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Quadratic structures associated to (multi)rings

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Abstract. We consider certain pairs (A, T) where A is a (multi)ring and $T \subseteq A$ is a multiplicative set that generates, by a convenient quotient construction, a (multi)structure that supports a quadratic form theory: with some natural hypotheses we generalize constructions previously presented in [3] and [6]. This also provides some steps towards an abstract formally real quadratic form theory (non necessarily reduced) were the forms have general coefficients (non only units).

1 Introduction

In [3], [5] and [6] are considered abstract theories of quadratic forms: special groups and real semigroups. The former treats simultaneously reduced and non-reduced theories but focuses on rings with a good amount of invertible coefficients to quadratic forms. The latter has the advantage of potentially consider general coefficients of a ring, but only addresses the

reduced case. Both are first-order theory, thus they allow the use of model theoretic methods.

M. Marshall in [11] introduced an approach to (reduced) theory of quadratic forms trough the concept of multiring (roughly, a ring with a multi valued sum): this seems more intuitive for an algebraist, encompassing some techniques of ordinary commutative algebra, encodes copies of special groups and real semigroups (see [14]), but still allows the use of model-theoretic tools.

The goal of the present paper is twofold:

- to describe interesting pairs (A, T) where A is a (multi)ring and $T \subseteq A$ is a certain multiplicative subset in such a way to obtain models of abstract theories of quadratic forms (special groups and real semigroups) via natural quotients - Marshall's quotient construction;

- use this construction to motivate a "non reduced" expansion of the theory of real semigroups to deal the formally real case, isolating axioms over pairs involving multrings and a subset with some properties.

Outline: Section 2 exposes the fundamental definitions and results on the (multi)structures that will be analysed in the present work: multirings, special groups and (formally) real semigroups. In Section 3 we introduce the concept of DM-multiring, that provides a generalization of the construction of special groups by Marshall's quotient construction obtained from certain pairs formed by a ring and a multiplicative subset. Section 4 establishes a relationship of our DM-multirings and the concept of quadratically presentable fields, recently introduced in [9]. In Section 5 we introduce the notions of quadratic pair, DP-multiring and quadratic multiring that provide examples of (formally) real semigroup via Marshall's quotient construction. We finish the work indicating some future themes of research motivated by the present paper.

2　Preliminaries

This section contains, basically, the fundamental definitions and results on multirings, hyperfields, special groups and real semigroups, included for the convenience of the reader; for more details, consult [14] and [11]. We introduce also the concepts of *formally real semigroup*.

2.1 Multirings and hyperfields

Definition 2.1 (Adapted from Definition 2.1 in [11])**.** A multiring is a sextuple $(R, +, \cdot, -, 0, 1)$ where R is a non-empty set, $+ : R \times R \to \mathcal{P}(R) \backslash \{\emptyset\}$, $\cdot : R \times R \to R$ and $- : R \to R$ are functions, 0 and 1 are elements of R satisfying:

i) $(R, +, -, 0)$ is a commutative multigroup;

ii) $(R, \cdot, 1)$ is a commutative monoid;

iii) $a.0 = 0$ for all $a \in R$;

iv) If $c \in a + b$, then $c.d \in a.d + b.d$. Or equivalently, $(a+b).d \subseteq a.d + b.d$.

Note that if $a \in R$, then $0 = 0.a \in (1 + (-1)).a \subseteq 1.a + (-1).a$, thus $(-1).a = -a$.

R is said to be an hyperring if for $a, b, c \in R$, $a(b + c) = ab + ac$.

A multring (respectively, a hyperring) R is said to be a multidomain (hyperdomain) if it has not zero divisors. A multring R will be a multifield if every non-zero element of R has multiplicative inverse; note that hyperfields and multifields coincide. We will use "hyperfield" since this is the prevailing terminology.

Example 2.2. (a) Suppose that $(G, +, 0)$ is an abelian group. Defining $a + b = \{a+b\}$ and $r(g) = -g$, we have that $(G, +, r, 0)$ is an abelian multigroup. In this way, every ring, domain and field is a multring, multidomain and hyperfield, respectively.

(b) $Q_2 = \{-1, 0, 1\}$ is hyperfield with the usual product (in $\mathbb{Z}$) and the multivalued sum defined by relations

$$\begin{cases} 0 + x = x + 0 = x, \text{ for every } x \in Q_2 \\ 1 + 1 = 1, \ (-1) + (-1) = -1 \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\} \end{cases}$$

(c) Let $K = \{0, 1\}$ with the usual product and the sum defined by relations $x + 0 = 0 + x = x$, $x \in K$ and $1 + 1 = \{0, 1\}$. This is a hyperfield called Krasner's hyperfield [10].

Now, another example that generalizes $Q_2 = \{-1, 0, 1\}$. Since this is a new one, we will provide the entire verification that it is a multring:

Example 2.3 (Kaleidoscope, Example 2.7 in [14])**.** Let $n \in \mathbb{N}$ and define

$$X_n = \{-n, ..., 0, ..., n\} \subseteq \mathbb{Z}.$$

We define the n-**kaleidoscope multiring** by $(X_n, +, \cdot, -, 0, 1)$, where $- : X_n \to X_n$ is restriction of the opposite map in $\mathbb{Z}$, $+ : X_n \times X_n \to \mathcal{P}(X_n) \setminus \{\emptyset\}$ is given by the rules:

$$a + b = \begin{cases} \{a\}, & \text{if } b \neq -a \text{ and } |b| \leqslant |a| \\ \{b\}, & \text{if } b \neq -a \text{ and } |a| \leqslant |b| \\ \{-a, ..., 0, ..., a\} & \text{if } b = -a \end{cases} ,$$

and $\cdot : X_n \times X_n \to X_n$ is is given by the rules:

$$a \cdot b = \begin{cases} \operatorname{sgn}(ab) \max\{|a|, |b|\} & \text{if } a, b \neq 0 \\ 0 & \text{if } a = 0 \text{ or } b = 0 \end{cases} .$$

In this sense, $X_0 = \{0\}$ and $X_1 = \{-1, 0, 1\} = Q_2$. For X_2, we have the following "multioperation" table for the sum:

+	-2	-1	0	1	2
-2	$\{-2\}$	$\{-2\}$	$\{-2\}$	$\{-2\}$	$\{-2, -1, 0, 1, 2\}$
-1	$\{-2\}$	$\{-1\}$	$\{-1\}$	$\{-1, 0, 1\}$	$\{2\}$
0	$\{-2\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{-2\}$	$\{-1, 0, 1\}$	$\{1\}$	$\{1\}$	$\{2\}$
2	$\{-2, -1, 0, 1, 2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$

and the following operation table for the product:

$\cdot$	-2	-1	0	1	2
-2	2	2	0	-2	-2
-1	2	1	0	-1	-2
0	0	0	0	0	0
1	-2	-1	0	1	2
2	-2	-2	0	2	2

With the above rules we have that $(X_n, +, \cdot, -, 0, 1)$ is a multiring.

Now, another example that generalizes $K = \{0, 1\}$.

Example 2.4 (H-hyperfield, Example 2.8 in [14])**.** Let $p \geqslant 1$ be a prime integer and $H_p := \{0, 1, ..., p-1\} \subseteq \mathbb{N}$. Now, define the binary multioperation and operation in H_p as follow:

$$a + b = \begin{cases} H_p \text{ if } a = b, \ a, b \neq 0 \\ \{a, b\} \text{ if } a \neq b, \ a, b \neq 0 \\ \{a\} \text{ if } b = 0 \\ \{b\} \text{ if } a = 0 \end{cases}$$

$$a \cdot b = k \text{ where } 0 \leqslant k < p \text{ and } k \equiv ab \text{ mod p.}$$

$(H_p, +, \cdot, -, 0, 1)$ is a hyperfield such that for all $a \in H_p$, $-a = a$. For example, considering $H_3 = \{0, 1, 2\}$, using the above rules we obtain these tables

+	0	1	2
0	{0}	{1}	{2}
1	{1}	{0, 1, 2}	{1, 2}
2	{2}	{1, 2}	{0, 1, 2}

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

In fact, these H_p is a kind of generalization of K, in the sense that $H_2 = K$.

We have to treat sums with some care when we are working with multirings. In order to use the multivalued sum without danger, we define recursively for $n \geqslant 2$:

$$a_1 + ... + a_n := \bigcup_{d \in a_2 + ... + a_n} a_1 + d.$$

In particular, for a multiring A, with $a_1, ..., a_n \in A$ and $\sigma \in S_n$, we have

$$a_1 + a_2 + ... + a_n = a_{\sigma(1)} + a_{\sigma(2)} + ... + a_{\sigma(n)}.$$

Now, we treat about morphisms:

Definition 2.5. Let A and B multirings. A map $f : A \to B$ is a morphism if for all $a, b, c \in A$:

   i) $c \in a + b \Rightarrow f(c) \in f(a) + f(b)$;

   ii) $f(-a) = -f(a)$;

   iii) $f(0) = 0$;

   iv) $f(ab) = f(a)f(b)$;

   v) $f(1) = 1$.

If A and B are multirings, a morphism $f : A \to B$ is a *strong morphism* if for all $a, b, c \in A$, if $f(c) \in f(a) + f(b)$, then there are $a', b', c' \in A$ with $f(a') = f(a), f(b') = f(b), f(c') = f(c)$ such that $c' \in a' + b'$. In the quadratic context, there is a more detailed analysis in Example 2.10 of [14].

The category of hyperfields (respectively multirings) and their morphisms will be denoted by $MField$ (respectively $MRing$). There are many others weaker versions for the notion of morphism of multiring.

Example 2.6. Consider the multiring X_n defined in 2.3. In fact, for $n \geqslant 1$, there is an injective morphism $i_n : X_n \to \mathcal{T}\mathbb{R}$, on the tropical real hyperfield defined in 7.2 of [15], induced by the inclusion map $X_n \hookrightarrow \mathbb{R}$.

There are many natural construction on the category of multrings as: products, directed inductive limits, quotients by an ideal, localizations by multiplicative subsets and quotients by ideals. Now, we present a construction that will be used several times below, that we call "Marshall's quotient":

Definition 2.7 (Example 2.6 in [11])**.** Fix a multiring A and a multiplicative subset S of A such that $1 \in S$. Define an equivalence relation $\sim$ on A by $a \sim b$ if and only if $as = bt$ for some $s, t \in S$. Denote by $\bar{a}$ the equivalence class of a and set $A/_m S = \{\bar{a} : a \in A\}$. Then, we define in agreement with Marshall's notation, $\bar{a} + \bar{b} = \{\bar{c} : cv \in as + bt, \text{ for some } s, t, v \in S\}$, $-\bar{a} = \overline{-a}$, and $\bar{a}\bar{b} = \overline{ab}$.

Then $A/_m S$ is a multiring. Moreover, if A is a hyperring, the same holds for $A/_m S$. The canonical projection $\pi : A \to A/_m S$ is a morphism.

