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Pre-image of functions in $C(L)$

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Abstract. Let $C(L)$ be the ring of all continuous real functions on a frame L and $S \subseteq \mathbb{R}$. An $\alpha \in C(L)$ is said to be an overlap of S, denoted by $\alpha \blacktriangleleft S$, whenever $u \cap S \subseteq v \cap S$ implies $\alpha(u) \leqslant \alpha(v)$ for every open sets u and v in $\mathbb{R}$. This concept was first introduced by A. Karimi-Feizabadi, A.A. Estaji, M. Robat-Sarpoushi in *Pointfree version of image of real-valued continuous functions* (2018). Although this concept is a suitable model for their purpose, it ultimately does not provide a clear definition of the range of continuous functions in the context of pointfree topology. In this paper, we will introduce a concept which is called pre-image, denoted by pim, as a pointfree version of the image of real-valued continuous functions on a topological space X. We investigate this concept and in addition to showing $\mathrm{pim}(\alpha) = \bigcap\{S \subseteq \mathbb{R} : \alpha \blacktriangleleft S\}$, we will see that this concept is a good surrogate for the image of continuous real functions. For instance, we prove, under some achievable conditions, we have $\mathrm{pim}(\alpha \vee \beta) \subseteq \mathrm{pim}(\alpha) \vee \mathrm{pim}(\beta)$, $\mathrm{pim}(\alpha \wedge \beta) \subseteq \mathrm{pim}(\alpha) \wedge \mathrm{pim}(\beta)$, $\mathrm{pim}(\alpha\beta) \subseteq \mathrm{pim}(\alpha)\mathrm{pim}(\beta)$ and $\mathrm{pim}(\alpha + \beta) \subseteq \mathrm{pim}(\alpha) + \mathrm{pim}(\beta)$.

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1 Introduction and preliminaries

A complete lattice L is said to be a frame if for any $a \in L$ and $B \subseteq L$, we have $a \wedge \bigvee B = \bigvee_{b \in B}(a \wedge b)$. We denote the top element and the bottom element of a frame L by **Top** and $\perp$, respectively. For every element a of a frame L the pseudocomplement of a is $a^* = \bigvee\{x \in L :\ x \wedge a = \perp\}$. Let L be a frame. The set of all prime ideals (respectively, maximal ideals) of L is denoted by Spec(L) (respectively, (Max(L)). An element $p \in L$ is called prime if $p <$ **Top**, and $a \wedge b \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$. Clearly, $a \in L$ is a prime element if and only if $\downarrow a = \{x \in L :\ x \leqslant a\}$ is a prime ideal of L. We denote by SpL the set of all prime element of L. For every $a \in L$, define $\mathfrak{h}^c(a) = \{p \in \mathrm{Sp}L :\ a \nleqslant p\}$. It is easily seen that $\{\mathfrak{h}^c(a) :\ a \in L\}$ is a topology on SpL. Here after we use SpL equipped with this topology.

Let X and Y be two partial ordered sets and $f : X \to Y$ and $g : Y \to X$ be two increasing maps. We say f is left adjoint of g (respectively, g is right adjoint of f) if $fg \leqslant I_Y$ and $gf \geqslant I_X$. It is easy to see that g is uniquely determined by f and vice versa. The right adjoint of a map $f : X \to Y$ (respectively, left adjoint of a map $g : Y \to X$), if there exists, is denoted by f_* (resp., g^*). Supposing X and Y are complete lattices, one can easily see that $f : X \to Y$ is a left adjoint map if and only if f preserves arbitrary joins and in this case $f_*(y) = \bigvee\{x \in X :\ f(x) \leqslant y\}$ for every $y \in Y$. A frame homomorphism is a map f from a frame L to a frame L' such that it preserves finite meets and arbitrary joins; clearly in this case we have $f(\perp) = \perp$ and $f(\mathbf{Top}) = \mathbf{Top}$. Obviously, every frame homomorphism is a left adjoint map. We denote by $\mathcal{O}X$ and $\mathcal{O}_x$ the frames of all open subsets of a topological space X and the set of all open neighborhoods of $x \in X$, respectively. If X and Y are two topological spaces, then for every continuous function $f : X \to Y$ we define $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ with $(\mathcal{O}f)(w) = f^{-1}(w)$ for every $w \in \mathcal{O}Y$. It is obvious that $\mathcal{O}$ is a contravariant functor from the category **Top** to the category **Frm**. Let L and L' be two frames. For every frame homomorphism $f : L \to L'$ we can define $\mathrm{Sp}f : \mathrm{Sp}L' \to \mathrm{Sp}L$ with $(\mathrm{Sp}f)(q) = f_*(q)$. For any $a \in L$, we can write

