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A new approach to tensor product of hypermodules

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Abstract. As an essential tool in homological algebra, tensor products play a basic role in classifying and studying modules. Since hypermodules are generalization of modules, it is important to generalize the concept of the tensor products of modules to the hypermodules. In this paper, in order to achieve this goal, we present a more general form of the definition of hypermodule. Based on this new definition, some of the required concepts and properties have been studied. By obtaining a free object in the category of hypermodules, the notion of tensor product of hypermodules is provided and some of its properties are studied.

1 Introduction and preliminaries

In the 8th Congress of Scandinavian Mathematicians in 1934, Marty introduced the hyperstructure theory, see [14], and stated that, the hypergroup is the generalization of groups. Marty showed that the characteristics of hypergroup can be used in solving some problems of groups, algebraic functions, and rational fractions. Surveys of the theory can be found in [8, 10].

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In the past few years, the hyperstructure theory has been studied in connection with various fields. Connes and Consani [4–6] have shown that there is a relationship between the hyperstructure theory to algebraic geometry and number theory

In [15], an interesting definition of homomorphism between hypermodules to get a free object presented. This new approach to the definition of homomorphism shows that, considering the appropriate morphism of hypermodules is very important and helpful in the categorical study. In [12, 17, 18], categories of hyper algebraic structures introduced and studied. The main purpose of presenting these categories was to obtain a free object by creating adjoint pairs. In [16], a new approach to the categorical study of hypermodules presented. Although the definition of hypermodule in [16] is not useful for this paper, we use the approach presented in Section 4 of that paper to define the homomorphism. This new definition of the homomorphism between hypermodules is very useful for the main purpose of this paper, which is to provide a definition of the tensor product of hypermodules.

In [2, 9] the authors tried to provide a definition of the tensor product of the (especial collections of) hyperstructures. In these references, the proposed methods for the definition of tensor products are not desirable for the following reasons.

In [2] the authors introduce and study tensor product of hypervector spaces based on Tallini hypervector spaces. In [19], Tallini defines the hypervector spaces as follows. Let K be a field and $(V,+)$ be an abelian group. A quadruplet $(V,+,\circ,K)$ is hypervector space over K, whenever $\circ : K \times V \longrightarrow \mathcal{P}^*(V)$, is a map such that the following conditions hold (for all $x, y \in V$ and $a, b \in K$):

(H$_1$) $a \circ (x+y) \subseteq a \circ x + a \circ y$;

(H$_2$) $(a+b) \circ x \subseteq a \circ x + b \circ x$;

(H$_3$) $a \circ (b \circ x) = (ab) \circ x$;

(H$_4$) $x \in 1 \circ x$.

Note that $\mathcal{P}^*(V)$ is the set of all the non-empty subsets of V. In fact, Tallini generalize the scalar multiplication of the vector space and the authors in [2] by focusing to the this special hyperstructures define tensor product. But the important point in [2] is that for the hypervector spaces V and W the map $i : V \times W \longrightarrow V \otimes_K W$ introduced in [2, page 713] is not a middle linear map. Because, by [2, Definition 16] middle linear maps are defined for vector spaces. Therefore, the presented approach in [2] to define the tensor product is not desirable.

In [9, Definition 1] the authors give the notions of left and right R-operator for a general hyperring R as follows. Let $(R, +, \cdot)$ be a general hyperring and X be a nonempty set. X is called a left R-operator if there is an external hyperoperation $\circ : R \times X \longrightarrow \mathcal{P}^*(X)$ satisfying $(r_1 \cdot r_2) \circ x = r_1 \circ (r_2 \circ x)$, for every $r_1, r_2 \in R$ and $x \in X$. Dually, the right R-operator is defined. In [9, Definition 2] the notion of (R_1, R_2)-cooperator as a left R_1-operator and right R_2-operator that satisfies the relation $(r_1 \circ x) \circ r_2 = r_1 \circ (x \circ r_2)$, for $r_1 \in R_1, r_2 \in R_2$ and $x \in X$ is presented. Accordingly, in the [9, Definition 4] for (R_1, R_2)-, (R_2, R_3)- and (R_1, R_3)-cooperators X, Y and Z, respectively, an (R_1, R_3)-map is defined as a map $\beta : X \times Y \longrightarrow Z$ that satisfies the relation $\beta(x \circ r, y) = \beta(x, r \circ y)$ for $x \in X, y \in Y$ and $r \in R_2$. Finally, in [9, Definition 5] for (R_1, R_2)-cooperator and (R_2, R_3)-cooperator X and Y, respectively, the authors give the notion of tensor product of X and Y over the hyperring R_2 as a pair (P, ψ). In this definition P is an (R_1, R_3)-cooperator and $\psi : X \times Y \longrightarrow P$ is an (R_1, R_3)-map such that for each (R_1, R_3)-cooperator Z and (R_1, R_3)-map $\beta : X \times Y \longrightarrow Z$, there exists a unique (R_1, R_3)-map $\bar{\beta} : P \longrightarrow Z$ satisfying $\bar{\beta} \circ \psi = \beta$. At the end of [9] for future work the authors state the next step by defining the tensor product based on the R-operators. What needs to be considered here is that, even if R is a ring, the tensor product defined by R-operators cannot be matched to the tensor product in the algebraic one. First of all, let X be an (R, S)-cooperator by defining maps $\circ : R \times X \longrightarrow \mathcal{P}^*(X)$ and $\circ' : X \times S \longrightarrow \mathcal{P}^*(X)$ satisfying the identities $r \circ x = \{x\}$ and $x \circ' s = \{x\}$ for $x \in X, r \in R$ and $s \in S$, respectively. Similarly, consider a set Y as (S, R)-cooperator. Let Z be any (R, R)-cooperator. It can be easily seen that each map $X \times Y \longrightarrow Z$ is a (R, R)-map. Now let (P, ψ) be the tensor product of X and Y over S. As before we can consider the set $X \times Y$ as an (R, R)-cooperator. Thus, the identity map over $X \times Y$ is a (R, R)-map and hence by definition of tensor product there exists a unique (R, R)-map $\bar{\beta} : P \longrightarrow X \times Y$ such that $\bar{\beta} \circ \psi = id_{X \times Y}$. This implies that $\mathrm{card}\,(X \times Y) \leq \mathrm{card}\, P$. Therefore, if we assume $R = S$ as the ring of integer numbers $\mathbb{Z}$, $X = \mathbb{Q}$ and $Y = \mathbb{Z}_n$, then the tensor product of $\mathbb{Q}$ and $\mathbb{Z}_n$ over

$\mathbb{Z}$ which described as above is an infinite set, while in the algebraic case is a zero group. Therefore, the method used to define the tensor product in [9] is not desirable.

In this article, we are looking to provide a suitable way to define a tensor product for hypermodules. First, we will give some basic definitions which will be used in the following sections, see [8] for more information. A *hypergroupoid* is a non-empty set H together with a mapping $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$. A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that for all a, b and c in H we have $(a \circ b) \circ c = a \circ (b \circ c)$. Let $(H, \circ)$ be a hypergroupoid. An element $e \in H$ is called an *identity* if for all $a \in H$, $a \in a \circ e \cap e \circ a$. A *quasihypergroup* is a hypergroupoid $(H, \circ)$ which satisfies the reproductive law, i.e., for all $a \in H$, $H \circ a = a \circ H = H$. A *hypergroup* is a semihypergroup which is also a quasihypergroup.

Suppose that $(H, \circ)$ is a semihypergroup and X is a nonempty subset of H. The semihypergroup generated by X is denoted by $< X >$ and we have, $< X >= \bigcup\{x_1 \circ ... \circ x_k \mid x_i \in X, k \in \mathbb{N}\}$. A non-empty subset A of H is called a *complete part* of H if for all $n \geqslant 2$ and for all $(x_1, x_2, ..., x_n) \in H^n$ we have the following implication:

$$\prod_{i=1}^{n} x_i \bigcap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A.$$

If $(H, \circ)$ is a semihypergroup and $\rho \subseteq H \times H$ is an equivalence, we set

$$A \overline{\overline{\rho}} B \Leftrightarrow a\rho b, \forall a \in A, \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of H^2. The relation ρ is called *strongly regular on the left* (*on the right*) if

$$x\rho y \Rightarrow a \circ x \overline{\overline{\rho}} a \circ y \; (x\rho y \Rightarrow x \circ a \overline{\overline{\rho}} y \circ a)$$

for all $(x, y, a) \in H^3$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 1.1. (Theorem 31, [7]). *If $(H, \cdot)$ is a semihypergroup (hypergroup) and ρ is a strongly regular relation on H, then the quotient H/ρ is a semigroup (group) under the operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \text{ for all } z \in x \cdot y.$$

For all $n > 1$ define the relation β_n on a semihypergroup H, as follows:

$$a\,\beta_n\,b \Leftrightarrow \exists(x_1,...,x_n) \in H^n \;:\; \{a,b\} \subseteq \prod_{i=1}^{n} x_i$$

and $\beta = \bigcup_{i=1}^{n} \beta_n$, where $\beta_1 = \{(x,x) \mid x \in H\}$ is the diagonal relation H. Let β^* be the transitive closure of β. If H is a hypergroup, then $\beta = \beta^*$, see [7, Theorem 81]. Let H be a hypergroup and $\pi : H \longrightarrow H/\beta$ be the canonical projection. The *heart* of H, which is denoted by ω_H, is the inverse image through π of the identity of the group H/β, that is, $\omega_H = \pi^{-1}(e_{H/\beta})$. If B is a non-empty subset of H, then $\omega_H \circ B = B \circ \omega_H = \mathscr{C}(B)$, where $\mathscr{C}(B)$ is the complete closure of B, see [7, Theorem 67].

A *quasicanonical* hypergroup (not necessarily commutative) is an algebraic structure $(H,+)$ satisfying the following conditions:

(i) for every $x,y,z \in H, x+(y+z) = (x+y)+z$;

(ii) there exists an element $0 \in H$ such that $0 + x = x$, for all $x \in H$;

(iii) for every $x \in H$; there exists a unique element $x' \in H$ such that $0 \in (x+x') \cap (x'+x)$ (we denote it by $-x$ and call it the opposite of x);

(iv) $z \in x+y$ implies that $y \in -x+z$ and $x \in z-y$.

We note that, if $x \in H$ and A, B are non-empty subsets of H, then by $-A$, $A+B$, $A+x$ and $x+B$ we mean that $-A = \{-a \mid a \in A\}$, $A+B = \bigcup\{a+b \mid a \in A, b \in B\}, A+x = A+\{x\}$ and $x+B = \{x\}+B$, respectively. Also, for all $x,y \in H$, we have $-(-x) = x$, $-0 = 0$, and $-(x+y) = -y-x$. A canonical hypergroup is a commutative quasicanonical hypergroup.

Remark 1.2. Let G be a canonical hypergroup and H be a canonical subhypergroup which is a complete part. Define the relation $\overset{H}{\equiv}$ on G by

$$\forall x,y \in G,\; x \overset{H}{\equiv} y \text{ iff } x-y \subseteq H.$$

Thus $\overset{H}{\equiv}$ is a strongly regular relation. Put $G/H := G/\overset{H}{\equiv}$ and $[x]$ is the equivalence class of $x \in G$. Thus $(G/H, \oplus)$ is an abelian group. Let $x+'y := \{[x]\oplus[y]\}$ for every $x,y \in G$. The canonical hypergroup $(G/H,+')$ is called *quotient canonical hypergroup*.