Proposition 2.8 (2.19 in [14]). *Let A, B be a multiring and $S \subseteq A$ a multiplicative subset of A. Then for every morphism $f : A \to B$ such that $f[S] = \{1\}$, there exist a unique morphism $\tilde{f} : A/_m S \to B$ such that the following diagram commute:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \pi\ } & A/_m S \\
 & {\scriptstyle f}\searrow & \downarrow {\scriptstyle !\tilde{f}} \\
 & & B
\end{array}
$$

where $\pi : A \to A/_m S$ is the canonical projection $\pi(a) = \bar{a}$.

Definition 2.9 (4.1 and 4.2 of [11]). A hyperfield F is said to be **real reduced** if $a^3 = a$ for all $a \in F$ and $a \in 1 + 1$ imply $a = 1$.

Definition 2.10 (7.5 and 7.6 of [11]). A multiring A is **real reduced** if is semi real and the following properties holds for all $a, b, c, d \in A$:

i) $1 \neq 0$;

ii) $a^3 = a$;

iii) $c \in a + ab^2 \Rightarrow c = a$;

iv) $c \in a^2 + b^2$ and $d \in a^2 + b^2$ implies $c = d$ (and from (iii), we conclude that this element $c \in a^2 + b^2$ is a square).

2.2 Special groups Let A be a set and $\equiv$ a binary relation on $A \times A$. We extend $\equiv$ to a binary relation $\equiv_n$ on A^n, by induction on $n \geqslant 1$, as follows:

i) $\equiv_1$ is the diagonal relation $\Delta_A \subseteq A \times A$

ii) $\equiv_2 = \equiv$.

iii) if $n \geq 3$, $\langle a_1, ..., a_n \rangle \equiv_n \langle b_1, ..., b_n \rangle$ if and only there are $x, y, z_3, ..., z_n \in A$ such that $\langle a_1, x \rangle \equiv \langle b_1, y \rangle$, $\langle a_2, ..., a_n \rangle \equiv_{n-1} \langle x, z_3, ..., z_n \rangle$ and

$$\langle b_2, ..., b_n \rangle \equiv_{n-1} \langle y, z_3, ..., z_n \rangle.$$

Whenever clear from the context, we frequently abuse notation and indicate the afore-described extension $\equiv$ by the same symbol.

Definition 2.11 (Special Group, 1.2 of [3])**.** A **special group** is an tuple $(G, -1, \equiv)$, where G is a group of exponent 2, that is, $g^2 = 1$ for all $g \in G$; -1 is a distinguished element of G, and $\equiv \subseteq G \times G \times G \times G$ is a relation (the special relation), satisfying the following axioms for all $a, b, c, d, x \in G$:

SG 0 $\equiv$ is an equivalence relation on G^2;

SG 1 $\langle a, b \rangle \equiv \langle b, a \rangle$;

SG 2 $\langle a, -a \rangle \equiv \langle 1, -1 \rangle$;

SG 3 $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow ab = cd$;

SG 4 $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow \langle a, -c \rangle \equiv \langle -b, d \rangle$;

SG 5 $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow \langle ga, gb \rangle \equiv \langle gc, gd \rangle$, for all $g \in G$.

SG 6 (3-transitivity) the extension of $\equiv$ for a binary relation on G^3 is a transitive relation.

A group of exponent 2, with a distinguished element -1, satisfying the axioms SG0-SG3 and SG5 is called a **proto special group**; a **pre special group** is a proto special group that also satisfyes SG4. Thus a **special group** is a pre-special group that satisfies SG6 (or, equivalently, for each $n \geq 1$, $\equiv_n$ is an equivalence relation on G^n.)

A n**-form** (or form of dimension $n \geqslant 1$) is an n-tuple of elements of a pre-SG G. An element $b \in G$ is **represented** on G by the form $\varphi = \langle a_1, ..., a_n \rangle$, in symbols $b \in D_G(\varphi)$, if there exists $b_2, ..., b_n \in G$ such that $\langle b, b_2, ..., b_n \rangle \equiv \varphi$.

A pre-special group (or special group) $(G, -1, \equiv)$ is:
- **formally real** if $-1 \notin \bigcup_{n \in \mathbb{N}} D_G(n\langle 1 \rangle)$;
- **reduced** if it is formally real and, for each $a \in G$, $a \in D_G(\langle 1, 1 \rangle)$ iff $a = 1$.

Now, some examples:

Example 2.12 (The trivial special relation, 1.9 of [3])**.** Let G be a group of exponent 2 and take -1 as any element of G different of 1. For $a, b, c, d \in G$, define $\langle a, b \rangle \equiv_t \langle c, d \rangle$ if and only if $ab = cd$. Then $G_t = (G, \equiv_t, -1)$ is a SG ([3]). In particular $2 = \{-1, 1\}$ is a reduced special group.

Example 2.13 (Special group of a field, Theorem 1.32 of [3])**.** Let F be a field. We denote $\dot{F} = F \setminus \{0\}$, $\dot{F}^2 = \{x^2 : x \in \dot{F}\}$ and $\Sigma \dot{F}^2 = \{\sum_{i \in I} x_i^2 :$

I is finite and $x_i \in \dot{F}^2\}$. Let $G(F) = \dot{F}/\dot{F}^2$. In the case of F is be formally real, we have $\Sigma \dot{F}^2$ is a subgroup of $\dot{F}$, then we take $G_{\mathrm{red}}(F) = \dot{F}/\Sigma \dot{F}^2$. Note that $G(F)$ and $G_{\mathrm{red}}(F)$ are groups of exponent 2. In [3] they prove that $G(F)$ and $G_{\mathrm{red}}(F)$ are special groups with the special relation given by usual notion of isometry, and $G_{\mathrm{red}}(F)$ is always reduced.

Definition 2.14 (1.1 of [3])**.** A map $(G, \equiv_G, -1) \xrightarrow{f} (H, \equiv_H, -1)$ between pre-special groups is a **morphism of pre-special groups or PSG-morphism** if $f : G \to H$ is a homomorphism of groups, $f(-1) = -1$ and for all $a, b, c, d \in G$

$$\langle a, b \rangle \equiv_G \langle c, d \rangle \Rightarrow \langle f(a), f(b) \rangle \equiv_H \langle f(c), f(d) \rangle$$

A **morphism of special groups or SG-morphism** is a pSG-morphism between the correspondents pre-special groups. f will be an isomorphism if is bijective and f, f^{-1} are PSG-morphisms.

It can be verified that a special group G is formally real iff it admites some SG-morphism $f : G \to 2$.

The category of special groups (respectively reduced special groups) and theirs morphisms will be denoted by $\mathcal{SG}$ (respectively $\mathcal{RSG}$). Now, we will analyze the connections between the $\mathcal{SG}$ and $MField$. For this, we need more results about special groups and their characterization. For this, we use the results proved in Lira's thesis [2]. Consider these axioms concerns about a group of exponent 2 with a distinguished element:

SG 7 $\forall a \, \forall a' \, \forall x \, \forall t \, \forall t' \, \forall y [(a, a') \equiv (x, t) \land (t, t') \equiv (1, y)]$
$\Rightarrow \exists a'' \, \exists s \, \exists s' [(a, a'') \equiv (y, s) \land (s, s') \equiv (1, x)].$

An equivalent statement for SG7 is

$$\bigcup_{t \in D_G(1,y)} D_G(x,t) = \bigcup_{s \in D_G(1,x)} D_G(y,s)$$

for all $x, y \in G$.

SG 8 For all forms $f_1, ..., f_n$ of dimension 3 and for all $a, a_2, a_3, b_2, b_3 \in G$,

$$\langle a, a_2, a_3 \rangle \equiv f_1 \equiv ... \equiv f_n \equiv \langle a, b_2, b_3 \rangle \Rightarrow \langle a_2, a_3 \rangle \equiv \langle b_2, b_3 \rangle.$$

SG 9 $\forall a \, \forall b \, \forall c \, \forall d [\langle a, b, ab \rangle \equiv \langle c, d, cd \rangle \Rightarrow \langle a, b, ab \rangle \equiv \langle d, c, cd \rangle]$

Proposition 2.15 (A. de Lima, [2]). *Let $(G, -1, \equiv)$ be a pre-special group. Then the following are equivalent:*

    (i) $G \models SG6$.
    (ii) $G \models SG7 \wedge SG8$.
    (iii) $G \models SG9$.

Proposition 2.16 (3.13 of [14]). *Let $(G, \equiv, -1)$ be a special group and define $M(G) = G \cup \{0\}$ where $0 := \{G\}^1$. Then $(M(G), +, -, \cdot, 0, 1)$ is a hyperfield, where*

- $a \cdot b = \begin{cases} 0 \text{ if } a = 0 \text{ or } b = 0 \\ a \cdot b \text{ otherwise} \end{cases}$

- $-(a) = (-1) \cdot a$

- $a + b = \begin{cases} \{b\} \text{ if } a = 0 \\ \{a\} \text{ if } b = 0 \\ M(G) \text{ if } a = -b, \text{ and } a \neq 0 \\ D_G(a, b) \text{ otherwise} \end{cases}$

Corollary 2.17 (3.14 of [14]). *The correspondence $G \mapsto M(G)$ extends to a faithful functor $M : \mathcal{SG} \to MField$.*

Proposition 2.18 (3.15 of [14]). *Let G be an SG and $M(G)$ as above. Then:*

    (i) $a^2 = 1$ *for all* $a \in M(G) \setminus \{0\}$;
    (ii) $1 \in 1 + a$ *for all* $a \in M(G)$;
    (iii) $1 + a$ *is closed by multiplication for all* $a \in M(G)$;
    (iv) *If exist* $p \in \dot{M}(G)$ *such that*

$$a \in c + cp$$
$$b \in p + ap$$
$$d \in p + cp.$$

---

[1]Here, the choice of the zero element was ad hoc. Indeed, we can define $0 := \{x\}$ for any $x \notin G$.

then exist $l \in \dot{M}(G)$ such that

$$a \in d + dl$$
$$b \in l + al$$
$$c \in l + dl.$$

Definition 2.19 (3.16 of [14])**.** A hyperfield F satisfying the properties i-iv of proposition 2.18 will be called a **special hyperfield**. Note that, if G is a special group, then $M(G)$ is a special hyperfield.

Proposition 2.20 (3.17 of [14])**.** *Every real reduced hyperfield is a special hyperfield.*

Theorem 2.21 (3.18 of [14])**.** *If F is a special hyperfield the $(\dot{F}, \equiv, -1)$ is a special group where $\langle a, b \rangle \equiv \langle c, d \rangle \Leftrightarrow ab = cd$ and $a \in c + d$.*

Corollary 2.22 (3.19 of [14])**.** *In the objects of $\mathcal{SMF}$, define $S(F) = \dot{F}$ as the special group as stated in theorem 2.21. Now, let $\sigma : F \to K$ be a SMF-morphism and define $S(\sigma) = \sigma|_{\dot{F}}$. Then $S : \mathcal{SMF} \to \mathcal{SG}$ is a functor.*

2.3   Real semigroups

Definition 2.23 (Ternary Semigroup, Definition 1.1 of [6])**.** A **ternary semigroup** (abbreviated TS) is a structure $(S, \cdot, 1, 0, -1)$ with individual constants $1, , 0, -1$ and a binary operation "$\cdot$" such that:

TS1 $(S, \cdot, 1)$ is a commutative semigroup with unity;

TS2 $x^3 = x$ for all $x \in S$;

TS3 $-1 \neq 1$ and $(-1)(-1) = 1$;

TS4 $x \cdot 0 = 0$ for all $x \in S$;

TS5 For all $x \in S$, $x = -1 \cdot x \Rightarrow x = 0$.

