Categories and
General Algebraic Structures
with Applications

Volume 16, Number 1, January 2022, 1-28.
https://doi.org/10.52547/cgasa.16.1.1

WWW.CGASA.ir

A new property of congruence lattices of slim, planar, semimodular lattices

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Dedicated to Béla Csákány on his forthcoming ninetieth birthday

Abstract. The systematic study of planar semimodular lattices started in 2007 with a series of papers by G. Grätzer and E. Knapp. These lattices have connections with group theory and geometry. A planar semimodular lattice L is *slim* if M_3 it is not a sublattice of L. In his 2016 monograph, "The Congruences of a Finite Lattice, A *Proof-by-Picture Approach*", the second author asked for a characterization of congruence lattices of slim, planar, semimodular lattices. In addition to distributivity, both authors have previously found specific properties of these congruence lattices. In this paper, we present a new property, the *Three-pendant Three-crown Property*. The proof is based on the first author's papers: 2014 (multifork extensions), 2017 ($\mathcal{C}_1$-diagrams), and a recent paper (lamps), introducing the tools we need.

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Keywords: Rectangular lattice, patch lattice, slim semimodular lattice, congruence lattice, lattice congruence, Three-pendant Three-crown Property.
Mathematics Subject Classification [2010]: 06C10.
Received: 8 March 2021, Accepted: 21 August 2021.
ISSN: Print 2345-5853, Online 2345-5861.

1 Introduction

1.1 The Main Theorem The book G. Grätzer [18] presents many results characterizing congruence lattices of various classes of finite lattices, spanning 80 years, up to 2015. In particular, in 1996, G. Grätzer, H. Lakser, and E. T. Schmidt [26] started looking at the class of semimodular lattices and were surprised: every finite distributive lattice can be represented as the congruence lattice of a *planar* semimodular lattice.

The sublattice M_3 played a crucial role in the Grätzer-Lakser-Schmidt construction, so it was natural to ask (see Problem 1 in G. Grätzer [19], originally raised in G. Grätzer [18]) what happens if, in addition to planarity and semimodularity, we also assume that the lattice is *slim*, that is, it does not have M_3 sublattices.

Open Problem 1.1. What are the congruence lattices of slim, planar, semimodular lattices?

We call a slim, planar, semimodular lattice an SPS lattice. A finite distributive lattice D is *representable by an SPS lattice L* (in short, *representable*) if D is isomorphic to the congruence lattice $\operatorname{Con} L$ of an SPS lattice L.

We say that a finite distributive lattice D satisfies the *Three-pendant Three-crown Property* if the ordered set R_3 of Figure 1 has no cover-preserving embedding into $\mathrm{J}(D)$.

Our paper continues the research in G. Czédli [7] that presented four new properties of $\operatorname{Con} L$. We provide one more.

Now we can state our result.

Main Theorem. Let L be a slim, planar, semimodular lattice. Then $\operatorname{Con} L$ satisfies the Three-pendant Three-crown Property.

We have one more theorem in this paper.

Theorem 1.2. *Let n be a positive integer number and $L_1, \dots, L_n$ be slim, planar, semimodular lattices with at least three elements. Then there exists a slim rectangular lattice H and a slim patch lattice L such that the following two isomorphisms hold:*

$$\operatorname{Con} H \cong \operatorname{Con} L_1 \times \cdots \times \operatorname{Con} L_n, \tag{1.1}$$

$$\operatorname{Con} L \cong \big(\operatorname{Con} L_1 \times \cdots \times \operatorname{Con} L_n\big) + \mathsf{B}_2, \tag{1.2}$$

In (1.2), the operation + stands for *ordinal sum*, also known as *linear sum* (the second summand is put on the top of the first one). We define rectangular lattices and patch lattices in Section 2.

1.2 Background G. Grätzer and E. Knapp [21]– [25] started the study of planar semimodular lattices. There are a number of surveys of this field, see the book chapter G. Czédli and G. Grätzer [10] in G. Grätzer and F. Wehrung, eds. [29], and G. Czédli and Á. Kurusa [11]. For the topic: congruences of planar semimodular lattices, see the book chapter G. Grätzer [15] in G. Grätzer and F. Wehrung, eds. [29].

This research have also led to results outside of lattice theory: to a group theoretical result by G. Czédli and E. T. Schmidt [13] and G. Grätzer and J. B. Nation [27], and to (combinatorial) geometric results by G. Czédli [4] and [6], K. Adaricheva and G. Czédli [1], and G. Czédli and Á. Kurusa [11]. G. Czédli and G. Makay [12] presented a computer game based on these lattices. G. Czédli [8] is a related model theoretic paper. Note that more than four dozen papers have been devoted to planar semimodular lattices and their applications since G. Grätzer and E. Knapp's 2007 paper [21]. The list of 48 of these papers is included in the appendix of G. Czédli [9]; see also http://www.math.u-szeged.hu/ czedli/m/listak/publ-psml.pdf for an updated list.

The next two theorems summarize what we know about congruence lattices of SPS lattices. (In both theorems, the covering relations are those of the ordered set $\mathrm{J}(\mathrm{Con}\, L)$ and not of the lattice $\mathrm{Con}\, L$.)

Theorem 1.3 (G. Grätzer [19] and [20]). *Let L be an SPS lattice with at least three elements.*

(i) *The ordered set* $\mathrm{J}(\mathrm{Con}\, L)$ *has at least two maximal elements.* (*Equivalently,* $\mathrm{Con}\, L$ *has at least two coatoms.*)

(ii) *Every element of the ordered set* $\mathrm{J}(\mathrm{Con}\, L)$ *has at most two covers.*

We know, see G. Grätzer [20], that the three-element chain C_3 cannot be isomorphic to the congruence lattice of an SPS lattice L, though $\mathrm{J}(\mathsf{C}_3)$ has only one maximal element. This shows that the necessary condition (1.3) for representability is not sufficient. G. Czédli [3] provides an eight element distributive lattice to show that the necessary condition (ii) for representability is not sufficient.

Our paper is a continuation of G. Czédli [7]. Here are some of the results of this paper.

Theorem 1.4 (G. Czédli [7]). *Let L be an SPS lattice with at least three elements.*

(i) *The set of maximal elements of the ordered set* $\mathrm{J}(\mathrm{Con}\, L)$ *can be represented as the disjoint union of two nonempty subsets such that no two distinct elements in the same subset have a common lower cover.*

(ii) *The ordered set R of Figure 1 cannot be embedded as a cover-preserving subset into the ordered set* $\mathrm{J}(\mathrm{Con}\, L)$ *provided that every maximal element of R is a maximal element of* $\mathrm{J}(\mathrm{Con}\, L)$.

(iii) *If* $x \in \mathrm{J}(\mathrm{Con}\, L)$ *is covered by a maximal element y of* $\mathrm{J}(\mathrm{Con}\, L)$, *then y is not the only cover of x in the ordered set* $\mathrm{J}(\mathrm{Con}\, L)$.

(iv) *Let* $x \neq y \in \mathrm{J}(\mathrm{Con}\, L)$, *and let z be a maximal element of* $\mathrm{J}(\mathrm{Con}\, L)$. *Assume that both x and y are covered by z in the ordered set* $\mathrm{J}(\mathrm{Con}\, L)$. *Then there is no element* $u \in \mathrm{J}(\mathrm{Con}\, L)$ *such that u is covered by x and y in* $\mathrm{J}(\mathrm{Con}\, L)$.

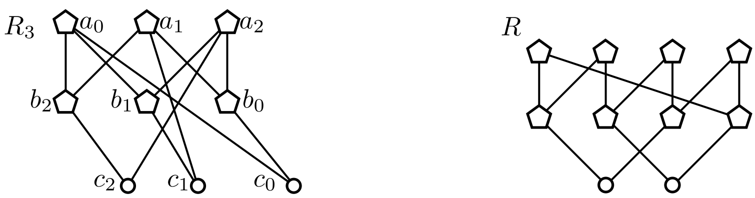

Figure 1: The Three-pendant Three-crown ordered set R_3 and the Two-pendant Four-crown ordered set R; the elements of the crowns are indicated by pentagons

Outline Section 2 recalls some concepts we need. Section 3 recalls some of the tools developed in G. Czédli [7] while we develop some new tools in Section 4. We prove our Main Theorem in Section 5. Finally, Section 6 proves Theorem 1.2 and discusses what we know about the congruence lattices of slim patch lattices.

2 Basic notation and concepts

All lattices in this paper are finite. We assume that the reader is familiar is with the rudiments of lattice theory. Most basic concepts and notation not defined in this paper are available in Part I of the monograph G. Grätzer [18], which is free to access. In particular,

$$\text{the } \textit{glued sum} \text{ of two lattices } A \text{ and } B \text{ is denoted by } A \dotplus B \tag{2.1}$$

(B is on the top of A with the unit element of A and the zero of B identified, so $\mathsf{C}_2 \dotplus \mathsf{C}_2$ is C_3). The n-element chain is C_n, the Boolean lattice with n atoms is B_n, and M_3 is the 5-element modular nondistributive lattice. The *set of maximal elements* of an ordered set P will be denoted by $\mathrm{Max}(P)$. In this paper, edges are synonymous with prime intervals.

