([\alpha;\beta]^{n,+}_{n+1,j}) = \beta$

also we require the following equalities:

- $\pi_{n+1}([\alpha;\beta]^{n,-}_{n+1,j}) = 1^{n,-}_{n+1,j}(\pi_n(\alpha)) = 1^{n,-}_{n+1,j}(\pi_n(\beta))$ and $\pi_{n+1}([\alpha;\beta]^{n,+}_{n+1,j}) = 1^{n,+}_{n+1,j}(\pi_n(\alpha)) = 1^{n,-+}_{n+1,j}(\pi_n(\beta))$
- $\forall \alpha \in M_n$, $[\alpha,\alpha]^{n,-}_{n+1,j} = 1^{n,-}_{n+1,j}(\alpha)$ and $[\alpha,\alpha]^{n,+}_{n+1,j} = 1^{n,+}_{n+1,j}(\alpha)$.

A morphism of cubical categorical stretchings

$$\mathbb{E} \xrightarrow{(m,c)} \mathbb{E}'$$

is given by commutative squares in ∞-$\mathbb{CM}\mathrm{ag_r}$:

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "\pi"'] & M' \arrow[d, "\pi'"] \\ C \arrow[r, "c"'] & C' \end{tikzcd}$$

such that for all $n \in \mathbb{N}$, and for all $(\alpha,\beta) \in \underline{M}_n$:

$$m_{n+1}([\alpha,\beta]^{n}_{n+1,j})=[m_n(\alpha),m_n(\beta)]^{n}_{n+1,j}\quad (j\in\{1,...,n+1\})$$

and for all $n\in\mathbb{N}$, and for all $(\alpha,\beta)\in\underline{M}_{n,j}$:

$$m_{n+1}([\alpha,\beta]^{n,\gamma}_{n+1,j})=[m_n(\alpha),m_n(\beta)]^{n,\gamma}_{n+1,j}\quad (j\in\{1,...,n\},\gamma\in\{-,+\})$$

The category of cubical categorical stretchings is denoted ∞-$\mathbb{CE}$tC. Consider the forgetful functor:

$$\infty\text{-}\mathbb{CE}\mathrm{tC}\xrightarrow{\quad U\quad}\mathbb{CS}\mathrm{ets}$$

defined on objects by:

$$(M,C,\pi,([-;-]^{n}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n\}},([-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n\};\gamma\in\{-,+\}})\mapsto M$$

Proposition 4.1. The functor U just above is right adjoint, thus produces a monad $\mathbb{W}=(W,\eta,\nu)$ on the category of cubical sets.

The proof is very similar to those in [12, 23]: Actually it is not difficult to see that the category ∞-$\mathbb{CE}$tC and the category $\mathbb{CS}$ets are both projectively sketchable. The sketch of the cubical sets is denoted by $\mathcal{E}_{\mathrm{S}}$ (see 2.1) and the sketch of the cubical categorical stretchings is denoted by $\mathcal{E}_{\mathrm{E}}$. The main parts of this sketch are described just below, and we see that $\mathcal{E}_{\mathrm{E}}$ contains $\mathcal{E}_{\mathrm{S}}$: $\mathcal{E}_{\mathrm{S}}\xhookrightarrow{\quad j\quad}\mathcal{E}_{\mathrm{E}}$, and is such that it induces a forgetful functor $\infty\text{-}\mathbb{CE}\mathrm{tC}\xrightarrow{\quad U\quad}\mathbb{CS}\mathrm{ets}$ such that we have a commutative diagram

$$\begin{tikzcd}
Mod(\mathcal{E}_{\mathrm{E}}) \arrow[r, "Mod(j)"] \arrow[d, "iso"'] & Mod(\mathcal{E}_{\mathrm{S}}) \arrow[d, "iso"] \\
\infty\text{-}\mathbb{CE}\mathrm{tC} \arrow[r, "U"] & \mathbb{CS}\mathrm{ets}
\end{tikzcd}$$

which shows that U is right adjoint by the theorem of Foltz [9].

Actually in Section 3 we described the sketch $\mathcal{E}_{\mathrm{M}}$ of cubical ∞-magmas, which was used to describe in 3.2 the main part of the sketch $\mathcal{E}_{\mathrm{C}}$ of cubical strict ∞-categories. Thus we have already some part of the sketch $\mathcal{E}_{\mathrm{E}}$ that we complete by sketching operations $[-;-]^n_{n+1,j}$ and $[-;-]^{n,\gamma}_{n+1,j}$ plus their axioms. With previous descriptions of sketches, and the one below, we shall see that we obtain the following inclusions of sketches:

$$\mathcal{E}_{\mathrm{S}} \hookrightarrow \mathcal{E}_{\mathrm{M}} \hookrightarrow \mathcal{E}_{\mathrm{C}} \hookrightarrow \mathcal{E}_{\mathrm{E}}$$

- Definition of the sort $\underline{M}_{n,j}$:

  We start with the following four limit cones:

$$\begin{tikzcd}
\underline{M_n} \arrow[r, "\pi^n_1"] \arrow[d, "\pi^n_0"'] & M_n \arrow[d, "\pi_n"] \\
M_n \arrow[r, "\pi_n"'] & Z_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}^s_{n,j} \arrow[r, "\pi^n_{1,s}"] \arrow[d, "\pi^n_{0,s}"'] & M_n \arrow[d, "s^n_{n-1,j}"] \\
M_n \arrow[r, "s^n_{n-1,j}"'] & M_{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M}^t_{n,j} \arrow[r, "\pi^n_{1,t}"] \arrow[d, "\pi^n_{0,t}"'] & M_n \arrow[d, "t^n_{n-1,j}"] \\
M_n \arrow[r, "t^n_{n-1,j}"'] & M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
& M_n \times M_n \arrow[dl, "p^n_0"'] \arrow[dr, "p^n_1"] & \\
M_n & & M_n
\end{tikzcd}$$