We shall write $-x$ for $(-1) \cdot x$. The semigroup verifying conditions [TS1] and [TS2] (no extra constants) will be called **3-semigroups**. We denote $\mathrm{Id}(S) = \{x \in S : x^2 = x\} = S^2$ and $S^* = \{x \in S : x^2 = 1\}$.

Example 2.24 (1.2(a) of [6])**.** The three-element structure $\mathbf{3} = \{1, 0, -1\}$ has an obvious ternary semigroup structure.

Here, we will enrich the language $\{\cdot, 1, 0, -1\}$ with a ternary relation D. We shall write $a \in D(b, c)$ instead of $D(a, b, c)$. We also set:

$$a \in D^t(b, c) \Leftrightarrow a \in D(b, c) \wedge -b \in D(-a, c) \wedge -c \in D(b, -a).$$

The relations D and D^t are called **representation** and **transversal representation** respectively.

Definition 2.25 (Real Semigroup, 2.1 of [6])**.** A **real semigroup** is a ternary semigroup together with a ternary relation D satisfying:

RS0 $c \in D(a, b)$ if and only if $c \in D(b, a)$.

RS1 $a \in D(a, b)$.

RS2 $a \in D(b, c)$ implies $ad \in D(bd, cd)$.

RS3 (Strong Associativity) If $a \in D^t(b, c)$ and $c \in D^t(d, e)$, then there exists $x \in D^t(b, d)$ such that $a \in D^t(x, e)$.

RS4 $e \in D(c^2 a, d^2 b)$ implies $e \in D(a, b)$.

RS5 If $ad = bd$, $ae = be$ and $c \in D(d, e)$, then $ac = bc$.

RS6 $c \in D(a, b)$ implies $c \in D^t(c^2 a, c^2 b)$.

RS7 (Reduction) $D^t(a, -b) \cap D^t(b, -a) \neq \emptyset$ implies $a = b$.

RS8 $a \in D(b, c)$ implies $a^2 \in D(b^2, c^2)$.

The theory of real semigroups can be alternatively axiomatized by the transversal relation D^t. In this case, we define

$$c \in D(a, b) \Leftrightarrow c \in D^t(c^2 a, c^2 b).$$

Example 2.26 (2.2 of [6])**.** (a) The three-element structure $\mathbf{3} = \{1, 0, -1\}$ has an obvious ternary semigroup structure.

(b) For any set X, the set $\mathbf{3}^X$ under pointwise operation and constant functions with values $1, 0, -1$, is a TS.

(c) The class of ternary semigroups is closed under direct product and substructures.

(d) Any group of exponent 2 is a 3-semigroup; the pointed group of exponent 2 with a distinguished element $-1 \neq 1$ underlying a RSG also verifies [TS3]. Any such group G, becomes a ternary semigroup by adding

a new absorbent element 0, that is, extending the operation by $x \cdot 0 = 0$ for $x \in G \cup \{0\}$. Note that the set of invertible elements of a 3-semigroup is a group of exponent 2.

(e) For any commutative ring A with 1, the set G_A of all functions $\overline{a} : \mathrm{Sper}(A) \to \mathbf{3}$, for $a \in A$, where

$$\overline{a}(\alpha) = \begin{cases} 1 \text{ if } a \in \alpha \setminus (-\alpha) \\ 0 \text{ if } a \in \alpha \cap (-\alpha) \\ -1 \text{ if } a \in (-\alpha) \setminus \alpha \end{cases}$$

with the operation induced by product in A is a TS.

Example 2.27 (RS and Rings, 2.2 of [6])**.** For any semi-real ring A, let the set G_A consist of all functions $\overline{a} : \mathrm{Sper}(A) \to \mathbf{3}$, for $a \in A$, where

$$\overline{a}(\alpha) = \begin{cases} 1 \text{ if } a \in \alpha \setminus (-\alpha) \\ 0 \text{ if } a \in \alpha \cap -\alpha \\ -1 \text{ if } a \in (-\alpha) \cap \alpha. \end{cases}$$

with the operation induced by product in A is a TS. More generally, given a (proper) preorder T of a ring A one can relativize the definition above to T, by considering functions $\overline{a}$ defined on $\mathrm{Sper}(A, T) = \{\alpha \in \mathrm{Sper}(A) : \alpha \supseteq T\}$, instead of $\mathrm{Sper}(A)$. The corresponding ternary semigroup will be denoted $G_{A,T}$.

Now, we will equip the ternary semigroup with the representation and transversal representation relations given by:

$$\overline{c} \in D_A(\overline{a}, \overline{b}) \Leftrightarrow \forall \alpha \in \mathrm{Sper}(A)[\overline{c}(\alpha) = 0 \vee \overline{a}(\alpha)\overline{c}(\alpha) = 1 \vee \overline{b}(\alpha)\overline{c}(\alpha) = 1].$$

$$\overline{c} \in D_A^t(\overline{a}, \overline{b}) \Leftrightarrow \forall \alpha \in \mathrm{Sper}(A)[(\overline{c}(\alpha) = 0 \wedge \overline{a}(\alpha) = \overline{-b}(\alpha)) \vee \overline{a}(\alpha)\overline{c}(\alpha) = 1 \vee \overline{b}(\alpha)\overline{c}(\alpha) = 1]$$

for $a, b, c \in A$. We have that G_A is a real semigroup. A similar definition with $\mathrm{Sper}(A)$ replaced by $\mathrm{Sper}(A, T)$ (T a proper preordering of A) also endows the ternary semigroup $G_{A,T}$ with a structure of real semigroup.

Example 2.28 (RS and RSG, 2.2 of [6])**.** The notion of a RS generalizes that of a reduced special group. Given a RSG G, we adding a absorbent

element 0 to give raise to a ternary semigroup $G_0 = G \cup \{0\}$. Extending the representation relation G to G_0 by

$$
D_{G_0}(a,b) = \begin{cases} \{a,b\} \text{ if } a = 0 \text{ or } b = 0; \\ D_G(a,b) \cup \{0\} \text{ if } a, b \in G, \end{cases}
$$

gives a representation relation to G_0. The axioms RS1-RS8 are immediate consequence of the special group axioms SG0-SG6 plus the following property: in a RSG we have

$$
a \in D(b,c) \Rightarrow -b \in D(-a,c),
$$

then D and D^t coincide on binary forms with entries in G.

The definition of morphism is quite standard: $f : (G, \cdot, 1, 0 - 1) \rightarrow (H, \cdot, 1, 0 - 1)$ is an **RS-morphism** if $f : G \rightarrow H$ is a morphism of semigroups, (that is, $f(ab) = f(a)f(b)$, $f(1) = 1$ and $f(0) = 0$); $f(-1) = -1$ and $a \in D(b,c) \Rightarrow f(a) \in D(f(b), f(c))$ (hence $a \in D^t(b,c) \Rightarrow f(a) \in D^t(f(b), f(c))$). The category of real semigroups and their morphisms will be denoted by $\mathcal{RS}$.

A fundamental ingredient in the theory of real semigroups is the following:

Theorem 2.29 (Separation Theorem, 4.4 of [6])**.** *Let G be a RS, and $a, b, c \in G$ and $X_G = Hom(G, \mathbf{3})$. Then:*
   (i) *$a \in D_G(b,c)$ if and only if for all $h \in X_G$, $h(a) \in D_{\mathbf{3}}(h(b), h(c))$.*
   (ii) *$a \in D_G^t(b,c)$ if and only if for all $h \in X_G$, $h(a) \in D_{\mathbf{3}}^t(h(b), h(c))$.*
   (iii) *If $a \neq b$, there is $h \in X_G$ such that $h(a) \neq h(b)$.*

The category of all real semigroups and the category of all real reduced multirings are isomorphic ([14]). In particular:

Theorem 2.30 (4.14 and 4.17 of [14])**.**
   (a) *Let $(G, \cdot, 1, 0, -1, D)$ be a real semigroup and define $+ : G \times G \rightarrow \mathcal{P}(G) \setminus \{\emptyset\}$, $a + b = D^t(a,b)$ and $- : G \rightarrow G$ by $-(g) = -1 \cdot g$. Then $(G, +, \cdot, -, 0, 1)$ is a real reduced multiring.*
   (b) *Let A be a real reduced multiring. Then $(A, \cdot, 1, 0, -1, D)$ is a real semigroup, where $d \in D(a,b) \Leftrightarrow d \in d^2 a + d^2 b$.*

In analogy with the theory of special groups (that contains the concepts of reduced special groups and formally real special groups), we propose the following "expansion" of the theory of real semigroups:

Definition 2.31. A **formally real semigroup** is a ternary semigroup together with a ternary relation D satisfying [RS0]-[RS3], [RS6] and:

RS7a (Zero) $D^t(0, a) = \{a\}$.

RS7b (Semi-reality) For all $n \geqslant 1$, $a_1, ..., a_n \in G$, $-1 \notin D^t(a_1^2, ..., a_n^2)$, with the conventions $D^t(a) = \{a\}$ and

$$D^t(a_1, ..., a_n) := \bigcup_{c \in D^t(a_2, ..., a_n)} D^t(a, c).$$

The definition of morphisms of a formally real semigroup is analogous. The category of formally real semigroups and their morphisms will be denoted by $\mathcal{FRS}$.

As an application of Separation Theorem for RS (2.29) we obtain:

Corollary 2.32. *Every real semigroup is a formally real semigroup.*

In section 5 below, we will relate formally real semigroups and multirings, in a very similar way of Theorem 2.30.

3   A special group associated to domains via Marshall quotient

Let F be a field. There is an almost canonical way to associate a special group to F (described in example 2.13): consider $G_F := \dot{F}/\dot{F}^2$ with the isometry given by the usual isometry provided by the algebraic theory of quadratic forms. As we have already seen, G_F is the multiplicative group of units of a special hyperfield, and in this sense,

$$M_F = G_F \cup \{0\} \cong F/_m \dot{F}^2.$$

In other words, we put in correspondence special groups and special hyperfields just adding (or erasing) a zero element.

One of the main purposes of this work is extend the above situation, $M_A \cong A/_mT$, where A is a commutative ring with unit and M_A is a *formally real semigroup*. This section deals with the case where A is a domain, that is, rings without zero divisors. Of course, we fatally need to impose some conditions to our structures:

Definition 3.1. An **hiperbolic multiring** is a multiring R such that $1 - 1 = R$.

Note that if R is hyperbolic and $a \in R^{\times}$, then $R = a - a$. For a ring R (that is, the sum is univalorated), R *never is hyperbolic*, since $1 - 1 = \{0\}$. However, this is not a problem, since the inclusion functor $Ring_2 \hookrightarrow MRing_2$ is not the most natural to be considered in the quadratic forms context. Considering the special group of a field $G(F) = \dot{F}/\dot{F}^2$ and its special hyperfield associated, $M(G(F)) = G(F) \cup \{0\}$, we get that $M(G(F))$ is hyperbolic. Hence, the desired functor to keep in mind is $M \circ G : Fields_2 \to SMF$.