For a finite lattice L, the set of (non-zero) join-irreducible elements and (non-unit) meet-irreducible elements will be denoted by $\mathrm{J}(L)$ and $\mathrm{M}(L)$, respectively, so $\mathrm{J}(L) \cap \mathrm{M}(L)$ is the set of *doubly irreducible* elements. We denote by x^* the unique cover of x for $x \in \mathrm{M}(L)$. For an element $a \in L$, let $\downarrow a = \{\, x \in L \mid x \leq a \,\}$ be the *principal ideal generated by* a and $\uparrow a = \{\, x \in L \mid x \geq a \,\}$ the *principal filter generated by* a.

A planar semimodular lattice is *slim* if it does not contain M_3 as a sublattice; see G. Grätzer and E. Knapp [21], [24], G. Czédli and E. T. Schmidt [13].

Let L be a planar lattice. A *left corner* $\mathrm{lc}(L)$ (resp., right corner $\mathrm{rc}(L)$) of L is a doubly-irreducible element in $L - \{0, 1\}$ on the left (resp., right) boundary of L. We define a *rectangular lattice* L as a planar semimodular lattice which has exactly one left corner, $\mathrm{lc}(L)$, and exactly one right corner, $\mathrm{rc}(L)$, and they are complementary, that is, $\mathrm{lc}(L) \vee \mathrm{rc}(L) = 1$ and $\mathrm{lc}(L) \wedge \mathrm{rc}(L) = 0$ (see G. Grätzer and E. Knapp [21]). Finally, a rectangular lattice in which both corners are coatoms are called a *patch lattice*.

3 Tools

We call the directions of $(1, 1)$ and $(1, -1)$ *normal* and any other direction $(\cos\alpha, \sin\alpha)$ with $\pi/2 < \alpha < 3\pi/2$ *steep*. (In [5] and other papers, the first author uses "*precipitous*" instead of "steep".) Edges and lines parallel to a steep vector are also called steep, and similarly for normal slopes.

The following definition and result are crucial in the study of SPS lattices.

Definition 3.1 (G. Czédli [5]). A diagram of the slim rectangular lattice L is a $\mathcal{C}_1$*-diagram* if it has the following two properties.

(i) If $x \in \mathrm{M}(L) - (\uparrow \mathrm{lc}(L) \cup \uparrow \mathrm{rc}(L))$, then the edge $[x, x^*]$ is steep.

(ii) Every edge not of the form $[x, x^*]$ as in (i) has a normal slope.

If, in addition,

(iii) any two edges on the lower boundary are of the same geometric length,

then the diagram is a $\mathcal{C}_2$*-diagram*.

Theorem 3.2 (G. Czédli [5]). *Every slim rectangular lattice L has a $\mathcal{C}_2$-diagram.*

In this section, L is a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram, as we shall soon define. The chains $\downarrow \mathrm{lc}(L)$, $\uparrow \mathrm{lc}(L)$, $\downarrow \mathrm{rc}(L)$, and $\uparrow \mathrm{rc}(L)$ are called the *bottom left boundary chain*, ..., *top right boundary chain*. These chains have normal slopes and they are the sides of a geometric rectangle, which we call the *full geometric rectangle* of L and denote it by $\mathrm{FulR}(L)$. The four vertices of this rectangle are 0, 1, $\mathrm{lc}(L)$, and $\mathrm{rc}(L)$. The *lower boundary* of L is $\downarrow \mathrm{lc}(L) \cup \downarrow \mathrm{rc}(L)$ and the *upper boundary* is $\uparrow \mathrm{lc}(L) \cup \uparrow \mathrm{rc}(L)$. With the exception of the corners, no meet-irreducible element belongs to the lower boundary of L.

The following is the central definition of G. Czédli [7].

Definition 3.3.

(A) Let L be a slim rectangular lattice. The edges $[x, y]$ of L with $x \in \mathrm{M}(L)$ are called *neon tubes*. We call a neon tube $[x, y]$ on the upper boundary of L a *boundary neon tube*; it is an *internal neon tube*, otherwise. Equivalently, neon tubes with normal slopes are boundary neon tubes, while steep neon tubes are internal.

In Figures 2, 11, and 12, we represent the neon tubes by thick edges.

(B) A boundary neon tube $\mathfrak{n} = [p, q]$ is also called a *boundary lamp*. This lamp I is an edge, the neon tube $\mathfrak{n}$ is the *neon tube of the lamp I*. Define $\mathrm{Foot}(I)$ as p and $\mathrm{Peak}(I)$ as q. If $\mathrm{Foot}(I)$ is on the top left boundary chain, then I is a *left boundary lamp*; similarly, we define *right boundary lamps*.

In Figure 2, the left boundary lamps and the right boundary lamps are $P_1, \dots, P_5$ and $Q_1, \dots, Q_6$, respectively, and $p_i = \mathrm{Foot}(P_i)$ and $q_j = \mathrm{Foot}(Q_j)$ for all i and j.

(C) Every steep (that is, internal) neon tube $\mathfrak{n} = [p, q]$ belongs to a unique *internal lamp* $I = [\beta_q, q]$, where β_q is the meet of all $p' \in L$ such that $[p', q]$ is a steep neon tube. For the lamp I, define the $\mathrm{Foot}(I)$ as β_q and the *peak* $\mathrm{Peak}(I)$ as q.

In Figure 2, there are five internal lamps, $A, \dots, E$ with $\mathrm{Foot}(A) = a$, $\mathrm{Foot}(B) = b$, and so on; also, $\mathrm{Peak}(A) = g$, $\mathrm{Peak}(B) = h$, and $\mathrm{Peak}(C) = z$; so $A = [a, g]$, $B = [b, h]$, and $C = [c, z]$.

(D) The set $\mathrm{Lamp}(L)$ consists of all lamps of L. For example, for the lattice L in Figure 2, there are 16 lamps in L.

(E) A lamp I determines a geometric region (as in David Kelly and I. Rival [30]) which we call the *body* of I, and denote it by $\mathrm{Body}(I)$. It has a geometric shape: it is either a line segment or a quadrilateral whose lower sides have normal slopes and whose upper sides are steep.

In Figure 2, the regions $\mathrm{Body}(A)$, $\mathrm{Body}(B)$, and $\mathrm{Body}(C)$ are colored dark-grey.

For later reference, we recall by G. Czédli [7, Lemma 3.1] that

$$\text{A lamp is uniquely determined by its foot.} \tag{3.1}$$

The feet of our lamps are black-filled in Figures 2–12; this helps us find them.

In the real world, lamps emit light. Our lamps do it in a special way: the light rays go from all points of the neon tubes of a lamp I downward with normal slopes. Next we give our definition of light emission. For an element $x \in L$, we define the line segment $\mathrm{Line}_{\mathrm{L}}(x)$ from x left and down, of normal slope to the lower-right boundary of L. Similarly, to the right, we have $\mathrm{Line}_{\mathrm{R}}(x)$.

So for a lamp I, we have the four line segments, from $\mathrm{Peak}(I)$ and $\mathrm{Foot}(I)$, left and right. We denote them by $\mathrm{LRoof}(I)$ (the left roof), $\mathrm{RRoof}(I)$ (the right roof), $\mathrm{LFloor}(I)$, (the left floor) and $\mathrm{RFloor}(I)$ (the right floor).

Definition 3.4 (G. Czédli [7])**.** For a lamp I of a slim rectangular lattice L, we define

(i) *the area left lit by I* (or, as in [7], *illuminated from the right by I*), denoted by $\mathrm{LeftLit}(I)$, is a quadrangle bounded by the line segments $\mathrm{Line}_{\mathrm{L}}(\mathrm{Peak}(I))$, $\mathrm{Line}_{\mathrm{L}}(\mathrm{Foot}(I))$, the upper right boundary of I, and the appropriate line segment of the lower left boundary of L.