  We consider the following commutative diagrams:

$$\begin{tikzcd}
& \underline{M_n} \arrow[ddl, "\pi^n_0"'] \arrow[d, dotted, "j^n"] \arrow[ddr, "\pi^n_1"] & \\
& M_n \times M_n \arrow[dl, "p^n_0"] \arrow[dr, "p^n_1"'] & \\
M_n & & M_n
\end{tikzcd}
\qquad
\begin{tikzcd}
& \underline{M}^s_{n,j} \arrow[ddl, "\pi^n_{0,s}"'] \arrow[d, dotted, "j^{n,s}"] \arrow[ddr, "\pi^n_{1,s}"] & \\
& M_n \times M_n \arrow[dl, "p^n_0"] \arrow[dr, "p^n_1"'] & \\
M_n & & M_n
\end{tikzcd}$$

$$\begin{tikzcd}
 & \underline{M}^t_{n,j} \arrow[ddl, "\pi^n_{0,t}"'] \arrow[d, dotted, "j^{n,t}"] \arrow[ddr, "\pi^n_{1,t}"] & \\
 & M_n \times M_n \arrow[dl, "p^n_0"] \arrow[dr, "p^n_1"'] & \\
M_n & & M_n
\end{tikzcd}$$

which gives the following limit cone (definition of $\underline{M}_{n,j}$):

$$\begin{tikzcd}
 & \underline{M}_{n,j} \arrow[dl] \arrow[d] \arrow[dr] & \\
\underline{M}^s_{n,j} \arrow[dr, "j^{n,s}"'] & \underline{M}^t_{n,j} \arrow[d, "j^{n,t}"] & \underline{M_n} \arrow[dl, "j^n"] \\
 & M_n \times M_n &
\end{tikzcd}$$

which also gives the arrows:

$$\underline{M}_{n,j} \xrightarrow{\quad j \quad} M_n \times M_n$$

We also have the following diagrams:

$$\begin{tikzcd}
 & & M_n \\
\underline{M}_{n,j} \arrow[urr, "q_0^n"] \arrow[r, "j"] \arrow[drr, "q_1^n"'] & M_n \times M_n \arrow[ur, "p_0^n"'] \arrow[dr, "p_1^n"] & \\
 & & M_n
\end{tikzcd}$$

and the operations:

$$\underline{M}_{n,j} \xrightarrow{[-;-]^{n,\gamma}_{n+1,j}} M_{n+1}$$

- Now we have the following commutative diagrams which define the "$s \times s$" for the sorts $\underline{M_n}$:
  - If $1 \leq i < j \leq n+1$
    for sources:

$$\begin{tikzcd}
\underline{M_n} \arrow[rr, "\pi_1^n"] \arrow[dr, dotted, "s^n_{n-1,i} \underset{\pi}{\times} s^n_{n-1,i}"] \arrow[dd, "\pi_0^n"'] & & M_n \arrow[dr, "s^n_{n-1,i}"] & \\
 & \underline{M_{n-1}} \arrow[rr, "\pi_1^{n-1}"] \arrow[dd, "\pi_0^{n-1}"'] & & M_{n-1} \arrow[dd, "\pi_{n-1}"] \\
M_n \arrow[dr, "s^n_{n-1,i}"'] & & & \\
 & M_{n-1} \arrow[rr, "\pi_{n-1}"'] & & Z_{n-1}
\end{tikzcd}$$

for targets:

$$\begin{tikzcd}
\underline{M_n} \arrow[rr, "\pi_1^n"] \arrow[dr, dotted, "t^n_{n-1,i} \underset{\pi}{\times} t^n_{n-1,i}"] \arrow[dd, "\pi_0^n"'] & & M_n \arrow[dr, "t^n_{n-1,i}"] & \\
& \underline{M_{n-1}} \arrow[rr, "\pi_1^{n-1}"] \arrow[dd, "\pi_0^{n-1}"'] & & M_{n-1} \arrow[dd, "\pi_{n-1}"] \\
M_n \arrow[dr, "t^n_{n-1,i}"'] & & & \\
& M_{n-1} \arrow[rr, "\pi_{n-1}"'] & & Z_{n-1}
\end{tikzcd}$$

– If $1 \leq j < i \leq n+1$

for sources:

$$\begin{tikzcd}
\underline{M_n} \arrow[rr, "\pi_1^n"] \arrow[dr, dotted, "s^n_{n-1,i-1} \underset{\pi}{\times} s^n_{n-1,i-1}"] \arrow[dd, "\pi_0^n"'] & & M_n \arrow[dr, "s^n_{n-1,i-1}"] & \\
& \underline{M_{n-1}} \arrow[rr, "\pi_1^{n-1}"] \arrow[dd, "\pi_0^{n-1}"'] & & M_{n-1} \arrow[dd, "\pi_{n-1}"] \\
M_n \arrow[dr, "s^n_{n-1,i-1}"'] & & & \\
& M_{n-1} \arrow[rr, "\pi_{n-1}"'] & & Z_{n-1}
\end{tikzcd}$$

for targets:

$$\begin{tikzcd}
\underline{M_n} \arrow[rr, "\pi_1^n"] \arrow[dr, dotted, "t^n_{n-1,i-1} \underset{\pi}{\times} t^n_{n-1,i-1}"] \arrow[dd, "\pi_0^n"'] & & M_n \arrow[dr, "t^n_{n-1,i-1}"] & \\
& \underline{M_{n-1}} \arrow[rr, "\pi_1^{n-1}"] \arrow[dd, "\pi_0^{n-1}"'] & & M_{n-1} \arrow[dd, "\pi_{n-1}"] \\
M_n \arrow[dr, "t^n_{n-1,i-1}"'] & & & \\
& M_{n-1} \arrow[rr, "\pi_{n-1}"'] & & Z_{n-1}
\end{tikzcd}$$

– If $1 \leq i = j \leq n+1$ we similarly obtain the following arrows:

$$\underline{M_n} \xrightarrow{s_j \times s_j} \underline{M_{n-1}}\ , \qquad \underline{M_n} \xrightarrow{t_j \times t_j} \underline{M_{n-1}}$$

- Now we are going to define the $s \times s$ and the $t \times t$ for $\underline{M}_{n,j}$.

