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Flatness properties of acts over semigroups

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Abstract. In this paper we study flatness properties (pullback flatness, limit flatness, finite limit flatness) of acts over semigroups. These are defined by requiring preservation of certain limits from the functor of tensor multiplication by a given act. We give a description of firm pullback flat acts using Conditions (P) and (E). We also study pure epimorphisms and their connections to finitely presented acts and pullback flat acts. We study these flatness properties in the category of all acts, as well as in the category of unitary acts and in the category of firm acts, which arise naturally in the Morita theory of semigroups.

Introduction

Acts over monoids have been studied actively since the beginning of 1970s. The monograph [16] contains a detailed overview of main properties that have been studied, and also a list of publications in this area.

From the very beginning, the so-called flatness properties have played an

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important role. These are defined by requiring the preservation of diagrams of certain types from the functor of tensor multiplication. For example, an act is called pullback flat if the functor of tensoring by it preserves pullbacks. Bo Stenström in his article [24] introduced a property which was later called strong flatness and which means preservation of both pullbacks and equalizers. He proved that strong flatness is equivalent to certain easy-to-check Conditions (P) and (E). Later, Sydney Bulman-Fleming [5] showed that strong flatness and pullback flatness coincide.

Much less has been said about flatness properties of acts over semigroups, partially because the study of these properties can be reduced to studying the monoid case. There are some papers about absolutely flat semigroups (e.g. [8]) where preservation of monomorphisms by tensor functor (usually called just flatness) has been considered. Also, unitary projective acts over semigroups have been studied in [10] and [9].

It appears that for an act the condition of being flat and unitary is notably different from merely being flat. Additionally, when considering flatness in terms of categories of unitary acts or firm acts — notions that naturally arise when studying the Morita theory of semigroups (for example, [21], where the term 'closed' was used instead of 'firm') — a unitary or firmness assumption on acts is natural. In the case of non-unital rings, firm flat modules have been studied for example in [23].

We show how to adapt the flatness results of [24] for the semigroup case. Following that we show that under some natural assumptions, these flatness conditions are equivalent to ones formulated in terms of unitary acts and in terms of firm acts. Pure epimorphisms of acts are used to give a characterization of pullback flatness independent of tensor products. Along the way, we show that under certain assumptions on a semigroup, the category of firm acts is locally finitely presentable.

1 Preliminaries

Throughout this paper S will stand for a (possibly empty) semigroup and S^1 will denote the monoid obtained from S by adjoining an external identity 1 (even if S has an identity element). A *right S-act* is a set A together with a mapping $A \times S \to A, (a, s) \mapsto as$ satisfying $(as)s' = a(ss')$ for all $a \in A$, $s, s' \in S$. Left S-acts are defined dually. We also allow acts to be empty, since we want the category Act_S of all right S-acts to have all limits. Clearly every S-act A_S can be considered as an S^1-act A_{S^1} if we define $a1 = a$ for each $a \in A$.

The tensor product $A \otimes_S M$ of a right S-act A_S and a left S-act ${}_SM$ is defined as the quotient set of $A \times M$ by the smallest equivalence relation identifying all pairs (as, m) and (a, sm), where $a \in A$, $s \in S$, $m \in M$. The equivalence class of (a, m) is denoted by $a \otimes m$. We will often write just $A \otimes M$ instead of $A \otimes_S M$. Note that $A \otimes_S M = A \otimes_{S^1} M$ for all S-acts A_S and ${}_SM$.

Definition 1.1. An act A_S is called *firm* if the mapping

$$\mu_A : A \otimes_S S \to A, \ \ a \otimes s \mapsto as,$$

is bijective.

Definition 1.2. An act A_S is called *unitary* if each $a \in A$ can be written as $a = a's$ for some $a' \in A$ and $s \in S$. This is equivalent to saying that μ_A is surjective.

We will denote by UAct_S (FAct_S) the full subcategory of Act_S generated by unitary (firm) acts. Note that if S is a monoid then the act A_S is unitary if and only if $a1 = a$ for every $a \in A$. Thus UAct_S coincides with the category of acts that is usually considered in the monoid case (see, e.g., [16]).

Let A_S be a right S-act. There exists *the functor* $A_S \otimes - : {}_S\mathsf{Act} \to \mathsf{Set}$ *of tensoring by* A_S (see [16]), defined by

$$(A_S \otimes -)({}_SM) := A \otimes M,$$

$$(A_S \otimes -)(f) := 1_A \otimes f$$

for every left S-act ${}_SM$ and every homomorphism f of left S-acts.

Definition 1.3. We call an act A_S *pullback flat* (*equalizer flat, finite limit flat, limit flat*), if the functor $A_S \otimes - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves pullbacks (equalizers, finite limits, limits).

The next lemma allows us to deduce several results about acts over semigroups from well-known results about unitary acts over monoids.

Lemma 1.4. *There is an isomorphism of categories* ${}_S\mathsf{Act} \to {}_{S^1}\mathsf{UAct}$ *which takes each* ${}_SB$ *to* ${}_{S^1}B$ *and each morphism* f *to* f.

Lemma 1.5. *For any category* $\mathcal{D}$, *the functor* $A\otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ *preserves* $\mathcal{D}$*-limits if and only if the functor* $A\otimes_{S^1} - : {}_{S^1}\mathsf{UAct} \to \mathsf{Set}$ *preserves* $\mathcal{D}$*-limits.*

Proof. If $F : {}_S\mathsf{Act} \to {}_{S^1}\mathsf{UAct}$ denotes the functor from Lemma 1.4, then

$$(A \otimes_{S^1} -) \circ F = A \otimes_S - ,$$

because $A \otimes_{S^1} B = A \otimes_S B$ for every ${}_SB \in {}_S\mathsf{Act}$. □

We will also use the following conditions (first two of them appearing first in [24] for the monoid case) for an S-act A_S:

(P): If $as = a's'$, $a, a' \in A$, $s, s' \in S$, then there exist $a'' \in A$, $u, v \in S$ such that

$$a = a''u, \;\; a' = a''v, \;\; us = vs'.$$

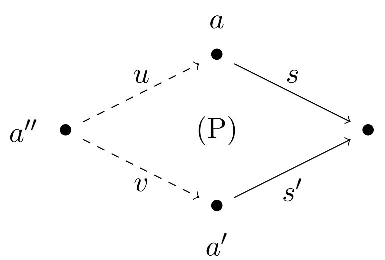
(E): If $as = as'$, $a \in A$, $s, s' \in S$, then there exist $a' \in A$, $u \in S$ such that

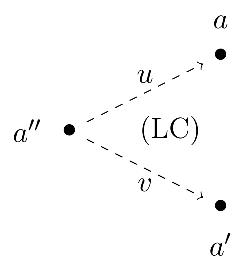
$$a = a'u, \;\; us = us'.$$

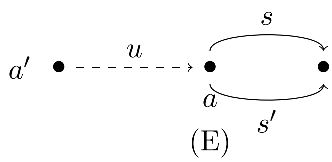
(LC): If $a, a' \in A$, then there exist $a'' \in A$, $u, v \in S$ such that

$$a = a''u, \;\; a' = a''v.$$

If S is a monoid, then Condition (LC) means precisely that A_S is locally cyclic in the sense of [7]. We also emphasize, that if A_S satisfies (LC) then A_S is unitary, but if A_{S^1} satisfies (LC) then it is not necessarily so.







(E)

Remark 1.6. Each nonempty act A_S can be identified with a semipresheaf (a contravariant semifunctor into Set) on a one-object semicategory (see Definition 2.1 in [13]), where morphisms are the elements of S. Such presheaf induces a semicategory of elements ([3, Definition 1.6.4]), which we denote by $\mathsf{El}(A_S)$ and which can be presented (up to equivalence) as follows:

- objects are the elements of the set A,
- morphisms $a \to a'$ are the elements $s \in S$ (or more formally triples (a, s, a')) such that $as = a'$,
- the composite $s' \circ s$ of morphisms $s : a \to a'$ and $s' : a' \to a''$ is the product $ss' \in S$.

Note that a nonempty act A_S

- satisfies (E) and (LC) if and only if $\mathsf{El}(A_S)$ is a cofiltered semicategory (cf. [3, Definition 2.13.1] or [22, page 211]),
- satisfies (E) and (P) if and only if $\mathsf{El}(A_S)$ is a pseudo-cofiltered semicategory (cf. [22, page 216]),
- satisfies (LC) if and only if any two objects of $\mathsf{El}(A_S)$ are joined by a span (cf. [3, Example 7.7.3]).

Lemma 1.7. *If A_S satisfies Condition* (P) ((E), (LC)), *then A_{S^1} has the same property.*

Conversely, if A_S is unitary and A_{S^1} satisfies Condition (P) ((E), (LC)), *then A_S has the same property.*

Proof. Let us prove the claim for (P) (for the other two conditions the proof is similar). Assume that A_S satisfies Condition (P) and $as = a's'$ where $a, a' \in A$ and $s, s' \in S^1$. There are four possibilities.