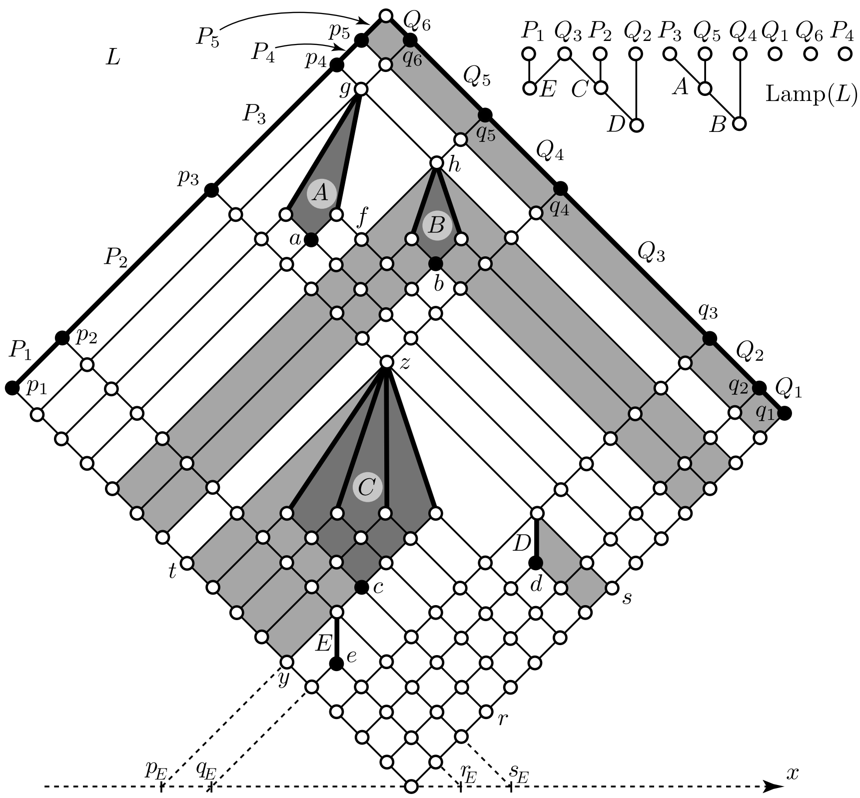

Figure 2: Lamps and related geometric objects

(ii) *the area right lit by* I, denoted by $\mathrm{RightLit}(I)$, is defined symmetrically.

(iii) *the area lit by* I, denoted by $\mathrm{Lit}(I)$ is defined as $\mathrm{LeftLit}(I) \cup \mathrm{RightLit}(I)$. The geometric (topological) interior of $\mathrm{Lit}(L)$ is denoted by $\mathrm{OLit}(L)$ and we call it the *open lit set* of I.

For example, in Figure 2, $\mathrm{LeftLit}(C)$, $\mathrm{RightLit}(D)$, and $\mathrm{Lit}(B)$ are shaded.

It follows from the statements (2.10) and (2.11) of G. Czédli [7] that, for every lamp I of L,

$$\text{the geometric (that is, topological) boundaries of the areas } \mathrm{Lit}(I),\ \mathrm{LeftLit}(I), \text{ and } \mathrm{RightLit}(I) \text{ consist of edges.} \tag{3.2}$$

Utilizing the concept of lit sets, we define some relations on $\mathrm{Lamp}(L)$; G. Czédli [7, Definition 2.9] defines eight relations but here we only need four.

Definition 3.5 (G. Czédli [7])**.** Let L be a slim rectangular lattice. We define four relations ρ_{Body}, ρ_{foot}, ρ_{infoot}, and ρ_{alg} on the set $\mathrm{Lamp}(L)$, by the following rules. For $I, J \in \mathrm{Lamp}(L)$,

(i) $(I, J) \in \rho_{\mathrm{Body}}$ if $I \neq J$, $\mathrm{Body}(I) \subseteq \mathrm{Lit}(J)$, and I is an internal lamp;

(ii) $(I, J) \in \rho_{\mathrm{foot}}$ if $I \neq J$, $\mathrm{Foot}(I) \in \mathrm{Lit}(J)$, and I is an internal lamp;

(iii) $(I, J) \in \rho_{\mathrm{infoot}}$ if $I \neq J$, $\mathrm{Foot}(I) \in \mathrm{OLit}(J)$, and I is an internal lamp;

(iv) $(I, J) \in \rho_{\mathrm{alg}}$ if $\mathrm{Peak}(I) \leq \mathrm{Peak}(J)$, $\mathrm{Foot}(I) \not\leq \mathrm{Foot}(J)$, and I is an internal lamp.

The significance of lamps becomes clear from the following statement, which is a part of the (Main) Lemma 2.11 of G. Czédli [7].

Lemma 3.6 (G. Czédli [7])**.** *Let L be a slim rectangular lattice. Then $\rho_{\mathrm{Body}} = \rho_{\mathrm{foot}} = \rho_{\mathrm{infoot}} = \rho_{\mathrm{alg}}$. Let ρ stand for any one (or all) of these relations and let $\leq$ be the reflexive transitive closure of ρ. Then $(\mathrm{Lamp}(L), \leq)$ is an ordered set and it is isomorphic to $\mathrm{J}(\mathrm{Con}\, L)$. Also, if $I, J \in \mathrm{Lamp}(L)$ and $I \prec J$ in $(\mathrm{Lamp}(L), \leq)$, then $(I, J) \in \rho$.*

This lemma is illustrated by Figure 2. The isomorphism

$$\mathrm{Lamp}(L) \cong \mathrm{J}(\mathrm{Con}\, L)$$

is witnessed by the map

$$\mathrm{Lamp}(L) \to \mathrm{J}(\mathrm{Con}\, L), \text{ defined by } I \mapsto \mathrm{con}(\mathrm{Foot}(I), \mathrm{Peak}(I)).$$

We also need the following statement.

Lemma 3.7 (G. Grätzer and E. Knapp [24]). *If K is a slim planar semimodular lattice with at least three elements, then there exists a slim rectangular lattice L such that* $\operatorname{Con} K \cong \operatorname{Con} L$.

G. Grätzer and E. Knapp [24] proved a stronger statement, which we do not require. See also G. Grätzer and E. T. Schmidt [28].

To verify the Three-pendant Three-crown Property, we have to work in $\mathrm{J}(\operatorname{Con} L)$. So by utilizing Lemmas 3.6 and 3.7, we can confine ourselves to investigate lamps in slim rectangular lattices.

4 Further tools and the Key Lemma

4.1 Coordinate quadruples

We start with some technical tools.

Definition 4.1. Let I be a lamp of a slim rectangular lattice L with a fixed $\mathcal{C}_1$-diagram. Assume that we choose the coordinate system of the plane $\mathbb{R}^2$ so that $(0,0)$ is the zero of L.

(i) Following G. Czédli [7], the *lit set* $\operatorname{Lit}(I)$ of an internal lamp I is bordered by the line segments $\operatorname{LRoof}(I)$ and $\operatorname{RRoof}(I)$, $\operatorname{LFloor}(I)$, and $\operatorname{RFloor}(I)$, and the appropriate segments on the lower boundary. If I is a boundary lamp, the above-mentioned line segments still border $\operatorname{Lit}(I)$. Any proper line segment lies on a line referred to as its *carrier line*.

(ii) Let $(p_I, 0)$, $(q_I, 0)$, $(r_I, 0)$ and $(s_I, 0) \in \mathbb{R}^2$ be the intersection points of the x-axis with the carrier lines of $\operatorname{LRoof}(I)$, $\operatorname{LFloor}(I)$, $\operatorname{RFloor}(I)$, and $\operatorname{RRoof}(I)$, respectively. Then (p_I, q_I, r_I, s_I) is called the *coordinate quadruple* of the lamp I.

(iii) Let $I, J \in \operatorname{Lamp}(L)$. Then I is *to the left* of J, in notation $I \mathrel{\lambda} J$, if $q_I \le p_J$ and $s_I \le r_J$.

In Figure 2, $\operatorname{LRoof}(C)$, $\operatorname{RRoof}(C)$, $\operatorname{LFloor}(C)$, and $\operatorname{RFloor}(C)$ are the line segments corresponding to the intervals (in fact, chains) $[t, z]$, $[s, z]$, $[y, c]$, and $[r, c]$, respectively. The coordinate quadruple of the lamp E is shown in Figure 2 and, for example, $E \mathrel{\lambda} D$ and $P_1 \mathrel{\lambda} C$; however, $P_2 \mathrel{\lambda} C$

and $A \lambda B$ fail. For $I \in \mathrm{Lamp}(L)$, the following observation follows from the definitions.

$$\begin{aligned}
&p_I < q_I < r_I < s_I \text{ if and only if } I \text{ is an internal lamp,}\\
&p_I = q_I < r_I < s_I \text{ if and only if } I \text{ is a left boundary lamp,}\\
&p_I < q_I < r_I = s_I \text{ if and only if } I \text{ is a right boundary lamp.}
\end{aligned} \tag{4.1}$$

Remark 4.2. Apart from an order isomorphism, $(-p_I, s_I)$ and $(-q_I, r_I)$ are the join-coordinates of $\mathrm{Peak}(I)$ and $\mathrm{Foot}(I)$ as in Czédli [5, Definition 4.2].

4.2 Key Lemma

The proof of the Main Theorem is based on the following key result.

Lemma 4.3 (Key Lemma)**.** *Let I and I' be lamps of a slim rectangular lattice L with a fixed $\mathcal{C}_1$-diagram. If $I \neq I'$ and they have a common lower cover in* $(\mathrm{Lamp}(L); \leq)$*, then either I is to the left of I' or I' is to the left of I.*

Proof. For later use, recall the following statement from G. Czédli [5, Corollary 6.1].