  – We treat only the case $1 \leq i < j \leq n+1$, because the other cases involves similar diagrams:

$$\begin{tikzcd}
\underline{M}^s_{n,j} \arrow[r, "\pi^n_{1,s}"] \arrow[d, "\pi^n_{0,s}"'] \arrow[ddrr, dotted, "s_i \times s_i"] & M_n \arrow[d, "s^n_{n-1,j}"] \arrow[ddrr, "s_i"] & & \\
M_n \arrow[r, "s^n_{n-1,j}"'] \arrow[ddrr, "s_i"'] & M_{n-1} \arrow[ddrr, dotted, "s_i"] & & \\
& & \underline{M}^s_{n-1,j} \arrow[r, "\pi^{n-1}_{1,s}"] \arrow[d, "\pi^{n-1}_{0,s}"'] & M_{n-1} \arrow[d, "s^{n-1}_{n-2,j}"] \\
& & M_{n-1} \arrow[r, "s^{n-1}_{n-2,j}"'] & M_{n-2}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \underline{M}^s_{n,j} \arrow[ddl, "\pi^n_{0,s}"'] \arrow[d, dotted, "j^{n,s}"] \arrow[ddr, "\pi^n_{1,s}"] \arrow[dddr, dotted, "s\times s"] & & \\
 & M_n \times M_n \arrow[dl, "p^n_0"] \arrow[dr, "p^n_1"'] \arrow[dddr, dotted, "s\times s"] & & \\
M_n \arrow[dddr, "s_i"'] & & M_n \arrow[dddr, dotted, "s_i"] & \\
 & & \underline{M}^s_{n-1,j} \arrow[ddl, "\pi^{n-1}_{0,s}"'] \arrow[d, dotted, "j^{n-1,s}"] \arrow[ddr, "\pi^{n-1}_{1,s}"] & \\
 & & M_{n-1} \times M_{n-1} \arrow[dl, "p^{n-1}_0"] \arrow[dr, "p^{n-1}_1"'] & \\
 & M_{n-1} & & M_{n-1}
\end{tikzcd}$$

In this diagram we have:

$$\begin{aligned}
p_0^{n-1} j^{n-1,s} s \times s &= \pi^{n-1}_{0,s} s \times s \\
&= s_i \pi^n_{0,s} \\
&= s_i p^n_0 j^{n-1,s} \\
&= p_0^{n-1} s \times s j^{n,s}
\end{aligned}$$

and also: $p_1^{n-1} j^{n-1,s} s \times s = p_1^{n-1} s \times s j^{n,s}$, thus $j^{n-1,s} s \times s = s \times s j^{n,s}$.
Thus we obtain the following commutative diagram (for $1 \leq i < j \leq n+1$):

$$\begin{tikzcd}
\underline{M}^s_{n,j} \arrow[r, "s_i\times s_i"] \arrow[d, "j^{n,s}"'] & \underline{M}^s_{n-1,j} \arrow[d, "j^{n-1,s}"] \\
M_n\times M_n \arrow[r, "s_i\times s_i"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}$$

and by using similar aguments we also obtain the following commutative diagrams:

$$\begin{tikzcd}
\underline{M}^s_{n,j} \arrow[r, "t_i\times t_i"] \arrow[d, "j^{n,s}"'] & \underline{M}^s_{n-1,j} \arrow[d, "j^{n-1,s}"] \\
M_n\times M_n \arrow[r, "t_i\times t_i"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}^t_{n,j} \arrow[r, "s_i\times s_i"] \arrow[d, "j^{n,t}"'] & \underline{M}^t_{n-1,j} \arrow[d, "j^{n-1,t}"] \\
M_n\times M_n \arrow[r, "s_i\times s_i"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M}^t_{n,j} \arrow[r, "t_i\times t_i"] \arrow[d, "j^{n,t}"'] & \underline{M}^t_{n-1,j} \arrow[d, "j^{n-1,t}"] \\
M_n\times M_n \arrow[r, "t_i\times t_i"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}_{n} \arrow[r, "s_i\times s_i"] \arrow[d, "j^{n}"'] & \underline{M}_{n-1} \arrow[d, "j^{n-1}"] \\
M_n\times M_n \arrow[r, "s_i\times s_i"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M}_{n} \arrow[r, "t_i\times t_i"] \arrow[d, "j^{n}"'] & \underline{M}_{n-1} \arrow[d, "j^{n-1}"] \\
M_n\times M_n \arrow[r, "t_i\times t_i"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}$$