1) $s, s' \in S$. Then Condition (P) for A_S provides the required a'', u, v.

2) $s \in S, s' = 1$. Then we take $a'' = a$, $u = 1$ and $v = s$.
3) $s = 1, s' \in S$. Then we take $a'' = a'$, $u = s'$ and $v = 1$.
4) $s = s' = 1$. Then we take $a'' = a$ and $u = v = 1$.

Assume now that A_S is unitary and A_{S^1} satisfies Condition (P). Suppose that $as = a's'$, where $a, a' \in A$ and $s, s' \in S$. Then there exist $a'' \in A$, $u, v \in S^1$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. Since A_S is unitary, we can find $\overline{a} \in A$ and $z \in S$ such that $a'' = \overline{a}z$. Hence

$$a = \overline{a}(zu), \quad a' = \overline{a}(zv), \quad (zu)s = (zv)s',$$

where $zu, zv \in S$. □

2 Acts satisfying Condition (P)

First we examine when an act satisfies Condition (P). Very similarly to Proposition 2.1 in [5] one can prove the following result.

Lemma 2.1. *Let S be a nonempty semigroup. If A_S satisfies Condition (P) then for any left act ${}_SM$ and elements $a, a' \in A$, $m, m' \in M$, if $a \otimes m = a' \otimes m'$ in $A \otimes M$ then there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$, $um = vm'$.*

Proposition 2.2. *If an act A_S satisfies Condition* (P) *then the functor $A_S \otimes - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves monomorphisms.*

Proof. Both in Set and ${}_S\mathsf{Act}$, the monomorphisms are the injective morphisms. Let $f : {}_SB \to {}_SC$ be a monomorphism in ${}_S\mathsf{Act}$. Suppose that $a \otimes f(b) = a \otimes f(b')$ in $A \otimes C$. By Lemma 2.1, there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u = a''v$ and $uf(b) = vf(b')$, whence $ub = vb'$, because f is injective. Now

$$a \otimes b = a''u \otimes b = a'' \otimes ub = a'' \otimes vb' = a''v \otimes b' = a \otimes b'$$

in $A \otimes B$. This means that the mapping $1_A \otimes f \colon A \otimes B \to A \otimes C$ is injective. □

Lemma 2.3. *For any right S-act A_S the multiplication map*

$$\mu_A^{S^1} : A \otimes S^1 \to A\,, \quad a \otimes s \mapsto as$$

is an isomorphism in Act_S*. Moreover, the functor* $- \otimes S^1 : \mathsf{Act}_S \to \mathsf{Act}_S$ *is naturally isomorphic to the identity functor of* Act_S*.*

Proof. The mapping $\mu_A^{S^1}$ is clearly a morphism of right S-acts. Since $A \otimes_S S^1 = A \otimes_{S^1} S^1$, it is the multiplication map of the unitary act A_{S^1} over the monoid S^1. Therefore it is bijective (see Proposition 2.5.13 in [16]). Clearly, μ is a natural transformation. □

Proposition 2.4. *If a right* S*-act* A_S *is unitary and the functor* $A_S \otimes - : {}_S\mathsf{Act} \to \mathsf{Set}$ *preserves monomorphisms, then* A_S *is firm.*

Proof. Let $i \colon S \to S^1$ denote the inclusion and let us view it as a morphism of left S-acts. Notice that $\mu_A \colon A \otimes_S S \to A$ factors as

$$A \otimes_S S \xrightarrow{1_A \otimes i} A \otimes_S S^1 = A \otimes_{S^1} S^1 \xrightarrow{\mu_A^{S^1}} A\,.$$

The map $\mu_A^{S^1}$ is bijective by Lemma 2.3 and the map $1_A \otimes i$ is injective by assumption. So the composite μ_A must be injective. Since A_S is assumed to be unitary, μ_A is surjective and therefore also bijective, meaning that A_S is firm. □

Corollary 2.5. *If* A_S *satisfies Condition* (P)*, then* A_S *is firm.*

In the theory of acts over monoids, the acts for which the tensor multiplication functor preserves monomorphisms are usually called *flat* (see [15], where flat acts were first defined, or [16], Definition 3.9.1). But one could also say “monomorphism flat”.

We also point out that equalizer flat acts are flat. This follows from the fact that each monomorphism in ${}_S\mathsf{Act}$ is regular, precisely as in the case of monoid acts (see [16, Theorem 2.2.44]).

Since ${}_S\mathsf{Act}$ is a variety of algebras in the sense of Universal Algebra, we know that it has all limits, including pullbacks. Note that, for some pairs of morphisms, their pullback may be the empty S-act.

Weak pullbacks are defined like pullbacks, but without the uniqueness requirement in the universal property (see Definition 2.2 in [14]). The proof of Theorem 2.8 in [14] carries over from Set-endofunctors to functors T : ${}_S\mathsf{Act} \to \mathsf{Set}$. Specifying that theorem for $T = A_S \otimes -$ gives us the following result.

Proposition 2.6. *For an act A_S, the functor $A_S \otimes - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves weak pullbacks if and only if for all morphisms $f : {}_SM \to {}_SQ$, $g : {}_SN \to {}_SQ$ and all pairs $(a \otimes m, a' \otimes n) \in (A \otimes M) \times (A \otimes N)$ with $a \otimes f(m) = a' \otimes g(n)$ there exists an element $a'' \otimes (u, v) \in A \otimes P$, where*

$$ {}_SP = \{(x, y) \in M \times N \mid f(x) = g(y)\}, $$

such that

$$ a'' \otimes u = a \otimes m \quad \text{and} \quad a'' \otimes v = a' \otimes n\,. $$

We have the following characterization of acts satisfying Condition (P) (cf. Lemma 2.2 in [5] for the monoid case).

Theorem 2.7. *For an act A_S over a nonempty semigroup S the following assertions are equivalent.*

1. *A_S satisfies Condition* (P)*.*
3. *A_S is firm and satisfies Condition* (P)*.*
2. *A_S is firm and the functor $A_S \otimes - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves weak pullbacks.*

Proof. If A_S is the empty act then all three conditions are satisfied. Consider a nonempty A_S.

(1) $\Rightarrow$ (2). This is a consequence of Corollary 2.5.

(2) $\Rightarrow$ (3). Assume that A_S is firm and satisfies Condition (P). We employ Proposition 2.6. Let $(a \otimes m, a' \otimes n) \in (A \otimes M) \times (A \otimes N)$ be such that $a \otimes f(m) = a' \otimes g(n)$ in $A \otimes Q$. By Lemma 2.1, there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $uf(m) = vg(n)$. It follows that $f(um) = g(vn)$, and therefore $(um, vn) \in P = \{(x, y) \in M \times N \mid f(x) = g(y)\}$. Moreover,

$$ a'' \otimes um = a''u \otimes m = a \otimes m \quad \text{and} \quad a'' \otimes vn = a''v \otimes n = a' \otimes n\,. $$

(3) $\Rightarrow$ (1). Suppose that $as = a's'$, where $a, a' \in A$ and $s, s' \in S$. Consider the left S-act morphisms

$$ \begin{aligned} f :& {}_SS \to {}_SS, \;\; x \mapsto xs, \\ g :& {}_SS \to {}_SS, \;\; x \mapsto xs'. \end{aligned} $$

Since A_S is unitary, there exist $a_1, a_1' \in A$, $r, r' \in S$ such that $a = a_1 r$ and $a' = a_1' r'$. Then $a_1 f(r) = a_1 rs = as = a's' = a_1' r's' = a_1' g(r')$ or $\mu_A(a_1 \otimes f(r)) = \mu_A(a_1' \otimes g(r'))$. Using injectivity of μ_A we conclude that $a_1 \otimes f(r) = a_1' \otimes g(r')$ in $A \otimes S$. By assumption and Proposition 2.6, there exists $a'' \otimes (u, v) \in A \otimes P$, where ${}_S P = \{(x, y) \in S \times S \mid xs = xs'\}$, such that

$$a'' \otimes u = a_1 \otimes r \quad \text{and} \quad a'' \otimes v = a_1' \otimes r'$$

in $A \otimes S$. This implies $a''u = a_1 r = a$ and $a''v = a_1' r' = a'$. Moreover, as $(u, v) \in P$, we have $us = vs'$. Thus A_S satisfies Condition (P). □

3 Pullback flat acts

The aim of this section is to give a characterization and examples of unitary pullback flat acts. First we make the following observation.

Proposition 3.1. *An act A_S is pullback flat if and only if the act A_{S^1} satisfies Condition* (P) *and Condition* (E).

Proof. This follows from Lemma 1.5 and Theorem 5.3 in [24]. □

Corollary 3.2. *The empty S-act is pullback flat.*

Pullback flat acts need not be unitary.

Example 3.3. Let $S = X^+$ be the free semigroup on an alphabet X. Then S_S is not a unitary act while the act S_{S^1} satisfies Condition (P) and Condition (E) and hence S_S is pullback flat.

