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(b, c)-inverse, inverse along an element, and the Schützenberger category of a semigroup

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Abstract. We prove that the (b, c)-inverse and the inverse along an element in a semigroup are actually genuine inverse when considered as morphisms in the Schützenberger category of a semigroup. Applications to the Reverse Order Law are given.

C Green's relations and the Schützenberger category of a semigroup

In this first section, we provide the reader with the necessary definitions and results regarding semigroups and categories. In particular, we recall the definition of the Schützenberger category of a semigroup and the interpretation of Green's relations in this setting. Section 2 then presents the main result of the article (Theorem C.7), that (b, c)-inverses (and inverses along an element) are genuine inverses when considered as morphisms in the corresponding Schützenberger category. Finally, applications to the Reverse Order Law are given in Section 3.

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C.1 Green's preorders and relations In this article, S denotes a semigroup, $E(S)$ its set of idempotents and S^1 the monoid generated by S. We recall below the definitions of *Green's preorders and relations* [5].

For any two elements $a, b \in S$:

$$a \leqslant_{\mathcal{L}} b \Longleftrightarrow S^1 a \subseteq S^1 b \Longleftrightarrow (\exists x \in S^1)\ a = xb;$$
$$a \leqslant_{\mathcal{R}} b \Longleftrightarrow aS^1 \subseteq bS^1 \Longleftrightarrow (\exists x \in S^1)\ a = bx;$$
$$a \leqslant_{\mathcal{H}} b \Longleftrightarrow \{a \leqslant_{\mathcal{L}} b \text{ and } a \leqslant_{\mathcal{R}} b\}.$$

If $\leqslant_{\mathcal{K}}$ is one of these preorders, then $a\mathcal{K}b \Leftrightarrow \{a \leqslant_{\mathcal{K}} b \text{ and } b \leqslant_{\mathcal{K}} a\}$, and $K_a = \{b \in S,\ b\mathcal{K}a\}$ denotes the $\mathcal{K}$-class of a.

In particular, $\mathcal{H} = \mathcal{L} \wedge \mathcal{R}$ is the meet of $\mathcal{L}$ and $\mathcal{R}$, and any non-empty intersection of a $\mathcal{L}$-class and a $\mathcal{R}$-class is a $\mathcal{H}$-class. Finally, $\mathcal{D} = \mathcal{L} \wedge \mathcal{R} = \mathcal{L} \circ \mathcal{R}$ denotes both the join and the relative product of $\mathcal{L}$ and $\mathcal{R}$ since the two relations commute:

$$a\mathcal{D}b \Longleftrightarrow (\exists x \in S)\ a\mathcal{L}x\mathcal{R}b \Longleftrightarrow (\exists y \in S)\ a\mathcal{R}y\mathcal{L}b.$$

The following cancellation property (and its dual) will prove very useful in the sequel. Let $a, b \in S$ be such that $a \leqslant_{\mathcal{L}} b$. Then for any $x, y \in S^1$, if $bx = by$ then $ax = ay$. In particular, if $a \leqslant_{\mathcal{L}} e$ with $e \in E(S)$, then $ae = a$.

C.2 Schützenberger category of a semigroup In order to study the semigroup S, various authors have introduced categories associated to this semigroup in a canonical way, mainly as subcategories of the category of right/left S-acts. This is the case for instance for K.S.S. Nambooripad in the case of regular semigroups [12] (Other kind of categories are also studied, notably in the case of inverse semigroups, see for instance [7] and references therein).

Then, it was remarked by Costa and Steinberg [2, Theorem 3.3] that the subcategory of left S-acts with principal left ideals as objects and inner equivariant maps (inner right translations) as morphisms is equivalent to a category constructed directly from S, which they call the *Schützenberger category* of the semigroup. We recall its definition below but differ from [2] in that we consider the opposite category.

Definition C.1. The *Schützenberger category* $\mathbb{D}(S)$ of a semigroup S has for objects the elements of S, and morphisms are triples $f = (a, x, b)$ with

$x \in aS^1 \cap S^1 b$. The domain of f is a, its codomain is b and we use the notation $f = a \xrightarrow{x} b$. If $x = au = vb$ and $g = (b, y, c) = b \xrightarrow{y} c$ is a morphism with $y = bw = rc$, then the composition is $g \circ f = a \xrightarrow{x} b \xrightarrow{y} c = a \xrightarrow{vy=xw} c$.

The Schützenberger category was named after Marcel-Paul Schützenberger who, in a seminal paper [13], associated to each $\mathcal{H}$-class of a semigroup a group (of inner translations of principal right ideals), a property before only known for $\mathcal{H}$-classes containing an idempotent (which form a maximal subgroup of the semigroup).

As observed in [2], $\mathbb{D}(S^{op}) \simeq \mathbb{D}(S)^{op}$, easing the use of duality (by duality, we always mean working in the opposite semigroup/category).

C.3 Categorical interpretation of Green's relations Let $a, b, x \in S$. By definition, $a \xrightarrow{x} b$ exists if and only if $x \leqslant_{\mathcal{L}} b$ and $x \leqslant_{\mathcal{R}} a$. Invertibility of such morphisms has been considered in [2, Lemma 3.6]. We propose here a slightly different version, with a direct proof.