$$\begin{aligned}(\mathrm{Sp}f)^{-1}(\mathfrak{h}^c(a)) &= \{q \in \mathrm{Sp}L' :\ f_*(q) \in \mathfrak{h}^c(a)\} \\ &= \{q \in \mathrm{Sp}L' :\ a \nleqslant f_*(q)\} \\ &= \{q \in \mathrm{Sp}L' :\ f(a) \nleqslant q\} = \mathfrak{h}^c(f(a)).\end{aligned}$$

Therefore, $\mathrm{Sp}f$ is a continuous map. It is easy to see that $\mathrm{Sp}I_L = I_{\mathrm{Sp}L}$ and $\mathrm{Sp}fg = \mathrm{Sp}g\mathrm{Sp}f$ whenever fg means the composition of f and g. Thus, $\mathrm{Sp} : \mathbf{Frm} \to \mathbf{Top}$ is a contravariant functor. In fact the functor Sp is a right adjoint of the functor $\mathcal{O}$.

Recall that an ordered ring is a ring A with a partial order $\leqslant$ such that for every $a, b, c \in A$, from $a \leqslant b$ it follows that $a + c \leqslant b + c$ and if $a, b \geqslant 0$, then $ab \geqslant 0$. An ordered ring is called a lattice-ordered ring if A is a lattice under the partial order on A. By an f-ring we mean a lattice-ordered ring R with this property that $a(b \wedge c) = ab \wedge ac$ and $(b \wedge c)a = ba \wedge ca$ for every $a \in R^+$ and every $b, c \in R$. An algebra (over a field F) is a structure consisting of a set A with two operations "+" and ".", and also a scaler multiplication such that $(A, +, .)$ is a ring and A with addition and scaler multiplication is a vector space (over F), and in addition, for every $x, y \in A$ and every $c \in F$, we have

$$1_F x = x \quad , \quad c(xy) = (cx)y = x(cy).$$

Finally, an f-algebra (over an ordered field) is an algebra with a partial order $\leqslant$ such that $(A, +, ., \leqslant)$ is an f-ring, and A with "+" and the scaler multiplication is a vector space (over F) in which $cx \geqslant 0$ for every $c \in F^+$ and every $x \in A^+$.

Suppose that A is a lattice-ordered ring and $a \in A$. The positive part of a, negative part of a, and $|a|$ are defined as $a^+ = a \vee 0$, $a^- = -a \vee 0$ and $|a| = a \vee -a$, respectively. Clearly, if A is an f-ring, then $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+a^- = 0$ and $|a|^2 = a^2$ for any $a \in A$.

In the present part of this paper, for convenience of readers, we give a short review of $C(L)$, at a slightly different perspective from what is stated in the main texts.

A frame homomorphism $\alpha : \mathcal{O}\mathbb{R} \to L$ is called continuous real function on a frame L and the set of all continuous real function on a frame L is denoted by $C(L)$. Although, this concept was first introduced by R.N. Ball and A.W. Hager in [1], B. Banaschewski studied this concept deeply in [2]; he also showed in [3] that $C(L)$ is a class which strictly contains $C(X)$. Note that we work under the axiomatic system of ZFC and in this system, we have $L(\mathbb{R}) \simeq \mathcal{O}\mathbb{R}$. In this axiomatic system $C(L)$ has a simpler representation.