In the next theorem we will give a characterization of unitary pullback flat acts. The following well-known result (see, e.g., paragraph 1.439 in [12]) will be needed.

Proposition 3.4. *Let $\mathcal{A}$ be a category with finite products. If a functor $G : \mathcal{A} \to \mathcal{B}$ preserves pullbacks then it preserves equalizers.*

Theorem 3.5. *The following assertions are equivalent for an act A_S over a nonempty semigroup S.*

(i) *A_S is unitary and pullback flat.*

(ii) A_S *is firm and pullback flat.*

(iii) A_S *is firm, pullback flat and equalizer flat.*

(iv) A_S *satisfies Condition* (P) *and Condition* (E).

Proof. The empty act satisfies all four conditions of the theorem. Consider a nonempty act A_S.

(i) $\Rightarrow$ (ii). If A_S is unitary and pullback flat then the functor $A \otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves monomorphisms. By Proposition 2.4, A_S is firm.

(ii) $\Rightarrow$ (iii). If A_S is firm and pullback flat then due to Proposition 3.4 A_S is also equalizer flat.

(iii) $\Rightarrow$ (iv). Let A_S be firm, pullback flat and equalizer flat. Then the functor $A \otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves pullbacks and equalizers. By Lemma 1.5, the functor $A \otimes_{S^1} - : {}_{S^1}\mathsf{UAct} \to \mathsf{Set}$ also preserves pullbacks and equalizers. By [24, Theorem 5.3], A_{S^1} satisfies conditions (P) and (E). Since A_S is unitary, Lemma 1.7 implies that A_S satisfies Conditions (P) and (E).

(iv) $\Rightarrow$ (i). If the act A_S satisfies Conditions (P) and (E), then also A_{S^1} satisfies these conditions by 1.7. Hence the functor $A \otimes_{S^1} - : {}_{S^1}\mathsf{UAct} \to \mathsf{Set}$ preserves pullbacks due to [24, Theorem 5.3]. By Lemma 1.5, A_S is pullback flat. It is unitary, because it satisfies Condition (P). □

Remark 3.6. If $S = \varnothing$ but $A \neq \varnothing$ then A_S has (P) and (E), but it cannot be unitary (firm), because the mapping $\mu_A : A \otimes \varnothing \to A$ is not surjective.

The next result, which follows from Theorem 2 in [10], describes all unitary acts that are projective objects in the category Act_S.

Theorem 3.7. *For an act P_S over a semigroup S the following assertions are equivalent.*

1. P_S *is unitary and projective in* Act_S.

2. P_S *is firm and projective in* Act_S.

3. $P_S \cong \coprod_{i \in I} e_i S$ *where* $e_i^2 = e_i \in S$.

Proof. Equivalence of (1) and (3) is proved in Theorem 2 in [10]. Obviously, (2) implies (1).

(3) $\Rightarrow$ (2). If $e \in S$ is an idempotent, then $est = es't'$ implies

$$es \otimes t = e \otimes est = e \otimes es't' = es' \otimes t'$$

in $eS \otimes S$ for all $s, t, s', t' \in S$. Thus acts of the form eS and also their coproducts are firm. □

Corollary 3.8. *If S is a semigroup then every unitary projective act in Act_S is pullback flat.*

Proof. A straightforward verification shows that acts of the form eS where e is an idempotent of S satisfy Conditions (P) and (E). The same is true for their coproducts. □

Given an arbitrary semigroup S without idempotents, do we have any examples of firm pullback flat S-acts? The answer to this question is not so obvious. For example the act S_S is unitary only if S is factorisable (that is, $S = SS$) and the act S^1_S is never unitary when S is nonempty. Still there is a general construction which produces firm pullback flat acts.

Construction* 3.9*: Consider any sequence $\mathbf{s} = (s_i)_{i \geqslant 1}$ of elements of S and construct a right S-act $M^{\mathbf{s}}_S$ as the quotient act of the coproduct (disjoint union)

$$\coprod_{k \geqslant 1} \{k\} \times S^1$$

over $k \in \{1, 2, \ldots\}$ of right S-acts S^1_S by the relation

$$(k, s) \sim (k', s') \iff (\exists n > k, k')\; s_n \ldots s_k s = s_n \ldots s_{k'} s' \,.$$

We denote the equivalence class of (k, s) by $[k, s]$. As in [17] one can show that $M^{\mathbf{s}}_S$ is a firm right S-act. A similar construction probably first appeared in the proof of Lemma 1 in [11], where left acts over monoids were considered. We call the acts $M^{\mathbf{s}}_S$ the *sequence acts over S*.

Lemma 3.10. *Sequence acts satisfy Condition* (P) *and Condition* (E)*.*

Proof. We prove (P), for (E) the proof is similar. Suppose that $[k, s]z = [k', s']z'$. Then $[k, sz] = [k', s'z']$ and there exists $n > k, k'$ such that

$$s_n \ldots s_k s z = s_n \ldots s_{k'} s' z' \,.$$

Putting $u = s_n \dots s_k s$ and $v = s_n \dots s_{k'} s'$ we have $uz = vz'$,

$$[k, s] = [n+1, u] = [n+1, 1]u \text{ and } [k', s'] = [n+1, v] = [n+1, 1]v,$$

because

$$(s_{n+2}s_{n+1} \dots s_k)s = (s_{n+2}s_{n+1})u, \quad (s_{n+2}s_{n+1} \dots s_{k'})s' = (s_{n+2}s_{n+1})v\,.$$

□

Corollary 3.11. *Sequence acts are firm and pullback flat.*

Proposition 3.12. *Let A_S be a unitary act. Then every $a \in A$ is in the image of some morphism $f\colon M_S^{\mathbf{s}} \to A_S$ for some $\mathbf{s} \in S^{\mathbb{N}}$.*

Proof. Let A_S be a unitary act and $a_0 \in A$. Then we can find sequences $(a_i)_{i\geqslant 1}$ and $(s_i)_{i\geqslant 1}$ of elements of A and S, respectively, such that

$$a_{i-1} = a_i s_i$$

for each $i \in \mathbb{N}$. Put $\mathbf{s} \coloneqq (s_i)_{i\geqslant 1}$ and define a mapping $f\colon M_S^{\mathbf{s}} \to A$ by

$$f([k, s]) \coloneqq a_{k-1}s.$$

(If s is the external identity in S^1 then by $a_{k-1}s$ we mean a_{k-1}.) To prove that f is well defined, suppose that $(k, s) \sim (k', s')$. Then $s_n \dots s_k s = s_n \dots s_{k'} s'$ for some $n > k, k'$. Hence

$$a_n s_n \dots s_k s = a_n s_n \dots s_{k'} s',$$

which reduces to $a_k s_k s = a_{k'} s_{k'} s'$ and finally to $a_{k-1}s = a_{k'-1}s'$, as needed.
It is clear that f is a homomorphism of right S-acts. Also

$$f([1, 1]) = a_0 1 = a_0\,.$$

□

The following is a semigroup theoretic analogue of Lemma 2.4 in [23].

Proposition 3.13. *Every unitary right S-act is a homomorphic image of a firm pullback flat S-act.*

Proof. Let A_S be a unitary act. By previous proposition, for every $a \in A$ we can find a sequence $\mathbf{s}_a \in S^{\mathbb{N}}$ and a homomorphism $f_a : M_S^{\mathbf{s}_a} \to A_S$ such that a lies in the image of f_a. The coproduct (disjoint union) of acts satisfying (P) and (E) satisfies (P) and (E). So taking

$$B_S := \coprod_{a \in A} M_S^{\mathbf{s}_a}$$

we obtain a firm pullback flat act which maps surjectively on A_S. □

Directed colimits play an important role in relation to flatness. We recall that a poset I is said to be *up-directed*, if it is non-empty and every two elements have an upper bound in I. We can view a poset I as a category in the standard way (see page 11 in [22]). An up-directed system $((A_i)_{i \in I}, (\varphi_{ij})_{i \leqslant j})$ in a category $\mathcal{C}$ is our way of presenting the data of a functor $I \to \mathcal{C}$, where $i \mapsto A_i$ and $(i \leqslant j) \mapsto \varphi_{ij}$.

Proposition 3.14. *In the category* Act_S *the directed colimit of any up-directed system* $((A_i)_{i \in I}, (\varphi_{ij})_{i \leqslant j})$ *exists, and may be constructed as* $(A/\theta, (\varphi_i)_{i \in I})$, *where*

1. $A = \coprod_{i \in I} A_i$;
2. $a \, \theta \, a'$ $(a \in A_i, a' \in A_j)$ *if and only if* $\varphi_{ik}(a) = \varphi_{jk}(a')$ *for some* $k \geqslant i, j$;
3. *for each* $i \in I$ *and* $a \in A_i$, $\varphi_i(a) = [a]_\theta$.

Lemma 3.15. *Directed colimit of acts satisfying Condition* (P) (*Condition* (E)) *also satisfies Condition* (P) (*Condition* (E)).