Let $(M, +)$ be a canonical hypergroups and

$$M(0) := \bigcup \{ \sum_{i=1}^{k} (a_i - a_i) \mid a_i \in G, k \in \mathbb{N} \},$$

that is, $M(0)$ is the subset of M which is the union of all sums of type zero. Thus $\omega_M = M(0)$, see [7, Theorem 131]. Also we have

$$\forall x, y \in M,\ x \overset{M(0)}{\equiv} y \text{ iff } x - y \subseteq \sum_{i=1}^{k} (a_i - a_i) \text{ for some } k \in \mathbb{N} \text{ and } a_i \in M.$$

Because, let $x \overset{M(0)}{\equiv} y$ and $z \in x - y$, so $z \in \sum_{i=1}^{k} (b_i - b_i)$ for some $b_i \in M$. Thus $x \in z + y \subseteq \sum_{i=1}^{k} (b_i - b_i) + y$ and hence $x - y \subseteq \sum_{i=1}^{k} (b_i - b_i) + (y - y)$.

Numerous definitions of homomorphisms between hypergroups are given in [7]. In the following we give a new definition of homomorphism between hypergroups. This perspective on defining homomorphism is very helpful in the categorical study of hypermodules and tensor product of hypermodules presented in Sections 2 and 3.

Definition 1.3. Suppose that $(H, \circ)$ and $(H', \circ)$ are two hypergroups. A *β-homomorphism* $f : H \xrightarrow{\beta} H'$ is a function $f : H \longrightarrow H'/\beta'$ such that for each $x, y \in H$ and $z \in x \circ y$, $f(z) = f(x) \otimes f(y)$.

Example 1.4. Let $(H, \circ)$ and $(H', \circ)$ be two hypergroups. In [7] a *homomorphism* from H to H' is defined as a function $f : H \longrightarrow H'$ satisfies in the following condition:

$$\forall (x, y) \in H^2, \quad f(x \circ y) \subseteq f(x) \circ' f(y).$$

If $f(x \circ y) = f(x) \circ' f(y)$, then f is called a *good homomorphism*. Suppose that $\pi' : H' \longrightarrow H'/\beta'$ is the canonical projection on H'. Therefore the function $\pi' f : H \longrightarrow H'/\beta'$ defines a β-homomorphism $\pi' f : H \xrightarrow{\beta} H'$, where $f : H \longrightarrow H'$ is a homomorphism of hypergroups.

Proposition 1.5. *Let $f : M \xrightarrow{\beta} N$ be a β-homomorphism between canonical hypergroups. If $x \in \beta(x')$, then $f(x) = f(x')$.*

Proof. Let $x - x' \subseteq \sum_{i=1}^{k}(m_i - m_i)$ and $t \in x - x'$. Therefore for each i there exists $a_i \in m_i - m_i$ such that $t \in \sum_{i=1}^{k} a_i$. Thus $f(t) = 0_{\frac{N}{N(0)}}$ and since $x \in t + x'$, $f(x) = f(x')$. □

Following [13] a *hyperring* is an algebraic hyperstructure $(R, +, \cdot)$, which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is, $0 \cdot x = x \cdot 0 = 0$, for all $x \in R$;

(3) The multiplication is distributive with respect to the hyperoperation "+" that is, $z \cdot (x + y) = z \cdot x + z \cdot y$ and $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$.

Let R_1 and R_2 be hyperrings. A mapping ψ from R_1 to R_2 is said to be a *good homomorphism*, see [10], if for all $a, b \in R_1$,

$$\psi(a + b) = \psi(a) + \psi(b),\ \psi(a.b) = \psi(a).\psi(b) \text{ and } \psi(0) = 0.$$

Let $(R, +, \cdot)$ be a hyperring and $\mathcal{U}$ be the set of all finite sums of products of elements of R. In [20] the relation γ^* on R is define by:

$a\gamma^* b$ iff $\exists z_1, \cdots, z_{n+1} \in R$ with $z_1 = a$, $z_{n+1} = b$ and $u_1, \cdots, u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$, for $i = 1, \cdots, n$. Then the relation γ^* is the smallest equivalence relation on R such that the quotient $(R/\gamma^*, \oplus, \odot)$ is a ring which is called *the fundamental ring*, see [20, Theorem 1]. Note that the sum $\oplus$ and the product $\odot$ in R/γ^* is defined in the usual manner:

$$\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c) \text{ for all } c \in \gamma^*(a) + \gamma^*(b)$$
$$\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d) \text{ for all } d \in \gamma^*(a) \cdot \gamma^*(b).$$

Proposition 1.6. *Let $(R, +, \cdot)$ be a hyperring and $\mathcal{R} = R/\gamma^*$ be its fundamental ring. Then*

(i) $1_{\mathcal{R}} = \gamma^*(1_R)$*;*

(ii) $0_{\mathcal{R}} = \gamma^*(0_R)$.

Proof. (i) Let $a \in R$ and $d \in \gamma^*(1_R) \cdot \gamma^*(a)$. Thus $s \in \gamma^*(1_R)$ and $t \in \gamma^*(a)$ exist such that $d \in s \cdot t$. So $\exists z_1, \cdots, z_{n+1}$ and $\exists z'_1, \cdots, z'_{m+1}$ with $z_1 = s$, $z_{n+1} = 1_R$, $z'_1 = t$, $z'_{m+1} = a$ and $u_1, \cdots, u_n, u'_1, \cdots, u'_m \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ and $\{z'_j, z'_{j+1}\} \subseteq u'_j$ for each $1 \le i \le n$ and $1 \le j \le m$. Without loss of the generality, we can assume $m \le n$. Put

$$\begin{cases} \forall 1 \le i \le m, \;\; z''_i := z'_i; \\ \forall m+1 \le i \le n, \;\; z''_i := z'_m; \\ z''_{n+1} := a. \end{cases} \qquad \begin{cases} \forall 1 \le i \le m-1, \;\; u''_i := u'_i; \\ \forall m \le i \le n-1, \;\; u''_i := z'_m; \\ u''_n := u'_m. \end{cases}$$

Thus for each $1 \le i \le n$, $(z_i \cdot z''_i) \cup (z_{i+1} \cdot z''_{i+1}) \subseteq u_i \cdot u''_i$, $d \in z_1 \cdot z''_1$ and $a \in z_{n+1} \cdot z''_{n+1}$. For every $2 \le i \le n$ choose an element $w_i \in z_i \cdot z''_i$ and put $w_1 := d$ and $w_{n+1} := a$. Therefore $\{w_i, w_{i+1}\} \subseteq u_i \cdot u''_i$ for each $1 \le i \le n$ and hence $a\gamma^* d$. This implies that $\gamma^*(1_R) \cdot \gamma^*(a) = \gamma^*(a)$.

(ii) The proof is similar to (i). $\square$

Following [7, 21] a (left) hypermodule over a hyperring R is a canonical hypergroup $(M, +)$ together with a scalar single-valued operation "$\cdot$", that is, a function which associates to any pair $(r, x) \in R \times M$ an element $rx \in M$ such that for all $(x, y) \in M^2$ and $(r, s) \in R^2$,

(i) $r \cdot (x + y) = r \cdot x + r \cdot y$;
(ii) $(r + s) \cdot x = r \cdot x + s \cdot x$;
(iii) $(rs) \cdot x = r \cdot (s \cdot x)$;
(iv) $0 \cdot x = 0$.

In fact, scalar multiplication is a map $\cdot : R \times M \longrightarrow \mathcal{P}^*(M)$ such that $r \cdot x$ is a singleton set. The important point to note is that the above definition depends on the sets which we assign to $A \cdot x$ and $r \cdot N$ for each $A \in \mathcal{P}^*(R)$, $N \in \mathcal{P}^*(M)$, $r \in R$ and $x \in M$. While, in algebraic mode, that is, when M is a module over a ring R, this ambiguity does not arise. These sets are not specified directly by the map "$\cdot$". So, it is necessary to define them. It is agreed that $A \cdot x = \{a \cdot x \mid a \in A\}$ and $r \cdot N = \{r \cdot x \mid x \in N\}$. Of course, this definition of hypermodule is restrictive, for example, it is not easy to give hypermodule structure to the canonical hypergroups. While each abelian groups takes a $\mathbb{Z}$-module structure canonically. Therefore, we need a more complete definition of hypermodule, which we will discuss it in Section 2.

2 The category of hypermodules

In this section, we generalize the definition of hypermodule, which is called ${}_R\star$-hypermodule, and study some of its properties. By choosing the appropriate homomorphism between ${}_R\star$-hypermodules, the category ${}_R\star-\mathbf{HM}$ is constructed, and at the end of this section it will be shown that there is an adjoint pair between ${}_R\star-\mathbf{HM}$ and a suitable category of modules.

Definition 2.1. Let $(R,+,\cdot)$ be a hyperring and $(M,+)$ be a canonical hypergroup. Also, let ${}_R\star_M : \mathcal{P}^*(R)\times M \longrightarrow \mathcal{P}^*(M)$ be a mapping such that for each $A,B\in\mathcal{P}^*(R)$ and $x\in M$ satisfying the following conditions:

$$\text{if } A\subseteq B, \text{ then } {}_R\star_M(A,x)\subseteq {}_R\star_M(B,x); \tag{sm1}$$
$${}_R\star_M(\{0\},x)=\{0\} \text{ and } {}_R\star_M(\{1_R\},x)=\{x\}. \tag{sm2}$$

A left *R-hypermodule* is a canonical hypergroup $(M,+)$ together with ${}_R\star_M$ and an external composition $\centerdot : R\times M\longrightarrow\mathcal{P}^*(M)$ with respect to ${}_R\star_M$ that defined by:

$$\centerdot(r,x)={}_R\star_M(\{r\},x)$$

satisfying the following conditions:

(a) $(r+s)\centerdot x=r\centerdot x+s\centerdot x$;

(b) $r\centerdot(x+y)=r\centerdot x+r\centerdot y$;

(c) $r\centerdot(s\centerdot x)=(rs)\centerdot x$;

(d) $r\centerdot M(0)\subseteq M(0)$ and $R(0)\centerdot x\subseteq M(0)$.

In which for every $A\in\mathcal{P}^*(R)$ and $N\in\mathcal{P}^*(M)$, $A\centerdot x:={}_R\star_M(A,x)$ and $r\centerdot N:=\bigcup\limits_{z\in N} r\centerdot z$.

From now on we say "M is a ${}_R\star$-hypermodule" instead of "M is a left R-hypermodule". Let N be a non-empty subset of M. N is called a ${}_R\star$-*subhypermodule* of M provided that N is a canonical subhypergroup of M and $r\centerdot y={}_R\star_M(\{r\},y)\subseteq N$ for all $r\in R$ and $y\in N$. Therefore $M(0)$ is a ${}_R\star$-subhypermodule of M.