Lemma C.2. *Let $a, b, x \in S$. Then $a \xrightarrow{x} b$ is invertible if and only if $a\mathcal{R}x\mathcal{L}b$. In this case, its inverse is $b \xrightarrow{b} x \xrightarrow{a} a = b \xrightarrow{y} a$ with $a\mathcal{L}y\mathcal{R}b$.*

Proof. (*If part*) Assume that $a\mathcal{R}x$ and $x\mathcal{L}b$. Then $a \xrightarrow{x} x$ and $x \xrightarrow{a} a$ are well defined, and composition proves that they are inverse of each other. By duality, so are $x \xrightarrow{x} b$ and $b \xrightarrow{b} x$, and finally $a \xrightarrow{x} b = a \xrightarrow{x} x \xrightarrow{x} b$ is invertible (by the 2-out-of-3 property of isomorphisms). Its inverse is given by the composite of the inverses $b \xrightarrow{b} x \xrightarrow{a} a$.
(*Only if part*) Assume that $a \xrightarrow{x} b$ is invertible with inverse $b \xrightarrow{y} a$. By duality we have only to prove that $a\mathcal{R}x$. First, $x \leqslant_{\mathcal{R}} a$ and $y \leqslant_{\mathcal{R}} b$, so that $y = bu$ for some $u \in S^1$. Second, as

$$a \xrightarrow{a} a = a \xrightarrow{x} b \xrightarrow{y} a = a \xrightarrow{xu} a$$

then $a = xu$ and $a \leqslant_{\mathcal{R}} x$, so that finally $a\mathcal{R}x$. □

Lemma C.2 gives a diagrammatic proof that $\mathcal{R}$ and $\mathcal{L}$ commute. Also, we recover directly and in a transparent way the following crucial result about trace products and existence of idempotents, due to Miller and Clifford.

Theorem C.3 ([10, Theorem 3]). *Let $a, b \in S$. Then $ab \in \mathcal{R}_a \cap \mathcal{L}_b$ (ab is a trace product) if and only if $\mathcal{R}_b \cap \mathcal{L}_a$ contains an idempotent.*

By Lemma C.2, the product ab is a trace product if and only if $a \xrightarrow{ab} b$ is invertible, and $\mathcal{R}_b \cap \mathcal{L}_a$ contains an idempotent if and only if there exists $e \in E(S)$ such that $b \xrightarrow{e} a$ is an invertible morphism. Therefore, the previous theorem subsumes to the following result (in the category $\mathbb{D}(S)$), where the relation between a, b and e is now explicit: $b \xrightarrow{e} a$ is nothing but the inverse of $a \xrightarrow{ab} b$ in the Schützenberger category $\mathbb{D}(S)$.

Corollary C.4. *Let $a, b \in S$. Then $a \xrightarrow{ab} b$ is invertible if and only if there exists $e \in E(S)$ such that $b \xrightarrow{e} a$ is an invertible morphism. In this case, the two morphisms are inverse of each other.*

Proof. (*If part*) Assume that $a \xrightarrow{ab} b$ is invertible and let $b \xrightarrow{e} a$ be its inverse. As $a \xrightarrow{ab} b \xrightarrow{e} a = a \xrightarrow{ae} a = a \xrightarrow{a} a$ then $ae = a$. But $e \leqslant_{\mathcal{L}} a$ and by cancellation $ee = e$ (e is idempotent).
(*Only if part*) Conversely, let $e \in E(S)$ be such that $b \xrightarrow{e} a$ is invertible. Then $b\mathcal{R}e\mathcal{L}a$ by Lemma C.2, and $ae = a$, $eb = b$. It follows that $b \xrightarrow{e} a$ is a morphism that satisfies $a \xrightarrow{ab} b \xrightarrow{e} a = a \xrightarrow{ae} a = a \xrightarrow{a} a$ and dually $b \xrightarrow{e} a \xrightarrow{ab} b = b \xrightarrow{b} b$. Finally $a \xrightarrow{ab} b$ is the inverse of $b \xrightarrow{e} a$. $\square$

Let $b \in S$. We finally consider the local monoid $\hom(b, b)$ (named local divisor by Diekert et. al. [3]). To this end, we recall the definition of *Mitsch's partial order* $\leqslant_{\mathcal{M}}$ [11]: for any $a, b \in S$,

$$a \leqslant_{\mathcal{M}} b \iff (\exists x, y \in S^1)\ a = ax = bx = ya = yb.$$

This partial order generalizes the natural partial order on the idempotents of S and the Nambooripad-Hartwig order on regular semigroups.

Proposition C.5. *Let $a, b \in S$. Then*
(i) $\hom(b, b) = \{b \xrightarrow{a} b | a \leqslant_{\mathcal{H}} b\}$;
(ii) $\mathrm{iso}(b, b) = \{b \xrightarrow{a} b | a\mathcal{H}b\}$;
(iii) $E(\hom(b, b)) = \{b \xrightarrow{a} b | a \leqslant_{\mathcal{M}} b\}$.

Proof. The first two statements follow directly from Lemma C.2. For the third one we let $a \in S$, $a \leqslant_{\mathcal{H}} b$. Then $a = bx = yb$ for some $x, y \in S^1$, so that

$$\begin{aligned} b \xrightarrow{a} b \xrightarrow{a} b &= b \xrightarrow{a} b \xrightarrow{bx} b \\ &= b \xrightarrow{ax} b \\ &= b \xrightarrow{yb} b \xrightarrow{a} b \\ &= b \xrightarrow{ya} b. \end{aligned}$$

Finally, under these assumptions, $b \xrightarrow{a} b$ is idempotent if and only if $ax = a = ya$ if and only if $a \leqslant_{\mathcal{M}} b$. □

C Inverse along an element and (b,c)-inverse

Let $a, x \in S$. One says that x is an *inner* (resp. *outer*, resp. *reflexive*) *inverse* of a if the equation $axa = a$ (resp. $xax = x$, resp. both) is satisfied. A reflexive inverse of a that commutes with a is unique if it exists, and called the group inverse of a. It is denoted by $a^{\#}$. In this case, one says that a is *group invertible* or *completely regular*. This happens if and only if $a^2\mathcal{H}a$ if and only if the $\mathcal{H}$-class H_a is a subgroup of S ([5, Theorem 7] or [10, Corollary 4]), and $a^{\#}$ is the inverse of a in this group.