For $u \neq v \in L$, the inequality $u < v$ holds if and only if the ordinate (that is, the vertical y-coordinate) of u is less than that of v and the geometric line through u and v is either steep or it has a normal slope. (4.2)

In the rest of this proof, assume that $I \neq I'$ are lamps of L and they have a common lower cover I'' and so incomparable, in notation, $I \parallel I'$. By Lemma 3.6, both (I'', I) and (I'', I') belong to ρ_{infoot}, that is, $\mathrm{Foot}(I'') \in \mathrm{OLit}(I)$ and $\mathrm{Foot}(I'') \in \mathrm{OLit}(I')$. Hence,

$$\mathrm{OLit}(I) \cap \mathrm{OLit}(I') \neq \varnothing. \tag{4.3}$$

As Figure 2 (for the lamp E) shows or alternatively, as Remark 4.2 yields,

(p_I, s_I), (q_I, r_I), and (p_I, q_I, r_I, s_I) determine $\mathrm{Peak}(I)$, $\mathrm{Foot}(I)$, and I, respectively; (4.4)

and similarly for I'. Since I and I' are distinct, it follows from (4.3) that

At least one of I and I' is not a left boundary lamp. Similarly, at least one is not a right boundary lamp. (4.5)

To make the proof more readable, we write (p, q, r, s) for (p_I, q_I, r_I, s_I) and (p', q', r', s') for $(p_{I'}, q_{I'}, r_{I'}, s_{I'})$.

Statement (4.5) and G. Czédli [7, Lemma 3.8] yield that

$$q \neq q' \quad \text{and} \quad r \neq r'. \tag{4.6}$$

We distinguish several cases.

Case 1: Both I and I' are internal lamps.

We need the following concept (which is based on the concept of circumscribed rectangles by G. Czédli [7, Definition 2.6]) as illustrated by Figure 2. For an internal lamp $J \in \mathrm{Lamp}(L)$, the *left shield* and the *right shield* of J are the left upper side and the right upper side of the circumscribed rectangle of J. So these shields are line segments. Namely, it follows from (2.8), (2.10), (2.14), and Definition 2.6 of G. Czédli [7] (and from the fact that $\mathrm{Foot}(J)$ is in the interior of the circumscribed rectangle of J) that

> the right shield of an internal lamp J is an edge of normal slope and this edge is longer than the geometric distance of (the carrier lines of) $\mathrm{LRoof}(J)$ and $\mathrm{LFloor}(J)$. Analogously for the left shield of J. (4.7)

Based on (4.7), there is another way to define the shields of an internal lamp J: the *left shield* of J is the unique edge of slope $(-1, -1)$ whose top is $\mathrm{Peak}(J)$; the *right shield* of J has slope $(1, -1)$ and its top is $\mathrm{Peak}(J)$. For example, in Figure 2, $[h, g]$ is the right shield of A while $[f, h]$ and $[y, \mathrm{Peak}(E)]$ are the left shields of B and E, respectively.

We know from (2.7) of G. Czédli [7] that distinct internal lamps have distinct peaks. This fact along with (3.1) and (4.4) yield that

$$(p, s) \neq (p', s') \quad \text{and} \quad (q, r) \neq (q', r'). \tag{4.8}$$

Next, we claim that

$$p \neq p'. \tag{4.9}$$

By way of contradiction, assume that $p = p'$. Since $q \neq q'$ by (4.6) and the role of I and I' is now symmetric, we can assume that $q < q'$. Since $p = p'$ and (4.8) yield that $s \neq s'$, we conclude that either $s < s'$ or $s' < s$.

Case 1A: $s' < s$. The situation (apart from the position of r') is illustrated by Figure 3, where $\mathrm{Lit}(I)$ is the grey area and $\mathrm{Lit}(I')$ is given by

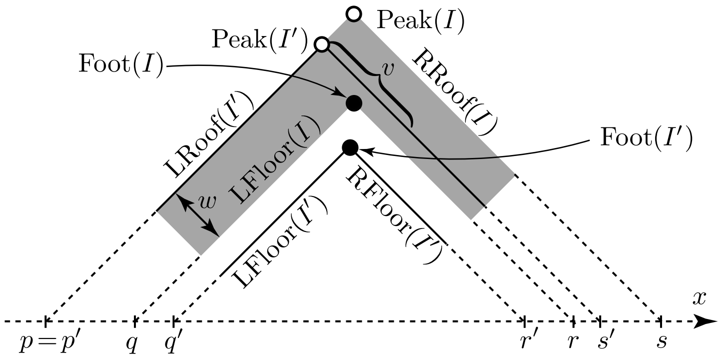

Figure 3: Proving (4.10)

its boundary line segments LRoof(I'), LFloor(I'), etc. The figure indicates the length v of the right shield of I', which is greater than the "width" $(q'-p')/\sqrt{2}$ of LeftLit(I') by (4.7), and the "width" $w=(q-p)/\sqrt{2}=(q-p')/\sqrt{2}$ of LeftLit(I). By (3.2), the geometric boundary of LeftLit(I) consists of edges (but these are not indicated in the figure between Foot(I) and Peak(I)). Since $v>w$, the geometric boundary of LeftLit(I) (consisting of edges) crosses the right shield of I'. But this contradicts the planarity of the diagram since this right shield is an edge by (4.7).

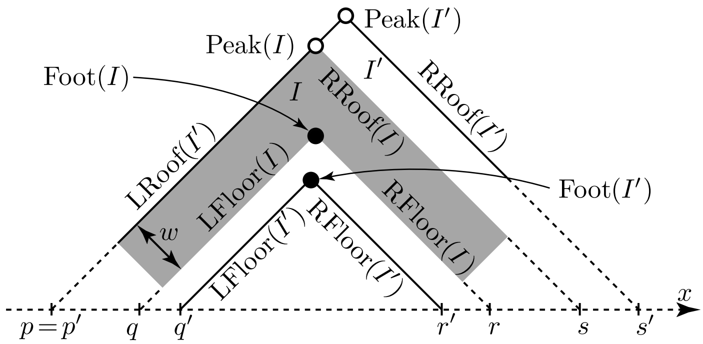

Figure 4: Still proving (4.10)

Case 1B: $s<s'$. This case is illustrated by Figure 4 (in which additional conditions hold, such as, $r'<r$). In this subcase, $q<q'$ yields that Foot(I), which is on a line with point $(q,0)$ and of slope $(1,1)$ is above the carrier

line of LFloor(I'). Hence, it is clear by the figure and, mainly by (4.2), that $\mathrm{Peak}(I) \leq \mathrm{Peak}(I')$ but $\mathrm{Foot}(I) \nleq \mathrm{Foot}(I')$. Thus, $(I, I') \in \rho_{\mathrm{alg}}$, contradicting that $I \parallel I'$. This completes *Case 1B* and proves (4.9).

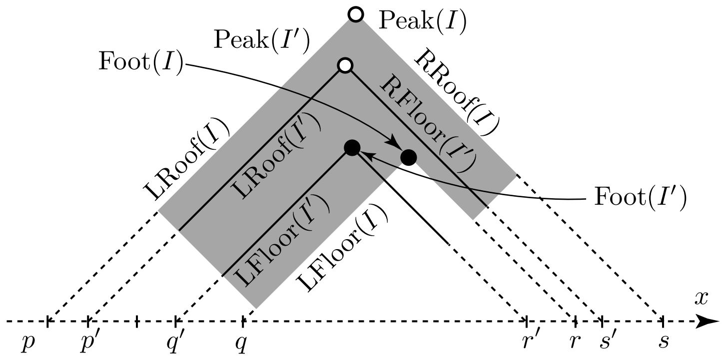

Figure 5: Case $p < p' < q$

Next, we are going to show that

$$\text{if } p \leq p', \text{ then } q \leq p'. \tag{4.10}$$

So assume that $p \leq p'$. Then we know from (4.9) that $p < p'$. By way of contradiction, assume that (4.10) fails, that is, $p' < q$. Then neither $q' = q$, nor $q' > q$ by Lemma 3.8 of G. Czédli [7]. So $p < p' < q' < q$, see Figure 5. Observe that the geometric boundary of LeftLit(I') cannot cross the right shield of I by (3.2) and (4.7). So we obtain from (4.2) that $\mathrm{Peak}(I') \leq \mathrm{Peak}(I)$. Note that Foot($I'$) is on the carrier line of LFloor(I'), which goes through the point $(q', 0)$; moreover, $q' < q$. Therefore, (4.2) also yields that $\mathrm{Foot}(I') \nleq \mathrm{Foot}(I)$. Hence, $(I', I) \in \rho_{\mathrm{alg}}$, contradicting that $I \parallel I'$. This contradicts that $p' < q$ and so proves the validity of (4.10).