– By using a similar argument for $1\leq j<i\leq n+1$, we obtain the following diagrams:

$$
\begin{tikzcd}
\underline{M}^{s}_{n,j} \arrow[r, "s_{i-1}\times s_{i-1}"] \arrow[d, "j^{n,s}"'] & \underline{M}^{s}_{n-1,j} \arrow[d, "j^{n-1,s}"] \\
M_n \times M_n \arrow[r, "s_{i-1}\times s_{i-1}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}^{s}_{n,j} \arrow[r, "t_{i-1}\times t_{i-1}"] \arrow[d, "j^{n,s}"'] & \underline{M}^{s}_{n-1,j} \arrow[d, "j^{n-1,s}"] \\
M_n \times M_n \arrow[r, "t_{i-1}\times t_{i-1}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\underline{M}^{t}_{n,j} \arrow[r, "s_{i-1}\times s_{i-1}"] \arrow[d, "j^{n,t}"'] & \underline{M}^{t}_{n-1,j} \arrow[d, "j^{n-1,t}"] \\
M_n \times M_n \arrow[r, "s_{i-1}\times s_{i-1}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}^{t}_{n,j} \arrow[r, "t_{i-1}\times t_{i-1}"] \arrow[d, "j^{n,t}"'] & \underline{M}^{t}_{n-1,j} \arrow[d, "j^{n-1,t}"] \\
M_n \times M_n \arrow[r, "t_{i-1}\times t_{i-1}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\underline{M}_{n} \arrow[r, "s_{i-1}\times s_{i-1}"] \arrow[d, "j^{n}"'] & \underline{M}_{n-1} \arrow[d, "j^{n-1}"] \\
M_n \times M_n \arrow[r, "s_{i-1}\times s_{i-1}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}_{n} \arrow[r, "t_{i-1}\times t_{i-1}"] \arrow[d, "j^{n}"'] & \underline{M}_{n-1} \arrow[d, "j^{n-1}"] \\
M_n \times M_n \arrow[r, "t_{i-1}\times t_{i-1}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
$$

- And also for $1 \leq i = j \leq n+1$ we have:

$$
\begin{tikzcd}
\underline{M}^{s}_{n,j} \arrow[r, "s_{j}\times s_{j}"] \arrow[d, "j^{n,s}"'] & \underline{M}^{s}_{n-1,j} \arrow[d, "j^{n-1,s}"] \\
M_n \times M_n \arrow[r, "s_{j}\times s_{j}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}^{s}_{n,j} \arrow[r, "t_{j}\times t_{j}"] \arrow[d, "j^{n,s}"'] & \underline{M}^{s}_{n-1,j} \arrow[d, "j^{n-1,s}"] \\
M_n \times M_n \arrow[r, "t_{j}\times t_{j}"'] & M_{n-1}\times M_{n-1}
\end{tikzcd}
$$

$$\begin{tikzcd} \underline{M}^t_{n,j} \arrow[r, "s_j \times s_j"] \arrow[d, "j^{n,t}"'] & \underline{M}^t_{n-1,j} \arrow[d, "j^{n-1,t}"] \\ M_n \times M_n \arrow[r, "s_j \times s_j"'] & M_{n-1} \times M_{n-1} \end{tikzcd} \qquad \begin{tikzcd} \underline{M}^t_{n,j} \arrow[r, "t_j \times t_j"] \arrow[d, "j^{n,t}"'] & \underline{M}^t_{n-1,j} \arrow[d, "j^{n-1,t}"] \\ M_n \times M_n \arrow[r, "t_j \times t_j"'] & M_{n-1} \times M_{n-1} \end{tikzcd}$$

$$\begin{tikzcd} \underline{M}_{n} \arrow[r, "s_j \times s_j"] \arrow[d, "j^{n}"'] & \underline{M}_{n-1} \arrow[d, "j^{n-1}"] \\ M_n \times M_n \arrow[r, "s_j \times s_j"'] & M_{n-1} \times M_{n-1} \end{tikzcd} \qquad \begin{tikzcd} \underline{M}_{n} \arrow[r, "t_j \times t_j"] \arrow[d, "j^{n}"'] & \underline{M}_{n-1} \arrow[d, "j^{n-1}"] \\ M_n \times M_n \arrow[r, "t_j \times t_j"'] & M_{n-1} \times M_{n-1} \end{tikzcd}$$

Thus with the diagram:

$$
\begin{tikzcd}
 & \underline{M}_{n,j} \arrow[dl] \arrow[d] \arrow[dr] \arrow[dddr, dotted, "s\times s"] & & \\
\underline{M}^s_{n,j} \arrow[dr, "j^{n,s}"'] \arrow[dddr, "s\times s"] & \underline{M}^t_{n,j} \arrow[d, "j^{n,t}"] \arrow[dddr, dotted, "s\times s"] & \underline{M}_n \arrow[dl, "j^n"] \arrow[dddr, dotted, "s\times s"] & \\
 & M_n \times M_n \arrow[dddr] & & \\
 & & \underline{M}_{n-1,j} \arrow[dl] \arrow[d] \arrow[dr] & \\
 & \underline{M}^s_{n-1,j} \arrow[dr, "j^{n-1,s}"'] & \underline{M}^t_{n-1,j} \arrow[d, "j^{n-1,t}"] & \underline{M}_{n-1} \arrow[dl, "j^{n-1}"] \\
 & & M_{n-1} \times M_{n-1} &
\end{tikzcd}
$$

it follows that we obtain:

- for $1 \leq i < j \leq n+1$

$$\underline{M}_{n,j} \xrightarrow{s_i \times s_i} \underline{M}_{n-1,j} \ , \qquad \underline{M}_{n,j} \xrightarrow{t_i \times t_i} \underline{M}_{n-1,j}$$

- for $1 \leq j < i \leq n+1$

$$\underline{M}_{n,j} \xrightarrow{s_{i-1} \times s_{i-1}} \underline{M}_{n-1,j} \ , \qquad \underline{M}_{n,j} \xrightarrow{t_{i-1} \times t_{i-1}} \underline{M}_{n-1,j}$$

– and for $1 \leq i = j \leq n+1$

$$\underline{M}_{n,j} \xrightarrow{s_j \times s_j} \underline{M}_{n-1,j}\ , \qquad \underline{M}_{n,j} \xrightarrow{t_j \times t_j} \underline{M}_{n-1,j}$$

- Now we can state some axioms with the following diagrams:

– If $1 \leq i < j \leq n+1$,

$$\begin{tikzcd} \underline{M}_n \arrow[r, "{[-;-]^n_{n+1,j}}"] \arrow[d, "s^n_{n-1,i} \underset{\pi}{\times} s^n_{n-1,i}"'] & M_{n+1} \arrow[d, "s^n_{n-1,i}"] \\ \underline{M}_{n-1} \arrow[r, "{[-;-]^{n-1}_{n,j-1}}"'] & M_n \end{tikzcd} \qquad \begin{tikzcd} \underline{M}_n \arrow[r, "{[-;-]^n_{n+1,j}}"] \arrow[d, "t^n_{n-1,i} \underset{\pi}{\times} t^n_{n-1,i}"'] & M_{n+1} \arrow[d, "t^n_{n-1,i}"] \\ \underline{M}_{n-1} \arrow[r, "{[-;-]^{n-1}_{n,j-1}}"'] & M_n \end{tikzcd}$$

– If $1 \leq i < j \leq n$,

$$\begin{tikzcd} \underline{M}_{n,j} \arrow[r, "{[-;-]^{n,\gamma}_{n+1,j}}"] \arrow[d, "s^n_{n-1,i} \times s^n_{n-1,i}"'] & M_{n+1} \arrow[d, "s^n_{n-1,i}"] \\ \underline{M}_{n-1} \arrow[r, "{[-;-]^{n-1,\gamma}_{n,j-1}}"'] & M_n \end{tikzcd} \qquad \begin{tikzcd} \underline{M}_{n,j} \arrow[r, "{[-;-]^{n,\gamma}_{n+1,j}}"] \arrow[d, "t^n_{n-1,i} \times t^n_{n-1,i}"'] & M_{n+1} \arrow[d, "t^n_{n-1,i}"] \\ \underline{M}_{n-1} \arrow[r, "{[-;-]^{n-1,\gamma}_{n,j-1}}"'] & M_n \end{tikzcd}$$

– If $1 \leq j < i \leq n+1$,

$$\begin{tikzcd} \underline{M}_n \arrow[r, "{[-;-]^n_{n+1,j}}"] \arrow[d, "s^n_{n-1,i-1} \underset{\pi}{\times} s^n_{n-1,i-1}"'] & M_{n+1} \arrow[d, "s^n_{n-1,i}"] \\ \underline{M}_{n-1} \arrow[r, "{[-;-]^{n-1}_{n,j}}"'] & M_n \end{tikzcd}$$

$$\begin{tikzcd}
\underline{M_n} \arrow[r, "{[-;-]^n_{n+1,j}}"] \arrow[d, "t^n_{n-1,i-1}\underset{\pi}{\times} t^n_{n-1,i-1}"'] & M_{n+1} \arrow[d, "t^n_{n-1,i}"] \\
\underline{M_{n-1}} \arrow[r, "{[-;-]^{n-1}_{n,j}}"'] & M_n
\end{tikzcd}$$

– If $2 \leq j+1 < i \leq n+1$,

$$\begin{tikzcd}
\underline{M}_{n,j} \arrow[r, "{[-;-]^{n,\gamma}_{n+1,j}}"] \arrow[d, "s^n_{n-1,i-1}\times s^n_{n-1,i-1}"'] & M_{n+1} \arrow[d, "s^n_{n-1,i}"] \\
\underline{M}_{n-1,j} \arrow[r, "{[-;-]^{n-1,\gamma}_{n,j}}"'] & M_n
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M}_{n,j} \arrow[r, "{[-;-]^{n,\gamma}_{n+1,j}}"] \arrow[d, "t^n_{n-1,i-1}\times t^n_{n-1,i-1}"'] & M_{n+1} \arrow[d, "t^n_{n-1,i}"] \\
\underline{M}_{n-1,j} \arrow[r, "{[-;-]^{n-1,\gamma}_{n,j}}"'] & M_n
\end{tikzcd}$$

– If $1 \leq i = j \leq n+1$,
for the operations $[-;-]^n_{n+1,j}$:

$$\begin{tikzcd}
\underline{M_n} \arrow[rr, "{[-;-]^n_{n+1,j}}"] \arrow[dr, "\pi^n_0"'] & & M_{n+1} \arrow[dl, "s^{n+1}_{n,j}"] \\
& M_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M_n} \arrow[rr, "{[-;-]^n_{n+1,j}}"] \arrow[dr, "\pi^n_1"'] & & M_{n+1} \arrow[dl, "t^{n+1}_{n,j}"] \\
& M_n &
\end{tikzcd}$$

for the operations $[-;-]^{n,\gamma}_{n+1,j}$ (if $1 \leq i = j \leq n$):

$$\begin{tikzcd}
\underline{M}_{n,j} \arrow[rr, "{[-;-]^{n,-}_{n+1,j}}"] \arrow[dr, "q_0^n"'] & & M_{n+1} \arrow[dl, "s^{n+1}_{n,j}"] \\
& M_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}_{n,j} \arrow[rr, "{[-;-]^{n,-}_{n+1,j}}"] \arrow[dr, "q_1^n"'] & & M_{n+1} \arrow[dl, "s^{n+1}_{n,j+1}"] \\
& M_n &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M}_{n,j} \arrow[r, "{[-;-]^{n,-}_{n+1,j}}"] \arrow[d, "t^n_{n-1,j}\times t^n_{n-1,j}"'] & M_{n+1} \arrow[d, "{t^n_{n-1,j},\, t^n_{n-1,j+1}}"] \\
\underline{M}_{n-1,j} \arrow[r, "{[-;-]^{n-1,-}_{n,j}}"'] & M_n
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M}_{n,j} \arrow[rr, "{[-;-]^{n,+}_{n+1,j}}"] \arrow[dr, "q_0^n"'] & & M_{n+1} \arrow[dl, "t^{n+1}_{n,j}"] \\
& M_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M}_{n,j} \arrow[rr, "{[-;-]^{n,+}_{n+1,j}}"] \arrow[dr, "q_1^n"'] & & M_{n+1} \arrow[dl, "t^{n+1}_{n,j+1}"] \\
& M_n &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M}_{n,j} \arrow[r, "{[-;-]^{n,+}_{n+1,j}}"] \arrow[d, "s^n_{n-1,j}\times s^n_{n-1,j}"'] & M_{n+1} \arrow[d, "{s^n_{n-1,j},\, s^n_{n-1,j+1}}"] \\
\underline{M}_{n-1,j} \arrow[r, "{[-;-]^{n-1,+}_{n,j}}"'] & M_n
\end{tikzcd}$$