In [8] a special outer inverse, called *inverse along an element*, was introduced, on the basement of Green's relation $\mathcal{H}$.

Definition C.1. Let $a, d \in S$. Then a is *invertible along* d if there exists $x \in S$ such that $xad = d = dax$ and $x \leqslant_{\mathcal{H}} d$.

If such an element exists then it is unique, and we denote it by a^{-d}.

Another characterization is the following ([8, Lemma 3] and [9, Theorem 2.2]).

Lemma C.2. *Let $a, d \in S$. Then a is invertible along d if and only if there exists $x \in S$ such that $xax = x$ and $x\mathcal{H}d$, and in this case $a^{-d} = x$. This happens if and only if $dad\mathcal{H}d$.*

That is, x is the only outer inverse of a in the $\mathcal{H}$-class of d (hence depends only on the $\mathcal{H}$-class).

In the same time, M.P. Drazin defined [4] the (b,c)-*inverse* (extending notably the *Bott-Duffin* (e,f)-*inverse*, which is recovered by letting $b = e$ and $c = f$ be idempotents).

Definition C.3. Let $a, b, c, x \in S$. Then x is a (b, c)-*inverse* of a if

(i) $x \in (bSx) \cap (xSc)$,

(ii) $xab = b$, $cax = c$.

A (b, c)-inverse, if it exists, is also unique and satisfies $xax = x$ ([4, Theorem 2.1]). We will denote it by $a^{-(b,c)}$ in the sequel.

It is proved in [4] and [8] that these two new notions generalize the classical generalized inverses (group inverse, Moore-Penrose inverse, Drazin inverse). It happens that they are actually equivalent notions.

C.1 Equivalence of the definitions It is commonly known that inverses along an element are a special case of (b, c)-inverses, since $a^{-d} = a^{-(d,d)}$. Surprisingly, that the converse is also true ((b, c)-inverses are inverses along an element) seems to have remained unnoticed by many scholars. The next result presents the exact relation between the two notions (and also their connexion with the Bott-Duffin inverse).

Theorem C.4. (i) *Let $a, b, c, x \in S$. If a is (b, c)-invertible with inverse x, then $b\mathcal{D}c$ and for all $d \in R_b \cap L_c$, a is invertible along d with inverse x.*

(ii) *Let $a, d \in S$. If a is invertible along d, then for all $b \in R_d$ and $c \in L_d$, a is (b, c)-invertible and $a^{-(b,c)} = a^{-d}$.*

(iii) *In particular, if $a, d \in S$ are such that a is invertible along d, then $e = a^{-d}a$ and $f = aa^{-d}$ are idempotents such that $e \leqslant_{\mathcal{R}} d$ and $f \leqslant_{\mathcal{L}} d$. But also $ed = d$ and $df = f$ by definition of the inverse along d, and $e \in R_d$, $f \in L_d$. Finally a is Bott-Duffin (e, f)-invertible and $a^{-(e,f)} = a^{-d}$ by (2).*

Proof. (i) Assume that a is (b, c)-invertible with (b, c)-inverse $x = a^{-(b,c)}$. Then $xab = b$, $cax = x$ and $b \leqslant_{\mathcal{R}} x$, $c \leqslant_{\mathcal{L}} x$. As also $x \in (bSx) \cap (xSc)$, then $x \leqslant_{\mathcal{R}} b$ and $x \leqslant_{\mathcal{L}} c$. Finally $b\mathcal{R}x\mathcal{L}c$ and $b\mathcal{D}c$. Also, $R_b \cap L_c = H_x$. Let $d \in R_b \cap L_c$. Then x satisfies $xax = x$ and $x\mathcal{H}d$, and $x = a^{-d}$ by Lemma C.2.

(ii) Let $a, d \in S$ such that a^{-d} exists, and let $b \in R_d$, $c \in L_d$. By cancellation properties, as $a^{-d}ad = d$ then $a^{-d}ab = b$ and as $daa^{-d} = d$ then $caa^{-d} = c$. Also, as $a^{-d}\mathcal{R}d\mathcal{R}b$ then $a^{-d} = bx$ for some $x \in S^1$. Then $a^{-d} = a^{-d}aa^{-d} = b(xa)a^{-d}$ and $a^{-d} \in bSa^{-d}$. By symmetry, $a^{-d} \in a^{-d}Sc$ and finally, a^{-d} is the (b, c)-inverse of a.

(iii) Let $a, d \in S$ such that a^{-d} exists. Then $e = a^{-d}a$ and $f = aa^{-d}$ are idempotents with $e \in R_d$, $f \in L_d$, so that a is (e,f)-invertible and $a^{-d} = a^{-(e,f)}$ by (2). □

This theorem shows that the three notions are essentially the same, and that $a^{-(d,d)} = a^{-d}$. However, while the using of two elements instead of one offers some flexibility (for instance one can choose b, c idempotents), other properties require $b = c$ (for instance the property $a^{-d} = d(ad)^{\#} = (da)^{\#}d$ [8, Theorem 7]).

As a consequence of Theorem C.4 and Lemma C.2, we see that the requirements in the definition of the (b,c)-inverse can be relaxed.

Corollary C.5. *Let $a, b, c, x \in S$. The following statements are equivalent:*

(i) *x is the (b,c)-inverse of a;*

(ii) *$xab = b, cax = c$, $x \leqslant_{\mathcal{R}} b$ and $x \leqslant_{\mathcal{L}} c$;*

(iii) *$xax = x$ and $x \in R_b \cap L_c$.*

Thus $x = a^{-(b,c)}$ is the only outer inverse of a in the $\mathcal{H}$-class $H = R_b \cap L_c$.