Supposing that $A, S \subseteq L$, we denote by $\downarrow A$ the set $\{x \in L : \exists a \in$

$A,\ x \leqslant a\}$; we use $\downarrow x$ instead of $\downarrow\{x\}$ and $\downarrow_S A$ instead of $S \cap \downarrow A$. Clearly, for any $S \subseteq L$, the map $\downarrow_S$: $L \to P(S)$ is a meet-homomorphism but not a join-homomorphism, see [15]. A subset B of L is said to be a base for L if $x = \bigvee \downarrow_B x$ for every $x \in L$. Let L and L' be two frames and B be a base for L. A map $f : B \to L'$ is said to be conditional homomorphism if for every $A \subseteq B$ and every finite $F \subseteq B$ we have $f(\bigvee A) = \bigvee f(A)$ and $f(\bigwedge F) = \bigwedge f(F)$, provided that $\bigvee A \in B$ and $\bigwedge F \in B$. Supposing that B is a base for a frame L, we call B a homomorphism maker if every conditional homomorphism from B to a frame L' has an extension homomorphism from L to L'.

Proposition 1.1. *Let B be a base for L closed under finite meets. Then B is homomorphism maker.*

Proof. Let $f : B \to L'$ be a conditional homomorphism. We define $\bar{f} : L \to L'$ with $\bar{f}(x) = \bigvee f(\downarrow_B (x))$ and prove that $\bar{f}$ is a homomorphism extension of f. Clearly, $\bar{f}$ is order preserving, $\bar{f}|_B = f$, $f(\bot) = \bot$ and $f(\mathbf{Top}) = \mathbf{Top}$. Assuming that $x_\lambda \in L$ for every $\lambda \in \Lambda$, since $\bar{f}$ is order preserving, we have $\bigvee_{\lambda\in\Lambda} \bar{f}(x_\lambda) \leqslant \bar{f}(\bigvee_{\lambda\in\Lambda} x_\lambda)$. Conversely, for every $b \in \downarrow_B (\bigvee_{\lambda\in\Lambda} x_\lambda)$,

$$b = \bigvee_{\lambda\in\Lambda} b \wedge x_\lambda = \bigvee_{\lambda\in\Lambda} \bigvee \{c \in B :\ c \leqslant b \wedge x_\lambda\},$$

which implies that

$$\begin{aligned} f(b) &= \bigvee_{\lambda\in\Lambda} \bigvee \{f(c) :\ c \in B,\ c \leqslant b \wedge x_\lambda\} \\ &\leqslant \bigvee_{\lambda\in\Lambda} \bigvee \{f(c) :\ c \in B,\ c \leqslant x_\lambda\} \\ &= \bigvee_{\lambda\in\Lambda} \bar{f}(x_\lambda), \end{aligned}$$

and this shows that

$$\bar{f}(\bigvee_{\lambda\in\Lambda} x_\lambda) = \bigvee \{f(b) :\ b \in \downarrow \bigvee_{\lambda\in\Lambda} x_\lambda\} \leqslant \bigvee_{\lambda\in\Lambda} \bar{f}(x_\lambda).$$

Therefore, $\bar{f}(\bigvee_{\lambda\in\Lambda} x_\lambda) = \bigvee_{\lambda\in\Lambda} \bar{f}(x_\lambda)$. Now, supposing that $x, y \in L$, clearly

$$\begin{aligned}\bar{f}(x\wedge y) &= \bigvee\{f(c):\ c\in B,\ c\leqslant x\wedge y\}\\ &= \bigvee\{f(c_1\wedge c_2):\ c_1,c_2\in B,\ c_1\leqslant x,\ c_2\leqslant y\}\\ &= \bigvee\{f(c_1)\wedge f(c_2):\ c_1,c_2\in B,\ c_1\leqslant x,\ c_2\leqslant y\}\\ &= \bigvee\{f(c_1):\ c_1\in B,\ c_1\leqslant x\}\wedge\bigvee\{f(c_2):\ c_2\in B,\ c_2\leqslant y\}\\ &= \bar{f}(x)\wedge\bar{f}(y).\end{aligned}$$

□

In the above proposition, the condition “closedness under finite meets" cannot be omitted. For example, suppose that $B = \{(a, b) :\ a, b \in \mathbb{Q},\ a < b\}$ and $f : B \to L$ with $f(a, b) = \mathbf{Top}$ for every $(a, b) \in B$. Obviously, B is a base for $\mathcal{O}\mathbb{R}$, f is conditional homomorphism and B is not homomorphism maker.