Let R be a ring without zero divisors. The main goal of this section is to describe conditions for a subset $T \subseteq R \setminus \{0\}$ of R in such a way that $R/_mT$ is a special hyperfield and therefore, (essentially) a special group. Of course, here is an abuse of notation: when we say that "$R/_mT$ is a special group" we mean that "the induced structure in $(R/_mT) \setminus \{0\}$ provides a special group strucuture".

We we seek for inspiration in the analogous conditions for the field case (see for instance, definition 1.28 of [3], and in particular, the "completing squares" lemma 1.29). After months of hard work, we obtained the following definition:

Definition 3.2. A **Dickmann-Miraglia multiring (or DM-multiring for short)** [2] is a pair (R, T) such that R is a multiring, $T \subseteq R$ is a multiplicative subset of $R \setminus \{0\}$, and (R, T) satisfy the following properties:

DM0 $R/_mT$ is hyperbolic.

DM1 If $\bar{a} \neq 0$ in $R/_mT$, then $\bar{a}^2 = \bar{1}$ in $R/_mT$. In other words, for all $a \in R \setminus \{0\}$, there are $r, s \in T$ such that $ar = s$.

---

[2]The name "Dickmann-Miraglia" is given in honor to professors Maximo Dickmann and Francisco Miraglia, the creators of the special group theory.

DM2 For all $a \in R$, $(\bar{1} - \bar{a})(\bar{1} - \bar{a}) \subseteq (\bar{1} - \bar{a})$ in $R/_m T$.

DM3 For all $a, b, x, y, z \in R \setminus \{0\}$, if

$$\begin{cases} \bar{a} \in \bar{x} + \bar{b} \\ \bar{b} \in \bar{y} + \bar{z} \end{cases} \quad \text{in } R/_m T,$$

then exist $\bar{v} \in \bar{x} + \bar{z}$ such that $\bar{a} \in \bar{y} + \bar{v}$ and $\overline{vb} \in \overline{xy} + \overline{az}$ in $R/_m T$.

If R is a ring, we just say that (R, T) is a DM-ring, or R is a DM-ring. A Dickmann-Miraglia hyperfield (or DM-hyperfield) F is a hyperfield such that $(F, \{1\})$ is a DM-multiring (satisfy DM0-DM3). In other words, F is a DM-hyperfield if F is hyperbolic and for all $a, b, v, x, y, z \in F^*$,

i) $a^2 = 1$.

ii) $(1 - a)(1 - a) \subseteq (1 - a)$.

iii) If $\begin{cases} a \in x + b \\ b \in y + z \end{cases}$ then exist $v \in x + z$ such that $a \in y + v$ and $vb \in xy + az$.

The condition $1 - 1 = R/_m T$ (axiom [DM0]) can be expanded to the realm of higher level orderings (and signatures). Various sufficient conditions for this higher level relation are presented in Section 3 of [8][3].

Remark 3.3. These axioms above deserves some explanation:

(i) Since R is a domain and $0 \notin T$, $\bar{a} = \bar{0}$ in $R/_m T$ iff $a = 0$.

(ii) DM1 entails that $R/_m T$ is a hyperfield.

(iii) In DM2, the expression $(1 - a)(1 - a)$ means **multiplication of sets**, that is,

$$(1 - a)(1 - a) := \{x \cdot y : x, y \in 1 - a\}.$$

(iv) Looking at the expression in DM3, from

$$\begin{cases} \bar{v} \in \bar{x} + \bar{z} \\ \bar{b} \in \bar{y} + \bar{z} \\ \bar{a} \in \bar{x} + \bar{b} \end{cases} \quad \text{in } R/_m T,$$

---

[3]We are grateful to the referee by pointing this reference and by suggesting this expansion. In particular, could be interesting investigate the relations between the H_p hyperfield in 2.4 and the hyperfield Q_{2^l} defined in Section 10 of [8], in the context of orderings and signatures of higher level.

and the properties of multiring, we obtain

$$\overline{vb} \in \overline{xy} + (\overline{xz} + \overline{yz} + \overline{z}^2) \supseteq \overline{xy} + \overline{z}(\overline{x} + \overline{y} + \overline{z}) \text{ in } R/_m T$$

and

$$\overline{a} \in \overline{x} + \overline{b} \subseteq \overline{x} + \overline{y} + \overline{z} \text{ in } R/_m T.$$

Hence, we can interpret the condition $\overline{vb} \in \overline{xy} + \overline{az}$ in $R/_m T$ as a way of "controlling" the product $\overline{vb}$ to "not escape so much" under the set $\overline{x} + \overline{y} + \overline{z}$. In the field case (when we can "change" $\in$ by $=$), under the Marshall's quotient the condition M3 is not necessary (see theorem 1.32 of [3]).

(v) In DM3, if $0 \in \{a, b, x, y, z\}$ the axiom is trivially valid.

Theorem 3.4. *Let (R, T) be a DM-multiring and denote $Sm(R, T) = (R/_m T)$. Then $Sm(R)$ is a special hyperfield (thus $Sm(R, T)^\times$ is a special group).*

Recall that a special hyperfield is a hyperfield F satisfying:

SMF1 $a^2 = 1$ for all $a \in \dot{F}$;

SMF2 $1 \in 1 + a$ for all $a \in F$;

SMF3 $1 + a$ is closed by multiplication for all $a \in \dot{F}$;

SMF4 For all $a, b, c \in \dot{F}$, If $\exists p \in \dot{F}$ such that $\begin{cases} a & \in c + cp \\ b & \in p + ap \quad \text{then} \\ d & \in p + cp. \end{cases}$

$\exists l \in \dot{F}$ such that $\begin{cases} a & \in d + dl \\ b & \in l + al \\ c & \in l + dl. \end{cases}$

Proof of Theorem 3.4. The properties [SMF1]-[SMF3] are imediately consequence of the axioms of sum in a multiring and [M0]-[M2] in the definition of DM-multirings. Then, we shall prove [SMF 4]:

We will rewrite the argument of theorem 1.32 in [3]. In order to do this, we will use the language of special groups. If we prove that $R/_m T$ is a special group, then we prove that it is a special hyperfield (since [SMF 4] is precisely the translation of the axiom [SG9] for special groups to the language of hyperfields).

Here, the special relation in $R/_mT$ is defined by the rule

$$\langle \overline{a}, \overline{b} \rangle \equiv \langle \overline{c}, \overline{d} \rangle \Leftrightarrow [\overline{ab} = \overline{cd} \text{ and } \overline{a} \in \overline{c} + \overline{d}] \text{ (in } R/_mT).$$

Translating this to a condition with coefficients in R, we have

$$\langle \overline{a}, \overline{b} \rangle \equiv \langle \overline{c}, \overline{d} \rangle \Leftrightarrow [abv = cdw \text{ and } ar \in cs + dt] \text{ for some } r, s, t, v, w \in R.$$

Using [SMF1]-[SMF3] and the multirings properties we obtain the validity of [SG0-SG5] (for more details, see theorem 3.18 of [14]).

Hence by 2.15 we only need to deal with [SG9] (see condition (5) in theorem 1.23 of [3]), and it is enough to show that

$$\langle \overline{a}, \overline{b}, \overline{c} \rangle \equiv \langle \overline{x}, \overline{y}, \overline{z} \rangle \text{ implies } \langle \overline{a}, \overline{b}, \overline{c} \rangle \equiv \langle \overline{y}, \overline{x}, \overline{z} \rangle.$$

Suppose $\langle \overline{a}, \overline{b}, \overline{c} \rangle \equiv \langle \overline{x}, \overline{y}, \overline{z} \rangle$. Then, there exist α, β, γ such that

$$\langle \overline{a}, \overline{\alpha} \rangle \equiv \langle \overline{x}, \overline{\beta} \rangle, \ \langle \overline{b}, \overline{c} \rangle \equiv \langle \overline{\alpha}, \overline{\gamma} \rangle \text{ and } \langle \overline{y}, \overline{z} \rangle \equiv \langle \overline{\beta}, \overline{\gamma} \rangle. \tag{3.1}$$

Then, there exists $p_a, q_a, r_a, p_\beta, q_\beta, r_\beta \in T$ such that

$$ap_a \in xq_a + \beta r_a. \tag{3.2}$$

$$\beta p_\beta \in yq_\beta + zr_\beta. \tag{3.3}$$

Therefore $\overline{a} \in \overline{x} + \overline{b}$ and $\overline{b} \in \overline{y} + \overline{z}$. Applying [DM3], there exists

$$\overline{v} \in \overline{x} + \overline{z}, \tag{3.4}$$

such that

$$\overline{a} \in \overline{y} + \overline{v}. \tag{3.5}$$

We discuss two cases.

Case I: $v = 0$. Then, from equation 3.5, we have $\overline{a} = \overline{y}$. Consequently, the third isometry in equation 3.1 can be written as $\langle \overline{a}, \overline{z} \rangle \equiv \langle \overline{\beta}, \overline{\gamma} \rangle$. This isometry, the first one in equation 3.1 and [SG4] yeld

$$\langle \overline{x}, -\overline{\alpha} \rangle \equiv \langle \overline{a}, -\overline{\beta} \rangle \equiv \langle -\overline{z}, \overline{\gamma} \rangle,$$

and so $\langle \overline{x}, -\overline{\alpha} \rangle \equiv \langle -\overline{z}, \overline{\gamma} \rangle$. Another application of [SG4] yelds $\langle \overline{x}, \overline{z} \rangle \equiv \langle \overline{\alpha}, \overline{\gamma} \rangle$, which toghether with the second isometry in equation 3.1, gives $\langle \overline{x}, \overline{z} \rangle \equiv \langle \overline{b}, \overline{c} \rangle$. Then, we have

$$\langle \overline{a}, \overline{x} \rangle \equiv \langle \overline{a}, \overline{x} \rangle, \ \langle \overline{b}, \overline{c} \rangle \equiv \langle \overline{x}, \overline{z} \rangle, \ \text{and} \ \langle \overline{x}, \overline{z} \rangle \equiv \langle \overline{x}, \overline{z} \rangle,$$

which shows that $\langle \overline{a}, \overline{b}, \overline{c} \rangle \equiv \langle \overline{a}, \overline{x}, \overline{z} \rangle$, as required.

Case II: $v \neq 0$. Equation 3.5 implies $\overline{a} \in \overline{y} + \overline{v}$, while equation 3.4 yields $\overline{v} \in \overline{x} + \overline{z}$. Therefore,

$$\langle \overline{a}, \overline{vay} \rangle \equiv \langle \overline{y}, \overline{v} \rangle \ \text{and} \ \langle \overline{v}, \overline{vxz} \rangle \equiv \langle \overline{x}, \overline{z} \rangle.$$

These isometries imply that, in order to prove that $\langle \overline{a}, \overline{b}, \overline{c} \rangle \equiv \langle \overline{y}, \overline{x}, \overline{z} \rangle$, it is enough to verify that $\langle \overline{vay}, \overline{vxz} \rangle \equiv \langle \overline{b}, \overline{c} \rangle$. From the isometries in equation 3.1 we get $\overline{\alpha} = \overline{ax\beta}$, $\overline{\gamma} = \overline{yz\beta}$ and $\langle \overline{b}, \overline{c} \rangle \equiv \langle \overline{\alpha}, \overline{\gamma} \rangle$. Then, we have $\langle \overline{b}, \overline{c} \rangle \equiv \langle \overline{ax\beta}, \overline{z\beta} \rangle$.