Proof. We use the description given in Proposition 3.14. To prove the claim for (P), suppose that $[a]s = [a']s'$ where $a \in A_i$ and $a' \in A_j$. Then there exists $k \geqslant i, j$ such that $\varphi_{ik}(as) = \varphi_{jk}(a's')$ in A_k. Since A_k satisfies (P), we can find $a'' \in A_k$ and $u, v \in S$ such that $\varphi_{ik}(a) = a''u$, $\varphi_{jk}(a') = a''v$ and $us = vs'$. Then $\varphi_{ik}(a) = \varphi_{kk}(a''u)$ and $\varphi_{jk}(a') = \varphi_{kk}(a''v)$, which implies

$$[a] = [a''u] = [a'']u \ \text{ and } \ [a'] = [a''v] = [a'']v \,.$$

For Condition (E) the proof is analogous. □

Corollary 3.16. *Any directed colimit of sequence acts over a semigroup S satisfies Condition* (P) *and Condition* (E)*.*

We end this section with a small observation about equalizer flat acts.

Proposition 3.17. *Unitary equalizer flat acts satisfy Condition* (E)*.*

Proof. If the functor $A \otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves equalizers, then, by Lemma 1.5, the functor $A \otimes_{S^1} - : {}_{S^1}\mathsf{UAct} \to \mathsf{Set}$ also preserves equalizers. Due to [16, Proposition 3.15.3], the act A_{S^1} satisfies Condition (E). Since A_S is unitary, Lemma 1.7 implies that A_S satisfies Condition (E). □

4 Finite limit flatness and limit flatness

We will now look at the stronger condition of finite limit flatness. Directly from the monoid case, we have the following.

Proposition 4.1. *A nonempty act A_S is finite limit flat if and only if the act A_{S^1} satisfies Condition* (E) *and is locally cyclic.*

Proof. This follows from Lemma 1.5 and Theorem 3.1 in [7]. □

Analogously to pullback flatness, if we add the assumption of unitarity, we can consider Conditions (E) and (LC) on A_S itself.

Theorem 4.2. *A nonempty act A_S over a nonempty semigroup S is unitary and finite limit flat if and only if it satisfies Conditions* (E) *and* (LC)*.*

Proof. **Necessity.** If $A \otimes_S -$ preserves finite limits then also $A \otimes_{S^1} -$ preserves finite limits. Hence A_{S^1} is locally cyclic and satisfies Condition (E) by Theorem 3.1 in [7]. Now Lemma 1.7 implies that A_S satisfies (E) and (LC).

Sufficiency. Assume that A_S satisfies Conditions (E) and (LC). Then A_{S^1} has the same properties by Lemma 1.7. By Theorem 3.1 in [7], $A \otimes_{S^1} -$ preserves finite limits. But then also $A \otimes_S -$ preserves finite limits. □

For any semigroup, there exist a number of unitary finite limit flat acts. The sequence acts, and any directed colimit thereof, provide a family of examples.

Proposition 4.3. *Any directed colimit of sequence acts is unitary and finite limit flat.*

Proof. A directed colimit of sequence acts satisfies Condition (E) by Corollary 3.16. Every sequence act $M_S^{\mathbf{s}}$ satisfies (LC), because it is a union of cyclic subacts

$$[1,1]S^1 \subset [2,1]S^1 \subset [3,1]S^1 \subset \dots .$$

To complete the proof we need to show that a directed colimit of locally cyclic acts is locally cyclic. We do this using the notation of Proposition 3.14, where $A_i, i \in I$, are assumed to satisfy (LC).

Take $[a_1]_\theta, [a_2]_\theta \in A/\theta$. Then there exist $i, j \in I$ such that $a_1 \in A_i$ and $a_2 \in A_j$. Since I is up-directed, there exists $k \geqslant i, j$. Using that A_k satisfies (LC), for its elements $\varphi_{ik}(a_1)$ and $\varphi_{jk}(a_2)$ there exist $a \in A_k$ and $s_1, s_2 \in S^1$ such that $\varphi_{ik}(a_1) = as_1$ and $\varphi_{jk}(a_2) = as_2$. Now, in A/θ, we have

$$[a_1]_\theta = \varphi_i(a_1) = (\varphi_k \varphi_{ik})(a_1) = \varphi_k(as_1) = \varphi_k(a)s_1 = [a]_\theta s_1 ,$$

and, similarly, $[a_2]_\theta = [a]_\theta s_2$. □

Unitary limit flat acts, contrary to unitary finite limit flat acts, need not exist. This turns out to be the case precisely when the semigroup S contains no idempotents.

Proposition 4.4. *A nonempty act A_S is unitary and limit flat if and only if $A_S \cong eS_S$ for some idempotent $e \in S$.*

Proof. **Necessity.** If $A \otimes_S -$ preserves limits then $A \otimes_{S^1} -$ preserves limits. By Theorem 3.5 in [7], $A_{S^1} \cong eS^1_{S^1}$ for some idempotent $e \in S^1$. Then also $A_S \cong eS^1_S$. If $e = 1$ then $eS^1_S = S^1_S$, which is not unitary, a contradiction. Hence $e \in S$ and $A_S \cong eS^1_S = eS_S$.

Sufficiency. If $A_S \cong eS_S$ for some idempotent $e \in S$, then also $A_{S^1} \cong eS^1_{S^1}$. By Theorem 3.5 in [7], $A \otimes_{S^1} -$ preserves limits and thus also $A \otimes_S -$ preserves limits. □

Remark 4.5. In some of the following results we have the assumption that S is a *firm semigroup*, meaning that S_S is a firm act, or equivalently that the multiplication map $S \otimes S \to S$ is invertible. We will note that all of these statements can be reformulated (applying the methods used in [19]) for the more general case of factorisable semigroups (meaning $SS = S$) with minor modifications to the statements and proofs. We have done this to improve the clarity of the text for the reader.

Knowing that the categories UAct_S and FAct_S are coreflective subcategories of Act_S will help us with calculating limits and colimits in these subcategories.

Lemma 4.6. *If S is a factorisable semigroup, then the inclusion functor $\mathsf{UAct}_S \to \mathsf{Act}_S$ has a right adjoint $U\colon \mathsf{Act}_S \to \mathsf{UAct}_S$, which maps an act A_S to the subact*

$$U(A) = AS = \{as \mid a \in A, s \in S\}\,.$$

On morphisms U acts by restricting the domain and the codomain. The coreflection functor U preserves directed colimits.

Proof. The functor U is the coreflection functor by [18, Proposition 3.10]. To see that U preserves directed colimits, let a colimit of an up-directed system $((A_i)_{i\in I}, (\varphi_{ij})_{i\leqslant j})$ in Act_S be constructed as in Proposition 3.14. Consider a cocone $(B_S, (\psi_i)_{i\in I})$ on the system $((A_iS)_{i\in I}, (U(\varphi_{ij}))_{i\leqslant j})$ in UAct_S. Define a mapping $\nu : (A/\theta)S \to B$ by putting

$$\nu([a]s) := \psi_i(as)$$

for every $[a]s \in (A/\theta)S$, where $a \in A_i$. If $[a]s = [a']s'$, $a \in A_i, a' \in A_j$, $s, s' \in S$, then $\varphi_{ik}(as) = \varphi_{jk}(a's')$ for some $k \geqslant i, j$. Hence

$$\psi_i(as) = (\psi_k U(\varphi_{ik}))(as) = (\psi_k U(\varphi_{jk}))(a's') = \psi_j(a's')\,,$$

so ν is well defined. Clearly ν is a morphism of right S-acts and unique with the property that $\nu U(\varphi_i) = \psi_i$ for all $i \in I$. □

Lemma 4.7. *If S is a firm semigroup, then the inclusion functor $\mathsf{FAct}_S \to \mathsf{Act}_S$ has a right adjoint $-\otimes S\colon \mathsf{Act}_S \to \mathsf{FAct}_S$, which is simply the functor of tensoring on the right with the biact ${}_SS_S$. The coreflection functor $-\otimes S$ preserves all colimits.*

Proof. The functor $-\otimes S$ is a coreflection functor according to [18, Proposition 3.9]. It preserves all colimits, because it has a right adjoint

$$\mathsf{Act}_S(S,-)\colon \mathsf{FAct}_S \to \mathsf{Act}_S$$

induced by the tensor-hom adjunction. □

We will recall how to calculate limits and colimits in coreflective subcategories.

Remark 4.8. Suppose $\mathcal{H}$ is a coreflective subcategory of a complete and cocomplete category $\mathcal{C}$ with a coreflector functor $F\colon \mathcal{C} \to \mathcal{H}$. Then, given a diagram $D\colon \mathcal{D} \to \mathcal{H}$ in the subcategory $\mathcal{H}$, we have

- $\lim^{\mathcal{H}}(D) \cong F(\lim^{\mathcal{C}}(D))$ and
- $\operatorname{colim}^{\mathcal{H}}(D) = \operatorname{colim}^{\mathcal{C}}(D)$.