Other diagrams for axioms:

– For operations $[-;-]^{n}_{n+1,j}$:

$$\begin{tikzcd} \underline{M}_n \arrow[r, "{[-;-]^{n}_{n+1,j}}"] \arrow[d, "\pi^n_0"'] & M_{n+1} \arrow[dd, "\pi_{n+1}"] \\ M_n \arrow[d, "\pi_n"'] & \\ Z_n \arrow[r, "1^{n}_{n+1,j}"'] & Z_{n+1} \end{tikzcd} \qquad \begin{tikzcd} \underline{M}_n \arrow[r, "{[-;-]^{n}_{n+1,j}}"] \arrow[d, "\pi^n_1"'] & M_{n+1} \arrow[dd, "\pi_{n+1}"] \\ M_n \arrow[d, "\pi_n"'] & \\ Z_n \arrow[r, "1^{n}_{n+1,j}"'] & Z_{n+1} \end{tikzcd}$$

– For operations $[-;-]^{n,\gamma}_{n+1,j}$:

$$\begin{tikzcd} \underline{M}_{n,j} \arrow[r, "{[-;-]^{n,\gamma}_{n+1,j}}"] \arrow[d, "q^n_0"'] & M_{n+1} \arrow[dd, "\pi_{n+1}"] \\ M_n \arrow[d, "\pi_n"'] & \\ Z_n \arrow[r, "1^{n,\gamma}_{n+1,j}"'] & Z_{n+1} \end{tikzcd} \qquad \begin{tikzcd} \underline{M}_{n,j} \arrow[r, "{[-;-]^{n,\gamma}_{n+1,j}}"] \arrow[d, "q^n_1"'] & M_{n+1} \arrow[dd, "\pi_{n+1}"] \\ M_n \arrow[d, "\pi_n"'] & \\ Z_n \arrow[r, "1^{n,\gamma}_{n+1,j}"'] & Z_{n+1} \end{tikzcd}$$

- The goal of the diagrams below is to exhibit the diagonal map:

$$M_n \xrightarrow{\delta^n_\gamma} \underline{M}_{n,j}$$

This diagonal is built with the following diagonals:

$$\begin{tikzcd}
M_n \arrow[dr, dotted, "\delta^n"] \arrow[drr, "id"] \arrow[ddr, "id"'] & & \\
& \underline{M_n} \arrow[r, "\pi_1^n"] \arrow[d, "\pi_0^n"] & M_n \arrow[d, "\pi_n"] \\
& M_n \arrow[r, "\pi_n"'] & Z_n
\end{tikzcd}$$

$$\begin{tikzcd}
M_n \arrow[dr, dotted, "\delta_s^n"] \arrow[drr, "id"] \arrow[ddr, "id"'] & & \\
& \underline{M}^s_{n,j} \arrow[r, "\pi_1^{n,s}"] \arrow[d, "\pi_0^{n,s}"] & M_n \arrow[d, "s^n_{n-1,j}"] \\
& M_n \arrow[r, "s^n_{n-1,j}"'] & M_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_n \arrow[dr, dotted, "\delta_t^n"] \arrow[drr, "id"] \arrow[ddr, "id"'] & & \\
& \underline{M}^t_{n,j} \arrow[r, "\pi_1^{n,t}"] \arrow[d, "\pi_0^{n,t}"] & M_n \arrow[d, "t^n_{n-1,j}"] \\
& M_n \arrow[r, "t^n_{n-1,j}"'] & M_{n-1}
\end{tikzcd}$$

Thus we get the following diagram:

$$\begin{tikzcd}
& \underline{M}_{n,j} \arrow[dl] \arrow[d] \arrow[dr] & & M_n \arrow[ll, dotted, "\delta_\gamma^n"'] \arrow[dlll, "\delta_s^n"] \arrow[dll, "\delta_t^n"] \arrow[dl, "\delta^n"] \\
\underline{M}^s_{n,j} \arrow[dr, "j^{n,s}"'] & \underline{M}^t_{n,j} \arrow[d, "j^{n,t}"] & \underline{M_n} \arrow[dl, "j^n"] & \\
& M_n \times M_n \arrow[d, shift right, "p_0^n"'] \arrow[d, shift left, "p_1^n"] & & \\
& M_n & &
\end{tikzcd}$$

The existence of the arrow δ^n_γ comes from the following fact:

$$\begin{aligned} p^n_0 j^{n,s}\delta^n_s &= \pi^n_{0,s}\delta^n_s \\ &= id \end{aligned}$$

and also

$$\begin{aligned} p^n_0 j^{n,t}\delta^n_t &= \pi^n_{0,t}\delta^n_t \\ &= id \end{aligned}$$

and we have $p^n_1 j^{n,s}\delta^n_s = p^n_1 j^{n,s}\delta^n_s$, which shows the equality:

$$j^{n,s}\delta^n_s = j^{n,t}\delta^n_t$$

We similarly show the equality $j^{n,t}\delta^n_t = j^n\delta^n$. Thus the existence of δ^n_γ.