Hence, what is needed is equivalent to $\langle \overline{ax\beta}, \overline{z\beta} \rangle \equiv \langle \overline{vay}, \overline{vxz} \rangle$. Since the discriminants are the same, it is enough to prove $\overline{ax\beta} \in \overline{vay} + \overline{vxz}$.

$$\overline{ax\beta} \in \overline{vay} + \overline{vxz} \Leftrightarrow \overline{ax\beta axv} \in \overline{vayaxv} + \overline{vxzaxv} \Leftrightarrow \overline{v\beta} \in \overline{xy} + \overline{az}.$$

then, it is enough verify that $\overline{v\beta} \in \overline{xy} + \overline{az}$. Moreover, axiom [DM3], already gave to us that $\overline{v\beta} \in \overline{xy} + \overline{az}$, which finalize the verification of [SG6].

$$\square$$

Example 3.5. Let X_n be the kaleidoscope multiring (as defined in 2.3). Of course, if $n \geqslant 2$, X_n is never a DM-hyperfield. However, considering $T = X_n^2 \setminus \{0\}$, since $X_n^2 = \{0, 1, 2, ..., n\}$ we get

$$K := X_n /_m T \cong X_1 = \{-1, 0, 1\}.$$

Since X_1 is a special hyperfield, (X_n, T) is a DM-multiring.

Example 3.6. Let p be a prime integer and consider the H_p as defined in 2.4 and $T = \sum H_p^2 \setminus \{0\}$. Then (H_p, T) is a DM-hyperfield since $H_p /_m T$ is a real reduced hyperfield.

The above theorem says that our DM-hyperfields are compatible with the special group strucuture obtained using Theorem 1.32 of [3].

Theorem 3.7. *Let A be a domain with $2 \neq 0$. Consider $T \subseteq A$ be a proper preordering or $T = A^2$ and denote $T^* = T \setminus \{0\}$. Then $A/_mT^*$ is a special hyperfield, and therefore $G_T(A) := (A/_mT^*) \setminus \{0\}$ is a special group with representation given by*

$$D_{G_A}(\bar{a}, \bar{b}) = \bar{a} + \bar{b} = \{\bar{c} : cr = as + bt \text{ for some } r, s, t \in T^*\}.$$

Moreover, $G_T(A)$ is reduced if and only if T is a proper preordering.

Proof. By theorem 3.4, we only need to proof that $A/_mT^*$ is a DM-hyperfield. First of all, note that

$$\text{For all } a, b \in A^*, \bar{a}, \bar{b} \in \bar{a} + \bar{b}. \tag{3.6}$$

If $a = \pm b$ is immediate (for example, $a(5a)^2 = a(4a)^2 + a(3a)^2$ or $a(3a)^2 = a(5a)^2 - a(4a)^2$, in the case where $3, 5 \neq 0$). If $a \neq \pm b$, then

$$a(a+b)^2 = a(a-b)^2 + b(2a)^2$$

and $a^2 + b^2, (a-b)^2, 2a^2 \in T^*$. Hence $\bar{a} \in \bar{a} + \bar{b}$. Similarly we conclude $\bar{b} \in \bar{a} + \bar{b}$.

Now, we verify the axioms [DM0]-[DM3].

DM0 Of course, $\bar{0} \in \bar{1} - \bar{1}$. If $a \neq 0$, and $a \neq \pm 1$, then

$$4a = (a+1)^2 - (a-1)^2,$$

and hence $\bar{a} \in \bar{1} - \bar{1}$. If $a = 1$ or $a = -1$, then

$$9 = 5^2 - 4^2 \text{ and } -9 = 4^2 - 5^2$$

testimony that $\bar{1}, -\bar{1} \in \bar{1} - \bar{1}$. Therefore $A/_mT^*$ is hyperbolic.

DM1 Let $\bar{a} \neq 0$ in $A/_mT$. Then $a^2 \in T$, hence $\bar{a}^2 = \bar{1}$.

DM2 Suppose whithout loss of generality that $a \in A^*$, $a \notin T$ (and hence $\bar{a} \notin \{-\bar{1}, \bar{0}, \bar{1}\}$). Now, let $\bar{\alpha}, \bar{\beta} \in \bar{1} + \bar{a}$, with $\alpha x = r + as$, $\beta y = t + aw$, for some $x, y, r, s, t, w \in T^*$. Then

$$(r + as)(t + aw) = (rt + a^2 sw) + (st + rw)a.$$

If T is a preordering, then $rt+a^2 sw \in T^*$ and $st+rw \in T^*$. If $T = A^2$, then $r = r_1^2$, $s = s_1^2$, $t = t_1^2$, $w = w_1^2$ for some $r_1, s_1, t_1, w_1 \in A^*$. Therefore

$$(r + as)(t + aw) = (rt + a^2 sw) + (st + rw)a$$
$$= a^2 sw + rt - 2r_1 s_1 t_1 w_1 a + 2r_1 s_1 t_1 w_1 a + (st + rw)a$$
$$= (a^2 sw - 2r_1 s_1 t_1 w_1 a + rt) + (st + 2r_1 s_1 t_1 w_1 + rw)a$$
$$= (a^2 s_1^2 w_1^2 - 2r_1 s_1 t_1 w_1 a + r_1^2 t_1^2) + (s_1^2 t_1^2 + 2r_1 s_1 t_1 w_1 +$$
$$r_1^2 w_1^2)a = (as_1 w_1 - r_1 t_1)^2 + (s_1 t_1 + r_1 w_1)^2 a.$$

If $(as_1 w_1 - r_1 t_1)^2 = (s_1 t_1 + r_1 w_1)^2 = 0$ we have $\overline{r + at} = \overline{0}$ or $\overline{s + aw} = \overline{0}$, and hence $r = -at$ or $s = -aw$, and both cases imply $-\overline{a} = \overline{1}$. If $(as_1 w_1 - r_1 t_1)^2, (s_1 t_1 + r_1 w_1)^2 \neq 0$ then $(as_1 w_1 - r_1 t_1)^2, (s_1 t_1 + r_1 w_1)^2 \in T^*$ and we are done. If $(as_1 w_1 - r_1 t_1)^2 = 0$, using 3.6

$$(r + as)(t + aw) = (s_1 t_1 + r_1 w_1)^2 a \Rightarrow \overline{\alpha}\overline{\beta} = \overline{a} \in 1 + \overline{a}.$$

If $(s_1 t_1 + r_1 w_1)^2 = 0$, using 3.6

$$(r + as)(t + aw) = (as_1 w_1 - r_1 t_1)^2 \Rightarrow \overline{\alpha}\overline{\beta} = \overline{1} \in 1 + \overline{a},$$

completing the proof.

DM3 Let

$$\begin{cases} \overline{a} \in \overline{x} + \overline{b} \\ \overline{b} \in \overline{y} + \overline{z} \end{cases} \quad \text{in } A/_m T,$$

with $\overline{a}, \overline{b}, \overline{x}, \overline{y}, \overline{z} \neq \overline{0}$. Then, there exists $p_a, q_a, r_a, p_b, q_b, r_b \in T$ such that

$$ap_a = xq_a + br_a. \tag{3.7}$$

$$bp_b = yq_b + zr_b. \tag{3.8}$$

Therefore

$$ap_a p_b = xp_b q_a + bp_b r_a = xp_b q_a + (yq_b + zr_b)r_a = xp_b q_a + yq_b r_a + zr_a r_b.$$

Now, consider

$$v = xp_b q_a + zr_a r_b. \tag{3.9}$$

Note that $\overline{v} \in \overline{x} + \overline{z}$ and

$$ap_a p_b = yq_b r_a + v, \tag{3.10}$$

with $\overline{a} \in \overline{y} + \overline{v}$. In order to complete the proof, we only need to verify that $\overline{vb} \in \overline{xy} + \overline{az}$. In fact,

$$
\begin{aligned}
vbp_b &= (xp_b q_a + zr_a r_b)(yq_b + zr_b) \\
&= xyp_b q_a q_b + xzp_b q_a r_b + yzq_b r_a r_b + z^2 r_a r_b^2 \\
&= xyp_b q_a q_b + z(xp_b q_a r_b + yq_b r_a r_b + zr_a r_b^2) \\
&= xyp_b q_a q_b + z(xp_b q_a r_b + (yq_b + zr_b)r_a r_b) \\
&= xyp_b q_a q_b + z(xp_b q_a r_b + bp_b r_a r_b) \\
&= xyp_b q_a q_b + (xq_a + br_a)zp_b r_b \\
&= xyp_b q_a q_b + ap_a zp_b r_b \\
&= xyp_b q_a q_b + azp_a p_b r_b,
\end{aligned}
$$

and hence, $\overline{vb} \in \overline{xy} + \overline{az}$.

$\square$

Corollary 3.8. *Let D be a domain with $2 \neq 0$ and consider the polynomial ring $D[x_1, ..., x_n]$. Let $T \subseteq D[x_1, ..., x_n]$ be a preordering or $T = (D[x_1, ..., x_n])^2$. Then $D[x_1, ..., x_n]/_m T^*$ is a special group.*

Theorem 3.9. *Let F be a hyperfield satisfying DM0-DM2. Then F satisfy DM3 if and only if satisfy SMF4. In other words, F is a DM-hyperfield if and only if it is a special hyperfield.*

Proof. After Theorem 3.4, we only need to prove that if F is a special hyperfield then F satisfy DM3. Let $a \in x + b$ and $b \in y + z$. Then by definition, $a \in x + y + z$, and then, there exist some $v \in x + z$ such that $a \in y + v$. We need to prove that $vb \in xy + az$. We discuss two cases.

Case I: $v = 0$. Then $a = y$ and $z = -x$. Moreover

$$0 = vb \in ax - ax = xy + az.$$

Case II: $v \neq 0$. Here we consider the special group structure in F^*. Moreover, for all $a, b \in F^*$, $a, b \in a + b$. Considering $a \in x + b$ and $b \in y + z$, we get the above isometries

$$\langle byz, x \rangle \equiv \langle x, byz \rangle, \ \langle axb, a \rangle \equiv \langle x, b \rangle \text{ and } \langle y, z \rangle \equiv \langle byz, b \rangle.$$

Then by definition $\langle byz, axb, a \rangle \equiv \langle x, y, z \rangle$.

Moreover, considering $a \in y + v$ and $v \in x + z$, we get the above isometries

$$\langle vxz, y \rangle \equiv \langle y, vxz \rangle, \ \langle ayv, a \rangle \equiv \langle y, v \rangle \text{ and } \langle x, z \rangle \equiv \langle vxz, v \rangle.$$

Then by definition $\langle vxz, ayv, a \rangle \equiv \langle y, x, z \rangle$. Since F^* is a special group, $\langle x, y, z \rangle \equiv \langle y, x, z \rangle$ and the isometry relation is 3-transitive. Then

$$\langle byz, axb, a \rangle \equiv \langle x, y, z \rangle \equiv \langle y, x, z \rangle \equiv \langle vxz, ayv, a \rangle,$$

and hence, $\langle byz, axb, a \rangle \equiv \langle vxz, ayv, a \rangle$. Using Witt's Cancellation, $\langle byz, axb \rangle \equiv \langle vxz, ayv \rangle$. Then,

$$vxz \in byz + axb \Rightarrow vbxz \in yz + ax \Rightarrow vb \in xy + az,$$

completing the proof.