In particular, this implies that when we are talking about a colimit of firm acts (unitary acts), we do not need to specify whether we are taking the colimit in FAct_S (UAct_S) or Act_S, since the colimits will coincide.

Lemma 4.9. *If S is a semigroup, A_S is a firm S-act and ${}_SB$ any left S-act, then we have an isomorphism $A \otimes SB \cong A \otimes B$ natural in $A \in \mathsf{FAct}_S$ and $B \in {}_S\mathsf{Act}$.*

Proof. Let $i\colon SB \to B$ denote the inclusion and let ν_B be the multiplication map $S \otimes B \to SB, s \otimes b \mapsto sb$.

Let us define the inverse $h\colon A \otimes B \to A \otimes SB$ of the mapping $1 \otimes i\colon A \otimes SB \to A \otimes B$ as the composite map

$$A \otimes B \xrightarrow{\mu_A^{-1} \otimes 1} A \otimes S \otimes B \xrightarrow{1 \otimes \nu} A \otimes SB\,.$$

Naturality of the map $1 \otimes i$ is trivial, so let us check that $1 \otimes i$ and h are mutually inverse. Take arbitrary $a \in A$ and $b \in B$. Since A_S is unitary, there exist $a' \in A$ and $s \in S$ such that $a = a's = \mu_A(a' \otimes s)$. Then

$$\begin{aligned}
((1 \otimes i)h)(a \otimes b) &= ((1 \otimes i)(1 \otimes \nu)(\mu_A^{-1} \otimes 1))(a \otimes b) \\
&= ((1 \otimes i)(1 \otimes \nu))(a' \otimes s \otimes b) \\
&= (1 \otimes i)(a' \otimes sb) \\
&= a' \otimes sb \\
&= a's \otimes b \\
&= a \otimes b\,.
\end{aligned}$$

Also, for any $a \in A$, $b \in B$ and $s \in S$ we have

$$\begin{aligned}(h(1 \otimes i))(a \otimes sb) &= ((1 \otimes \nu)(\mu_A^{-1} \otimes 1))(a \otimes sb) \\ &= ((1 \otimes \nu)(\mu_A^{-1} \otimes 1))(as \otimes b) \\ &= (1 \otimes \nu)(a \otimes s \otimes b) \\ &= a \otimes sb\,. \end{aligned}$$

□

We are now ready to prove that for a firm semigroup, the notion of pullback flatness (finite limit flatness) could just as well be defined with respect to the category of firm acts.

Theorem 4.10. *The following statements are equivalent for a firm semigroup S, a firm right S-act A_S and a small category $\mathcal{D}$:*

1. *the functor* $A \otimes -\colon {}_S\mathsf{Act} \to \mathsf{Set}$ *preserves $\mathcal{D}$-limits,*
2. *the functor* $A \otimes -\colon {}_S\mathsf{UAct} \to \mathsf{Set}$ *preserves $\mathcal{D}$-limits,*
3. *the functor* $A \otimes -\colon {}_S\mathsf{FAct} \to \mathsf{Set}$ *preserves $\mathcal{D}$-limits.*

Proof. (1) $\Rightarrow$ (2). By the left-right dual of Lemma 4.6 we know that ${}_S\mathsf{UAct}$ is a coreflective subcategory of ${}_S\mathsf{Act}$, with the coreflection functor $U\colon {}_S\mathsf{Act} \to {}_S\mathsf{UAct}$ given by $B \mapsto SB = \{sb \mid s \in S, b \in B\}$.

Now, let A_S be such that $A \otimes -\colon {}_S\mathsf{Act} \to \mathsf{Set}$ preserves $\mathcal{D}$-limits. Given a diagram $D\colon \mathcal{D} \to {}_S\mathsf{UAct}$, we can calculate as follows:

$$\begin{aligned} A \otimes \lim_i^{{}_S\mathsf{UAct}} D(i) &\cong A \otimes S(\lim_i^{{}_S\mathsf{Act}} D(i)) && \text{(by Remark 4.8)} \\ &\cong A \otimes \lim_i^{{}_S\mathsf{Act}} D(i) && \text{(by Lemma 4.9)} \\ &\cong \lim_i^{\mathsf{Set}} (A \otimes D(i)). && \text{(by assumption)} \end{aligned}$$

(2) $\Rightarrow$ (1). Let A_S be such that $A \otimes -\colon {}_S\mathsf{UAct} \to \mathsf{Set}$ preserves $\mathcal{D}$-limits.

Then for a diagram $D\colon \mathcal{D} \to {}_S\mathsf{Act}$ we have

$$\begin{aligned} A \otimes \lim_i^{{}_S\mathsf{Act}} D(i) &\cong A \otimes U(\lim_i^{{}_S\mathsf{Act}} D(i)) && \text{(by Lemma 4.9)} \\ &\cong A \otimes \lim_i^{{}_S\mathsf{UAct}} D(i) && \text{(by Remark 4.8)} \\ &\cong \lim_i^{\mathsf{Set}}(A \otimes D(i))\,. && \text{(by assumption)} \end{aligned}$$

(1) $\Leftrightarrow$ (3). Analogous to the proof of (1) $\Leftrightarrow$ (2), with the identity $A \otimes SB \cong A \otimes B$ replaced by $A \otimes (S \otimes B) \cong A \otimes B$, since the coreflection functor ${}_S\mathsf{Act} \to {}_S\mathsf{FAct}$ is given by ${}_SB \mapsto S \otimes B$ (Lemma 4.7). □

How far is pullback flatness from being firm and pullback flat? In general, we do not know a canonical way of turning a pullback flat act into a firm act while retaining pullback flatness. When S is firm, then the canonical firm act associated with an act A_S is $A \otimes S$. We observe that this assignment need not always preserve pullback flatness.

Proposition 4.11. *The functor* $- \otimes S\colon \mathsf{Act}_S \to \mathsf{Act}_S$ *preserves pullback flatness* (*equalizer flatness, finite limit flatness*) *if and only if* S_S *is a pullback flat* (*equalizer flat, finite limit flat*) *right* S*-act.*

Proof. We will prove this for pullback flatness. The proof is similar for the other flatness properties.

Necessity. Suppose that $- \otimes S\colon \mathsf{Act}_S \to \mathsf{Act}_S$ preserves pullback flatness. By the dual of Lemma 2.3, the functor ${}_SS^1 \otimes - : {}_S\mathsf{Act} \to {}_S\mathsf{Act}$ is naturally isomorphic to the identity functor of ${}_S\mathsf{Act}$, in particular it preserves pullbacks. Then the functor $S^1 \otimes - : {}_S\mathsf{Act} \to \mathsf{Set}$ also preserves pullbacks, that is, S^1_S is a pullback flat right S-act. By assumption, the right S-act $S^1 \otimes S \cong S_S$ (the dual of Lemma 2.3) is pullback flat.

Sufficiency. Assume that the functor $S \otimes -\colon {}_S\mathsf{Act} \to \mathsf{Set}$ preserves pullbacks. Then $S \otimes -\colon {}_S\mathsf{Act} \to {}_S\mathsf{Act}$ also preserves pullbacks. Therefore, whenever A_S is a right S-act and $A \otimes -\colon {}_S\mathsf{Act} \to \mathsf{Set}$ preserves pullbacks, the functor

$$(A \otimes S) \otimes - \cong A \otimes (S \otimes -) = (A \otimes -) \circ (S \otimes -) : {}_S\mathsf{Act} \to \mathsf{Set}$$

preserves pullbacks, meaning that $A \otimes S \in \mathsf{Act}_S$ is pullback flat. □

Example 4.12. The following four-element semigroup S is firm but the act S_S does not satisfy Conditions (E), (P) or (LC):

	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
c	0	0	b	c

Indeed,

- S is firm, because c is a right identity element;
- S_S does not satisfy (LC), because the Cayley table has no row containing both a and b (there is no span connecting a and b in the semicategory $\mathsf{El}(S_S)$);
- S_S does not satisfy (P), because a and b are connected by a cospan in $\mathsf{El}(S_S)$ ($a0 = b0$), but not by a span;
- S_S does not satisfy (E), because for the parallel morphisms $a, b \colon a \to 0$ in $\mathsf{El}(S_S)$ ($aa = ab = 0$), there is only one arrow into a, namely $c \colon a \to a$ ($ac = a$), but $ca = 0 \neq b = cb$.

On the other hand, if a semigroup has a left identity element e, then S is firm and $S_S = eS_S$ has (LC), (E) and (P) (see the diagram in Section 8).

5 Pure epimorphisms

In this section we look at a description of firm pullback flat acts using pure epimorphisms. The advantage of this description is that it can be formulated in purely category theoretic terms (which we will do in Section 6). The approach is that of Section 4 of [24], which we will translate to the semigroup case using the isomorphism of categories in Lemma 1.4, starting with finitely presentable acts.

An act is finitely presentable if it can be given a presentation with a finite number of generators and a finite number of equations. To be more precise, an act is finitely presentable if it is isomorphic to a quotient of a free act on a finite set by a congruence generated by a finite relation. We will now explain in more detail what this means.