There is a similar definition of a *right R-hypermodule*, or briefly $\star_R$-*hypermodule*, with respect to ${}_M\star_R$ in which the elements of R are written on the right.
Let R and S be hyperrings and M be canonical hypergroup. M is called an (R,S)-*bihypermodule*, whenever

(a) M is a $\star_R$-hypermodule and is a ${}_S\star$-hypermodule;

(b) For all $r \in R, s \in S$ and $x \in M$ we have $(r \centerdot x) \centerdot s = r \centerdot (x \centerdot s)$, i.e,

$$\bigcup_{y \in {}_R\star_M(\{r\},x)} {}_M\star_S(y,\{s\}) = \bigcup_{z \in {}_M\star_S(m,\{s\})} {}_R\star_M(\{r\},z);$$

Example 2.2. (1) Let $(M,+)$ be a canonical hypergroup and $(\mathbb{Z},+,\cdot)$ be the hyperring derived from the ring of integer numbers $(\mathbb{Z},\oplus,\cdot)$, that is, $m+n=\{m\oplus n\}$ for every $m,n\in\mathbb{Z}$. Define ${}_{\mathbb{Z}}\star_M : \mathcal{P}^*(\mathbb{Z}) \times M \longrightarrow \mathcal{P}^*(M)$ by ${}_{\mathbb{Z}}\star_M(A,x) = \bigcup\limits_{n\in A} nx$, where

$$nx := \begin{cases} \underbrace{x + x + \cdots + x}_{n \text{ times}} & \text{if } n > 0 \\ \{0\} & \text{if } n = 0 \\ \underbrace{(-x) + (-x) + \cdots + (-x)}_{-n \text{ times}} & \text{if } n < 0. \end{cases}$$

Thus ${}_{\mathbb{Z}}\star_M$ satisfies the conditions (sm1) and (sm2). Considering the external composition $\centerdot : \mathbb{Z} \times M \longrightarrow \mathcal{P}^*(M)$ with respect to ${}_{\mathbb{Z}}\star_M$, that is, $n \centerdot x = {}_{\mathbb{Z}}\star_M(\{n\},x)$. Therefore M is ${}_{\mathbb{Z}}\star$-hypermodule.

This example shows that every ${}_R\star$-hypermodule ($\star_R$-hypermodule) is a $(R,\mathbb{Z})$-bihypermodule ($(\mathbb{Z},R)$-bihypermodule).

(2) Let R be a hyperring. For the non-empty set X define the set $\mathcal{F}_R(X)$ as follows:

$$\mathcal{F}_R(X) := \{X \xrightarrow{f} R \mid f \text{ is vanish almost everywhere}\}.$$

In [15] it has been shown that $(\mathcal{F}_R(X),+)$ is a canonical hypergroup such that the hyperoperation "+" defined by

$$f+g := \{h \in \mathcal{F}_R(X) \mid (\forall x \in X), h(x) \in f(x)+g(x)\}.$$

Define a map ${}_R\star_{\mathcal{F}_R(X)} : \mathcal{P}^*(R) \times \mathcal{F}_R(X) \longrightarrow \mathcal{P}^*(\mathcal{F}_R(X))$ by:

$$ {}_R\star_{\mathcal{F}_R(X)}(A, f) := \left\{ g \in \mathcal{F}_R(X) \mid (\forall x \in X), g(x) \in \bigcup_{a \in A} \{af(x)\} \right\}. $$

Thus ${}_R\star_{\mathcal{F}_R(X)}$ satisfies in the conditions (sm1) and (sm2). Suppose that $\centerdot : R \times \mathcal{F}_R(X) \longrightarrow \mathcal{P}^*(\mathcal{F}_R(X))$ is the external composition with respect to ${}_R\star_{\mathcal{F}_R(X)}$, that is, $r \centerdot f = {}_R\star_{\mathcal{F}_R(X)}(\{r\}, f) = \{{}_rf\}$, where ${}_rf : X \longrightarrow R$ is defined by ${}_rf(x) = rf(x)$. For each $r, s \in R$ and $f, g \in \mathcal{F}_R(X)$ we have:

$$ \begin{aligned} (r+s) \centerdot f &= {}_R\star_{\mathcal{F}_R(X)}(r+s, f) \\ &= \{k \in \mathcal{F}_R(X) \mid (\forall x \in X), k(x) \in \bigcup_{a \in r+s} \{af(x)\}\} \\ &= \{k \in \mathcal{F}_R(X) \mid (\forall x \in X), k(x) \in (r+s)f(x)\} \\ &= \{k \in \mathcal{F}_R(X) \mid (\forall x \in X), k(x) \in rf(x) + sf(x)\} \\ &= {}_rf + {}_sf. \end{aligned} $$

Let $h \in {}_rf + {}_rg$. Therefore for every $x \in X$ we have $h(x) \in rf(x) + rg(x) = r(f(x) + g(x))$ and hence there exists $a_x \in f(x) + g(x)$ such that $h(x) = ra_x$. Suppose that $\mathbf{c} : \mathcal{P}^*(R) \longrightarrow R$ is the choice function such that $\mathbf{c}(A) \in A$, for all $A \in \mathcal{P}^*(R)$. Let A_h be a finite subset of X such that for each $x \notin A_h$, $h(x) = 0$ and put $s_x := \mathbf{c}(\{t \in f(x) + g(x) \mid h(x) = rt\})$, for all $x \in A_h$. Define $l : X \longrightarrow R$ by $l(x) := \begin{cases} s_x & x \in A_h \\ 0 & x \notin A_h. \end{cases}$

Thus $l \in f + g$ and for each $x \in X$, ${}_rl(x) = rl(x) = h(x)$. This implies that $h \in r\centerdot(f+g)$. Therefore $r\centerdot f + r\centerdot g \subseteq r\centerdot(f+g)$. The proof of the converse of the inclusion is easy. Thus $r \centerdot f + r \centerdot g = r \centerdot (f+g)$. The verification of the third axiom is not difficult. Now let $f \in (\mathcal{F}_R(X))(0)$, so $f \in \sum_{i=1}^{k}(f_i - f_i)$, where $f_i \in \mathcal{F}_R(X)$. Thus for $x \in X$ and $r \in R$, $rf(x) \in \sum_{i=1}^{k}(rf_i(x) - rf_i(x))$. Therefore ${}_rf \in \sum_{i=1}^{k}({}_rf_i - {}_rf_i)$ and hence $r \centerdot (\mathcal{F}_R(X))(0) \subseteq (\mathcal{F}_R(X))(0)$. Let $g \in R(0) \centerdot f = {}_R\star_{\mathcal{F}_R(X)}(R(0), f)$ and $A_g := \{x \in X \mid g(x) \neq 0\}$. Thus

A_g is a finite subset of X. If $A_g = \emptyset$, then $g = 0 \in (\mathcal{F}_R(X))(0)$. Let $A_g = \{x_1, \cdots, x_n\}$, so there exists $a_j \in R(0)$ such that $g(x_j) = a_j f(x_j)$ for each $j \in \{1, \cdots, n\}$. It can be assumed that $a_j \in \sum_{i=1}^{k}(r_{ij} - r_{ij})$ and hence $g(x_j) \in \sum_{i=1}^{k}(({}_{r_{ij}}f)(x_j) - ({}_{r_{ij}}f)(x_j))$ for $1 \le j \le n$. Define $h_i : X \longrightarrow R$ by $h_i(x) := \begin{cases} ({}_{r_{ij}}f)(x_j) & x = x_j,\ 1 \le j \le n \\ 0 & \text{otherwise} \end{cases}$ for $i \in \{1, \cdots, k\}$. This implies that $g \in \sum_{i=1}^{k}(h_i - h_i)$ and so $g \in (\mathcal{F}_R(X))(0)$. Therefore $\mathcal{F}_R(X)$ is a ${}_R\star$-hypermodule.

(3) Let $(R, \oplus, \cdot)$ be a ring and $(M, \oplus)$ be a R-module. Also, let $(M, +)$ $((R, +, \cdot))$ be a canonical hypergroup (hyperring) which is derived from $(M, \oplus)$ $((R, \oplus, \cdot))$, that is, $x + y = \{x \oplus y\}$ for each $x, y \in M$ $(x, y \in R)$. Define ${}_R\star_M : \mathcal{P}^*(R) \times M \longrightarrow \mathcal{P}^*(M)$ by ${}_R\star_M (A, x) := \{rx \mid r \in A\}$. Let $\centerdot : R \times M \longrightarrow \mathcal{P}^*(M)$ be the external composition with respect to ${}_R\star_M$ which is defined by $r \centerdot x = {}_R\star_M(\{r\}, x) = \{rx\}$. Then M is a ${}_R\star$-hypermodule. This says that every R-module can be considered as a ${}_R\star$-hypermodule in a natural way.

Remark 2.3. Let $(\mathcal{F}_R(X), +)$ be the canonical hypergroup as in the Theorem 2.2(2). For every $r \in R$ and $f \in \mathcal{F}_R(X)$ consider the scalar multiplication $\cdot : R \times \mathcal{F}_R(X) \longrightarrow \mathcal{F}_R(X)$ by $r \cdot f := {}_rf$. Thus $r \centerdot f = \{r \cdot f\}$. We see that by this scalar multiplication $\mathcal{F}_R(X)$ does not apply to the definition of the hypermodule provided in [7]. Because, from this perspective for each $A \in \mathcal{P}^*(R)$ and $f \in \mathcal{F}_R(X)$ we have $A \cdot f = \{{}_af \mid a \in A\}$ and so,

$$(r + s) \cdot f \subseteq r \cdot f + s \cdot f.$$

While in the Theorem 2.2(2), $(r+s) \centerdot f = r \centerdot f + s \centerdot f$. Therefore the definition of ${}_R\star$-hypermodule M is completely related to the map ${}_R\star_M$.

Proposition 2.4. *Let M be a ${}_R\star$-hypermodule. Then for each $x \in M$, $(-1_R) \centerdot x = \{-x\}$.*

Proof. Let $z \in (-1_R) \centerdot x$, so $(-1_R) \centerdot z \subseteq (-1_R) \centerdot ((-1_R) \centerdot x) = 1_R \centerdot x = \{x\}$. Thus $(-1_R) \centerdot z = \{x\}$ and hence $\{z\} = (-1_R) \centerdot ((-1_R) \centerdot z) = (-1_R) \centerdot \{x\} =$

$(-1_R) \centerdot x$. Since $0_R \in 1_R + (-1_R)$, $0 \in 1_R \centerdot x + (-1_R) \centerdot x = \{x\} + \{z\}$. This implies that $0 \in x + z$ and so $z = -x$. Therefore $(-1_R) \centerdot x = \{-x\}$. $\square$

Let $(M, +)$ be a ${}_R\star$-hypermodule ($\star_R$-hypermodule). Put $\frac{M}{M(0)} := M/\overset{M(0)}{\equiv}$ and denotes the elements of $\frac{M}{M(0)}$ by $[x]_{_RM}$ ($[x]_{M_R}$) for each $x \in M$. Thus $(\frac{M}{M(0)}, \oplus)$ is an abelian group.