$\square$

Theorem 3.10. *Let* $(G, \equiv, 1, -1)$ *be a pre-special group. Then the following are equivalent:*

(1) *G is special, that is, satisfy (for example) SG6.*
(2) *$M(G)$ (the hyperfield associated to G) satisfy DM3.*
(3) *G satisfy the following condition for all $a, b, x, y, z \in G$:*

If $a \in D_G(x, b)$ and $b \in D_G(y, z)$ then there exist $v \in D_G(x, z)$ such that $a \in D_G(y, v)$ and $vb \in D_G(xy, az)$.

4   DM-multirings and quadratically presentable fields

Let (R, T) be a DM-multiring and $G(R, T) := (R/_m T) \setminus \{0\}$. Since $G(R, T)$ is a special group, we can provide a theory of quadratic forms for R inherited from $G(R, T)$: Let $\equiv$ be the isometry relation on $G(R, T)^2$ given by $\langle a, b \rangle \equiv \langle c, d \rangle$ iff $ab = cd$ in $G(R, T)$ and $a \in c + d \setminus \{0\}$. We extend $\equiv$ to a binary relation $\equiv_n$ on $G(R, T)^n$, by induction on $n \geqslant 2$, as follows:

i) $\equiv_2 = \equiv$.

ii) $\langle a_1, ..., a_n \rangle \equiv_n \langle b_1, ..., b_n \rangle$ if and only there are $x, y, z_3, ..., z_n \in A$ such that $\langle a_1, x \rangle \equiv \langle b_1, y \rangle$, $\langle a_2, ..., a_n \rangle \equiv_{n-1} \langle x, z_3, ..., z_n \rangle$ and $\langle b_2, ..., b_n \rangle \equiv_{n-1} \langle y, z_3, ..., z_n \rangle$.

Since $G(R, T)$ is a special group, $\equiv_n$ is transitive for all $n \geqslant 2$ (in fact, this is the content of axiom SG6). Whenever clear from the context, we frequently abuse notation and indicate the aforedescribed extension $\equiv$ by the same symbol.

A **form** φ on $G(R, T)$ is an n-tuple $\langle a_1, ..., a_n \rangle$ of elements of $G(R, T)$; n is called the **dimension** of φ, $\dim(\varphi)$. We also call φ a n-**form**.

By convention, two forms of dimension 1 are isometric if and only if they have the same coefficients. If $\varphi = \langle a_1, ..., a_n \rangle$ is a form on $G(R, T)$, define

a) The **set of elements represented by** φ as

$$D_{G(R,T)}(\varphi) := \{b \in G(R, T) : \exists z_2, ..., z_n \in G(R, T) \text{ such that } \varphi \equiv \langle b, z_2, ..., z_n \rangle\}.$$

b) The **discriminant** of φ as $d(\varphi) = \prod\limits_{i=1}^{n} a_i$.

c) **Direct sum** as $\varphi \oplus \theta = \langle a_1, ..., a_n, b_1, ..., b_m \rangle$.

d) **Tensor product** as $\varphi \otimes \theta = \langle a_1 b_1, ..., a_i b_j, ..., a_n b_m \rangle$. If $a \in G(R, T)$, $\langle a \rangle \otimes \varphi$ is written $a\varphi$.

A form φ on $G(R, T)$ is **isotropic** if there is a form ψ over $G(R, T)$ such that $\varphi \equiv \langle 1, -1 \rangle \oplus \psi$; otherwise it is said to be **anisotropic**. We say that φ is **universal** if $D_{G(R,T)}(\varphi) = G(R, T)$.

In this sense, **Witt Ring** $W(R,T)$ of (R,T) is defined as the Witt ring $W(G(R,T))$ of $G(R,T)$. We can go further, and define a **form** $\varphi = \langle a_1, ..., a_n \rangle$ on (R,T) by considering the form $\overline{\varphi} := \langle \overline{a_1}, ..., \overline{a_n} \rangle$ on $G(R,T)$ and so on.

Moreover, this quadratic form theory inherited from $G(R,T)$ is compatible with the more general Witt rings described by P. Gladki and K. Worytkiewicz in [9]:

Definition 4.1 (Presentable monoid, group, ring [9])**.** Let $(A, \leqslant, 0)$ be a pointed poset (that is, a poset with a distinguished element $0 \in A$).

a) $(A, \leqslant, 0, +)$ is a **presentable monoid** if the distinguished element 0 is supercompact and $+ : M \times M \to M$ is a suprema-preserving binary operation such that for all $a, b, c \in M$

    (a) $a + (b + c) = (a + b) + c$;
    (b) $a + 0 = 0 + a = a$;
    (c) $a + b = b + a$.

b) $(A, \leqslant, 0, +, -)$ is a **presentable group** if $(A, \leqslant, 0, +)$ is a presentable monoid and $- : G \to G$ is a suprema preserving involutive homomorphism (called **inversion**) such that $s \leqslant t + u$ imply $t \leqslant s + (-u)$ for all $s, t, u \in \mathcal{S}_G$ (here $\mathcal{S}_G$ denote the set of G's minimal elements).

c) $(A, \leqslant, 0, 1, +, -, \cdot)$ is a **presentable ring** if $(A, \leqslant, 0, +, -)$ is a presentable group, $(A, 1, \cdot)$ is a commutative monoid such that the element 1 is supercompact, $\cdot$ is compatible with $\leqslant$ and $-$ (that is, $a \leqslant b$ imply $a \cdot c \leqslant b \cdot c$ and $a \cdot (-b) = -(a \cdot b)$ for all $a, b, c \in A$), $\cdot$ is distributive with respect to $+$, $0 \cdot a = 0$ for all $a \in R$ and $\cdot$ satisfy

$$\mathcal{S}_{a \cdot b} = \{s \cdot t : s \in \mathcal{S}_a,\, t \in \mathcal{S}_b\}.$$

Here $\mathcal{S}_a := \downarrow a \cap \mathcal{S}_A$ for all $a \in A$, that is, $\mathcal{S}_a$ is the set of all minimal elements below $a \in A$.

d) $(A, \leqslant, 0, 1, +, -, \cdot)$ is a **presentable field** if is a presentable ring such that every non-zero element is invertible.

Now we recall the concept of quadratically presentable fields (in the sense of definitions 5.1, 5.5 and 5.7 of [9]). A presentable field $(A, \leqslant, 0, 1, +, -, \cdot)$ is **pre-quadratically presentable** whenever

i) $a \leqslant a + b$ for all $a \in \mathcal{S}_A^*$, $b \in \mathcal{S}_A$;

ii) $a \leqslant 1 + b$ and $a \leqslant 1 + c$ imply $a \leqslant 1 - bc$ for all $a, b, c \in \mathcal{S}_A$;

iii) $a^2 = 1$ for all $a \in \mathcal{S}_A \setminus \{0\}$.

A **form** on a pre-quadratically presentable field A is an n-tuple $\langle a_1, ..., a_n \rangle$ of elements of $\mathcal{S}_A^*$. The relation $\cong$ of **isometry** of forms of the same dimension is given by induction: (i) $\langle a \rangle \cong \langle b \rangle$ iff $a = b$; (ii) $\langle a_1, a_2 \rangle \cong \langle b_1, b_2 \rangle$ iff $a_1 a_2 = b_1 b_2$ and $b_1 \leqslant a_1 + a_2$; (iii) finally, for $n \geqslant 3$

$$\langle a_1, ..., a_n \rangle \cong \langle b_1, ..., b_n \rangle \text{ iff there exists } x, y, c_3, ..., c_n \in \mathcal{S}_A^* \text{ such that } \langle a_1, x \rangle$$
$$\cong \langle b_1, y \rangle, \langle a_2, ..., a_n \rangle \cong \langle x, c_3, ..., c_n \rangle, \langle a_b, ..., b_n \rangle \cong \langle y, c_3, ..., c_n \rangle.$$

A pre-quadratically presentable field is **quadratically presentable** whenever the isometry relation defined above is an equivalence relation on the set of all forms of the same dimension.

Let (R, T) be a DM-multiring. Let $K := R/_m T$ and consider $\mathcal{P}^*(K)$, the pierced powerset of the set K (that is, its set of nonempty subsets). Then $(\mathcal{P}^*(K), \subseteq, \{0\}, \{1\}, +, -, \cdot)$ is a presentable field ([9], Example 4.5), where the operations in $\mathcal{P}^*(K)$ are defined for $A, B \in \mathcal{P}^*(K)$ by

$$-A := \bigcup_{a \in A} \{-a\}, \ A + B := \bigcup_{a \in A, b \in B} a + b \text{ and } A \cdot B := \bigcup_{a \in A, b \in B} \{a \cdot b\}.$$

Following 5.18 [9], we obtain:

Theorem 4.2. *Let (R, T) be a DM-multiring. Let $K := R/_m T$ and $(\mathcal{P}^*(K), \subseteq, \{0\}, \{1\}, +, -, \cdot)$ be the induced presentable field. Then:*
(1) $\mathcal{P}^*(K)$ is a quadratically presentable field.
(2) $W(\mathcal{P}^*(K)) \cong W(K) = W(R, T)$, where $W(\mathcal{P}^*(K))$ is the Witt ring defined in 5.13 [9].

Proof. (1) This follows, essentially, from the same argument of 3.4, since K is a special hyperfield.

(2) Just repeat the arguments used in 7.1, 7.2 and 7.3 of [9].

For the readers comfortable with theory of special groups, the proof of this theorem is just a translation of axiom SG6. $\square$

In 7.4 of [9] it is asked:

"It is an open question when the resulting pre-quadratically presentable field is quadratically presentable."

We finish this section arguing that such question is, in principle, non void. More precisely:

Proposition 4.3. *There exists a pre-quadratically presentable field that is not quadratically presentable.*

Proof. We will show that $pQPF$ is a cocomplete category but QPF is not a cocomplete category.

- In 5.18 of [9] are established equivalences of categories:
  quadratically presentable fields (QPF) $\leftrightsquigarrow$ special groups (SG);
  pre-quadratically presentable fields ($pQPF$) $\leftrightsquigarrow$ pre-special groups (pSG).

- $pQPF$ ($\simeq pSG$) is a cocomplete category.

  According the definition of pre-special group (Definition 1.2 in [3]), it is axiomatizatied by a universal Horn Theory (definition 5.10 in [1]) thus it is a limit theory (Definition 5.7 in [1]). By Theorem 5.9 in [1], pSG is a finitely locally presentable category, (Definition 1.9 in [1]), thus it is a cocomplete category.

- QPF ($\simeq SG$) is not a cocomplete category.