First we take a look at free acts on sets. Let us consider an assignment $\mathbf{F}$ which takes a nonempty set X to the right S-act $X \times S^1$ with the action

$$(x, s)t := (x, st)$$

and a nonempty mapping $f : X \to Y$ to a right S-act morphism

$$\mathbf{F}(f) : X \times S^1 \to Y \times S^1, \ \ (x, s) \mapsto (f(x), s)\,.$$

Also, $\mathbf{F}$ will take the empty set to the empty act, and empty mappings to empty mappings. It is easy to check that $\mathbf{F} : \mathsf{Set} \to \mathsf{Act}_S$ is a functor.

Lemma 5.1. *The functor* $\mathbf{F}$ *is left adjoint to the forgetful functor* $U : \mathsf{Act}_S \to \mathsf{Set}$*.*

Proof. The unit $\eta : 1_{\mathsf{Set}} \Rightarrow U\mathbf{F}$ and the counit $\varepsilon : \mathbf{F}U \Rightarrow 1_{\mathsf{Act}_S}$ are defined by

$$\begin{aligned} \eta_X &: X \to X \times S^1, \ \ x \mapsto (x, 1), \\ \varepsilon_{A_S} &: A \times S^1 \to A, \ \ (a, s) \mapsto as. \end{aligned}$$

A straightforward verification shows that η and ε are natural transformations. The triangle equalities hold because

$$\begin{aligned} (\varepsilon_{X\times S^1} \circ \mathbf{F}(\eta_S))(x, s) &= \varepsilon_{X\times S^1}((x, s), 1) = (x, s)1 = (x, s1) = (x, s), \\ (U(\varepsilon_A) \circ \eta_{U(A)})(a) &= \varepsilon_A(a, 1) = a1 = a. \end{aligned}$$

□

Therefore the following definition is justified.

Definition 5.2. A right S-act is called *free* on a set X if it is isomorphic to the act $X \times S^1$. *Finitely generated free S-acts* are those isomorphic to

$$\{1, \dots, n\} \times S^1$$

for some $n \in \mathbb{N}$, and the empty act.

Proposition 5.3. *Every free right S-act is projective in* Act_S*.*

Proof. Consider a free S-act $X \times S^1$, an epimorphism (surjective morphism) $\pi : A_S \to B_S$ and a morphism $f : X \times S^1 \to B_S$ in Act_S. For every $x \in X$ we choose $a_x \in A$ such that $\pi(a_x) = f(x, 1)$. Defining $g : X \times S^1 \to A$ by

$$g(x, s) := a_x s$$

we have $\pi g = f$. □

Definition 5.4. We say that an S-act B_S is *finitely presentable* if there exist a finitely generated free act F_S and a finite subset $H \subseteq F \times F$ such that

$$B_S \cong F/\rho(H)\,,$$

where $\rho(H)$ is the smallest congruence on F_S containing the set H.

Example 5.5. Every free act on a finite set is finitely presentable, since $\rho(\varnothing)$ is the equality relation and $F/\rho(\varnothing) = F_S$.

Next we consider pure epimorphisms of S-acts. Note that epimorphisms in Act_S are the surjective homomorphisms.

Definition 5.6 ([4])**.** A surjective S-act morphism $\varphi : B_S \to A_S$ is called a *pure epimorphism* if for every finitely presentable S-act C_S and morphism $\psi : C_S \to A_S$ there exists a morphism $\mu : C_S \to B_S$ such that $\varphi\mu = \psi$.

The following proposition is a version of Proposition 4.3 from [24] for acts over semigroups.

Proposition 5.7. *Let $\varphi : B_S \to A_S$ be a surjective morphism of S-acts. Then the following assertions are equivalent:*

1. *φ is a pure epimorphism.*
2. *Whenever $a_1, ..., a_n \in A$ and*

$$\begin{array}{rcl} a_{\alpha_1} s_1 & = & a_{\beta_1} t_1 \\ & \cdots & \\ a_{\alpha_m} s_m & = & a_{\beta_m} t_m \end{array},$$

where $\alpha_j, \beta_j \in \{1, ..., n\}$ and $s_j, t_j \in S^1$, there exist $b_1, ..., b_n \in B$ such that $\varphi(b_i) = a_i$ for every $i \in \{1, \ldots, n\}$ and

$$\begin{array}{rcl} b_{\alpha_1} s_1 & = & b_{\beta_1} t_1 \\ & \cdots & \\ b_{\alpha_m} s_m & = & b_{\beta_m} t_m \end{array}.$$

Proof. We observe that A_S is free or finitely presentable if and only if A_{S^1} is free or finitely presented in the sense of [24]. Thus $\varphi : B_S \to A_S$ is a pure epimorphism if and only if it is a pure epimorphism in UAct_{S^1} in the sense of [24]. By Proposition 4.3 of that article, the latter is equivalent to the fact that A_{S^1} satisfies condition (2). But A_{S^1} satisfies condition (2) if and only if A_S does so. □

Corollary 5.8. *For any S-act A_S, the morphism $\mu_A \colon A \otimes S \to A$ is a pure epimorphism if and only if it is an isomorphism (that is, when A_S is firm).*

Proof. Sufficiency being clear, let us prove necessity. If μ_A is a pure epimorphism then it is surjective. We need to check that it is injective. Suppose that $as = a's'$, $a, a' \in A$, $s, s' \in S$. By Proposition 5.7, there exist $a_1 \otimes u, a_2 \otimes v \in A \otimes S$ such that $(a_1 \otimes u)s = (a_2 \otimes v)s'$, $a = \mu_A(a_1 \otimes u) = a_1 u$ and $a' = \mu_A(a_2 \otimes v) = a_2 v$. Hence

$$\begin{aligned} a \otimes s = a_1 u \otimes s = a_1 \otimes us = (a_1 \otimes u)s = (a_2 \otimes v)s' = a_2 \otimes vs' \\ = a_2 v \otimes s' = a' \otimes s' \end{aligned}$$

in $A \otimes S$. □

Thus if A_S is a unitary act which is not firm then μ_A is an epimorphism which is not pure. Such acts indeed exist (see Example 2.12 in [20]).

We will now translate the purity-based description of pullback flatness given in Theorem 5.3 of [24] to the setting of acts over semigroups.

Proposition 5.9. *A right S-act A_S is pullback flat if and only if every surjection $B_S \to A_S$ is a pure epimorphism.*

Proof. Theorem 5.4 in [24] applied to the monoid S^1 tells us that A_{S^1} is a pullback flat unitary act over the monoid S^1 precisely when every surjection $B_S \to A_S$ is a pure epimorphism in UAct_{S^1}. We can observe that the isomorphism of categories in Lemma 1.4 preserves and reflects pullback flatness (Lemma 1.5) and pure epimorphisms, so the result for A_S follows. □

6 Purity and flatness in the category of firm acts

In this section our goal is to characterize flatness using pure epimorphisms in the category FAct_S of firm right S-acts. We would like to do so without the definitions being dependent on FAct_S being a subcategory of Act_S. Unfortunately we can not translate Definition 5.4 directly to the case of FAct_S, since the forgetful functor from FAct_S to Set need not have a left adjoint (cf. Lemma 5.1), meaning that we do not have a notion of a free firm act on a set. We will switch to a definition of "finitely presentable" that does not depend on the notion of freeness. In the setting of *locally finitely presentable categories* (standard reference: [1]), aspects of pure epimorphisms were studied in [4]. To borrow their definition of flatness, we will start by recounting basic definitions of that setting.

Definition 6.1 ([1, Def. 1.1])**.** An object A in a category $\mathcal{C}$ is said to be *finitely presentable* if the representable functor $\mathcal{C}(A,-)\colon \mathcal{C} \to \mathsf{Set}$ preserves directed colimits.

Put into more elementary terms, if $((A_i)_{i\in I}, (\varphi_{ij})_{i\leqslant j})$ is an up-directed system in a category $\mathcal{C}$ with a colimit $(A, (\varphi_i)_{i\in I})$, then a representable functor $\mathcal{C}(C,-)\colon \mathcal{C} \to \mathsf{Set}$ preserves that colimit if and only if

1. each morphism $C \to A$ factors through $\varphi_i\colon A_i \to A$ for some $i \in I$,
2. for each $i \in I$ and $g, h\colon C \to A_i$, if $\varphi_i g = \varphi_i h$, then $\varphi_{ij} g = \varphi_{ij} h$ for some $j \geqslant i$.

In any variety in the sense of Universal Algebra, an object satisfies Definition 6.1 precisely when it satisfies a condition in the style of Definition 5.4 ([1, Theorem 3.12]), so our two notions of being finitely presentable are compatible. Using this new definition, a *pure epimorphism* can be defined in any category as an epimorphism $B \to A$, through which every morphism $C \to A$, with C finitely presentable, factors.

In Act_S, as well as in any other variety of algebras, finitely presentable objects play a special role, in that we can fix some set of them, such that every object is a directed colimit of objects in that set. A cocomplete category with that property is said to be *locally finitely presentable.* Any variety is an example of this (Corollary 3.7 of [1]), so we have the following.