C.2 Interpretation through the variant semigroup For any $a \in S$ we let $S_a = (S, \cdot_a)$ be the variant semigroup at a, with multiplication $\cdot_a : (s,t) \mapsto sat$.

Theorem C.6. *Let $a, b, c, d \in S$. Then:*

(i) *a is (b,c)-invertible if and only if $c \cdot_a b$ is a trace product in S_a, in which case $a^{(b,c)}$ is the unique idempotent in $R^a_b \cap L^a_c$ (where $\mathcal{K}^a$ denotes Green's relation $\mathcal{K}$ in S_a).*

(ii) *a is invertible along d if and only if d is completely regular in S_a (H^a_d is a subgroup of S_a), in which case a^{-d} is the identity of the group H^a_d.*

C.3 Categorical interpretation of the (b,c)-inverse In this section, we prove that the (b,c)-inverse (hence also the Bott-Duffin (e,f)-inverse and the inverse along an element) actually corresponds to a genuine inverse, but in the category $\mathbb{D}(S)$.

Theorem C.7. *Let $a, b, c \in S$. Then a is (b,c)-invertible if and only if $c \xrightarrow{cab} b$ is an isomorphism of $\mathbb{D}(S)$ ($cab \in R_c \cap L_b$), in which case its inverse morphism is $b \xrightarrow{a^{-(b,c)}} c$.*

Proof. (*If part*) Assume that $c \xrightarrow{cab} b$ is invertible and let $b \xrightarrow{x} c$ be its inverse. Then $b\mathcal{R}x\mathcal{L}c$ by Lemma C.2. Also $b \xrightarrow{b} b = b \xrightarrow{x} c \xrightarrow{cab} b = b \xrightarrow{xab} b$ so that $xab = b$, and dually $cax = c$. It follows that x is the (b, c)-inverse of a by Corollary C.5.
(*Only if part*) Conversely, let $x = a^{-(b,c)}$ be the (b, c)-inverse of a. Then $b \xrightarrow{x} c$ is well-defined and satisfies $b \xrightarrow{x} c \xrightarrow{cab} b = b \xrightarrow{xab} b = b \xrightarrow{b} b$. Dually $c \xrightarrow{cab} b \xrightarrow{x} c = c \xrightarrow{cax} c = c \xrightarrow{c} c$, so that $c \xrightarrow{cab} b$ is an invertible morphism with inverse $b \xrightarrow{x} c$. □

Decompose $c \xrightarrow{cab} b$ as $c \xrightarrow{cab} cab \xrightarrow{cab} b$. We deduce from Lemma C.2 and Theorem C.7 the following corollary.

Corollary C.8. *Let $a, b, c \in S$. Then the following statements are equivalent:*

(i) *a is (b, c)-invertible;*

(ii) *$c \xrightarrow{cab} b$ is invertible;*

(iii) *$c \xrightarrow{cab} cab$ and $cab \xrightarrow{cab} b$ are invertible; in which case*

$$b \xrightarrow{a^{-(b,c)}} c = b \xrightarrow{b} cab \xrightarrow{c} c.$$

Proof. By Theorem C.7, (i) ⇔ (ii), and (iii) ⇒ (ii) by composition. Finally (ii) ⇒ $c\mathcal{R}cab\mathcal{L}b$ ⇒ (iii) by Lemma C.2. Under these conditions, the equality follows from $b \xrightarrow{b} cab$ being the inverse of $cab \xrightarrow{cab} b$ and $cab \xrightarrow{c} c$ being the inverse of $c \xrightarrow{cab} cab$. □

To close this section, we give a diagrammatic proof of [8, Theorem 7].

Theorem C.9 ([8, Theorem 7])**.** *Let $a, d \in S$. Then a is invertible along d if and only if $ad\mathcal{L}d$ and H_{ad} is a group. In this case $a^{-d} = d(ad)^{\#}$.*

Proof. Assume that a is invertible along d. Then $dad\mathcal{H}d$ and, in particular, $ad\mathcal{L}d$. It follows that $d \xrightarrow{d} ad$ is well-defined and invertible, with inverse $ad \xrightarrow{ad} d$. But then

$$d \xrightarrow{dad} d = d \xrightarrow{d} ad \xrightarrow{(ad)^2} ad \xrightarrow{ad} d$$

is invertible if and only if $ad \xrightarrow{(ad)^2} ad$ is invertible, that is, $(ad)^2\mathcal{H}ad$ by Proposition C.5, or equivalently if and only if H_{ad} is a group. To obtain the formula, just invert the previous morphisms and compose, with the inverse of $ad \xrightarrow{(ad)^2} ad$ being $ad \xrightarrow{(ad)^\# ad} ad$, as seen next:

$$\begin{aligned} d \xrightarrow{a^{-d}} d &= (d \xrightarrow{dad} d)^{-1} \\ &= \left(d \xrightarrow{d} ad \xrightarrow{(ad)^2} ad \xrightarrow{ad} d\right)^{-1} \\ &= d \xrightarrow{d} ad \xrightarrow{(ad)^\# ad} ad \xrightarrow{ad} d \\ &= d \xrightarrow{d} ad \xrightarrow{ad(ad)^\#} d \\ &= d \xrightarrow{d(ad)^\#} d. \end{aligned}$$

□

We will hereafter also use the dual statement: *a is invertible along d if and only if $da\mathcal{R}d$ and H_{da} is a group, in which case $a^{-d} = (da)^\# d$.*

C Application to the Reverse Order Law

As an application, we consider a common problem regarding generalized inverses, the so-called Reverse Order Law (ROL), that aims to generalize the well-known property of (genuine) inverses: *if a, b are invertible, then so is ab and $(ab)^{-1} = b^{-1}a^{-1}$.*

The main results about ROLs for (b,c)-inverses can be found in [1].