  * Consider RSG the full subcategory of SG of all reduced special groups, that is, a special group G such that for each $a \in G$, $\langle a, a \rangle \equiv \langle 1, 1 \rangle$ iff $a = 1$. This is a slightly variation on the notion of reduced special group (Definition 1.2 in [3]) since we not exclude the case where $G = \{1\}$ (equivalently, we not impose $-1 \neq 1$). Following the proofs of the results in Chapter 10, Section 3, in [3], the category RSG of all reduced special groups (including the trivial special group $\{1\}$)) misses some binary coproducts, thus is not cocomplete.

  * The full subcategory $\iota : RSG \hookrightarrow SG$ is reflexive, that is, it has a left adjoint $S : SG \to RSG$, $G \in Obj(SG) \mapsto G/Sat(G) \in Obj(RSG)$, where the unity of adjunction is $(G \overset{q_G}{\twoheadrightarrow} S(G) := G/Sat(G))_{G \in Obj(SG)}$. This follows from a combination of results in [3]: Remark (iii) just below Definition 2.7; Remark 2.16 and Proposition 2.21.

* Let $\Gamma : \mathcal{I} \to RSG$ be a small diagram that does not have a colimit in RSG. Suppose that $\iota \circ \Gamma : \mathcal{I} \to SG$ has a colimit $(\gamma_i : \Gamma(i) \to G_\infty)_{i \in obj(\mathcal{I})}$ in SG. Then it is easy to check that $(q_{G_\infty} \circ \gamma_i : \Gamma(i) \to S(G_\infty))_{i \in Obj(\mathcal{I})}$ satisfies the universal property of being the colimit of $\Gamma : \mathcal{I} \to RSG$ in RSG, a contradiction.

$\square$

5 Quadratic multirings and (formally) real semigroup associated to semi real rings via Marshall quotient

Paraphrasing M. Marshall, "when we change fields for rings, we are in deep water" ([12])! For example, let R be a generic commutative ring and $T \subseteq R$ be a multiplicative set containing 1. From now on, we denote

$$zd(R) := \{a \in R : a \text{ is a zero divisor}\}$$
$$nzd(R) := R \setminus zd(R) = \{a \in R : a \text{ is not a zero divisor}\}.$$

If $a, b \in T \setminus \{0\}$ with $ab = 0$ (that is, a, b are zero-divisors), then $R/_m T^* \cong \{0\}$: in fact for all $x \in R$, $x(ab) = 0 \cdot 1$ with $ab, 1 \in T$, and hence $\overline{x} = \overline{0}$.

Even in the case $T \subseteq nzd(R)$, if $a \in zd(R)$, say $ab = 0$ for some $b \in zd(R)$ then $\overline{a}\overline{b} = \overline{0}$, so $(\overline{ab})^2 = 0 \neq \overline{1}$, and in particular, $R/_m T$ is not a hyperfield.

Then, *if $zd(R) \neq \emptyset$, $R/_m T^*$ will never be a special group, since will never be a hyperfield.* Because this, we will seek for conditions for a pair (R, T) with R a ring and $T \subseteq nzd(R)$ multiplicative provide a (formally) real semigroup structure in $R/_m T$.

In this context we christen the following definition:

Definition 5.1. Let R be a multiring and $T \subseteq nzd(R)$ be a multiplicative subset containing 1. We say that (R, T) is a **quadratic pair** if

Q1 $R/_m T$ is semi real.

Q2 If $a \in R$ and $a^2 \notin zd(R)$, then $a^2 \in T$.

Q3 For all $a \in R$, then $\overline{a}^3 = \overline{a}$ in $R/_m T$.

Q4 For all $a, b \in R$, there exists $r, s, t \in T$ such that $ar \in a^3 s + a^2 bt$.

Let's look closey to the axioms in definition 5.1. In this sense, Q1 is a kind of generalization of the semireal condition and Q2 is a weakness of $A^2 \subseteq T$. Q4 is saying that "$\overline{a} + \overline{ab^2} = \{\overline{a}\}$ in $A/_m T$". The following theorem is immediate:

Theorem 5.2. *Let (R, T) be a quadratic pair and define for all $a, b, c \in R$ the following relations:*

$$\overline{c} \in D^t(\overline{a}, \overline{b}) \text{ if and only if } \overline{c} \in \overline{a} + \overline{b}$$
$$\overline{c} \in D(\overline{a}, \overline{b}) \text{ if and only if } \overline{c} \in D^t(\overline{c}^2 a, \overline{c}^2 b).$$

Then $(R/_m T, D, D^t)$ is a formally real semigroup. Conversely, if (G, D, D^t) is a formally real semigroup such that a^2 is a zero divisor or $a^2 = 1$. Define

$$c \in a + b \text{ if and only if } c \in D^t(a, b).$$

Then $(G, \{1\})$ is a quadratic pair.

Proof. Let (R, T) be a quadratic pair. Axiom RS7b is consequence of Q1 and axiom RS1 is consequence of Q4. The other axioms of formally realsemigroup are consequence of basic properties of multiring and so on.

Conversely, if (G, D, D^t) is a formally real semigroup such that a^2 is a zero divisor or $a^2 = 1$, we automatically have Q2. Q1 is consequence of RS7b, Q3 is consequence of G be a ternary semigroup and Q4 is consequence of RS1. The fact of $(G, +, \cdot, 0, 1)$ be a multiring is consequence of the another axioms of formally realsemigroup (and ternary semigroup). $\qquad\square$

Now is time to deal with the real semigroup case. We define the following:

Definition 5.3. A **Dickmann-Petrovich multiring (or DP-multiring for short)**[4] is a quadratic pair (R, T) satisfy the following properties:

DP1 $1 + T \subseteq T$.

DP2 For all $a \in R$, exist $t \in T$ such that $1 + a^2 t \in T$.

DP3 For all $a, b \in R$, $\overline{a}^2 + \overline{b}^2$ is a singleton set in $R/_m T$.

---

[4]The name "Dickmann-Petrovich" is given in honor to professors Max Dickmann and Alejandro Petrovich, who are the creators of realsemigroup theory.

Theorem 5.4. *Let (R, T) be a DP-ring and denote $Rs(R) = (R/_m T)$. Then $Rs(R)$ is a real reduced multiring (thus it is a real semigroup).*

Proof. Since $T \subseteq nzd(R)$, $\bar{1} \neq \bar{0}$ in $Rs(R)$. Moreover, by (Q4) we get $\overline{a^3} = \bar{a}$ in $Rs(R)$.

Note that since T is multiplicative, [Q0] and [DP1] imply $T \cdot T = T$ and

$$T + T = T + T \cdot T = T \cdot (1 + T) \subseteq T \cdot T = T,$$

then we have that $T + T \subseteq T$ which imply that $\bar{a} + \bar{a} = \{\bar{a}\}$ for all $\bar{a} \in Rs(R)$.

From (DP2) we get $\overline{1} + \overline{b}^2 = \{\bar{1}\}$ for all $b \in R$, which imply $\bar{a} + \overline{ab^2} = \{\bar{a}\}$ for all $a, b \in R$. Finally, [DP3] says that $\overline{a}^2 + \overline{b}^2$ is a singleton set in $R/_m T$, completing the proof that $R/_m T$ is a real semigroup. $\square$

Example 5.5. Let (R, T) be a DM-multiring. Then (R, T) is also a quadratic pair.

Example 5.6. Let (R, T) be a DM-ring such that $T + T \subseteq T$. Then (R, T) is also a DP-ring.

With definition 5.1 and theorem 5.2, we generalize the real reduced multirings:

Definition 5.7. A multiring A is said to be **quadratic** if it satisfies the following properties:

QM0 $-1 \notin \sum A^2$.

QM1 for all $a \in A$, $a \in 1 - 1$.

QM2 for all $a \in A$, $a^3 = a$.

QM3 for all $a, b \in A$, $a \in a + a^2 b$.

Example 5.8. Let p be a prime integer and consider H_p as in 2.4. Since $a^2 = 1$ and $a = -a$ for all $a \in H_p$ and $a + a = H_p$ for all $a \neq 0$, we have that H_p is not a quadratic multiring.

But H_p satisfy QM1, QM2 and QM3. Then, consider the produt multiring $R = X_1 \times H_p$, where $X_1 = \{-1, 0, 1\}$. Since X_1 is a DM-hyperfield (and hence a DP-multiring) and the operations and multioperation in R is defined coordinatewise, we have that R satisfy QM1, QM2 and QM3. Since $(a, b) \in R^2$ if and only if $a \in \{0, 1\}$ and $b \in H_p$, we have $-1_R = (-1, 1) \notin R^2$. Hence R is a quadratic multiring.

Example 5.9 (Constructions)**.** (i) (Products) Let $\{R_i\}_{i \in I}$ be a class of quadratic multiring and let $R = \prod_{i \in I} R_i$. Since the operations and multioperation in R is defined coordinatewise, we have that R is a quadratic multiring. More generally, supose that R_i satisfy QM1, QM2 and QM3 for all $i \in I$. If there is an index $i_0 \in I$ such that R_{i_0} is a quadratic multiring, then R is a quadratic multiring.

(ii) (Directed Colimits) If $(I, \leq)$ is an upward directed poset and $(f_{ij} : R_i \to R_j)_{i \leq j}$ is a diagram of quadratic multirings, then $colim_{i \in I} R_i$ is a quadratic multring. More generaly, if $(f_{ij} : R_i \to R_j)_{i \leq j}$ is an upward directed diagram of multirings such that $\{i \in I : R_i$ is a quadratic multring$\}$ is a cofinal subset of I, then $colim_{i \in I} R_i$ is a quadratic multring.

(iii) (Reduced Products and Ultraproducts) The class of quadratic multirings can be axiomatized by certain kind of first-order formulas (in a convenient relational language) that shows that this subclass of the class of multirings is closed under reduced products (and ultraproducts, in particular). This result can be achieved more directly by the description of reduced product of a family of (quadratic) multirings, modulo some filter on the index set, as the directed colimit of products of the members of the family indexed by some member of the filter: $\prod_{i \in I} R_i / \mathcal{F} \cong colim_{J \in \mathcal{F}} \prod_{i \in J} R_i$.

Example 5.10 (Special Groups)**.** Let G be a special group, and consider $F = M(G) := G \cup \{0\}$ its special hyperfield associated. Of course, F satisfy conditions QM1-QM3 in 5.7. F satisfy DM0 iff F is formally real, that is, if $-1 \notin \sum F^2$, which occurs iff G is formally real, that is,

$$-1 \notin D_G(n \otimes \langle 1 \rangle) \text{ for all } n \geqslant 1.$$

Example 5.11. Let A be a von Neumann regular semi-real ring such that $2 \in A^\times$. Then $A/_m A^{\times 2}$ is a quadratic multiring. In fact, first observe that

(i) If F is a field with $2 \in F^\times$, then $F/_m F^{\times 2}$ is a multiring that satisfies **QM1-QM3** as indicate examples 2.13 and 5.10. This means that F satisfies the following Horn-geometric sentences:

- $\forall a \exists x, y, x', y'(xx' = yy' = 1 \wedge a = x^2 - y^2)$.
- $\forall a \exists x, y, x', y'(xx' = yy' = 1 \wedge a^3 x^2 = ay^2)$.
- $\forall a, b \exists x, y, z, x', y', z'(xx' = yy' = zz' = 1 \wedge ax^2 = ay^2 + a^2 bz^2)$.