Corollary 1.2. *Let $B = \{(r, s) :\ r, s \in \mathbb{Q}\} \cup \{\mathbb{R}\}$. Clearly, B is a base for $\mathcal{O}\mathbb{R}$ and closed under finite meets. Hence, B is a homomorphism maker. In other words, a map $f : B \to L$ has an extension homomorphism $\alpha \in C(L)$ if and only if f has the following properties.*

(R1) $f((p, q) \wedge (r, s)) = f(p, q) \wedge f(r, s)$, *whenever* $p, q, r, s \in \mathbb{Q}$ *and* $(p, q) \wedge (r, s) \neq \emptyset$.

(R2) $f((p, q) \vee (r, s)) = f(p, q) \vee f(r, s)$, *whenever* $p, q, r, s \in \mathbb{Q}$ *and* $p \leqslant r < q \leqslant s$.

(R3) $f(p, q) = \bigvee\{f(r, s) :\ r, s \in \mathbb{Q},\ p < r < s < q\}$ *for every* $p, q \in \mathbb{Q}$.

(R4) $\mathbf{Top} = f(\mathbb{Q}) = \bigvee\{f(p, q) :\ p, q \in \mathbb{Q}\}$.

Suppose that $\diamond$ is an operation such as “+", “.", “$\vee$" and “$\wedge$". For every $\alpha, \beta \in C(L)$ and every $p, q \in \mathbb{Q}$, we define

$$(\alpha\diamond\beta)(p,q) = \bigvee\{\alpha(r,s)\wedge\beta(t,u):\ r,s,t,u\in\mathbb{Q},\ (r,s)\diamond(t,u)\subseteq(p,q)\},$$

where $(r, s) \diamond (t, u) = \{a \diamond b :\ a \in (r, s),\ b \in (t, u)\}$. It can be proved that $\alpha \diamond \beta$ is a conditional homomorphism on $B = \{(r, s) :\ r, s \in \mathbb{Q}\} \cup \{\mathbb{R}\}$ and hence $\alpha \diamond \beta \in C(L)$, see [2] and [14]. Also, for every $r \in \mathbb{R}$ it is defined that

$\mathbf{r}(w) = \mathbf{Top}$ if $r \in w$ and $\mathbf{r}(w) = \perp$ if $r \notin w$. It is clear to see that $\mathbf{r} \in C(L)$. Now, $r.\alpha$ is defined by $\mathbf{r}\alpha$. Consequently $C(L)$ is an f-algebra with these operations.

Proposition 1.3. *For every $\alpha, \beta \in C(L)$ and every $w \in \mathcal{O}\mathbb{R}$, we have*

$$\begin{aligned}(\alpha \diamond \beta)(w) &= \bigvee \left\{\alpha(r,s) \wedge \beta(t,u) : \ r,s,t,u \in \mathbb{Q}, \ (r,s) \diamond (t,u) \subseteq w\right\} \\ &= \bigvee \left\{\alpha(w_1) \wedge \beta(w_2) : \ w_1, w_2 \in \mathcal{O}\mathbb{R}, \ w_1 \diamond w_2 \subseteq w\right\},\end{aligned}$$

where $w_1 \diamond w_2 = \{a \diamond b : \ a \in w_1, \ b \in w_2\}$.

Proof. Assume that

$$A_{a,b} = \{\alpha(r,s) \wedge \beta(t,u) : \ r,s,t,u \in \mathbb{Q}, \ (r,s) \diamond (t,u) \subseteq (a,b)\} \ ,$$

$$A_w = \{\alpha(r,s) \wedge \beta(t,u) : \ r,s,t,u \in \mathbb{Q}, \ (r,s) \diamond (t,u) \subseteq w\}$$

and

$$B_w = \{\alpha(w_1) \wedge \beta(w_2) : \ w_1, w_2 \in \mathcal{O}\mathbb{R}, \quad w_1 \diamond w_2 \subseteq w\}.$$

Since $(\alpha \diamond \beta) \in C(L)$, it follows that

$$\begin{aligned}(\alpha \diamond \beta)(w) &= (\alpha \diamond \beta)\left(\bigcup \left\{(a,b) \in \mathcal{O}\mathbb{R} : \ a,b \in \mathbb{Q}, \ (a,b) \subseteq w\right\}\right) \\ &= \bigvee \left\{(\alpha \diamond \beta)(a,b) : \ a,b \in \mathbb{Q}, \ (a,b) \subseteq w\right\} \\ &= \bigvee \left\{\bigvee A_{a,b} : \ (a,b) \subseteq w\right\}.\end{aligned}$$