Proposition 6.2. Act_S *is a locally finitely presentable category.*

In [4] an object A in a locally finitely presentable category is called flat if every strong epimorphism $B \to A$ is a pure epimorphism. For convenience, in the following this property will be referred to as *BR-flatness*. To see that in the category Act_S this condition coincides with the one in Proposition 5.9, we will need to observe that strong epimorphisms in Act_S are precisely the surjective act morphisms.

Proposition 6.3. *If S is a firm semigroup, then epimorphisms and strong epimorphisms in* Act_S *and* FAct_S *are precisely the surjective act morphisms.*

Proof. By the dual of Lemma 1.4, the category Act_S is isomorphic to UAct_{S^1}, which is the category of contravariant functors from S^1 (viewed as a one object category) to Set. Colimits and limits in a functor category are calculated pointwise, which implies that epimorphisms in UAct_{S^1}, and by extension in Act_S, are precisely the surjective maps, and all epimorphisms are regular and strong (see Example 4.3.10.g in [3]).

Since FAct_S is a coreflective subcategory of Act_S by Lemma 4.7, a morphism $f\colon A_S \to B_S$ is an epimorphism in FAct_S precisely when it is an epimorphism in Act_S, meaning that f is an epimorphism in FAct_S precisely when it is surjective.

Since strong epimorphisms in FAct_S are epimorphisms, they must be surjective. Conversely, suppose $f\colon A_S \to B_S$ is a surjective morphism in FAct_S. Then it is a regular epimorphism in Act_S, meaning that it is the coequalizer of some pair of morphisms $u, v\colon C_S \to A_S$ in Act_S. Since the coreflection functor $-\otimes S\colon \mathsf{Act}_S \to \mathsf{FAct}_S$ preserves all colimits (Lemma 4.7), the morphism $f \otimes 1_S\colon A \otimes S \to B \otimes S$ is the coequalizer of the pair of morphisms $u \otimes 1_S, v \otimes 1_S\colon C \otimes S \to A \otimes S$ in FAct_S, which means that $f \otimes 1_S$ is a regular epimorphism in FAct_S.

From this it follows that f is a regular epimorphism in FAct_S, since the naturality of μ implies that $\mu_B(f \otimes 1) = f\mu_A$, in which μ_A and μ_B are isomorphisms, because A_S and B_S are firm acts. This concludes the proof, since regular epimorphisms are strong epimorphisms (Proposition 4.3.6 in [3]). □

Proposition 6.4. *If S is a firm semigroup then*

1. *a unitary right S-act is finitely presentable in* UAct_S *if and only if it is finitely presentable in* Act_S*,*

2. *a firm right S-act is finitely presentable in* FAct_S *if and only if it is finitely presentable in* Act_S*.*

Proof. (1) Suppose A_S is a unitary act finitely presentable in UAct_S and suppose $((C_i)_{i\in I}, (\varphi_{ij})_{i\leqslant j})$ is an up-directed system in Act_S. Letting $I\colon \mathsf{UAct}_S \to \mathsf{Act}_S$ denote the inclusion functor (left adjoint to the functor $U\colon \mathsf{Act}_S \to \mathsf{UAct}_S$), we have

$$\begin{aligned}
\mathsf{Act}_S(A, \overset{\mathsf{Act}_S}{\operatorname{colim}}\, C_i) &= \mathsf{Act}_S(IA, \overset{\mathsf{Act}_S}{\operatorname{colim}}\, C_i) && (I \text{ is the inclusion})\\
&\cong \mathsf{UAct}_S(A, U(\overset{\mathsf{Act}_S}{\operatorname{colim}}\, C_i)) && (I \dashv U)\\
&\cong \mathsf{UAct}_S(A, \overset{\mathsf{UAct}_S}{\operatorname{colim}}\, U(C_i)) && \\
& && (U \text{ pres. dir. colimits by Lemma 4.6})\\
&\cong \overset{\mathsf{Set}}{\operatorname{colim}}\, \mathsf{UAct}_S(A, U(C_i)) && \\
& && (A \text{ is finitely presentable in } \mathsf{UAct}_S)\\
&\cong \overset{\mathsf{Set}}{\operatorname{colim}}\, \mathsf{Act}_S(IA, C_i) && (I \dashv U)\\
&= \overset{\mathsf{Set}}{\operatorname{colim}}\, \mathsf{Act}_S(A, C_i)\,, && (I \text{ is the inclusion})
\end{aligned}$$

meaning that the representable functor $\mathsf{Act}_S(A, -)\colon \mathsf{Act}_S \to \mathsf{Set}$ preserves directed colimits.

Conversely, if a A_S is a unitary act such that $\mathsf{Act}_S(A, -)\colon \mathsf{Act}_S \to \mathsf{Set}$ preserves directed colimits of arbitrary acts, it will surely preserve directed colimits of unitary acts.

(2) This can be proved similarly, using the fact that the coreflection functor $-\otimes S\colon \mathsf{Act}_S \to \mathsf{FAct}_S$ preserves arbitrary colimits, since it is left adjoint to the functor $\mathsf{Act}_S(S, -)\colon \mathsf{FAct}_S \to \mathsf{Act}_S$ via the tensor-hom adjunction (see Proposition 3.9 and the diagram on page 260 in [18]). □

Example 6.5. In the following results, we have the condition of S_S being finitely presentable in Act_S. Here are some examples of semigroups that satisfy this condition:

1. finite semigroups,
2. monoids,
3. right zero semigroups (semigroups satisfying the identity $xy = y$).

Proposition 6.6. *Let S be a firm semigroup. Then the functor $-\otimes S\colon \mathsf{Act}_S \to \mathsf{FAct}_S$ takes finitely presentable acts to finitely presentable acts if and only if S_S is finitely presentable in Act_S.*

Proof. **Necessity.** By the left-right dual of Lemma 2.3 we have that $S_S \cong S^1 \otimes S$ in Act_S. Since S^1_S is the free right S-act on one generator, it is finitely presentable in Act_S, which implies that $S_S \cong S^1 \otimes S$ is finitely presentable in FAct_S by assumption, and by Proposition 6.4 also finitely presentable in Act_S.

Sufficiency. Let A_S be a finitely presentable act in Act_S and let $((C_i)_{i\in I}, (\varphi_{ij})_{i\leqslant j})$ be an up-directed system in FAct_S. Given a biact ${}_TB_S$, notice that if the representable functor $\mathsf{Act}_S(B_S, -)\colon \mathsf{Act}_S \to \mathsf{Set}$ preserves some specific colimit, then also $\mathsf{Act}_S({}_TB_S, -)\colon \mathsf{Act}_S \to \mathsf{Act}_T$ preserves that colimit. Then

$$\begin{aligned}
\mathsf{FAct}_S(A\otimes S, \overset{\mathsf{FAct}_S}{\operatorname{colim}}\, C_i) &= \mathsf{Act}_S(A\otimes S, \overset{\mathsf{Act}_S}{\operatorname{colim}}\, C_i) && \text{(Remark 4.8)}\\
&\cong \mathsf{Act}_S(A, \mathsf{Act}_S(S, \overset{\mathsf{Act}_S}{\operatorname{colim}}\, C_i)) && \text{(tensor-hom adjunction)}\\
&\cong \mathsf{Act}_S(A, \overset{\mathsf{Act}_S}{\operatorname{colim}}\, \mathsf{Act}_S(S, C_i)) && (S_S \text{ is finitely presentable})\\
&\cong \overset{\mathsf{Set}}{\operatorname{colim}}\, \mathsf{Act}_S(A, \mathsf{Act}_S(S, C_i)) && (A_S \text{ is finitely presentable})\\
&\cong \overset{\mathsf{Set}}{\operatorname{colim}}\, \mathsf{Act}_S(A\otimes S, C_i) && \text{(tensor-hom adjunction)}\\
&= \overset{\mathsf{Set}}{\operatorname{colim}}\, \mathsf{FAct}_S(A\otimes S, C_i)\,. \quad \square
\end{aligned}$$

Corollary 6.7. *If S is a firm semigroup such that S_S is finitely presentable in Act_S, then FAct_S is a locally finitely presentable category.*

Proof. By Proposition 6.2 Act_S is locally finitely presentable, meaning any firm act A_S is a directed colimit of some system $((A_i)_{i\in I}, (\varphi_{ij})_{i\leqslant j})$ of acts finitely presentable in Act_S. By Proposition 6.6, $A_S \cong A \otimes S$ is the colimit (in FAct_S) of the directed system $((A_i \otimes S)_{i\in I}, (\varphi_{ij} \otimes 1_S)_{i\leqslant j})$ of firm acts finitely presentable in FAct_S. □

Proposition 6.8. *If S is a firm semigroup and S_S is finitely presentable in Act_S, then a morphism $f\colon A_S \to B_S$ in FAct_S is a pure epimorphism in FAct_S precisely when f is a pure epimorphism in Act_S.*

Proof. **Necessity.** By Proposition 3 of [2], a morphism in a locally finitely presentable category $\mathcal{C}$ with pullbacks is a pure epimorphism if and only if it is a directed colimit of split epimorphisms, taken in the category of arrows of $\mathcal{C}$ (see Example 1.55(1) in [1]). The category of arrows of $\mathcal{C}$ is the category of functors from the category with two objects and one nontrivial arrow between them, to the category $\mathcal{C}$. This means we can calculate colimits in the category of arrows pointwise (separately for the domain and the codomain of the arrow).