(ii) The Proposition 5.6 of [4] shows that the von Neumann regular ring A is the ring of global sections over a Boolean space where the sheaf has fields with 2 invertible as stalks.

Thus, the Proposition 3.2-(d), [4], applied to the sheaf of item ii) above implies that formulas of item i) are valid in A. Therefore $A/_m A^{\times 2}$ is a quadratic multiring.

Example 5.12 (Faithfully Quadratic Rings)**.** Now, we relate our DM-multrings, DP-multrings and quadratic multirings with faithfully quadratic rings as presented in [5]: let A be a semi-real ring with $2 \in \dot{A}$, T be a preordering of A or $T = A^2$. A **T-subgroup** of A is a multiplicative subset S of $\dot{A}$ containing $\{-1\} \cup \dot{T}$. For $a, b \in S$, denote

$$D_{S,T}^v(a,b) := \{c \in S : c = as + bt \text{ for some } s, t \in T\}.$$
$$D_{S,T}^t(a,b) := \{c \in S : c = as + bt \text{ for some } s, t \in \dot{T}\}.$$

The triple (A, T, S) is **faithfully quadratic** if (among other things) satisfy $D_{S,T}^v(a,b) = D_{S,T}^t(a,b)$ for all $a, b \in S$ (see for instance, definition 3.1 in [5]). Denote

$$a^T = b^T \text{ iff } ab \in \dot{T} \text{ iff } b = at \text{ for some } t,$$

and consider $G_T(S) = \{a^T : a \in S\}$. Define the binary isometry $\equiv_T^S$ by

$$\langle a^T, b^T \rangle \equiv \langle c^T, d^T \rangle \text{ iff } a^T b^T = c^T d^T \text{ and } D_{S,T}^v(a,b) = D_{S,T}^v(c,d).$$

In general, $(G_T(S), \equiv_T^S, -1^T)$ is a proto-special group If (A, T, S) is faithfully quadratic, then Dickmann and Miraglia showed (see theorem 3.5 [5]) that $G_T(S)$ is a special group.

Now, consider (A, T, S) and let $R = A/_m(T \cap nzd(A))$. Then $D_{S,T}^t(a,b) \subseteq \bar{a} + \bar{b}$ for all $a, b \in A$. Moreover, if $A^2 \subseteq nzd(A)$, or more generally, if (A, T) is a quadratic ring, then R is a quadratic multiring containing the proto special group $G_T(S)$. This is particularly useful given that (A, T, S) is not necessarily faithfully quadratic.

Definition 5.13. Let (X, τ) be a topological space. The topology τ is called perfectly normal if it is normal and every closed set is G_δ-set. The topology τ is called T_6 if it is Hausdorff and perfectly normal.

Example 5.14. • A T_1 topological space X is perfectly normal if, and only if, for every closed set F exists a continuous function $f\colon X \to \mathbb{R}$ such that $F = f^{-1}(0)$ (Theorem 1.5.19 of [7]).
  • Every metric space is T_6 (Corollary 4.1.13 of [7]).

Example 5.15 (The ring of continuous functions)**.** Let X be T_6 topological space and consider $A = C(X, \mathbb{R})$, the ring of continuous functions $f : X \to \mathbb{R}$. Let $T = A^2 \cap nzd(A)$. In the following, is proved that $C(X, \mathbb{R})/_m T$ is a real reduced multiring (in particular, a quadratic multiring). Before that, consider the remarks:

- Since X is perfectly normal, given a open set $U \subseteq X$ there is a continuous function $g\colon X \to \mathbb{R}$ such that $g\big|_U$ is strictly positive and $Z(g) = U^c$.

- $f \in C(X, \mathbb{R})$ is zero divisor if, and only if, $Z(f)$ has non-empty interior. In fact, if $U \subseteq Z(f)$ is non-empty interior, then exists $g \in C(X, \mathbb{R})$ such that $Z(g) = U^c$; thus g is a non-zero function and $fg = 0$. Reciprocally, if $Z(f)$ has empty interior and $g \in C(X, \mathbb{R})$ satisfies $fg = 0$, then $Z(f)^c$ is open and dense while $Z(f)^c \subseteq Z(g)$. Since g is continous, $g = 0$ and so f is non-zero divisor.

- By the preceding item,

$$T = \{f \in C(X, \mathbb{R})\colon f \text{ is non-negative and } Z(f) \text{ has empty interior}\}.$$

Before proceding with the proof, a notation: given $h \in C(X, \mathbb{R})$, denote by $p_h \in C(X, \mathbb{R})$ any function satisfying:

- $Z(p_h)$ has empty interior (that is, p_h is a non-zero divisor).

- p_h is non-negative over $Z(h)$.

- For all $x \notin Z(h)$, $p_h(x) = h(x)$.

A possible construction is to consider a positive function $p \in C(X, \mathbb{R})$ with $Z(p) = (int(Z(h)))^c$ and set $p_h := h + p$.

Claim. *Let $f, g \in C(X, \mathbb{R})$ be two functions and $D \subseteq X$ a dense subset such that for all $x \in D$, $sgn(f(x)) = sgn(g(x))$. Then $\overline{f} = \overline{g}$ in $C(X, \mathbb{R})/_m T$.*

Proof. Assume that for all $x \in D$, $\mathrm{sgn}(f(x)) = \mathrm{sgn}(g(x))$. Then for all $x \in D$ we have $f(x) \cdot p_{|g|}(x) = g(x) \cdot p_{|f|}(x)$ $(*)$ Since D is a dense subset of X, the equality $(*)$ is true for all real number. Thus, since $p_{|f|}, p_{|g|} \in T$, we have $\overline{f} = \overline{g}$ in $A/_m T$. $\qquad\square$

To finalize this example, we have to prove the axioms of real reduced multiring:

- Since $0 \notin T$, we have $\overline{1} \neq \overline{0}$ in $A/_m T$.

- For all $x \in \mathbb{R}$, we have $\mathrm{sgn}(f^3(x)) = \mathrm{sgn}(f(x))$. Thus by the above claim $\overline{f}^3 = \overline{f}$ in $A/_m T$.

- Let $f, g \in A$ and $\overline{h} \in \overline{f} + \overline{f}\overline{g}^2$ in $A/_m T$. Then exists $s_1, s_2, s_3 \in T$ such that $hs_1 = fs_2 + fg^2 s_3$. Thus, for all $x \in Z(s_1)^c \cap Z(s_2)^c \cap Z(s_3)^c$, we have

    . if $f(x) = 0$, then $h(x) = 0$;

    . if $f(x) > 0$, then $h(x) > 0$;

    . if $f(x) < 0$, then $h(x) < 0$.

  Since $Z(s_1)^c \cap Z(s_2)^c \cap Z(s_3)^c$ is a dense subset, by above claim, $\overline{h} = \overline{f}$.

- Let $f, g \in A$ and $\overline{h_1}, \overline{h_2} \in \overline{f} + \overline{g}$ in $A/_m T$. By an argument similiar of the preceding item, the signals of h_1, h_2 are equal in dense subset and thus $\overline{h_1} = \overline{h_2}$.

6 Final remarks and future works

We emphasize that DM-multirings and DP-multirings provide a new way to look at the abstract theories of quadratic forms.

In fact, for special groups (or more generally, theories that generalizes the field case), we obtain an easy way to describe the axiom SG6 in the theory of special groups (3.10). For real semigroups (or theories that generalizes the ring case), we obtain a new example of real semigroup (Example 5.15) in addition with an entire new quadratic structure, the quadratic multirings, that are categorically equivalent to formally real semigroup. We hope that quadratic multirings could give a raise a non reduced theory of quadratic

forms for more general rings (maybe, with some controlling restrictions in the set of nonzero divisors).

After that, we glance these roads to follow:

1. We intend to analyse further the introduced notions of formally real semigroups, formally real multirings and quadratic multirings.

2. With Example 5.15 as a prototype, specialize the study of quadratic multirings where every element is the product of a non-zero divisor and an idempotent. This could give some hint about the structure of invertible elements in real semigroups, which until today is not known to be a reduced special group in general.

3. In [13] is constructed a von Neumann hull functor from multiring category and that, when restricted to in semi-real rings, it commutes with real semigroup functor. This allows us to obtain some quadratic forms properties of a semi-real ring by looking to its von Neumann regular hull. It would be interesting to determine what kind of property in the von Neumann hull of a quadratic multiring return to the original structure.

4. The definition and analysis of the structure of Witt ring of more general quadratic structures (non only obtained from special groups): this subject have already appeared in section 4, in connection with [9].

References

[1] Adámek, J. and Rosicky, J., "Locally Presentable and Accessible Categories", London Math. Soc. Lecture Note Ser. 189, Cambridge University Press, 1994.

[2] Lima, A., *Les Groupes Speciaux. Aspects Algebriques et Combinatoires de la Theorie des Espaces d'Ordres Abstraits*, Ph.D. Thesis, University Paris 7, 1996.

[3] Dickmann, M. and Miraglia, F., "Special Groups: Boolean-theoretic Methods in the Theory of Quadratic Forms", Mem. Amer. Math. Soc. 145(689), American Mathematical Society, 2000.

[4] Dickmann, M. and Miraglia, F., *Quadratic form theory over preordered von neumann-regular rings*, J. Algebra 319(4) (2008), 1696-1732.

[5] Dickmann, M. and Miraglia, F., "Faithfully Quadratic Rings", Mem. Amer. Math. Soc. 238(1128), American Mathematical Society, 2015.

[6] Dickmann, M. and Petrovich, A., *Real semigroups and abstract real spectra*, I., *Algebraic and arithmetic theory of quadratic forms*, Contemp. Math. 344 (2004), 99-120.

[7] Engelking, R., "General Topology", Sigma Series in Pure Mathematics, Heldermann Verlag, 1989.

[8] Gladki, P. and Marshall, M., *Orderings and signatures of higher level on multirings and hyperfields*, J. K-Theory: K-Theory Appl. Algebra Geom. Topol. 10(3) (2012), 489-518.

[9] Gladki, P. and Worytkiewicz, K., *Witt rings of quadratically presentable fields*, Categ. General Alg. Struct. Appl. 12(1) (2020), 1-23.

[10] Jun, J., *Algebraic geometry over hyperrings*, Adv. Math. 323 (2018), 142-192.

[11] Marshall, M., *Real reduced multirings and multifields*, J. Pure Appl. Algebra 205(2) (2006), 452-468.

[12] Marshall, M., "Spaces of Orderings and Abstract Real Spectra", Springer, 1996.

[13] Ribeiro, H.R.d.O. and Mariano, H.L., *von Neumann regular hyperrings and applications to real reduced multirings*, arXiv:2101.06527, 2021.

[14] Ribeiro, H.R.d.O., Roberto, K.M.d.A., and Mariano, H.L., *Functorial relationship between multirings and the various abstract theories of quadratic forms*, São Paulo J. Math. Sci., 2020, `https://doi.org/10.1007/s40863-020-00185-1`.

[15] Viro, O., *Hyperfields for tropical geometry I., hyperfields and dequantization*, arXiv:1006.3034, 2010.

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