Theorem C.1 ([1, Theorems 2.3 and 2.7, Corollary 2.4])**.**
Let $a, w, b, s, t, c \in S$.

(i) *Assume that $a^{-(t,c)}$ and $w^{-(b,s)}$ exist. Then $(aw)^{-(b,c)}$ exists and equals $w^{-(b,s)}a^{-(t,c)}$ if and only if $b = w^{-(b,s)}a^{-(t,c)}awb$ and $c = caww^{-(b,s)}a^{-(t,c)}$.*

(ii) *In particular if $a^{-(t,c)}$ and $w^{-(b,s)}$ exist and $a \leqslant_{\mathcal{L}} s$, $w \leqslant_{\mathcal{R}} t$ then $(aw)^{-(b,c)}$ exists and equals $w^{-(b,s)}a^{-(t,c)}$.*

(iii) *Assume that $a^{-(t,c)}$ and $\left(a^{-(t,c)}aw\right)^{-(b,s)}$ exist. Then $(aw)^{-(b,c)}$ exists and equals $\left(a^{-(t,c)}aw\right)^{-(b,s)} a^{-(t,c)}$ if and only if $c = caw\left(a^{-(t,c)}aw\right)^{-(b,s)} a^{-(t,c)}$.*

We slightly improve these results by using the 2-out-of-3 and 2-out-of-6 properties of isomorphisms, and show that they rely on st being a trace product ($st \in R_s \cap L_t$).

Theorem C.2. *Let $a, w, b, s, t, c \in S$ be such that $a^{-(t,c)}$ and $w^{-(b,s)}$ exist. Then $(aw)^{-(b,c)}$ exists and equals $w^{-(b,s)}a^{-(t,c)}$ if and only if there exists $e \in E(S)$ such that*

(i) $t \xrightarrow{e} s$ *is an invertible morphism;*

(ii) $caewb = cawb$.

In this case, st is a trace product.

Proof. Assume that $a^{-(t,c)}$ and $w^{-(b,s)}$ exist. Then

$$b \xrightarrow{w^{-(b,s)}} s \xrightarrow{st} t \xrightarrow{a^{-(t,c)}} c = b \xrightarrow{w^{-(b,s)}a^{-(t,c)}} c$$

is invertible if and only if $s \xrightarrow{st} t$ is invertible, if and only if st is a trace product by Lemma C.2. By Corollary C.4 this occurs if and only if $t \xrightarrow{e} s$ is an invertible morphism for some $e \in E(S)$ (which is then precisely the inverse of $s \xrightarrow{st} t$). In this case, it holds that

$$(b \xrightarrow{w^{-(b,s)}a^{-(t,c)}} c)^{-1} = c \xrightarrow{cat} t \xrightarrow{e} s \xrightarrow{swb} b = c \xrightarrow{caewb} b$$

and we conclude by Theorem C.7 and unicity of the inverse. □

The following corollaries are then straightforward (in the second one, we just let $s = b$ and $c = t$).

Corollary C.3. *Let $a, w, b, s, t, c \in S$ be such that $a^{-(t,c)}$ and $w^{-(b,s)}$ exist, st is a trace product and either $a \leqslant_{\mathcal{L}} s$ or $w \leqslant_{\mathcal{R}} t$. Then aw has a (b, c)-inverse and $(aw)^{-(b,c)} = w^{-(b,s)}a^{-(t,c)}$.*

Proof. As st is a trace product, then $t \xrightarrow{e} s$ is an invertible morphism for some $e \in E(S)$ by Corollary C.4, and $t\mathcal{R}e\mathcal{L}s$ by Lemma C.2. Assume that $a \leqslant_{\mathcal{L}} s$ (the other case is dual). Then $a \leqslant_{\mathcal{L}} s\mathcal{L}e$ hence $a \leqslant_{\mathcal{L}} e$ by transitivity. Thus $ae = a$ by cancellation and we conclude by Theorem C.2. □

Corollary C.4. *Let $a, w, b, c \in S$ be such that a^{-c} and w^{-b} exist. Then $(aw)^{-(b,c)}$ exists and equals $w^{-b}a^{-c}$ if and only if $c \xrightarrow{e} b$ is an invertible morphism for some $e \in E(S)$ (bc is a trace product) such that $caewb = cawb$. Moreover, in this case*

$$(aw)^{-(b,c)} = (aw)^{-bc} = w^{-b}a^{-c}.$$

Proof. The nontrivial part of the corollary is the equality $(aw)^{-(b,c)} = (aw)^{-(bc)}$. Assume that bc is a trace product. Then by definition $b\mathcal{R}bc\mathcal{L}c$ and by Theorem C.4.(i), $(aw)^{-(b,c)} = (aw)^{-bc}$. $\square$

Theorem C.5. *Let $a, w, b, s, t, c \in S$ be such that $a^{-(t,c)}$ and $\left(a^{-(t,c)}aw\right)^{-(b,s)}$ exist. Then $(aw)^{-(b,c)}$ exists and equals $\left(a^{-(t,c)}aw\right)^{-(b,s)} a^{-(t,c)}$ if and only if $s \xrightarrow{st} t$ is an invertible morphism (st is a trace product).*

Proof. Assume that $a^{-(t,c)}$ and $\left(a^{-(t,c)}aw\right)^{-(b,s)}$ exist. Then

$$b \xrightarrow{\left(a^{-(t,c)}aw\right)^{-(b,s)}} s \xrightarrow{st} t \xrightarrow{a^{-(t,c)}} c = b \xrightarrow{\left(a^{-(t,c)}aw\right)^{-(b,s)}a^{-(t,c)}} c$$

is invertible if and only if $s \xrightarrow{st} t$ is invertible. In this case its inverse is

$$c \xrightarrow{cat} t \xrightarrow{e} s \xrightarrow{sa^{-(t,c)}awb} b = c \xrightarrow{caea^{-(t,c)}awb} b$$