Proposition 2.5. *Let R be a hyperring and $\mathcal{R} = R/\gamma^*$ be its fundamental ring. If $(M, +)$ is a ${}_R\star$-hypermodule, then $(\frac{M}{M(0)}, \oplus)$ is an $\mathcal{R}$-module.*

Proof. For $a \in R$ and $x \in M$ define $\gamma^*(a) \cdot [x]_{_RM} = [y]_{_RM}$, where $y \in a \centerdot x$. Let $\gamma^*(a) = \gamma^*(a')$, $[x]_{_RM} = [x']_{_RM}$ and $y' \in a' \centerdot x'$. Thus $\exists r_1, \cdots, r_{n+1} \in R$ with $r_1 = a$, $r_{n+1} = a'$ and $u_1, \cdots, u_n \in \mathcal{U}$ such that $\{r_i, r_{i+1}\} \subseteq u_i$ for $1 \leq i \leq n$ and also $x - x' \subseteq \sum\limits_{i=1}^{k}(x_i - x_i)$. Thus we have

$$\begin{aligned} a + r_2 &\subseteq u_1 + u_2, \\ &\vdots \\ -r_i + r_{i+1} &\subseteq -u_{i-1} + u_{i+1}, \qquad \text{for all } \ 2 \leq i \leq n-1 \\ &\vdots \\ -r_n - a' &\subseteq -u_{n-1} - u_n. \end{aligned}$$

Therefore $a - a' \subseteq \sum\limits_{i=1}^{n}(u_i - u_i)$ and so

$$a \centerdot x - a' \centerdot x' \subseteq \sum_{i=1}^{n}(u_i - u_i) \centerdot x + \sum_{i=1}^{k} a' \centerdot (x_i - x_i). \tag{2.1}$$

Suppose that $u_i = \sum\limits_{j=1}^{n_i} \prod\limits_{k=1}^{m_j} r_{jki}$ for $i \in \{1, \cdots, n\}$ and hence we have $(u_i - u_i) \centerdot x = \sum\limits_{j=1}^{n_i} t_{ji} \centerdot (x - x)$, where $t_{ji} := \prod\limits_{k=1}^{m_j} r_{jki}$. Thus eq. (2.1) implies that $y - y' \subseteq \sum\limits_{i=1}^{n} \big(\sum\limits_{j=1}^{n_i} t_{ji} \centerdot (x - x)\big) + \sum\limits_{i=1}^{k} a' \centerdot (x_i - x_i)$, so $[y]_{_RM} = [y']_{_RM}$.

Therefore $\cdot : \mathcal{R} \times \frac{M}{M(0)} \longrightarrow \frac{M}{M(0)}$ is a scalar multiplication and it is easy to see that $\frac{M}{M(0)}$ is an $\mathcal{R}$-module. $\square$

Suppose that $(R, +, \cdot)$ is a hyperring and $(N, +)$ is a ${}_R\star$-hypermodule. Define ${}_R\star_{\frac{N}{N(0)}} : \mathcal{P}^*(R) \times \frac{N}{N(0)} \longrightarrow \mathcal{P}^*(\frac{N}{N(0)})$ by:

$$ {}_R\star_{\frac{N}{N(0)}}(A, [x]_{{}_RN}) := \{[z]_{{}_RN} \mid z \in {}_R\star_N(\{r\}, x) = r \centerdot x \text{ for some } r \in A\}. $$

Note that if $z' \in_R \star_N(\{r\}, x) = r \centerdot x$, then $[z]_{{}_RN} = [z']_{{}_RN}$. Considering external composition $\centerdot' : R \times \frac{N}{N(0)} \longrightarrow \mathcal{P}^*(\frac{N}{N(0)})$ with respect to ${}_R\star_N$ that defined by $r \centerdot' [x]_{{}_RN} = {}_R\star_{\frac{N}{N(0)}}(\{r\}, [x]_{{}_RN}) = \{[x']_{{}_RN}\}$ such that $x' \in r \centerdot x$. It can be easily seen that $(\frac{N}{N(0)}, +')$ is ${}_R\star$-hypermodule, where $[x]_{{}_RN} +' [y]_{{}_RN} := \{[x]_{{}_RN} \oplus [y]_{{}_RN}\}$ for $[x]_{{}_RN}, [y]_{{}_RN} \in \frac{N}{N(0)}$. So we have the following definition.

Definition 2.6. Suppose that M and N are two ${}_R\star$-hypermodules. A ${}_R\star$-*homomorphism* $f : M \xrightarrow{\star} N$ is a β-homomorphism $f : M \xrightarrow{\beta} N$ between canonical hypergroups M and N such that $\{f(z)\} = r \centerdot' f(x)$ for all $x \in M$, $r \in R$ and $z \in r \centerdot x$.

Remark 2.7. (a) If $f : M \xrightarrow{\star} N$ is a ${}_R\star$-homomorphism, then for each $x, y \in M$, $f(x) \oplus f(y) \subseteq f(x) + f(y) + N(0)$. Also, if $g, h : M \xrightarrow{\star} N$ are a ${}_R\star$-homomorphism such that $h(x) \subseteq f(x) + g(x)$, then $h(x) = f(x) \oplus g(x)$.

(b) There is a similar definition of homomorphism between $\star_R$-hypermodules in which condition (ii) of Theorem 2.6 replace with $f(z) = f(x) \centerdot r$ for each $z \in x \centerdot r$. If M, N are (R, S)-bihypermodules, then $f : M \xrightarrow{\star} M'$ is said to be a homomorphism of (S, R)-bihypermodules, whenever f is both a ${}_R\star$-homomorphism and a $\star_S$-homomorphism.

(c) Let M and N be canonical hypergroups. Every β-homomorphism $f : M \xrightarrow{\beta} N$ is a ${}_{\mathbb{Z}}\star$-homomorphism between ${}_{\mathbb{Z}}\star$-hypermodules as described in the Theorem 2.2(1).

Let $f : M \xrightarrow{\star} N$ and $g : N \xrightarrow{\star} K$ be two ${}_R\star$-homomorphisms. For each $x \in M$ define,

$$ g \bullet f(x) := g(y), \text{ where } y \in f(x). $$

Thus $g \bullet f : M \xrightarrow{\star} K$ is a ${}_R\star$-homomorphism. Define $\overline{1}_M : M \longrightarrow \frac{M}{M(0)}$ by $\overline{1}_M(x) = [x]_{{}_RM}$ for $x \in M$. So $\overline{1}_M : M \xrightarrow{\star} M$ is a ${}_R\star$-homomorphism and for every ${}_R\star$-homomorphisms $f : M \xrightarrow{\star} N$ and $g : K \xrightarrow{\star} M$ we have $\overline{1}_M \bullet g = g$ and $f \bullet \overline{1}_M = f$. Therefore $\overline{1}_M$ is an identity ${}_R\star$-homomorphism on M. The collection of ${}_R\star$-hypermodules together with ${}_R\star$-homomorphisms forms a category, which is denoted by ${}_R\star-\mathbf{HM}$.

Given two ${}_R\star$-hypermodules M and N, the set of ${}_R\star$-homomorphism from M to N is denoted by $\mathrm{Hom}_R(M, N)$. For each $f, g \in \mathrm{Hom}_R(M, N)$ and $x \in M$ define $(f \boxplus g)(x) := f(x) \oplus g(x)$. This turns $(\mathrm{Hom}_R(M, N), +)$ into a canonical hypergroup, where $f + g := \{f \boxplus g\}$. This implies that $\mathrm{Hom}_R(M, -) : {}_R\star-\mathbf{HM} \longrightarrow \mathbf{CHg}$ is a functor, where $\mathbf{CHg}$ is the category of canonical hypergroups and whose morphisms are good homomorphisms.

By ${}_{R^\star}M$ ($M_{\star_S}$, ${}_{R^\star}M_{\star_S}$) we means that M is a ${}_R\star$-hypermodule ($\star_S$-hypermodule, (R, S)-bihypermodule, respectively).

Proposition 2.8. *Let R and S be hyperrings.*

(i) *If ${}_{R^\star}M_{\star_S}$ and ${}_{R^\star}N$, then $Hom_R(M, N)$ is a ${}_S\star$-hypermodule.*

(ii) *If ${}_{R^\star}M$ and ${}_{R^\star}N_{\star_S}$, then $Hom_R(M, N)$ is a $\star_S$-hypermodule.*

Proof. (i) Let $s \in S$ and $f \in \mathrm{Hom}_R(M, N)$. For $x \in M$ put $f_s(x) := f(y)$, where $y \in x \centerdot s$. If $y' \in x \centerdot s$, then $y - y' \subseteq (x - x) \centerdot s \subseteq M(0) \centerdot s$. Since $M(0)$ is a $\star_S$-subhypermodule of M, $y - y' \subseteq M(0)$ and hence $f(y) = f(y')$. We can see that $f_s : M \xrightarrow{\star} N$ is a ${}_R\star$-homomorphism. Suppose that $H := \mathrm{Hom}_R(M, N)$ define ${}_S\star_H : \mathcal{P}^*(S) \times H \longrightarrow \mathcal{P}^*(H)$ by ${}_S\star_H(A, f) := \{f_a \mid a \in A\}$. Thus ${}_S\star_H$ satisfies in the conditions (sm1) and (sm2). Let $\centerdot : S \times H \longrightarrow \mathcal{P}^*(H)$ be the external composition with respect to ${}_S\star_H$, that is, $s \centerdot f ={}_R \star_H(\{s\}, f) = \{f_s\}$. It is easy to see that H is a ${}_S\star$-hypermodule.

(ii) Let $s \in S$ and $f \in \mathrm{Hom}_R(M, N)$. For $x \in M$ put $\hat{f}_s(x) := [y]_{{}_RN}$, where $y \in y_f \centerdot s$ such that $f(x) = [y_f]_{{}_RN}$. Let $[y_f]_{{}_RN} = [z]_{{}_RN}$ and $y' \in z \centerdot s$. Therefore $y_f - z \subseteq \sum_{i=1}^{k}(z_i - z_i)$ for $k \in \mathbb{N}$ and $z_i \in N$. Since $N(0)$ is a $\star_S$-subhypermodule of N, for each $i \in \{1, \cdots, k\}$ we have $(z_i - z_i) \centerdot s \subseteq N(0)$. This implies that $y - y' \subseteq (y_f - z) \centerdot s \subseteq N(0)$ and hence $[y]_{{}_RN} = [y']_{{}_RN}$.

We can see that $\hat{f}_s : M \xrightarrow{\star} N$ is a ${}_R\star$-homomorphism. Similar to part (a) $\mathrm{Hom}_R(M, N)$ is a $\star_S$-hypermodule. $\square$

Remark 2.9. (a) In the Theorem 2.8 the other cases for M and N are similar to the classical one and their proofs are similar to (i) and (ii).

(b) From (i) and (ii) of the Theorem 2.8, we obtain the functors

$$\mathrm{Hom}_R(M, -) : {}_R\star-\mathbf{HM} \longrightarrow {}_S\star-\mathbf{HM}\ ; \ \ \mathrm{Hom}_R(M, -) : {}_R\star-\mathbf{HM} \longrightarrow \star_S - \mathbf{HM}$$

respectively.

From now on, when we say $f : M \longrightarrow N$ is a morphism in ${}_R\star-\mathbf{HM}$ we mean $f : M \xrightarrow{\star} N$ is a ${}_R\star$-homomorphism.