As a variant of (4.10), observe that

$$\text{if } s' \leq s, \text{ then } s' \leq r. \text{ Also, if } s \leq s', \text{ then } s \leq r'. \tag{4.11}$$

Indeed, the first part of (4.11) follows from (4.10) by left-right symmetry while its second part follows from the first part by interchanging the role of I and I'.

Next, we claim that

$$\text{if } p \leq p', \ \text{ then } \ I \mathrel{\lambda} I'. \tag{4.12}$$

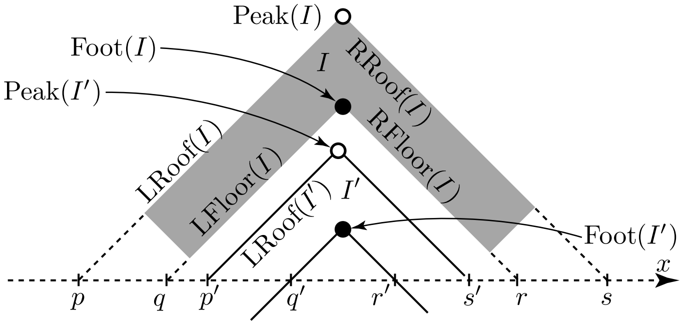

Figure 6: Case $p < p'$ and $s' < s$

So assume that $p \le p'$. By (4.10), we have that $q \le p'$. We claim that $s \le s'$. Assume, to the contrary, that $s' < s$. By (4.11), $s' \le r$; see Figure 6. Hence, $\mathrm{OLit}(I) \cap \mathrm{OLit}(I') = \varnothing$, contradicting (4.3). This shows that $s \le s'$. Applying (4.11), we obtain that $s \le r'$. Now, as part ((iii)) of Definition 4.1 shows, $q \le p'$ and $s \le r'$ complete the argument proving (4.12).

Since I and I' play a symmetric role, we can assume that $p \le p'$. Thus, (4.12) yields the validity of the lemma for the case of internal lamps.

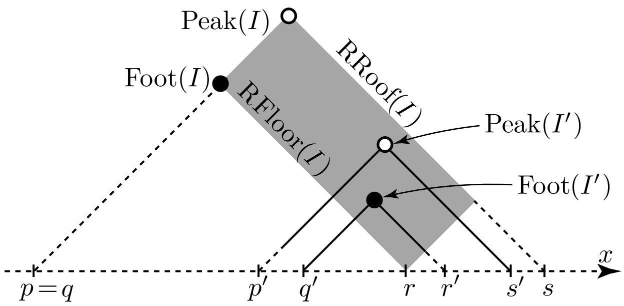

Figure 7: I is a left boundary lamp and $r < r' < s$

Case 2: Of the two lamps, I and I', one is a boundary lamp and the other one is internal. By symmetry, we can assume that I is a left boundary lamp and I' is an internal lamp. By (4.1), $p = q$. Since this is clearly the least possible value, $q \le p'$. Hence, to show that $I \;\lambda\; I'$, we need to show that $s \le r'$.

Suppose, for a contradiction, that $r' < s$. If $r' < r$, then we also have that $s' \leq r$; otherwise, $\mathrm{RFloor}(I)$ would cross the left shield of I' (see Figure 6 after collapsing p and q). So if $r' < r$, then $s' \leq r$, but then $\mathrm{OLit}(I) \cap \mathrm{OLit}(I') = \varnothing$ (similarly to Figure 6 but now $p = q$ and $\mathrm{LeftLit}(I)$ reduces to a line segment), and this equality contradicts (4.3). This rules out that $r' < r$. Since $r' = r$ is also ruled out by Lemma 3.8 of G. Czédli [7], we have that $r < r'$.

So we have that $r < r' < s$; see Figure 7. Combining (4.2) and $r < r'$, we obtain that $\mathrm{Foot}(I') \not\leq \mathrm{Foot}(I)$. Thus $\mathrm{Peak}(I') \not\leq \mathrm{Peak}(I)$; indeed, otherwise we would have that $(I', I) \in \rho_{\mathrm{alg}}$ and so $I' \leq I$ would contradict $I \parallel I'$. Since $s' \leq s$ (together with the trivial $p \leq p'$) would imply that $\mathrm{Peak}(I') \leq \mathrm{Peak}(I)$, which has just been excluded, we obtain that, as opposed to what Figure 7 shows, $s < s'$. However, then $r' < s < s'$ and $\mathrm{RRoof}(I)$ crosses the left shield of I', which contradicts (3.2), (4.7), and the planarity of L. We have shown that $I \mathrel{\lambda} I'$, as required.

Case 3: Both I and I' are boundary lamps. If they both were left boundary lamps, then $\mathrm{OLit}(I) \cap \mathrm{OLit}(I')$ would contradict (4.3). We would have the same contradiction if both were right boundary lamps. Hence one of them, say I, is a left boundary lamp while the other, I', is a right boundary lamp, and the required $I \mathrel{\lambda} I'$ trivially holds. This completes the proof of Lemma 4.3. $\square$

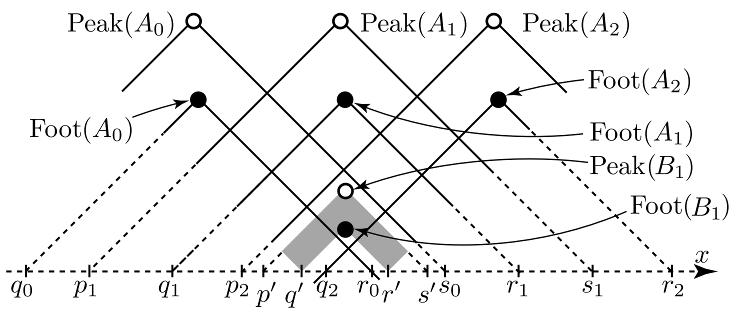

Figure 8: A_0, A_1, A_2, and B_1

5 Proving the Main Theorem

Now we are ready to prove our main result.

Proof of the Main Theorem. The theorem is trivial for lattices with less than three elements. Hence, by Lemma 3.7, it suffices to prove the theorem for slim rectangular lattices. By way of contradiction, assume that L is a slim rectangular lattice that fails the 3P3C-property. Then by Lemma 3.6, R_3 is a cover-preserving ordered subset of $\mathrm{Lamp}(L)$. Let X_i be the lamp corresponding to $x_i \in R_3$. It follows from Lemma 4.3 that for any $i \neq j \in \{0, 1, 2\}$, either A_i is to the left of A_j (in notation, $A_i \mathrel{\lambda} A_j$), or $A_j \mathrel{\lambda} A_i$. Therefore, since any permutation of $\{A_0, A_1, A_2\}$ extends to an automorphism of R_3, we can assume that $A_0 \mathrel{\lambda} A_1$ and $A_1 \mathrel{\lambda} A_2$; see Figure 8, where the coordinate quadruple of A_i is (p_i, q_i, r_i, s_i). By Definition 4.1((iii)), it follows that

$$q_0 \leq p_1,\ s_0 \leq r_1,\ q_1 \leq p_2,\ s_1 \leq r_2,\ \text{and } p_i \leq q_i \leq r_i \leq s_i \tag{5.1}$$

for every $i \in \{0, 1, 2\}$. Note that A_0 is either an internal lamp such as in the figure, or it is a left boundary lamp and then $\mathrm{LeftLit}(A_0)$ is only a line segment, and analogously for A_2. Let (p', q', r', s') denote the coordinate tuple of B_1; note that $\mathrm{Lit}(B_1)$ is grey in the figure. It follows from (5.1) and from trivial properties of $\mathcal{C}_1$-diagrams that $\mathrm{OLit}(A_1) \cap \mathrm{OLit}(B_1) = \varnothing$. On the other hand, $C_1 \prec A_1$ and $C_1 \prec B_1$ give that $(C_1, A_1) \in \rho_{\mathrm{infoot}}$ and $(C_1, B_1) \in \rho_{\mathrm{infoot}}$ by Lemma 3.6. It follows that $\mathrm{Foot}(C_1) \in \mathrm{OLit}(A_1) \cap \mathrm{OLit}(B_1) = \varnothing$, which is a contradiction, completing the proof of the Main Theorem. $\square$

6 Rectangular and patch lattices

6.1 Distance-free geometry of distributive 4-cells We assume some familiarity with the multifork extensions of G. Czédli [2]. As a preparation for the proof of Theorem 1.2, let us have a closer look at the "geometry" of distributive 4-cells of a $\mathcal{C}_1$-diagram of a slim rectangular lattice L. Let $C = \{u, a, b, v\}$ with $u \prec a \prec v$ and $u \prec b \prec v$ be a 4-cell of a $\mathcal{C}_1$-diagram. We use the notations $1_C = v$ and $0_C = u$ for the *top* and the *bottom* of this 4-cell. Following G. Czédli [2], C is a *distributive* 4-*cell* if the principal ideal $\downarrow 1_C$ is a distributive lattice. In this case, C is a *normal* 4-*cell* in the sense that its four edges are of normal slopes, and so is every 4-cell in $\downarrow 1_C$ since $\downarrow 1_C$ is a grid by G. Czédli [5, Lemma 5.8]; by definition, a *grid* is (the

$\mathcal{C}_1$-diagram of) the direct product of two finite chains. In particular, $\downarrow 1_C$ is a slim rectangular lattice.