Note that both Act_S and FAct_S have pullbacks. Additionally, Act_S is locally finitely presentable by Proposition 6.2 and FAct_S is locally finitely presentable by Proposition 6.7. Since the inclusion $\mathsf{FAct}_S \to \mathsf{Act}_S$ preserves split epimorphisms and all colimits, it will preserve pure epimorphisms.

Sufficiency. Suppose $f\colon A \to B$ is a morphism of FAct_S, which is a pure epimorphism in Act_S. This means that every morphism $P \to B$ in Act_S with P finitely presentable factors through f. Since any object finitely presentable in FAct_S is also finitely presentable in Act_S by Proposition 6.4, f is also a pure epimorphism in FAct_S. □

Lemma 6.9 (Observation 11 in [2])**.** *If $\mathcal{C}$ is a locally finitely presentable category and $f\colon A \to B$ and $g\colon B \to C$ are morphisms of $\mathcal{C}$ such that gf is a pure epimorphism, then g must be a pure epimorphism.*

We can now prove that for suitable firm semigroups, the pullback flatness condition from Proposition 5.9 also works in the context of the category of firm acts, similarly to how pullback flatness extends to the context of firm acts via Theorem 4.10.

Theorem 6.10. *Let S be a firm semigroup such that S_S is finitely presentable. Then a firm act A_S is BR-flat in FAct_S if and only if it is BR-flat in Act_S.*

Proof. **Necessity.** Given a firm act A_S which is BR-flat in FAct_S, we will show that A_S is also BR-flat in Act_S. Let B_S be any right S-act along with a surjective morphism $f\colon B_S \to A_S$. Then the mapping $f \otimes 1_S\colon B \otimes S \to A \otimes S$ is also surjective and by firmness of A_S, the map $\mu_A\colon A \otimes S \to A$ is surjective. Therefore we have a surjective morphism $f\mu_B = \mu_A(f \otimes 1)\colon B \otimes S \to A$, which lies in FAct_S, because $B \otimes S$ is firm by Corollary 3.4 in [18].

$$\begin{array}{ccc} B \otimes S & \overset{f \otimes 1}{\dashrightarrow} & A \otimes S \\ {\scriptstyle \mu_B}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ B & \underset{f}{\longrightarrow} & A\,. \end{array}$$

By assumption $f\mu_B$ must be a pure epimorphism in FAct_S and by Proposition 6.8 also a pure epimorphism in Act_S. Lemma 6.9 now implies that f is a pure epimorphism in Act_S. We have shown that A_S is BR-flat in Act_S.

Sufficiency. This is clear because of Proposition 6.8 and Proposition 6.3. □

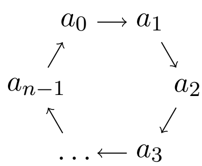
7 Flatness properties of acts over the free semigroup on one generator

As an example, let us consider the case where $S = (\mathbb{N}, +)$ is the free semigroup on one generator $1 \in S$. Then a right $\mathbb{N}$-act $A_{\mathbb{N}}$ can be pictured using a directed graph (where loops are allowed) with elements of A as vertices and an arrow $a \to a'$ existing precisely when $a1 = a'$. This graph is obtained from the semicategory $\mathsf{El}(A_{\mathbb{N}})$ by discarding all arrows s, where $s \geqslant 2$. Two acts $A_{\mathbb{N}}$ and $B_{\mathbb{N}}$ are equal or isomorphic precisely when the corresponding graphs are.

By an *infinite chain* we mean a graph of the form

$$\ldots \to a_{n-2} \to a_{n-1} \to a_n \to a_{n+1} \to a_{n+2} \to \ldots .$$

By a *cycle* we mean a graph



including the length 1 cycles (loops). In such a case $A_{\mathbb{N}}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with $\mathbb{N}$-action $(\overline{z}, n) \mapsto \overline{z+n}$.

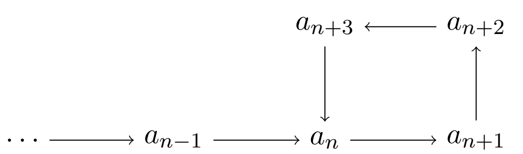
Proposition 7.1. *A nonempty act* $A_{\mathbb{N}}$

1. *is unitary if and only if every vertex of its graph has at least one incoming arrow,*
2. *satisfies Condition* (E) *if and only if its graph has no cycles,*
3. *satisfies Condition* (LC) *if and only if its graph is an infinite chain, a cycle, or a downwards infinite chain with a terminating cycle on top (as in Example 7.3(1)).*

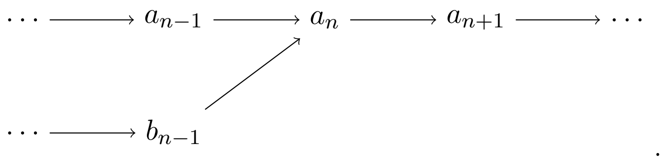
Proposition 7.2. *For a nonempty act* $A_{\mathbb{N}}$ *the following conditions are equivalent:*

1. $A_{\mathbb{N}}$ *satisfies Condition* (P)*,*
2. $A_{\mathbb{N}}$ *is firm,*
3. *every vertex of the graph of* $A_{\mathbb{N}}$ *has precisely one incoming arrow,*
4. $A_{\mathbb{N}}$ *is a disjoint union of subacts whose graph is either an infinite chain or a cycle.*

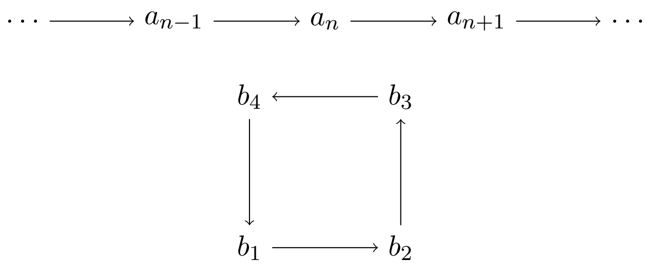
Example 7.3. 1. An act satisfying (LC), but neither (E) or (P):



2. An act satisfying Condition (E), but neither (P) or (LC):

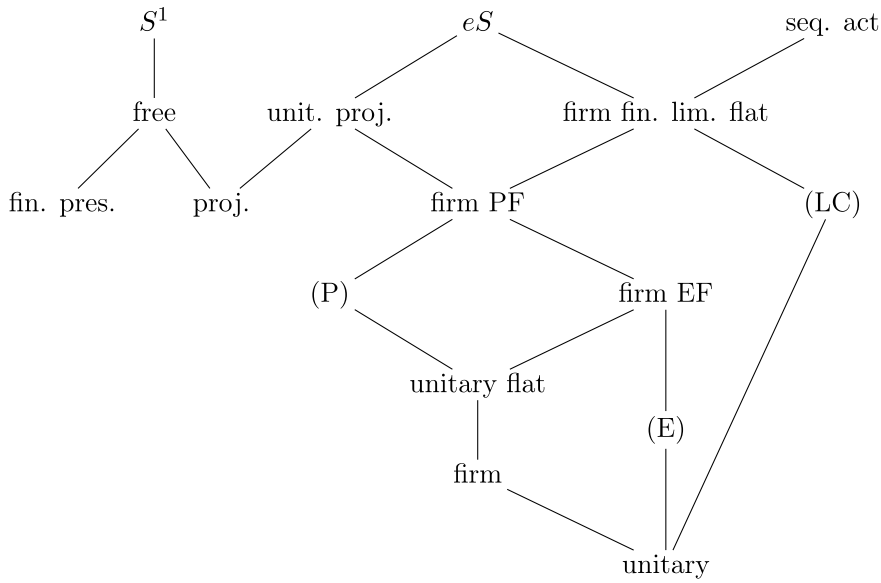


3. An act satisfying Condition (P), but neither (E) or (LC):



8 Conclusion

We end with a diagram that shows the relationships between properties of nonempty acts over a nonempty semigroup S. The lines represent inclusions between classes of acts, bigger classes being lower. We point out that, by [9, Corollary 4.4], acts of the form eS where $e^2 = e \in S$ can be described up to isomorphism as unitary S-acts A_S such that the hom-functor $\mathsf{Act}_S(A, -) : \mathsf{Act}_S \to \mathsf{Set}$ preserves Rees exact sequences. We also mention that in some sense the simplest act S_S need not have any of the properties shown here, in sharp contrast with the monoid case.



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