Proposition 2.10. *Let $M \xrightarrow{f} N$ be a morphism in ${}_R\star-\mathbf{HM}$. Then*

(i) *f is an epimorphism in ${}_R\star-\mathbf{HM}$ if and only if $N = I_f$, where $I_f := \bigcup\limits_{x\in M} f(x)$.*

(ii) *f is a monomorphism in ${}_R\star-\mathbf{HM}$ if and only if $Ke(f) = \{0\}$, where $Ke(f) := \{x \in M \mid f(x) = [0]_{{}_RN}\}$.*

(iii) *f is an isomorphism in ${}_R\star-\mathbf{HM}$ if and only if f is both monomorphism and epimorphism.*

Proof. (i) It is easy to see that I_f is a ${}_R\star$-subhypermodule of N and for each $y \in N$, $y - y \subseteq f(0) \subseteq \mathrm{I}_f$. Define the relation $\overset{I_f}{\equiv}$ on N by

$$\forall y, y' \in N,\ \ y \overset{\mathrm{I}_f}{\equiv} y' \text{ iff } y - y' \subseteq \mathrm{I}_f.$$

It can be easily proved that $\overset{\mathrm{I}_f}{\equiv}$ is a strongly regular relation and $\frac{N}{\mathrm{I}_f} := N/\overset{I_f}{\equiv}$ is a ${}_R\star$-hypermodule. Denote the elements of $\frac{N}{\mathrm{I}_f}$ by $[y]$ for all $y \in N$. Consider the following diagram in ${}_R\star-\mathbf{HM}$

$$M \xrightarrow{f} N \underset{z}{\overset{p}{\rightrightarrows}} \frac{N}{\mathrm{I}_f},$$

where for each $y \in N$, $p(y) = [y]$ and $z(y) = [0]$. Thus $p \bullet f = z \bullet f$ and hence $N = \mathrm{I}_f$. Conversely, let

$$M \xrightarrow{f} N \underset{h}{\overset{g}{\rightrightarrows}} K,$$

be a diagram in ${}_R\star-\mathbf{HM}$ such that $g \bullet f = h \bullet f$. Since $\mathrm{I}_f := \bigcup\limits_{x \in M} f(x)$, for each $y \in N$ there exists $x \in M$ such that $y \in f(x)$. Thus $g(y) = g \bullet f(x) = h \bullet f(x) = h(y)$ and hence $g = h$.

(ii) It is obvious.

(iii) Suppose that f is both monomorphism and epimorphism, so by (i) $N = \mathrm{I}_f$. Define $g : N \longrightarrow M$ by $g(n) = [m]_{{}_RM}$ such that $n \in f(m)$ for $n \in N$. If $n \in f(m')$ for another $m' \in M$, then $f(m) = f(m') = [n]_{{}_RN}$. Thus for each $x \in m - m'$ we have $f(x) = f(m) \oplus (-f(m')) = [0]_{{}_RN}$. Therefore $m - m' \subseteq \mathrm{Ke}(f)$. Since $\mathrm{Ke}(f) = \{0\}$, $m = m'$ and so g is a well defined map. It is easy to see that g is a ${}_R\star$-homomorphism, $f \bullet g = \overline{1}_N$ and $g \bullet f = \overline{1}_M$. This implies that f is an isomorphism. The proof of the converse is obvious. $\square$

In the category ${}_R\star-\mathbf{HM}$ a sequence of two morphisms $M \xrightarrow{f} N \xrightarrow{g} K$ is called *exact* whenever $\mathrm{I}_f = \mathrm{Ke}(g)$.

Remark 2.11. For morphism $f : M \longrightarrow N$ in ${}_R\star-\mathbf{HM}$ we have:

(a) $0 \longrightarrow M \xrightarrow{f} N$ is exact if and only if f is a monomorphism;

(b) $M \xrightarrow{f} N \longrightarrow 0$ is exact if and only if f is an epimorphism.

Theorem 2.12. *Let $(R, +, \cdot)$ be a hyperring and $\mathcal{R} := R/\gamma^*$ be its fundamental ring. Then there is an adjunction from ${}_\mathcal{R}\boldsymbol{M}$, the category of left $\mathcal{R}$-modules, to ${}_R\star-\mathbf{HM}$, that is, there is a triple*

$$< \mathbf{F}, \mathbf{G}, \varphi >: {}_\mathcal{R}\boldsymbol{M} \rightharpoonup {}_R\star-\mathbf{HM}$$

where $\mathbf{F}$ *and* $\mathbf{G}$ *are functors*

$$\begin{tikzcd} {}_\mathcal{R}\boldsymbol{M} \arrow[r, shift left, "\mathbf{F}"] & {}_R\star-\mathbf{HM} \arrow[l, shift left, "\mathbf{G}"] \end{tikzcd}$$

while φ is a function which assigns to each pair of objects $(M, \oplus) \in {}_\mathcal{R}\boldsymbol{M}$ and $(N, +) \in {}_R\star-\mathbf{HM}$ a bijection of sets

$$\varphi = \varphi_{M,N} : {}_R\star-\mathbf{HM}\Big(\mathbf{F}(M, \oplus), (N, +)\Big) \cong {}_\mathcal{R}\boldsymbol{M}\Big((M, \oplus), \mathbf{G}(N, +)\Big)$$

which is a natural in M and N. Also $\boldsymbol{G}$ is full and faithful.

Proof. Let $(M,\oplus) \in {}_{\mathcal{R}}\boldsymbol{M}$ and $(M,+)$ be the canonical hypergroup which is derived from $(M,\oplus)$. So $M(0) = \{0\}$, see Theorem 2.2(3). Define ${}_{R}\star_{M} : \mathcal{P}^*(R) \times M \longrightarrow \mathcal{P}^*(M)$ by ${}_{R}\star_{M}(A,x) := \{\gamma^*(a)\cdot x \mid a \in A\}$. Theorem 1.6 implies that ${}_{R}\star_{M}$ satisfying in the conditions (sm1) and (sm2). Let $\centerdot : R \times M \longrightarrow \mathcal{P}^*(M)$ be the external composition with respect to ${}_{R}\star_{M}$ which is defined by $r \centerdot x = {}_{R}\star_{M}(\{r\},x) = \{\gamma^*(r)\cdot x\}$. We can see that $(M,+)$ is a ${}_{R}\star$-hypermodule. Now define the functor $\boldsymbol{F} : {}_{\mathcal{R}}\boldsymbol{M} \longrightarrow {}_{R}\star\!-\mathbf{HM}$ by

$$\begin{tikzcd} (M,\oplus) \arrow[r, mapsto] \arrow[d, "f"] & (M,+) \arrow[d, "\boldsymbol{F}(f)"] \\ (N,\oplus) \arrow[r, mapsto] & (N,+), \end{tikzcd}$$

where for each $x \in M$, $\boldsymbol{F}(f)(x) := \{f(x)\} = [f(x)]_{{}_RN}$. Theorem 2.5 implies that $\boldsymbol{G} : {}_{R}\star\!-\mathbf{HM} \longrightarrow {}_{\mathcal{R}}\boldsymbol{M}$ defined by

$$\begin{tikzcd} (M,+) \arrow[r, mapsto] \arrow[d, "g"] & (\frac{M}{M(0)},\oplus) \arrow[d, "\boldsymbol{G}(g)"] \\ (N,+) \arrow[r, mapsto] & (\frac{N}{N(0)},\oplus), \end{tikzcd}$$

where for each $x \in M$, $\boldsymbol{G}(g)([x]_{{}_RM}) := g(x)$ is a functor. It is easy to see that $\boldsymbol{G}$ is full and faithful. For each $g \in {}_{R}\star\!-\mathbf{HM}(\mathbf{F}(M,\oplus),(N,+))$ define $(M,\oplus) \xrightarrow{\varphi(g)} \boldsymbol{G}(N,+)$ by $(\varphi(g))(x) = g(x)$. Since $(M,+) = \mathbf{F}(M,\oplus)$, $x+y = \{x\oplus y\}$ for $x,y \in M$ and so $\varphi(g)$ is a $\mathcal{R}$-homomorphism. Also we can see that φ is a natural isomorphism and hence the proof is complete. $\square$

3 Tensor product of ${}_{R}\star$-hypermodules

In this section the free object on a set X in the category ${}_{R}\star\!-\mathbf{HM}$ is introduced and, by using it, the tensor product of the ${}_{R}\star$-hypermodules is defined. Then some of its properties are given, including having a universal property in a particular category and its relation to Hom functor.

From now on we show $r\centerdot x$ by rx. Let R be a hyperring. Define the functor $\boldsymbol{G}_R : {}_R\star\!-\mathbf{HM} \longrightarrow \mathbf{Set}$ by

$$\begin{array}{ccc} M & \longmapsto & \frac{M}{M(0)} \\ \Big\downarrow f & & \Big\downarrow \boldsymbol{G}_R(f) \\ N & \longmapsto & \frac{N}{N(0)} \end{array}$$

where for each set $[x]_{_RM} \in \frac{M}{M(0)}$, $\boldsymbol{G}_R(f)([x]_{_RM}) := f(x)$. Note that if $[x]_{_RM} = [x']_{_RM}$, then there exists $z \in x - x' \cap \sum\limits_{i=1}^{k}(x_i - x_i)$ for some $x_i \in M$. Since $f(z) = [0]_{_RN}$, $f(x) = f(x')$. Thus $\boldsymbol{G}_R(f)$ is well defined. It is easy to see that $\boldsymbol{G}_R$ is a forgetful functor and hence ${}_R\star\!-\mathbf{HM}$ is concrete over $\mathbf{Set}$.

Remark 3.1. By Theorem 2.2(1) for canonical hypergroup M, $\boldsymbol{G}_{\mathbb{Z}}(M) = M/\beta$. So, if $(M, +)$ is a canonical hypergroup derived from an abelian group $(M, \oplus)$, then $\beta(x) = \{x\}$ and hence there exists a group isomorphism $(\boldsymbol{G}_{\mathbb{Z}}(M), \oplus) \overset{\Gamma}{\cong} (M, \oplus)$ such that $\Gamma(\beta(x)) = x$.