Next, let $C_1 = \{u_1, a_1, b_1, v_1\}$ and $C_2 = \{u_2, a_2, b_2, v_2\}$ be distinct distributive 4-cells of a $\mathcal{C}_1$-diagram of a slim rectangular lattice L. The notation is chosen so that, for $i \in \{1, 2\}$ $u_i = 0_{C_i}$, $v_i = 1_{C_i}$, and a_i is to the left of b_i. Although $[a_1, v_1]$ is not a neon tube in (the diagram of) L in general, it becomes a neon tube and a boundary lamp in the subdiagram $\downarrow 1_{C_1}$. We write $C_1 \searrow C_2$ to denote that C_2 belongs to the subdiagram $\downarrow 1_{C_1}$ and it is included in the area $\mathrm{RightLit}([a_1, v_1])$ in this subdiagram; see Definition 3.4. For example, in Figure 9, $E \searrow F$ and $A \searrow D$. Similarly, $C_1 \swarrow C_2$ means that C_2 belongs to the subdiagram $\downarrow 1_{C_1}$ and it is included in the area $\mathrm{LeftLit}([b_1, v_1])$ in this subdiagram. For example, in Figure 9, $E \swarrow A$ and $G \swarrow C$. Since $\downarrow 1_{C_1}$ is a grid, if $C_1 \searrow C_2$, then v_1, b_1, v_2, b_2 lie on the same geometrical line of slope $(1, -1)$, and analogously in case $C_1 \swarrow C_2$. Note that in case of $C_1 \searrow C_2$, the 4-cells C_1 and C_2 can be adjacent (that is, $u_1 = a_2$ and $b_1 = v_2$), but they can be distant from each other; we are not interested in their geometrical distance now. We say that the 4-cells C_1 and C_2 are *geometrically parallel*, in notation $C_1 \parallel_{\mathrm{g}} C_2$, if they are distinct and none of $C_1 \swarrow C_2$, $C_2 \swarrow C_1$, $C_1 \searrow C_2$, and $C_2 \searrow C_1$ holds. For example, if $v_2 \le u_1$, then $C_1 \parallel_{\mathrm{g}} C_2$ and, in Figure 9, $B \parallel_{\mathrm{g}} H$. The geometrical parallelism is a symmetrical relation. Clearly,

for any two distinct distributive 4-cells of L, exactly one of the alternatives $C_1 \swarrow C_2$, $C_2 \swarrow C_1$, $C_1 \searrow C_2$, $C_2 \searrow C_1$, and $C_1 \parallel_{\mathrm{g}} C_2$ holds. (6.1)

A *translation of the plane* is a map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (x + a, y + b)$ where $(a, b) \in \mathbb{R}^2$ is a constant. We need the following definition.

Definition 6.1. Let L and L' be slim rectangular lattices, with a fixed $\mathcal{C}_1$-diagram each. Let Z and Z' be sets of pairwise disjoint distributive 4-cells of L and of L', respectively. We say that Z and Z' have the *same distance-free geometry* if there is a bijective map $f \colon Z \to Z'$ with action denoted by $C \mapsto C' := f(C)$ such that

— for every $C \in Z$, there is a translation of the plane that maps C to C' (in particular, C and C' are congruent with respect to the usual Euclidean metric), and

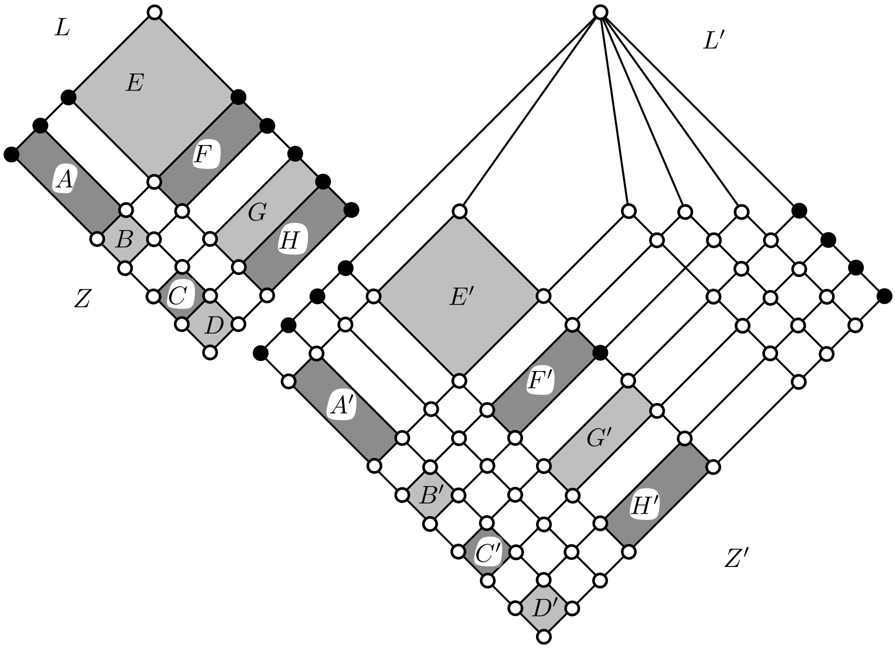

Figure 9: Z and Z' have the same distance-free geometry

— for any two distinct C_1 and C_2 in Z, the same alternative of (6.1) holds for (C_1, C_2) as for (C'_1, C'_2).

For example, in Figure 9, $Z = \{A, B, \dots, H\}$ and $Z' = \{A', B', \dots, H'\}$ have the same distant-free geometry.

By G. Czédli [2, Theorem 3.7], L is obtained from a grid G by a sequence of multifork extensions at distributive 4-cells. Furthermore, we know from (2.10) of G. Czédli [7] that each internal lamp comes to existence by a multifork extension while G is changing to L in several steps. This is why the following lemma will be important later.

Lemma 6.2. *Let Z and Z' be as in Definition 6.1, $t \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, and $E \in Z$. Insert a t-fold multifork into E (in other words, take a t-fold multifork extension of L at E) to obtain a larger slim rectangular lattice K. Denote by Y the collection of 4-cells of (the $\mathcal{C}_1$-diagram of) K that are in the areas determined by the 4-cells of Z. (In other words, Y consist of those*

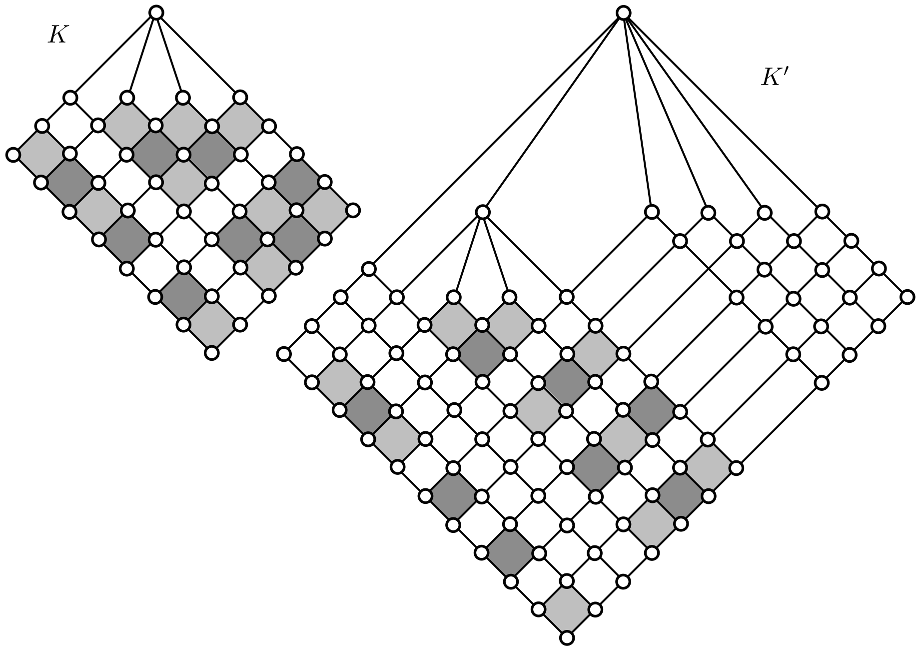

Figure 10: Illustrating Lemma 6.2

4-cells of Z that are 4-cells of K and those 4-cells of K that have come to existence by dividing 4-cells of Z at the multifork extension.) Then it is possible to perform a t-fold multifork extension of L' at E' to obtain K' such that with the analogously defined Y', Y and Y' have the same distance-free geometry.