(where $t \xrightarrow{e} s$, $e \in E(S)$ is the inverse $s \xrightarrow{st} t$). But then $a^{-(t,c)}\mathcal{R}t\mathcal{R}e$ and $ea^{-(t,c)} = a^{-(t,c)}$, so that $caea^{-(t,c)} = caa^{-(t,c)} = c$. We deduce that

$$c \xrightarrow{cat} t \xrightarrow{e} s \xrightarrow{sa^{-(t,c)}awb} b = c \xrightarrow{cawb} b$$

and by Theorem C.7 and unicity of the inverse,

$$(aw)^{-(b,c)} = \left(a^{-(t,c)}aw\right)^{-(b,s)} a^{-(t,c)}.$$

$\square$

Corollary C.6. *Let $a, w, b, c \in S$ be such that a^{-c} and $(a^{-c}aw)^{-b}$ exist. Then the following statements are equivalent:*

(i) *$b \xrightarrow{bc} c$ is an invertible morphism;*

(ii) *$(aw)^{-(b,c)}$ exists and $(aw)^{-(b,c)} = (a^{-c}aw)^{-b} a^{-c}$;*

(iii) *$(aw)^{-(b,c)}$ and $(aw)^{-bc}$ exist, and $(aw)^{-(b,c)} = (aw)^{-bc}$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is Theorem C.5 with $s = b$ and $c = t$.

(ii) $\Rightarrow$ (iii) Assume (ii). As (ii) $\Leftrightarrow$ (i) then $b \xrightarrow{bc} c$ is an invertible morphism. By Lemma C.2, $bc \in R_b \cap L_c$ and by Theorem C.4.(i), $(aw)^{-(bc)}$ exists and equals $(aw)^{-(b,c)}$.

(iii) $\Rightarrow$ (i) Assume (iii). As by definition $(aw)^{-(b,c)} \in R_b \cap L_c$, $(aw)^{-bc} \in H_{bc}$ and by assumption $(aw)^{-(b,c)} = (aw)^{-bc}$ then $bc \in R_b \cap L_c$, and by Lemma C.2 $b \xrightarrow{bc} c$ is an invertible morphism. $\square$

We deduce the following equivalences involving the Bott-Duffin inverse and the inverse along a trace product.

Corollary C.7. *Let $w \in S$ and $e, f \in E(S)$. Consider the following statements:*
(i) $(fw)^{-e}$ *exists;*
(i') $(we)^{-f}$ *exists;*
(ii) $(fw)^{-e}$ *exists,* $w^{-(e,f)}$ *exists and* $w^{-(e,f)} = (fw)^{-e} f$*;*
(ii') $(we)^{-f}$ *exists,* $w^{-(e,f)}$ *exists and* $w^{-(e,f)} = e\,(we)^{-f}$*;*
(iii) $w^{-(e,f)}$ *exists;*
(iv) $w^{-(e,f)}$ *and* w^{-ef} *exist and are equal;*
(v) ef *is a trace product.*
Then

$$\text{(i)} \wedge \text{(i}'\text{)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(ii}'\text{)} \Leftrightarrow \text{(iii)} \wedge \text{(v)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(i)} \wedge \text{(v)} \Leftrightarrow \text{(i}'\text{)} \wedge \text{(v)}.$$

Under these assumptions:

$$w^{-(e,f)} = w^{-ef} = (fw)^{-e} f = e\,(we)^{-f} = e(fwe)^{\#} f.$$

Proof. First, by passing to the opposite semigroup and considering the symmetry that exchanges e and f, we observe that the symmetric dual of (i) (resp. (ii)) is (i') (resp. (ii')) whereas the symmetric duals of (iii), (iv) and (v) are themselves (for instance, $w^{-(e,f)}$ exists in S if and only if $f\mathcal{R}fwe\mathcal{L}e$ if and only if $f\mathcal{L}^{op}(e \times w \times f)\mathcal{R}^{op}e$ in the opposite semigroup $S^{op} = (S, \times = .^{op})$, that is $w^{-(f,e)}$ exists in the opposite semigroup, which is the symmetric of (iii)). Second, as in Corollary C.6, by definition of the (b, c)-inverse and the inverse along an element, and by Lemma C.2, (iv) $\Rightarrow$ (v). And conversely, by Theorem C.4.(i), (v) $\wedge$ (iii) $\Rightarrow$ (iv). Finally, (ii) $\Rightarrow$ (i) $\wedge$ (iii) and (iv) $\Rightarrow$ (iii) are tautological. Thus, it is sufficient to prove that (i)$\wedge$(i$'$) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (v), (iii) $\wedge$ (v) $\Rightarrow$ (i) and (i) $\wedge$ (v) $\Rightarrow$ (ii).
To prove that (ii) $\Rightarrow$ (v) and (i) $\wedge$ (v) $\Rightarrow$ (ii), we work in S^1 and observe that, as f is idempotent, then 1 is invertible along f with $1^{-f} = f$. We then

use Corollary C.6 with $a = 1, b = e$ and $c = f$.
We finally prove the last two implications.