Theorem 3.2. *Let R be a hyperring. For each non-empty set X, the ${}_R\star$ $-\mathbf{HM}$-object $\mathcal{F}_R(X)$ is free on X.*

Proof. Let $M \in {}_R\star\!-\mathbf{HM}$ and $\psi : X \longrightarrow \boldsymbol{G}_R(M)$ be a morphism in $\mathbf{Set}$. Let $\bar{\iota}_X : X \longrightarrow \boldsymbol{G}_R(\mathcal{F}_R(X))$ be the mapping that defined by $\bar{\iota}_X(x) = \{\delta_x\}$, where $\delta_x : X \longrightarrow R$ is defined by $\delta_x(y) := \begin{cases} 1_R & \text{if } x = y \\ 0_R & \text{otherwise} \end{cases}$ for all $x, y \in X$. Thus for any $f \in \mathcal{F}_R(X)$, $\{f\} = \sum\limits_{x \in A_f} f(x)\delta_x$, where $A_f := \{x \in X \mid f(x) \neq 0\}$ is a finite subset of X. Consider the map $\psi : X \longrightarrow \boldsymbol{G}_R(M)$. Define $\widehat{\psi} : \mathcal{F}_R(X) \longrightarrow \frac{M}{M(0)}$ by

$$\widehat{\psi}(f) := \begin{cases} \bigoplus\limits_{x \in A_f} f(x)\psi(x) & \text{if } A_f \neq \emptyset \\ [0]_{_RM} & \text{if } A_f = \emptyset. \end{cases}$$

Now we show that $\widehat{\psi} : \mathcal{F}_R(X) \longrightarrow M$ is a ${}_R\star$-homomorphism. For this reason, let $f, g \in \mathcal{F}_R(X)$ and $h \in f+g$. For each $x \in X$ since $h(x) \in f(x)+g(x)$, $A_h \subseteq A_f \cup A_g$ and $h(x)\psi(x) = f(x)\psi(x) \oplus g(x)\psi(x)$. We have two cases:

Case 1: If $A_h \neq \emptyset$, then

$$\begin{aligned}\sum_{x\in A_h} f(x)\psi(x) + \sum_{x\in A_h} g(x)\psi(x) &\subseteq \sum_{x\in A_f} f(x)\psi(x) + \sum_{x\in A_g} g(x)\psi(x)\\ &= \{\bigoplus_{x\in A_f} f(x)\psi(x)\} + \{\bigoplus_{x\in A_f} f(x)\psi(x)\}\\ &= \{\widehat{\psi}(f)\} + \{\widehat{\psi}(g)\}\\ &= \{\widehat{\psi}(f) \oplus \widehat{\psi}(g)\}.\end{aligned}$$

Thus $\{\bigoplus_{x\in A_h} h(x)\psi(x)\} = \sum_{x\in A_h} h(x)\psi(x) \subseteq \sum_{x\in A_h} f(x)\psi(x) + \sum_{x\in A_h} g(x)\psi(x) = \{\widehat{\psi}(f) \oplus \widehat{\psi}(g)\}$ and hence $\widehat{\psi}(h) = \widehat{\psi}(f) \oplus \widehat{\psi}(g)$.

Case 2: If $A_h = \emptyset$, then for each $x \in X$, $0 \in f(x) + g(x)$ and hence $f(x) = -g(x)$. This implies that $h(x)\psi(x) = [0]_{{}_RM}$. Since $A_f = A_g$, $\widehat{\psi}(f) = -\widehat{\psi}(g)$. Thus $\widehat{\psi}(h) = \widehat{\psi}(f) \oplus \widehat{\psi}(g) = [0]_{{}_RM}$.

It is easy to see that for each $r \in R$, $f \in \mathcal{F}_R(X)$ and $g \in rf$ the equality $r\widehat{\psi}(f) = \widehat{\psi}(g)$ holds. Therefore $\widehat{\psi}$ is a ${}_R\star$-homomorphism and the equalities $\boldsymbol{G}_R(\widehat{\psi}) \circ \bar{\iota}_X(x) = \boldsymbol{G}_R(\widehat{\psi})([\delta_x]_M) = \widehat{\psi}_0([\delta_x]_M) = \widehat{\psi}(\delta_x) = \psi(X)$ hold for each $x \in X$. Thus $\boldsymbol{G}_R(\widehat{\psi}) \circ \bar{\iota}_X = \psi$. Now let $\lambda : \mathcal{F}_R(X) \xrightarrow{\star} M$ be a ${}_R\star$-homomorphism such that $\boldsymbol{G}_R(\lambda) \circ \bar{\iota}_X = \psi$, so $\lambda(\delta_x) = \psi(x)$ for each $x \in X$. Let $h \in \mathcal{F}_R(X)$. If $A_h \neq \emptyset$, then $\lambda(h) = \bigoplus_{x\in A_h} h(x)\lambda(\delta_x) = \bigoplus_{x\in A_h} h(x)\psi(x) = \widehat{\psi}(h)$. If $A_h = \emptyset$, then $h \equiv 0$, that is, for every $x \in X$, $h(x) = 0$. Since λ is a ${}_R\star$-homomorphism, $\lambda(h) = [0]_{{}_RM}$ and by definition of $\widehat{\psi}$ we have $\widehat{\psi}(h) = [0]_{{}_RM}$. Therefore $\lambda = \widehat{\psi}$ and hence the proof is complete. □

Remark 3.3. Let $(R, \oplus, \cdot)$ be a ring and $(\mathcal{F}(X), \oplus)$ be the free R-module generated by X. If $(R, +, \cdot)$ is the hyperring derived from the ring $(R, \oplus, \cdot)$, then $\mathcal{F}_R(X) = \mathcal{F}(X)$ and for each $f, g \in \mathcal{F}_R(X)$, $f + g = \{f \oplus g\}$.

Corollary 3.4. *For each ${}_R\star$-hypermodule M there exist a free ${}_R\star$-hypermodule F and an epimorphism $F \longrightarrow M$ in ${}_R\star-$**HM**.*

Proof. Let X be a generating set of M. Define $\psi : X \longrightarrow \frac{M}{M(0)}$ by $\psi(x) = [x]_{_RM}$ for $x \in M$. Thus, Theorem 3.2 implies that there exists a unique $_R\star$-homomorphism $\widehat{\psi} : \mathcal{F}_R(X) \longrightarrow M$ such that $\boldsymbol{G}_R(\widehat{\psi}) \circ \bar{\iota}_X = \psi$. Now consider the diagram $\mathcal{F}_R(X) \xrightarrow{\widehat{\psi}} M \underset{\eta}{\overset{\lambda}{\rightrightarrows}} N$ in $_R\star-\mathbf{HM}$, such that $\lambda \bullet \widehat{\psi} = \eta \bullet \widehat{\psi}$. Let $m \in M$ and so $m \in \sum_{i=1}^{k} r_i x_i$ for some $r_i \in R$ and $x_i \in X$. Thus the equality $[m]_M = \bigoplus_{i=1}^{k} r_i [x_i]_{_RM}$ holds. Define $h : X \longrightarrow R$ by

$$h(x) := \begin{cases} r_i & \text{if } x \in \{x_1, \cdots, x_k\} \\ 0_R & \text{otherwise.} \end{cases}$$

Therefore $[m]_M = \bigoplus_{i=1}^{k} h(x_i)\psi(x_i) = \widehat{\psi}(h)$. Since $m \in \widehat{\psi}(h)$, the equalities $\lambda(m) = \lambda \bullet \widehat{\psi}(m) = \eta \bullet \widehat{\psi} = \eta(m)$ hold. This implies that $\widehat{\psi}$ is an epimorphism in $_R\star-\mathbf{HM}$. $\square$

Let M be a $\star_R$-hypermodule and N be a $_R\star$-hypermodule, C be a canonical hypergroup and $\mathbb{Z}$ be the hyperring derived from the ring of integer numbers, see Theorem 2.2(1). A *middle linear β-map* from $M \times N$ to C, which is denoted by $f : M \times N \xrightarrow{\mathcal{M}} C$, is a function $f : M \times N \longrightarrow \boldsymbol{G}_{\mathbb{Z}}(C)$ such that for all $x, x_i \in M$, $y, y_i \in N$, $r \in R$:

$$f(w, y) = f(x_1, y) \oplus f(x_2, y), \quad (\text{ for each } w \in x_1 + x_2); \tag{mh1}$$
$$f(x, l) = f(x, y_1) \oplus f(x, y_2), \quad (\text{ for each } l \in y_1 + y_2); \tag{mh2}$$
$$f(u, y) = f(x, v), \quad (\text{ for each } u \in xr \text{ and } v \in ry). \tag{mh3}$$

Let $\mathscr{M}_H(M, N)$ be the category whose object are all middle linear β-maps on $M \times N$. By definition, a morphism in $\mathscr{M}_H(M, N)$ from $f : M \times N \xrightarrow{\mathcal{M}} C$ to $g : M \times N \xrightarrow{\mathcal{M}} D$ is a β-homomorphism $h : C \xrightarrow{\beta} D$ such that $h \blacklozenge f = g$, where $h \blacklozenge f$ is defined by $h \blacklozenge f(x, y) := h(c)$ for some $c \in f(x, y)$. Note that if c' is another element in $f(x, y)$, then $\beta(c) = \beta(c')$. Thus $c - c' \subseteq \sum_{i=1}^{n} (c_i - c_i)$ for some $n \in \mathbb{N}$ and $c_i \in C$. Therefore $h(c) = h(c')$.

Definition 3.5. Let M be a $\star_R$-hypermodule and N be a ${}_R\star$-hypermodule. Put $\mathcal{K} :=< K >$, where K is the union of all subsets of the following forms (for all $x, x' \in M; y, y' \in N; r \in R$):

$$\delta_{(w,y)} - \delta_{(x,y)} - \delta_{(x',y)} \quad (\text{ for each } w \in x + x'); \tag{ht1}$$

$$\delta_{(x,l)} - \delta_{(x,y)} - \delta_{(x,y')} \quad (\text{ for each } l \in y + y'); \tag{ht2}$$

$$\delta_{(u,y)} - \delta_{(x,v)}, \qquad (\text{ for each } u \in xr \text{ and } v \in ry). \tag{ht3}$$

Where the function $\delta_{(x,y)} : M \times N \longrightarrow \mathbb{Z}$ is defined as in Theorem 3.2. The quotient canonical hypergroup $(\mathcal{F}_{\mathbb{Z}}(M \times N)/\mathcal{K}, +)$ derived from the abelian group $(\mathcal{F}_{\mathbb{Z}}(M \times N)/\mathcal{K}, \oplus)$ is called the *tensor product* of M and N; it is denoted $M \circledast_R N$.

The equivalence $[\delta_{(x,y)}] \in \mathcal{F}_{\mathbb{Z}}(M \times N)/\mathcal{K}$ is denoted by $x \circledast_R y$.

Remark 3.6. In the Theorem 3.5,

(a) $< K >= \bigcup\{\sum_{i=1}^{n} k_i \mid k_i \in K, n \in \mathbb{N}\}$;

(b) By Theorem 3.3 we have $\omega_{\mathcal{F}_{\mathbb{Z}}(M \times N)} = (\mathcal{F}_{\mathbb{Z}}(M \times N))(0) = \{0\}$. So $\mathcal{K} = \mathscr{C}(< K >)$ and hence $\mathcal{K}$ is a complete part canonical subhypergroup of $\mathcal{F}_{\mathbb{Z}}(M \times N)$.

(c) Let M be a $\star_R$-hypermodule and N be a ${}_R\star$-hypermodule. By Theorem 3.1 there exists a group isomorphism

$$(\boldsymbol{G}_{\mathbb{Z}}(M \circledast_R N), \oplus) \cong (M \circledast_R N, \oplus).$$

Thus the map $\boldsymbol{i} : M \times N \longrightarrow M \circledast_R N$ defined by

$$(x, y) \longmapsto x \circledast_R y$$

is a middle linear β-map and it is called the *canonical middle linear β-map*;

(d) Every sets (ht1) to (ht3) are a singleton set.

(e) If $f \in \mathcal{F}_{\mathbb{Z}}(M \times N)$, then $\{f\} = \sum_{(x,y) \in A_f} f(x, y)\delta_{(x,y)}$, where $A_f = \{(x, y) \in M \times N \mid f(x, y) \neq 0\}$ is a finite subset of $M \times N$. Thus $\{[f]\} = \sum_{(x,y) \in A_f} f(x, y)x \circledast_R y = \left\{ \bigoplus_{(x,y) \in A_f} f(x, y)x \circledast_R y \right\}$.