Lemma 6.2 is illustrated by Figures 9 and 10, where each of Z, Z', Y, Y' consists of the grey-filled 4-cells of the corresponding diagram. Note that, say, E is not in Y since it has been split into six 4-cells, three of which are distributive in K and belong to Y. Lemma 6.2 is a trivial consequence of definitions.

Let (A, ρ) and (A', ρ') be ordered sets. Their *cardinal sum* will be denoted by $(A, \rho) \dot\cup (A', \rho')$; it is $(A \sqcup A', \rho \sqcup \rho')$ where $\sqcup$ stands for disjoint union. The operation $\dot+$ for *glued sum* was defined at the beginning of Section 2.

6.2 Proving Theorem 1.2 For arbitrary lattices $L_1, \dots, L_n$ with 0 and 1, $\mathrm{Con}\,(L_1 \dot{+} \dots \dot{+} L_n)$ is obviously isomorphic to $\mathrm{Con}\,L_1 \times \dots \times \mathrm{Con}\,L_n$; see (2.1). Since the glued sum of two SPS lattices is clearly an SPS lattice, we conclude (1.1).

Next, to prove (1.2), Lemma 3.7 allows us to assume that $L_1, \dots, L_n$ are slim rectangular lattices and their diagrams are $\mathcal{C}_2$-diagrams. Recall that the *grid* of the slim rectangular lattice L_i is its sublattice generated by the upper boundary of L_i. This grid will be denoted by G_i; it is a distributive lattice with all if its edges of normal slopes. We denote by Z_i the set of 4-cells of G_i.

For $i \in \{1, \dots, n\}$, let t_i be the number of boundary lamps of L_i, and let $t = t_1 + \dots + t_n + 2$. We start our construction by taking $S_7^{(t)}$; see at the middle right of Figure 11, where $n = 2$, $t_1 = 3$, $t_2 = 2$, and $t = 7$. The feet of the lamps are black-filled in Figure 11. Also, the feet of the internal neon tubes of $S_7^{(7)}$ are indicated by pentagons. We assume that $S_7^{(7)}$ is given by a $\mathcal{C}_2$-diagram.

Let U and V be the left boundary lamp and the right boundary lamp, respectively, of $S_7^{(7)}$, and let W be its unique internal lamp. In Figure 11, the bottom of a lamp denoted by a capital letter is denoted by the corresponding lower-case letter.

Let $A, B, C, \dots$ be the list of $(t-2)$ boundary lamps consisting, in this order, of the left boundary lamps of L_1, the right boundary lamps of L_1, the left boundary lamps of L_2, the right boundary lamps of L_2, ..., the left boundary lamps of L_n, and the right boundary lamps of L_n. Disregarding the leftmost one and the rightmost one, we label the feet of the neon tubes of W by a', b', c', ..., from left to right, in this order.

By G. Czédli [2, Proposition 3.3], $S_7^{(t)}$ is a slim patch lattice. All the elements a', b', c', ... are the tops of distributive 4-cells as defined in G. Czédli [2]. Insert a fork (that is, a 1-fold multifork) into each of these cells; the lattice we obtain is denoted by K; see the upper half of Figure 11; the elements of $K - S_7^{(7)}$ (that is, the new elements) are oval. We know that K is a slim patch lattice, see G. Czédli [2, Proposition 3.3]. The top edges of the forks just inserted are neon tubes and also 1-tube lamps; let $a, b, c, \dots$ denote their feet. We can assume that these new neon tubes are vertical. For each $i \in \{1, \dots, n\}$, for each left boundary lamp X of L_i and for each right boundary lamp Y of L_i, turn the intersection $\mathrm{RightLit}(X) \cap \mathrm{LeftLit}(Y)$,

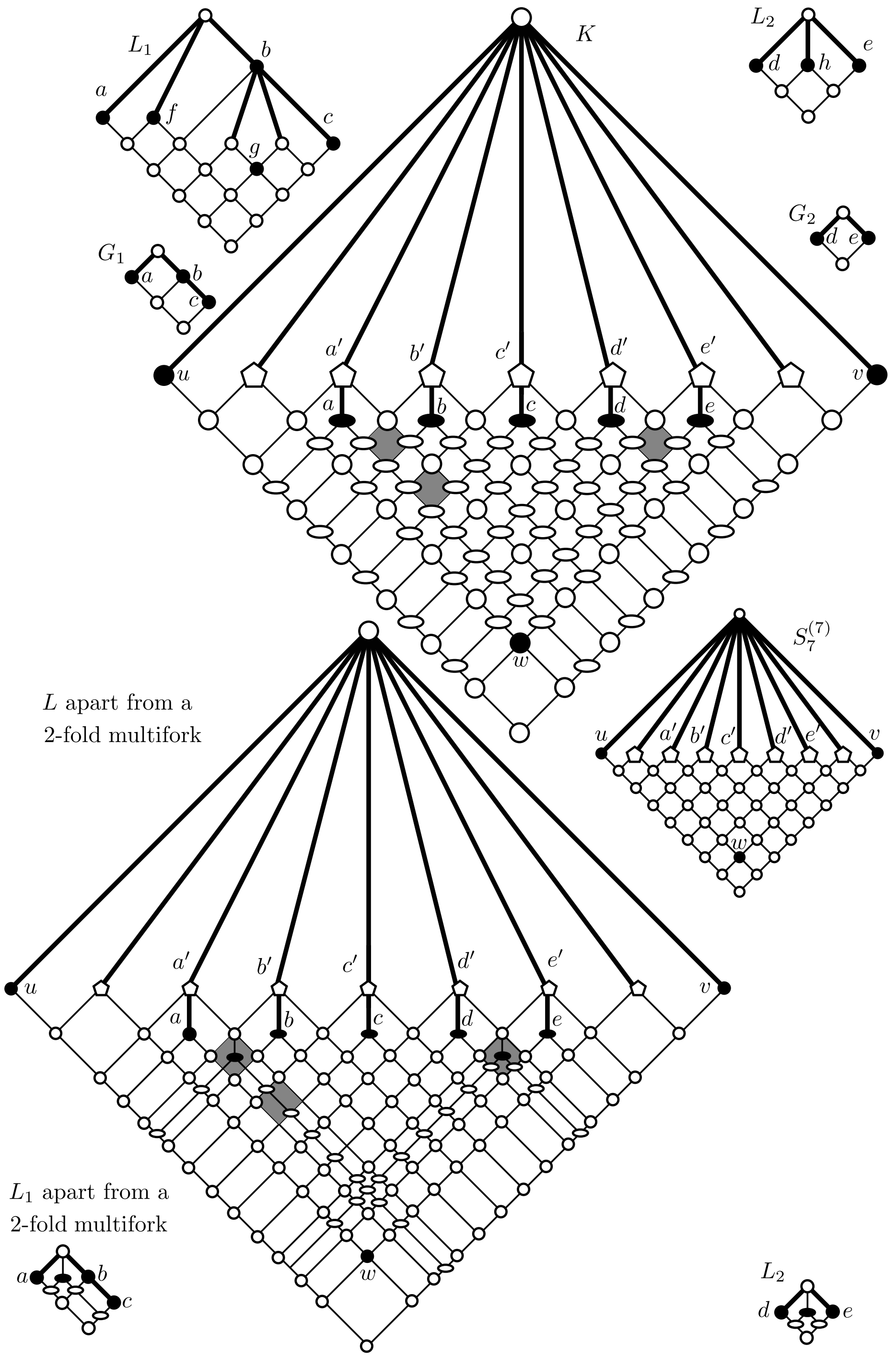

Figure 11: Constructing K from L_1 and L_2 and constructing L from K apart from a 2-fold multfork

which is a 4-cell, into grey; see Figure 11 again.