(i) $\wedge$ (i$'$) $\Rightarrow$ (ii) Assume (i) and (i'). Then by Theorem C.9 and its dual $e\mathcal{L}fwe\mathcal{R}f$, so that w is (e,f)-invertible by Theorem C.7. Let $x = (fw)^{-e} f$. Then $x = e(fwe)^{\#} f = e(we)^{-f}$ by Theorem C.9. Still by Theorem C.9, as $(we)^{-f}$ exists then $wef\mathcal{L}f$, and $f = ywef$ for some $y \in S^1$. And as $(fw)^{-e}$ exists, then $e\mathcal{L}fwe$ and fwe is group invertible, so that $e = zfwe = z(fwe)^2(fwe)^{\#}$ for some $z \in S^1$. It follows that $f = ywef = ywz(fwe)^2(fwe)^{\#} f = (ywzfwefw)x$, so that $f\mathcal{L}x$. By dual arguments $e\mathcal{R}x$, and $x \in R_e \cap L_f$. Also $xwx = (fw)^{-e} fw\,(fw)^{-e} f = (fw)^{-e} f = x$ and by Corollary C.5, $x = w^{-(e,f)}$.

(iii) $\wedge$ (v) $\Rightarrow$ (i) Assume (ii) and (v). Then $R_f \cap L_e$ is a group by (v) and Theorem C.3 that contains fwe by (iii) and Theorem C.7. Thus $(fw)e\mathcal{L}e$ and $H_{(fw)e}$ is a group, which is equivalent to (i) by Theorem C.9.

Finally, under these assumptions, $w^{-(e,f)} = w^{-ef} = (fw)^{-e} f = e\,(we)^{-f}$, and the last equality follows from Theorem C.9: $(fw)^{-e} = e(fwe)^{\#}$. $\square$

Example C.8. Let $S = T_3$ be the full transformation semigroup, which consists of all functions from the set $\{1,2,3\}$ to itself with multiplication $(f,g) \mapsto g \circ f$. We write (abc) for the function which sends 1 to a, 2 to b, and 3 to c.
The egg-box diagram form T_3 is as follows ($\mathcal{R}$-classes are rows, $\mathcal{L}$-classes columns and $\mathcal{H}$-classes are squares; bold elements are idempotents).

(1 1 1)	**(2 2 2)**	**(3 3 3)**				
			(1 2 2), (2 1 1)	**(1 3 3)**, (3 1 1)	(2 3 3), (3 2 2)	
			(2 1 2), **(1 2 1)**	(3 1 3), (1 3 1)	**(3 2 3)**, (2 3 2)	
			(2 2 1), (1 1 2)	(3 3 1), **(1 1 3)**	(3 3 2), **(2 2 3)**	
						(1 2 3), (2 3 1), (3 1 2), (1 3 2), (3 2 1), (2 1 3)

Let $w = (213)$, $e = \mathbf{(122)}$ and $f = \mathbf{(323)}$. Then $ef = (322) \in R_e \cap L_f$ and ef is a trace product. Also $fw = (313)$ is invertible along e since $efwe = (211)\mathcal{H}e$ with inverse $(fw)^{-e} = (211)$. It then follows from Corollary C.7 that

$$w^{-(e,f)} = w^{-(ef)} = (213)^{-(322)} = (fw)^{-e} f = (211)\mathbf{(323)} = \mathbf{(233)}.$$

We also verify that $fwe = (212)\mathcal{H}\mathbf{(121)}$ is group invertible with group inverse (212), that $(we)^{-f}$ exists and is equal to (232) and that

$$w^{-(e,f)} = \mathbf{(233)} = e\,(we)^{-f} = \mathbf{(122)}(232) = e(fwe)^{\#} f = \mathbf{(122)}(212)\mathbf{(323)}.$$

Also, we observe these simple ROLs that involve triple products.

Theorem C.9. *Let $a, w, b, r, c \in S$. Consider the following statements:*

(i) *a has a (r, c)-inverse;*

(ii) *w has a (b, r)-inverse;*

(iii) *arw has a (b, c)-inverse.*

Then any two of the three properties imply the third. Moreover, in case the inverses exist then:

(i) *if $b = r$, then $(arw)^{-(r,c)} = (rw)^{\#} a^{-(r,c)}$;*

(ii) *if $c = r$, then $(arw)^{-(b,r)} = w^{-(b,r)}(ar)^{\#}$;*

(iii) *if $b = c = r$, then $(arw)^{-r} = (rw)^{\#} r(ar)^{\#} = r(wr)^{\#}(ar)^{\#} = (rw)^{\#}(ra)^{\#} r$;*

(iv) *if r is idempotent, then $(arw)^{-(b,c)} = w^{-(b,r)} a^{-(r,c)}$;*

(v) *if bc is a trace product and $r = e$ is the idempotent in $R_c \cap L_b$ then $(aew)^{-bc} = w^{-b} a^{-c}$.*

Proof. We just use the 2-out-of-3 property of isomorphisms on the composition

$$c \xrightarrow{car} r \xrightarrow{rwb} b = c \xrightarrow{carwb} b\cdot$$

For the other statements, we pass to the inverse. In the first case, $w^{-(r,r)} = w^{-r} = (rw)^{\#} r$ by Theorem C.9 so that

$$r \xrightarrow{w^{-(r,r)}} r \xrightarrow{a^{-(r,c)}} c = r \xrightarrow{(rw)^{\#} r} r \xrightarrow{a^{-(r,c)}} c = r \xrightarrow{(rw)^{\#} a^{-(r,c)}} c.$$

The second case is dual and the third combines the previous two. For the fourth one, as r is idempotent then

$$b \xrightarrow{w^{-(b,r)}} r \xrightarrow{a^{-(r,c)}} c = b \xrightarrow{w^{-(b,r)} a^{-(r,c)}} c\cdot$$

The fifth statement is then straightforward (see Theorem C.4.(ii)). $\square$

Example C.10. We consider the setting of Example C.8. Let $a = (231)$, $w = (221)$, $b = r = (232)$ and $c = (211)$. Then $w = (221)$ is invertible along $r = (232)$ with inverse $w^{-r} = (323)$, and a is (r,c)-invertible (since $car = (233) \in R_c \cap L_r$) with inverse $a^{-(r,c)} = (212)$. It holds that $rw = (212) = (rw)^{\#}$ and $arw = \mathbf{(122)}$. We deduce from Theorem C.9 that arw is (r,c)-invertible and

$$(arw)^{-(r,c)} = (rw)^{\#}a^{-(r,c)} = (212)(212) = \mathbf{(121)}.$$

We verify this equality: $x(arw)r = \mathbf{(121)(122)}(232) = (232) = r$, $c(arw)x = (211)\mathbf{(122)(121)} = (211) = c$ and $x = \mathbf{(121)} \in R_r \cap L_c$.