(f) $M(0) \circledast_R N = M \circledast_R N(0) = 0$.

(g) Let R be a ring. Let $(M,+)$ be a $\star_R$-hypermodule and $(N,+)$ be a ${}_R\star$-hypermodule derived from right R-modules $(M,\oplus)$ and left R-module $(N,\oplus)$, respectively. Then $M\otimes_R N \cong M\circledast_R N$, where $M\otimes_R N$ is the tensor product of the modules M and N

Theorem 3.7. *Let R be a hyperring and M be a $\star_R$-hypermodule and N be a ${}_R\star$-hypermodule. Then $i : M\times N \xrightarrow{\mathcal{M}} M\circledast_R N$ is a universal object in the category $\mathscr{M}_H(M,N)$.*

Proof. Let C be a canonical hypergroup and $\psi : M\times N \xrightarrow{\mathcal{M}} C$ be a middle linear β-map. By Theorem 2.2(1), C is ${}_{\mathbb{Z}}\star$-hypermodule and, by Theorem 3.2, we have the commutative diagram

$$\begin{tikzcd}
M\times N \arrow[r, "\bar{\iota}_{M\times N}"] \arrow[d, "\psi"'] & \mathcal{F}_{\mathbb{Z}}(M\times N) \arrow[dl, dotted, "\boldsymbol{G}_R(\widehat{\psi})"] \\
C &
\end{tikzcd}$$

where $\widehat{\psi} : \mathcal{F}_{\mathbb{Z}}(M\times N) \xrightarrow{\beta} C$ is the unique ${}_{\mathbb{Z}}\star$-homomorphism, which is a β-homomorphism, such that $\boldsymbol{G}_{\mathbb{Z}}(\widehat{\psi})\bullet\bar{\iota}_{M\times N} = \psi$. Thus $\widehat{\psi}(\delta_{(x,y)}) = \psi(x,y)$ for each $(x,y)\in M\times N$. Since ψ is a middle linear β-map, the image of every sets (ht1) to (ht3) under $\widehat{\psi}$ is $[0]_{{}_{\mathbb{Z}}C}$ and so $\mathcal{K}\subseteq \mathrm{Ke}(\widehat{\psi})$. Consider the mapping $\alpha : \mathcal{F}_{\mathbb{Z}}(M\times N)/\mathcal{K} \longrightarrow \boldsymbol{G}_{\mathbb{Z}}(C)$ that defined by $\alpha([f]) := \widehat{\psi}(f)$. But $\{f\} = \sum\limits_{(x,y)\in A_f} f(x,y)\delta_{(x,y)}$, where $A_f = \{(x,y)\in M\times N \mid f(x,y)\neq 0\}$ is a finite subset of $M\times N$. If $A_f\neq\emptyset$, then

$$\widehat{\psi}(f) = \bigoplus_{(x,y)\in A_f} f(x,y)\widehat{\psi}(\delta_{(x,y)}) = \bigoplus_{(x,y)\in A_f} f(x,y)\psi(x,y).$$

If $A_f=\emptyset$, then $\widehat{\psi}(f) = [0]_{{}_{\mathbb{Z}}C}$. Hence

$$\alpha([f]) = \begin{cases} \bigoplus\limits_{(x,y)\in A_f} f(x,y)\psi(x,y) & \text{if } A_f\neq\emptyset \\ [0]_{{}_{\mathbb{Z}}C}. & \text{if } A_f=\emptyset \end{cases}$$

For any $f, g \in \mathcal{F}_{\mathbb{Z}}(M \times N)/\mathcal{K}$, $f + g = \{h\}$ such that for each $(x, y) \in M \times N$, $h(x, y) = f(x, y) + g(x, y)$. Also $A_h \subseteq A_f \cup A_g$. If $A_h \neq \emptyset$, then $\alpha([f]) \oplus \alpha([g]) = \widehat{\psi}(f) \oplus \widehat{\psi}(g) = \widehat{\psi}(h) = \alpha([h])$. If $A_h = \emptyset$, then $h \equiv 0$ and hence $f \equiv -g$. Thus $A_f = A_g$ and so $\alpha([h]) = \alpha([f]) \oplus \alpha([g]) = [0]_{\mathbb{Z}^C}$. But $\mathcal{F}_{\mathbb{Z}}(M \times N)/\mathcal{K} = M \circledast_R N$ and $[\delta_{(x,y)}] = x \circledast_R y$. Therefore $\alpha : M \circledast_R N \xrightarrow{\beta} C$ is a β-homomorphism such that $\alpha \blacklozenge i = \psi$. Let $\lambda : M \circledast_R N \xrightarrow{\beta} C$ be any β-homomorphism such that $\lambda \blacklozenge i = \psi$ and $[h] \in \mathcal{F}_{\mathbb{Z}}(M \times N)/\mathcal{K}$. Since $\{h\} = \sum\limits_{(x,y) \in A_h} h(x, y)\delta_{(x,y)}$, we have

$$\begin{aligned}
\lambda([h]) &= \bigoplus_{(x,y) \in A_h} h(x, y)\lambda([\delta_{(x,y)}]) \\
&= \bigoplus_{(x,y) \in A_h} h(x, y)\lambda \blacklozenge i(x, y) \\
&= \bigoplus_{(x,y) \in A_h} h(x, y)\psi(x, y) \\
&= \bigoplus_{(x,y) \in A_h} h(x, y)\alpha \blacklozenge i(x, y) \\
&= \alpha([h]).
\end{aligned}$$

Thus $\alpha = \lambda$ and the proof is complete. $\square$

Lemma 3.8. *Let M be a $\star_R$-hypermodule and N be a ${}_R\star$-hypermodule. For all $x, x' \in M; y, y' \in N$ we have:*

(i) *If $[x]_{M_R} = [x']_{M_R}$, then $\delta_{(x,y)} - \delta_{(x',y)} \subseteq \mathcal{K}$;*

(ii) *If $[y]_{{}_RN} = [y']_{{}_RN}$, then $\delta_{(x,y)} - \delta_{(x,y')} \subseteq \mathcal{K}$;*

(iii) *If $t \in [x]_{M_R} + [x']_{M_R}$, then $\delta_{(t,y)} - \delta_{(x,y)} - \delta_{(x',y)} \subseteq \mathcal{K}$;*

(iv) *If $s \in [y]_{{}_RN} + [y']_{{}_RN}$, then $\delta_{(x,s)} - \delta_{(x,y)} - \delta_{(x,y')} \subseteq \mathcal{K}$.*

Proof. (i) Since $[x]_{M_R} = [x']_{M_R}$, there exists $(z_1, \cdots, z_k) \in M^k$ for some $k \in \mathbb{N}$ such that $x - x' \subseteq \sum\limits_{i=1}^{k}(z_i - z_i)$. Let $z \in x - x' \cap \sum\limits_{i=1}^{k}(z_i - z_i)$. Therefore $x \in x' + z$ and hence $\delta_{(x,y)} - \delta_{(x',y)} - \delta_{(z,y)} \subseteq \mathcal{K}$. Also $z \in \sum\limits_{i=1}^{k} z'_i$ such that $z'_i \in z_i - z_i$ for each $1 \leq i \leq k$. Thus $\delta_{(z,y)} - \left(\sum_{i=1}^{k} \delta_{(z'_i,y)}\right) \subseteq \mathcal{K}$ and $z_i \in z'_i + z_i$. Therefore $\delta_{(z_i,y)} - \delta_{(z'_i,y)} - \delta_{(z_i,y)} \subseteq \mathcal{K}$ and hence $\delta_{(z'_i,y)} \in \mathcal{K}$. This implies that $\delta_{(z,y)} \in \mathcal{K}$. So

$$\begin{aligned}
\delta_{(x,y)} - \delta_{(x',y)} &\subseteq \delta_{(x,y)} - \delta_{(x',y)} + (\delta_{(z,y)} - \delta_{(z,y)}) \\
&\subseteq (\delta_{(x,y)} - \delta_{(x',y)} - \delta_{(z,y)}) + \delta_{(z,y)} \\
&\subseteq \mathcal{K} + \mathcal{K} \subseteq \mathcal{K}.
\end{aligned}$$

(ii) The proof is similar to (i).

(iii) Let $x_1 \in [x]_{M_R}$ and $x'_1 \in [x']_{M_R}$ such that $t \in x_1 + x'_1$. Therefore $\delta_{(t,y)} - \delta_{(x_1,y)} - \delta_{(x'_1,y)} \subseteq \mathcal{K}$ and (i) implies that $\delta_{(x_1,y)} - \delta_{(x,y)} \subseteq \mathcal{K}$ and $\delta_{(x'_1,y)} - \delta_{(x',y)} \subseteq \mathcal{K}$. Thus

$$\begin{aligned}
\delta_{(t,y)} - \delta_{(x,y)} - \delta_{(x',y)} &\subseteq \delta_{(t,y)} - \delta_{(x,y)} - \delta_{(x',y)} + (\delta_{(x_1,y)} - \delta_{(x_1,y)}) + (\delta_{(x'_1,y)} - \delta_{(x'_1,y)}) \\
&= (\delta_{(t,y)} - \delta_{(x_1,y)} - \delta_{(x'_1,y)}) - (\delta_{(x_1,y)} - \delta_{(x,y)}) - (\delta_{(x'_1,y)} - \delta_{(x',y)}) \\
&\subseteq \mathcal{K}.
\end{aligned}$$

(iv) The proof is similar to (iii). □

Corollary 3.9. *Let R be a hyperring and M, M' be a $\star_R$-hypermodule and N, N' be a ${}_R\star$-hypermodule. If $f : M \xrightarrow{\star} M'$ is a $\star_R$-homomorphism and $g : N \xrightarrow{\star} N'$ is a ${}_R\star$-homomorphism, then there is a unique β-homomorphism $M \circledast_R N \xrightarrow{\beta} M' \circledast_R N'$ such that*

$$x \circledast_R y \longmapsto x_f \circledast_R y_g,$$

where $x_f \in f(x)$ and $y_g \in g(y)$ for all $x \in M, y \in N$.

Proof. Define $\psi : M \times N \longrightarrow M' \circledast_R N'$ by $\psi(x, y) := x_f \circledast_R y_g$ such that $x_f \in f(x)$ and $y_g \in g(y)$. Let $p \in f(x)$ and $q \in g(y)$, so $[x_f]_{M'_R} = [p]_{M'_R}$ and $[y_g]_{{}_R N'} = [q]_{{}_R N'}$. Theorem 3.8 implies that $x_f \circledast_R y_g = p \circledast_R y_g$ and $p \circledast_R y_g = p \circledast_R q$. Thus ψ is well defined (that is, independent of choice x_f and y_g). Now we show that ψ is a middle linear β-map.

(mh1) If $w \in x + x'$, then $f(w) = f(x) \oplus f(y)$. So there exists $p \in f(x)$ and $p' \in f(x')$ such that $w_f \in p + p'$. Since $[x_f]_{M'_R} = [p]_{M'_R}$ and $[x'_f]_{M'_R} = [p']_{M'_R}$, $w_f \circledast_R y_g = p \circledast_R y_g \oplus p' \circledast_R y_g = x_f \circledast_R y_g \oplus x'_f \circledast_R y_g$.