- Then we obtain the following commutative diagrams which express the axioms of reflexivity of the operations $[-;-]^n_{n+1,j}$:

$$\begin{tikzcd} \underline{M}_n \arrow[rr, "{[-;-]^n_{n+1,j}}"] & & M_{n+1} \\ M_n \arrow[u, "\delta^n"] \arrow[urr, "1^n_{n+1}"'] & & \end{tikzcd}$$

and the following commutative diagrams which express the axioms of reflexivity of the operations $[-;-]^{n,\gamma}_{n+1,j}$:

$$\begin{tikzcd} \underline{M}_{n,j} \arrow[rr, "{[-;-]^{n,\gamma}_{n+1,j}}"] & & M_{n+1} \\ M_n \arrow[u, "\delta^n_\gamma"] \arrow[urr, "1^{n,\gamma}_{n+1}"'] & & \end{tikzcd}$$

Definition 4.2. Cubical weak ∞-categories with connections are algebras for the monad $\mathbb{W}$ above.

Let us show with a simple example how cubical weak ∞-categories provide a richer weakened structure than globular weak ∞-categories: for simplicity we show it inside an object $\mathbb{E}$ of ∞-$\mathbb{C}\mathbb{E}$tC:

$$\mathbb{E} = (M, C, \pi, ([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n+1\}}, ([-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n\};\gamma\in\{-,+\}})$$

Consider the following string in $M(1)$:

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

and take the 1-cells $x = (h\circ g)\circ f$ and $y = h\circ(g\circ f)$. Because $(x,y)\in \underline{M}_1\cap\underline{M}_{1,0}$ we get the following 2-cells:

$$\begin{tikzcd} a \arrow[r, "(h\circ g)\circ f"] \arrow[d, "1_a"'] \arrow[dr, phantom, "{[x,y]^1_{2,1}}"] & d \arrow[d, "1_d"] \\ a \arrow[r, "h\circ(g\circ f)"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "(h\circ g)\circ f"'] \arrow[dr, phantom, "{[x,y]^1_{2,2}}"] & a \arrow[d, "h\circ(g\circ f)"] \\ d \arrow[r, "1_d"'] & d \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "(h\circ g)\circ f"] \arrow[d, "h\circ(g\circ f)"'] \arrow[dr, phantom, "{[x,y]^{1,-}_{2,1}}"] & d \arrow[d, "1_d"] \\ d \arrow[r, "1_d"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] \arrow[dr, phantom, "{[x,y]^{1,+}_{2,1}}"] & d \arrow[d, "h\circ(g\circ f)"] \\ d \arrow[r, "(h\circ g)\circ f"'] & d \end{tikzcd}$$

Remark 4.3. We could have defined cubical categorical stretchings slightly differently than those above by using just the operations:

$$([-;-]^n_{n+1,j})_{n\in\mathbb{N};j\in\{1,\ldots,n\}}$$

to weakened the structure of cubical strict ∞-categories. Denote by ∞-$\mathbb{C}\mathbb{E}$tC$'$ the category of these slightly impoverished structures. We also have a forgetful functor:

$$\infty\text{-}\mathbb{C}\mathbb{E}\text{tC}' \xrightarrow{\quad U' \quad} \mathbb{C}\mathbb{S}\text{ets}$$

which is right adjoint and which produces another monad $\mathbb{W}' = (W', \eta', \nu')$ whose algebras could be also considered as interesting models of cubical weak ∞-categories without incorporating the structure of connections. Also we have an evident forgetful functor:

$$\infty\text{-}\mathbb{C}\mathbb{E}\text{tC} \xrightarrow{\quad U \quad} \infty\text{-}\mathbb{C}\mathbb{E}\text{tC}'$$

which is right adjoint and which produces a functor:

$$\mathbb{W}\text{-}\mathbb{A}\text{lg} \xrightarrow{\quad U \quad} \mathbb{W}\text{-}\mathbb{A}\text{lg}'$$

which shows that the models that we have chosen for our article are also models for these impoverished structures. And our choice to add operations:

$$([-;-]^{n,\gamma}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n\};\gamma\in\{-,+\}}$$

to get our models of cubical weak ∞-categories is similar to the choice of cubical strict ∞-categories with, rather than without, connections. We believe that our choice gives not only more refined models than those of the category $\mathbb{W}\text{-}\mathbb{A}\text{lg}'$ but also is necessary for a good approach to cubical weak ∞-categories, where formalism of connections is implicit and used in our weakened structures.

Remark 4.4. In [5] the authors have proved that the category of cubical strict ∞-categories is equivalent to the category of globular strict ∞-categories. We suspect that such a phenomenon is still true in the world of weak models. Let us be more precise about what we are saying: denote by $\mathbb{P}$ the Penon monad on the category of globular sets (see [23]) whose algebras are particularly nice models of globular weak ∞-categories (see for example [3, 7]). It is suspected (see [25]) that its category of algebras $\mathbb{P}\text{-}\mathbb{A}\text{lg}$ can be equipped with a *canonical Quillen model structure* similar to the one described in [20] for strict globular ∞-categories, and we also suspect that $\mathbb{W}\text{-}\mathbb{A}\text{lg}$ can be equipped with such a canonical Quillen model structure. Thus a weak version of the article [5] should be that the category $\mathbb{P}\text{-}\mathbb{A}\text{lg}$ is Quillen equivalent to $\mathbb{W}\text{-}\mathbb{A}\text{lg}$ when these categories are equipped with their canonical Quillen model structures.