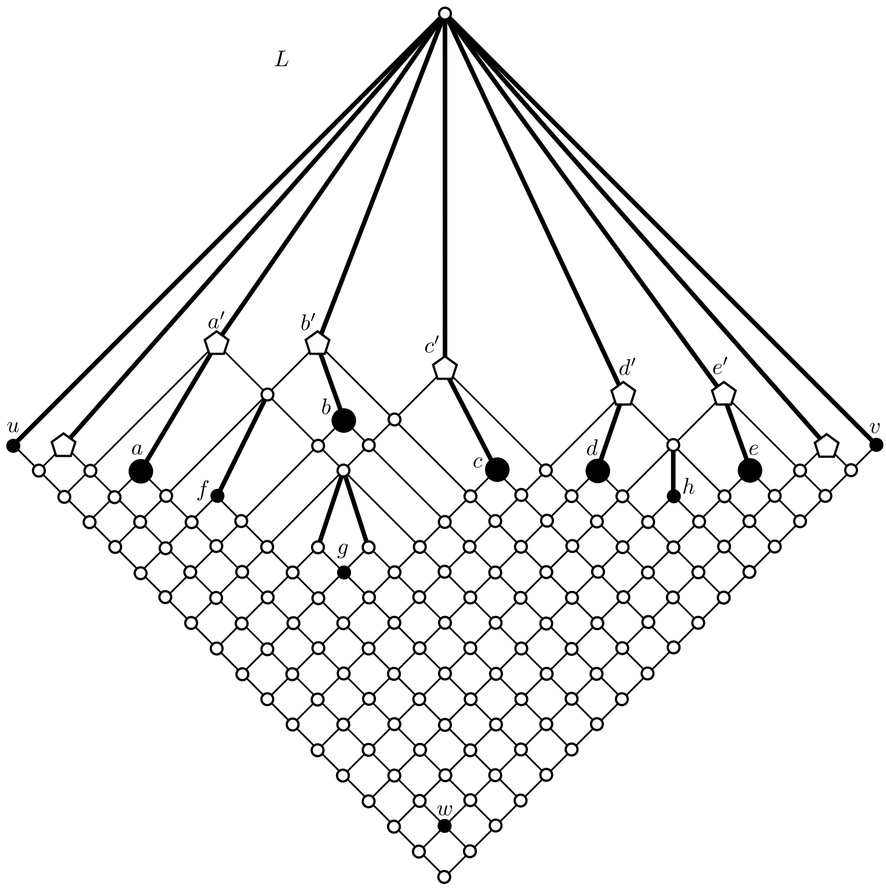

Figure 12: A $\mathcal{C}_2$-diagram of L

For a given i, we denote the set of these grey-filled cells by Z'_i. Since we constructed the diagram of K from a $\mathcal{C}_2$-diagram of $S_7^{(t)}$ and since the new neon tubes are vertical, $Z' := Z'_1 \cup \dots \cup Z'_n$ consists of pairwise geometrically congruent squares. Note that any two normal squares of the same size differ only up to a translation of the plane. By rescaling the diagrams of L_i and, thus, G_i for $i \in \{1, \dots, n\}$, we can assume that the 4-cells of Z and those of Z' are all of the same size. By construction, Z and Z' have the same distance-free geometry. Furthermore,

$$\begin{aligned}&\text{for all } 1 \le i < j \le n,\ A \in Z'_i, \text{ and}\\ &B \in Z'_j, \text{ we have that } A \parallel_{\mathrm{g}} B.\end{aligned} \tag{6.2}$$

By G. Czédli [2, Theorem 3.7], L_i is obtained from G_i by a sequence of multifork extensions at distributive 4-cells for $i \in \{1, \dots, n\}$. While doing so, the set Z_i of 4-cells of G_i changes first to $Z_{i,1}$, then to $Z_{i,2}$, and so on. Lemma 6.2 allows us to perform, apart from "distance-free geometric isomorphism" the same multifork extensions in K (we give more explanation a bit later). In this way, Z'_i changes to $Z'_{i,1}$, then it changes to $Z'_{i,2}$, and so on. Of course, if the first multifork added to G_i is a t_1-fold multifork added to a 4-cell C, then we add a t_1-fold multifork to $C' \in Z_i$. In the next step, when we add a t_2-fold multifork to a distributive 4-cell D of the lattice diagram just obtained from G_i, then (using that $Z_{i,1}$ and $Z'_{i,1}$ have the same distance-free geometry) we add a t_2-fold multifork at $D' \in Z'_{i,1}$, etc. It follows from (6.2) that

the multifork extensions at 4-cells of Z' and its descendants do not interfere with those at 4-cells of Z'_j and its descendants provided $i \neq j$. (6.3)

The lower half of Figure 11 shows where we are after insterting one multifork into each of G_1 and G_2; the new elements are oval-shaped. To obtain L from the lattice in the lower half of the figure, one more multifork extension is only necessary. Note that we have reshaped the diagram of L into the $\mathcal{C}_2$-diagram given in Figure 12, since otherwise the figure would have been too crowded and unreadable. The role of Lemma 6.2 is to show by induction that

since Z_i and Z'_i have the same distance-free geometry, so have $Z_{i,1}$ and $Z'_{i,1}$, then so have $Z_{i,2}$ and $Z'_{i,2}$, and so on. (6.4)

Let L denote the lattice we obtain from K after the multifork extensions described above. Since $\mathrm{Lit}(W) = \mathrm{FulR}(K) = \mathrm{FulR}(L)$ gives that (A, W), (B, W), $(C, W), \dots$ belong to ρ_{foot}, we obtain (by Lemma 3.6) that

the inequalities $A < W$, $B < W$, $C < W$, $\dots$ hold in $\mathrm{Lamp}(L)$. (6.5)

Let H_i denote the set of lamps that are (in the geometric sense) in the grey 4-cells of Z'_i, $Z'_{i,1}$, $Z'_{i,2}$, ..., that is in the geometrical area determined by Z'_i. Then H_i is an ordered subset of $\mathrm{Lamp}(L)$. It follows from (6.4) that $H_i \cong \mathrm{Lamp}(L_i)$. Since light only goes in the directions $(-1,-1)$ and $(1,-1)$, or because of (6.3), we obtain that no lamp of H_i lights up any $\mathrm{Foot}(H_j)$ for $i \neq j$. Thus we obtain that $\mathrm{Lamp}(L) - \{U, V, W\} = H_1 \dot\cup \cdots \dot\cup H_n$. This equality, $H_i \cong \mathrm{Lamp}(L_i)$, and (6.5) yield that

$$\mathrm{Lamp}(L) \cong \big(\mathrm{Lamp}(L_1) \dot\cup \cdots \dot\cup \mathrm{Lamp}(L_n)\big) + \{U, V, W\}, \qquad (6.6)$$

where $W \prec U$, $W \prec V$, and $U \parallel V$. Finally, (6.6) and the Representation Theorem of Finite Distributive Lattices imply the validity of (1.2) and complete the proof of Theorem 1.2.

6.3 Patch lattices G. Grätzer [19, Problem 3] asks to characterize the congruence lattices of slim patch lattices. We now summarize what we know about these congruence lattices but Problem 3 of G. Grätzer [19] remains open. We start with an observation.

Lemma 6.3. *If L is a slim rectangular lattice, then the following three conditions are equivalent.*

(i) *L is a slim patch lattice.*

(ii) $\mathrm{J}(\mathrm{Con}\, L)$ *has exactly two maximal elements.*

(iii) *There is a finite distributive lattice D_0 such that* $\mathrm{Con}\, L \cong D_0 \dot+ \mathsf{B}_2$.

Proof. By G. Czédli [7, Lemma 3.2], the maximal elements of $\mathrm{Lamp}(L)$ are exactly the boundary lamps. Hence, Lemma 3.6 implies that (i) is equivalent to (ii). This equivalence also easily follows from the Swing Lemma, see G. Grätzer [16]. Also, the fact that (ii) equivalent to (iii) holds by the Structure Theorem of Finite Distributive Lattices. □

The (1.2)-part of Theorem 1.2 establishes a new connection between slim rectangular lattices and slim patch lattices; other connections have been explored by G. Czédli [2] and G. Czédli and E. T. Schmidt [14].

The four element boolean lattice B_2 and the glued sum construction in part (iii) of Lemma 6.3 are well understood. So we focus on D_0 to describe the known properties of congruence lattices of slim patch lattices. The next statement reduces seven *known* conditions that hold for congruence lattices of slim, planar, semimodular lattices by Theorems 1.3–1.4 and the Main Theorem to four.

Corollary 6.4. *Let* $D = \operatorname{Con} L$ *be the congruence lattice of a slim patch lattice* L. *Then the following four statements hold.*

(i) *There exists a unique finite distributive lattice* D_0 *such that* $D = D_0 \dot{+} \mathsf{B}_2$.

In the next three statements, D_0 *refers to the distributive lattice defined in* (6.4).

(ii) *Every element of the ordered set* $\mathrm{J}(D_0)$ *has at most two covers.*

(iii) *Two distinct maximal elements of the ordered set* $\mathrm{J}(D_0)$ *have no common lower cover.*

(iv) *The ordered set* $\mathrm{J}(D_0)$ *satisfies the Three-pendant Three-crown Property.*

Furthermore, if L *is a finite lattice,* $D = \operatorname{Con} L$, *and* D *satisfies* (6.4), (6.4) *and* (6.4) *above, then* L *also satisfies all the six properties listed in Theorems 1.3 and 1.4.*

Proof. Part (6.4) follows from Lemma 6.3. Let A_2 denote the two element antichain. Observe that if $D = D_0 \dot{+} \mathsf{B}_2$, then $\mathrm{J}(D) = \mathrm{J}(D_0) \dot{+} \mathsf{A}_2$. Hence, applying Theorem 1.3(ii), Theorem 1.4(iv), and the Main Theorem, we obtain parts (6.4), (6.4), and (6.4), respectively. The rest of the corollary is a trivial consequence of $\mathrm{J}(D) = \mathrm{J}(D_0) \dot{+} \mathsf{A}_2$. □

Acknowledgement

The first author thanks the National Research, Development and Innovation Fund of Hungary, funding scheme K 134851, for supporting this research. Also, the authors are indebted to the referee for his useful comments.

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