Finally, by letting $s = a$ and $t = w$ in Theorems C.2 and C.5, or $r = e$ be the idempotent in $R_w \cap L_a$ in Theorem C.9, we obtain the following result.

Theorem C.11. *Let $a, w, b, c \in S$. Then the following statements are equivalent:*

(i) $a^{-(w,c)}$, $w^{-(b,a)}$ *and* $(aw)^{-(b,c)}$ *exist, and* $(aw)^{-(b,c)} = w^{-(b,a)}a^{-(w,c)}$;

(ii) $a^{-(w,c)}$, $w^{-(b,a)}$ *exist;*

(iii) ca, aw *and* wb *are trace products.*

Proof. (i) $\Rightarrow$ (ii) Straightforward.

(ii) $\Rightarrow$ (iii) Assume (ii). Then $caw \in R_c \cap L_w$ and $awb \in R_a \cap L_b$ by Corollary C.8 so that $aw \in R_a \cap L_w$ is a trace product, and $a \xrightarrow{aw} w$ is invertible. Let $w \xrightarrow{e} a$, $e \in E(S)$ be it inverse. Then $e \in R_w \cap L_a$ hence $ae = a$ and dually $ew = w$. By Theorem C.7, $c \xrightarrow{caw} w$ is invertible hence so is $c \xrightarrow{caw} w \xrightarrow{e} a = c \xrightarrow{cae} a = c \xrightarrow{ca} a$ and ca is a trace product. We conclude by duality that wb is also a trace product.

(iii) $\Rightarrow$ (i) Assume (iii) and consider the following composition:

$$\begin{aligned} c \xrightarrow{cawb} b &= c \xrightarrow{caw} w \xrightarrow{e} a \xrightarrow{awb} b \\ &= c \xrightarrow{ca} a \xrightarrow{awb} b \\ &= c \xrightarrow{caw} w \xrightarrow{wb} b. \end{aligned}$$

By the 2-out-of-6 property, all morphisms are isomorphisms and $a^{-(w,c)}$, $w^{-(b,a)}$ $(aw)^{-(b,c)}$ exist. Moreover, by inverting the previous morphisms the following equality is satisfied.

$$\begin{aligned} b \xrightarrow{(aw)^{-(b,c)}} c &= b \xrightarrow{w^{-(b,a)}} a \xrightarrow{aw} w \xrightarrow{a^{-(w,c)}} c \\ &= b \xrightarrow{w^{-(b,a)}a^{-(w,c)}} c. \end{aligned}$$

$\square$

(Indubitably, a direct proof based on Theorem C.3 is also possible).

Example C.12. Let S be a semigroup with involution. It is known that $a \in S$ is Moore-Penrose invertible (MP-invertible) if and only if aa^*, a^*a are trace products [8, Corollary 12], in which case the MP-inverse a^+ satisfies $a^+ = a^{-a^*}$ [8, Theorem 11]. It is also known [6] that $(aw)^+ = w^+a^+$ if and only if a, w are MP-invertible and $aww^* \leqslant_{\mathcal{L}} a$ and $a^*aw \leqslant_{\mathcal{R}} w$. Let $a, w \in S$ be MP-invertible elements. Then a^*a and ww^* are trace products. By Theorem C.11 with $c = a^*, b = w^*$, aw is a trace product if and only if $a^{-(w,a^*)}$ and $w^{-(w^*,a)}$ exist (if and only if $aww^* \in R_a \cap L_{w^*}$ and $a^*aw \in R_{a^*} \cap L_w$), and in this case

$$(aw)^+ = (aw)^{-(w^*,a^*)} = w^{-(w^*,a)}a^{-(w,a^*)}$$

since $(aw)^* \in R_{w^*} \cap L_{a^*}$. Observe that this formula is very different from Greville's in general.

Let $S = \mathcal{M}_2(\mathbb{Q})$, $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $W = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Then $A^+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $W^+ = \frac{1}{10}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. Also $AW = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in R_A \cap L_W$ ($\frac{1}{2}AWA = A$ and $\frac{1}{2}WAW = W$). Thus AW is a trace product and

$$(AW)^+ = (AW)^{-(W^*,A^*)} = W^{-(W^*,A)}A^{-(W,A^*)}$$

where all terms exist. Pose $X = \frac{1}{2}A$. Then $AWX = A$ and $XWW^* = W^*$. As also $\frac{1}{6}W^*A = X$ then $X \in R_{W^*} \cap L_A$ and $X = W^{-(W^*,A)}$. Pose $Y = \frac{1}{2}W$. Then $YAW = W$ and $A^*AY = A^*$. As also $YA^* = Y$ then $Y \in R_W \cap L_{A^*}$ and $Y = A^{-(W,A^*)}$. Finally, the ROL of Theorem C.11 gives

$$(AW)^+ = XY = \frac{1}{4}AW = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

One can then check directly that the Moore-Penrose equations are satisfied. On the other hand, Greville's formula does not hold since $A^*AW = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \nleqslant_{\mathcal{R}} W$, and indeed $(AW)^+ \neq W^+A^+$.

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