(mh2) It is valid as (mh1).

(mh3) Let $u \in xr$ and $v \in ry$. Thus $[u_f]_{M'_R} = [p]_{M'_R}$ and $[v_g]_{{}_RN'} = [q]_{{}_RN'}$, where $p \in x_f r$ and $q \in r y_g$. Therefore $u_f \circledast_R y_g = p \circledast_R y_g = x_f \circledast_R q = x_f \circledast_R v_g$.

So ψ induces a unique β-homomorphism $\widehat{\psi} : M \circledast_R N \xrightarrow{\beta} M' \circledast_R N'$ such that $\widehat{\psi}(x \circledast_R y) = x_f \circledast_R y_g$. $\square$

The unique β-homomorphism $M \circledast_R N \xrightarrow{\beta} M' \circledast_R N'$ in the above corollary is called the *induced map by f and g* and it is denoted by $f \circledast g$.

The part (i) and (iii) of the following theorem uses tensor product to extend scalars.

Theorem 3.10. *Let R and S be a hyperring and given hypermodules ${}_{S^\star}M_{\star_R}$, ${}_{S^\star}M'_{\star_R}$, ${}_{R^\star}N$, ${}_{R^\star}N'$, $P_{\star_R}$, $P'_{\star_R}$, ${}_{R^\star}Q_{\star_S}$, ${}_{R^\star}Q'_{\star_S}$.*

(i) *$M \circledast_R N$ is a ${}_S\star$-hypermodule.*

(ii) *Let $f : M \xrightarrow{\star} M'$ be a homomorphism of (S, R)-bihypermodules. If $g : N \xrightarrow{\star} N'$ is a ${}_R\star$-homomorphism, then*

$$f \circledast g : M \circledast_R N \xrightarrow{\star} M' \circledast_R N',$$

that is the induced map by f and g, is a ${}_S\star$-homomorphism.

(iii) *$P \circledast_R Q$ is $\star_S$-hypermodule.*

(iv) *Let $h : P \xrightarrow{\star} P'$ be a $\star_R$-homomorphism. If $k : Q \xrightarrow{\star} Q'$ is a homomorphism of (R, S)-bihypermodules, then*

$$h \circledast k : P \circledast_R Q \xrightarrow{\star} P' \circledast_R Q',$$

that is, the induced map by h and k, is a $\star_S$-homomorphism.

Proof. (i) Let $s \in S$. Define $\psi_s : M \times N \longrightarrow M \circledast_R N$ by $\psi_s(x, y) := z \circledast_R y$ for some $z \in sx$. If $z' \in sx$, then $z - z' \subseteq s(x - x)$. Since $(x - x) \subseteq M(0)$ and $M(0)$ is a ${}_S\star$-subhypermodule of M, $s(x - x) \subseteq M(0)$. Thus $z - z' \subseteq M(0)$.

Let $z_1 \in z - z'$, so $z_1 \in \sum_{i=1}^{k}(z_i - z_i)$ for some $z_i \in M$. This implies that $[z]_{M_R} = [z']_{M_R} \oplus [z_1]_{M_R}$ and $[z_1]_{M_R} = [0]_{M_R}$. Therefore $[z]_{M_R} = [z']_{M_R}$ and by Theorem 3.8(i) we have $\delta_{(z,y)} - \delta_{(z',y)} \subseteq \mathcal{K}$. Thus $z \circledast_R y = z' \circledast_R y$ and hence ψ_s is well defined (that is, independent of choice $z \in sx$). Now we show that ψ_s is a middle linear β-map. Let $x, x_i \in M$, $y, y_i \in N$, $r \in R$.

(mh1) If $w \in x_1 + x_2$, then $\psi_s(w, y) = z \circledast_R y$, $\psi_s(x_1, y) = z_1 \circledast_R y$ and $\psi_s(x_2, y) = z_2 \circledast_R y$, where $z \in sw \subseteq sx_1 + sx_2$, $z_1 \in sx_1$ and $z_2 \in sx_2$. Thus $z \in z_1' + z_2'$ for some $z_1' \in sx_1$ and $z_2' \in sx_2$. As above $[z_1']_{M_R} = [z_1]_{M_R}$, $[z_2']_{M_R} = [z_2]_{M_R}$ and hence $z \circledast_R y = z_1' \circledast_R y \oplus z_2 \circledast_R y = z_1 \circledast_R y \oplus z_2 \circledast_R y$.

(mh2) If $l \in y_1 + y_2$, then $\psi_s(x, l) = z \circledast_R l$, $\psi_s(x, y_1) = z_1 \circledast_R y_1$ and $\psi_s(x, y_2) = z_2 \circledast_R y_2$, where $z, z_1, z_2 \in sx$. Therefore $[z]_{M_R} = [z_1]_{M_R} = [z_2]_{M_R}$ and hence $z \circledast_R l = z \circledast_R y_1 \oplus z \circledast_R y_2 = z_1 \circledast_R y_1 \oplus z_2 \circledast_R y_2$.

(mh3) If $u \in xr$ and $v \in ry$, then $\psi_s(u, y) = z_1 \circledast_R y$ and $\psi_s(x, v) = z_2 \circledast_R v$, where $z_1 \in su$ and $z_2 \in sx$. Therefore $z_1 \in (sx)r$, and so there exists $z' \in sx$ such that $z_1 \in z'r$. Thus $[z']_{M_R} = [z_2]_{M_R}$. Since $z_1 \circledast_R y = z' \circledast_R v$ and $z' \circledast_R v = z_2 \circledast_R v$, $z_1 \circledast_R y = z_2 \circledast_R v$.

So ψ_s induces a unique β-homomorphism $\widehat{\psi_s} : M \circledast_R N \xrightarrow{\beta} M \circledast_R N$ such that $\widehat{\psi_s}(x \circledast_R y) = z \circledast_R y$, where $z \in sx$.

Define ${}_S\star_{(M \circledast_R N)} : \mathcal{P}^*(S) \times (M \circledast_R N) \longrightarrow \mathcal{P}^*(M \circledast_R N)$ by

$$ {}_R\star_{(M \circledast_R N)}(A, [f]) := \{\widehat{\psi_s}([f]) \mid s \in A\}. $$

Thus ${}_S\star_{(M \circledast_R N)}$ satisfies in the conditions (sm1) and (sm2). Suppose that $\centerdot : S \times (M \circledast_R N) \longrightarrow \mathcal{P}^*((M \circledast_R N)$ is the external composition with respect to ${}_S\star_{(M \circledast_R N)}$, that is, for $[f] = \bigoplus_{(x,y) \in A_f} f(x, y)x \circledast_R y \in M \circledast_R N$ we have $s \centerdot [f] = {}_S\star_{(M \circledast_R N)}(\{s\}, [f]) = \{\widehat{\psi_s}([f])\} = \{\bigoplus_{(x,y) \in A_f} f(x, y)z \circledast_R y\}$, where $z \in sx$. It is easy to see that this external composition, yields $M \circledast_R N$ as a ${}_S\star$-hypermodule.

(ii) It is enough to show that for each $s \in S, x \in M$ and $y \in N$, $\{f \circledast g(z \circledast_R y)\} = s(f \circledast g(x \circledast_R y))$, where $z \in sx$. Which is equivalent to prove that $z_f \circledast_R y_g = z' \circledast_R y_g$ such that $z' \in sx_f$. Since f is a ${}_S\star$-homomorphism, $[z_f]_{{}_SM'} = f(z) = [x']_{{}_SM'}$ for some $x' \in sx_f$. Therefore

$[z_f]_{{}_SM'} = [x']_{{}_SM'} = [z']_{{}_SM'}$ and hence $z_f - z' \subseteq \sum_{i=1}^{k}(x'_i - x'_i)$ for some $x'_i \in M'$. This implies that $z_f - z' \subseteq M'(0)$ and so $[z_f]_{M'_R} = [z']_{M'_R}$. Thus $z_f \circledast_R y_g = z' \circledast_R y_g$.

The proof of (iii) and (iv) is similar to (i) and (ii), respectively. $\square$

Remark 3.11. Suppose that M is a $\star_R$-hypermodule. Therefore $M(0)$ is a ${}_R\star$-hypermodule by a canonical way and hence Theorem 3.10 implies that $M \circledast_R - : {}_R\star\!-\mathbf{HM} \longrightarrow \mathbf{CHg}$ is a functor.

Theorem 3.12. (Adjoint Isomorphism for Hypermodules) *Given hypermodules ${}_{S^\star}M_{\star_R}$, ${}_{R^\star}N$ and ${}_{S^\star}K$, where R and S are hyperrings. Then there is a natural isomorphism:*

$$\tau_{N,M,K} : Hom_R(N, Hom_S(M,K)) \longrightarrow Hom_S(M \circledast_R N, K)$$

Proof. Let $f \in \mathrm{Hom}_R(N, \mathrm{Hom}_S(M,K))$ so $f(y) = [f_y]_{{}_RH} = \{f_y\}$, where $H := \mathrm{Hom}_S(M,K)$ and $y \in N$. Define $\psi_f : M \times N \longrightarrow \frac{K}{K(0)}$ by $\psi_f(x,y) := f_y(x)$. To prove that ψ_f is a middle linear β-map we only check the condition (mh3). Let $u \in xr$ and $v \in ry$. Thus $\{f(v)\} = r \centerdot' f(y) = r \centerdot' [f_y]_{{}_RH} = \{[g]_{{}_RH}\}$ such that $g \in r \centerdot f_y = \{(f_y)_r\}$, where $(f_y)_r : M \longrightarrow K$ is defined by $(f_y)_r(z) = f_y(q)$ such that $q \in rz$, see Theorem 2.8(a). Therefore $f_v = (f_y)_r$ and hence $\psi_f(x,v) = f_v(x) = (f_y)_r(x) = f_y(u) = \psi_f(u,y)$ as desired. Thus there exists a unique β-homomorphism $\widehat{\psi_f} : M \circledast_R N \longrightarrow K$ such that $\widehat{\psi_f} \blacklozenge \boldsymbol{i} = \psi_f$. Now, let $s \in S$ and $[f] = \bigoplus_{(x,y)\in A_f} f(x,y)x \circledast_R y$ so $s \centerdot [f] = \{\bigoplus_{(x,y)\in A_f} f(x,y)z \circledast_R y\}$, where $z \in sx$. Also $\widehat{\psi_f}(z \circledast_R y) = \psi_f(z,y) = f_y(z)$. Since $f : M \longrightarrow K$ is a ${}_S\star$-homomorphism, $s \centerdot' \widehat{\psi_f}(x \circledast_R y) = s \centerdot' f_y(x) = \{f_y(z)\}$. This implies that $\{\widehat{\psi_f}(z \circledast_R y)\} = s \centerdot' \widehat{\psi_f}(x \circledast_R y)$ and hence $\widehat{\psi_f}$ is a ${}_S\star$-homomorphism. Define $\tau_{N,M,K}(f) ;= \widehat{\psi_f}$. It is easy to see that $\tau_{N,M,K}$ is natural and bijective map and it is a good homomorphism between canonical hypergroups. $\square$

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