4.2 Magmatic properties of cubical weak ∞-categories Consider a cubical weak ∞-category $W(C) \xrightarrow{v} C$. In this monadic presentation, $W(C)$ has to be thought as the free cubical weak ∞-category representing the underlying syntax with which all algebras with underlying cubical set C are interpretations of it via their structural morphisms. For example here v is the structural morphism which plays the role of interpreting in C the "syntax" $W(C)$, and thus puts a structure of $\mathbb{W}$-algebra on C. We shall well distinguish notations of operation inside $W(C)$ and inside C in order to separate the syntactic part from the model part of our algebras. For example, the operations of composition shall be denoted $\circ^n_j$ in the models, whereas we shall use the notation $\star^n_j$ instead when we work in the free models. The reflexions are denoted $\iota^n_{n+1,j}$ in the models and $1^n_{n+1,j}$ in the free models. The connections are denoted $\iota^{n,\alpha}_{n+1,j}$ in the models and $1^{n,\alpha}_{n+1,j}$ in the free models. Definitions of operations for models use those for free models and the interpretative nature of $W(C) \xrightarrow{v} C$ emerges then with the axiomatic of algebras for monads: for example we consider first the following definition of operations on C:

(i) If a, $b \in C(n)$ are such that $s^n_j(b) = t^n_j(a)$ for $j \in \{1, ..., n\}$ then we put $a \circ^n_j b = v_n(\eta(a) \star^n_j \eta(b))$

(ii) If $a \in C(n)$ is an n-cell then we put $\iota^n_{n+1,j}(a) = v_{n+1}(1^n_{n+1,j}(\eta(a)))$, $n \in \mathbb{N}$, $j \in [\![1, n+1]\!]$

(iii) If $a \in C(n)$ is an n-cell then we put $\iota^{n,\gamma}_{n+1,j}(a) = v_{n+1}(1^{n,\gamma}_{n+1,j}(\eta(a)))$, $n \geq 1$, $j \in [\![1, n]\!]$, $\gamma \in \{-, +\}$

Thus v puts on C a cubical ∞-magma structure and its interpretative nature is primarily expressed by the fact that it is a morphism of cubical ∞-magmas between the free cubical ∞-magma $W(C)$ and this cubical ∞-magma on C. It is the axioms of algebras which show us such an important fact: actually we need to show that $v(a \star^n_j b) = v(a) \circ^n_j v(b)$, $v(1^n_{n+1,j}(a)) = \iota^n_{n+1,j}(v(a))$,

$v(1^{n,\gamma}_{n+1,j}(a)) = \iota^{n,\gamma}_{n+1,j}(v(a))$. Let us show the first equality:

$$\begin{aligned}
v(a) \circ^n_j v(b) &= v(\eta(v(a)) \star^n_j \eta(v(b))) \\
&= v(W(v)\eta_{W(C)}(a) \star^n_j W(v)\eta_{W(C)}(b)) \\
&= v(W(v)(\eta_{W(C)}(a) \star^n_j \eta_{W(C)}(b))) \\
&= v(\nu(C)(\eta_{W(C)}(a) \star^n_j \eta_{W(C)}(b))) \\
&= v(\nu(C)(\eta_{W(C)}(a)) \star^n_j \nu(C)(\eta_{W(C)}(b))) \\
&= v(a \star^n_j b)
\end{aligned}$$

Other equalities are shown similarly. In [23] J.Penon called such properties of algebras *magmatic*. In particular these shall be useful for concrete computations in any $\mathbb{W}$-algebras.

4.3 Computations for low dimensions

Definition 4.5. Consider a reflexive cubical set $C \in \mathbb{C}_{\mathrm{r}}\mathbb{S}\mathrm{ets}$. It has dimension $p \in \mathbb{N}$ for reflexions if all its q-cells $x \in C(q)$ for which $q > p$ are of the form $x = 1^{q-1}_{q,j}(y)$ and if there is at least one p-cell which is not of this form. It has dimension $p \in \mathbb{N}$ for connections if all its q-cells $x \in C(q)$ for which $q > p$ are of the form $x = 1^{q-1,\gamma}_{q,j}(y)$ and if there is at least one p-cell which is not of this form. It has dimension $p \in \mathbb{N}$, if it has dimension $p \in \mathbb{N}$ for reflexions and connections.

Definition 4.6. Consider a $\mathbb{W}$-algebra (C, v). It has dimension $p \in \mathbb{N}$ for reflexion if its underlying reflexive set produced by its underlying ∞-magma structure (see 4.2) has dimension $p \in \mathbb{N}$ for reflexion. It has dimension $p \in \mathbb{N}$ for connection if its underlying reflexive set produced by its underlying ∞-magma structure has dimension $p \in \mathbb{N}$ for connections. It has dimension $p \in \mathbb{N}$, if it has dimension $p \in \mathbb{N}$ for reflexions and connections.

In [26] the author had defined weak double categories, also known as cubical bicategories or cubical weak 2-category. We suspect that they are 2-dimensional $\mathbb{W}$-algebras. However it is possible to show that 2-dimensional $\mathbb{W}$-algebras are weak double categories in the sense of [26]. The proof uses the magmatic properties of $\mathbb{W}$-algebras explained in 4.2 but is rather long. We leave it as an exercise for the reader.

Acknowledgement

I thank mathematicians of my team AGA[11], and the good ambience provided by the team in the lab, especially I want to mention Elisabeth Gassiat, Olivier Schiffmann, Christophe Breuil, Valentin Hernandez, Philip Boalch, François Charles and Benjamin Hennion. I want also to mention mathematicians who help me a lot: Maxim Kontsevich, Pierre Cartier, Ronnie Brown, Vasily Pestun, Jordi Lopez-Abad, Bertrand Monthubert, Ghislain Rémy, Michael Batanin, Ross Street and Mark Weber. I thank mathematicians of the IRIF where I started to write this paper[12]. Especially I want to mention Paul-André Méllies, Mai Gehrke, Pierre Louis Curien and Thomas Ehrhard.

Finally I want to mention other wonderful persons: Stéf Bonnot-Briey, Frédérique Vidal, Sophie Cluzel, Jean-Philippe Bourgoin, Alexandra Van Cauteren, Françoise Dorocq and Jean-Pierre Ledru.

I dedicate this work to my daughters, Leïli and Amina-Tassadit, with Love.

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