Therefore, clearly, $(\alpha \diamond \beta)(w) \leqslant \bigvee A_w \leqslant \bigvee B_w$. Now, suppose that $\alpha(r,s) \wedge \beta(t,u) \in A_w$. Obviously, there exist $a, b \in \mathbb{Q}$ such that $(r,s) \diamond (t,u) \subseteq (a,b) \subseteq w$. Hence, $\alpha(r,s) \wedge \beta(t,u) \in A_{a,b}$ and consequently $\bigvee A_w \leqslant \bigvee A_{a,b} \leqslant (\alpha \diamond \beta)(w)$ and so $\bigvee A_w = (\alpha \diamond \beta)(w)$. Finally, assume that $\alpha(w_1) \wedge \beta(w_2) \in B_w$, where $w_1 \diamond w_2 \subseteq w$. Clearly, $w_1 = \bigcup_{i \in I}(r_i, s_i)$ and $w_2 = \bigcup_{j \in J}(t_j, u_j)$, where $r_i, s_i, t_j, u_j \in \mathbb{Q}$ for every $i \in I$ and every $j \in J$. Thus,

$$\bigcup_{i \in I} \bigcup_{j \in J} (r_i, s_i) \diamond (t_j, u_j) = \bigcup_{i \in I} (r_i, s_i) \diamond \bigcup_{j \in J} (t_j, u_j) = w_1 \diamond w_2 \subseteq w$$

and so $(r_i, s_i) \diamond (t_j, u_j) \subseteq w$ for every $i \in I$ and every $j \in J$. Therefore, it is easy to see that $\alpha(w_1) \wedge \beta(w_2) = \bigvee_{i \in I} \bigvee_{j \in J} \alpha(r_i, s_i) \wedge \beta(t_j, u_j) \leqslant \bigvee A_w$. Hence, $\bigvee B_w \leqslant \bigvee A_w$ and so $\bigvee B_w = \bigvee A_w$

□

Throughout the paper, the notations L and $C(L)$ stand for a frame and the f-algebra of all continuous real functions on the frame L, respectively. The reader is referred to [2], [14], and [12], for more information about frames and $C(L)$. Also, see [4], [5], [11], [15], and [10] for more information about general lattice theory and rings of continuous functions, respectively.

We need the following proposition which can be found in the literature.

Proposition 1.4. *Let $\alpha, \beta \in C(L)$ and $a \in \mathbb{R}$. The following statements hold.*

(a) *If $\alpha \geqslant 0$, then $\alpha(-\infty, x) = \perp$ for every $x \leqslant 0$.*

(b) If $\alpha \geqslant 0$, then $\alpha(x, +\infty) = \mathbf{Top}$ for every $x < 0$.

(c) $(\alpha \vee \beta)(x, +\infty) = \alpha(x, +\infty) \vee \beta(x, +\infty)$ *for every $x \in \mathbb{R}$.*

(d) $(\alpha \vee \beta)(-\infty, x) = \alpha(-\infty, x) \wedge \beta(-\infty, x)$ *for every $x \in \mathbb{R}$.*

(e) $(\alpha \wedge \beta)(x, +\infty) = \alpha(x, +\infty) \wedge \beta(x, +\infty)$ *for every $x \in \mathbb{R}$.*

(f) $(\alpha \wedge \beta)(-\infty, x) = \alpha(-\infty, x) \vee \beta(-\infty, x)$ *for every $x \in \mathbb{R}$.*

(g) $(\mathbf{c}\alpha)(w) = \alpha(\frac{1}{c}w)$ *for every $w \in \mathcal{O}\mathbb{R}$ and each $c \neq 0$, where $bw = \{bx : \ x \in w\}$.*

(h) $(\mathbf{c} + \alpha)(w) = \alpha(w - c)$ *for each $w \in \mathcal{O}\mathbb{R}$ and each $c \in \mathbb{R}$, where $w + b = \{x + b : \ x \in w\}$.*

2 Pre-image of a continuous real function on L

In [13], although it does not introduce a determined definition for pointfree version of the “image" of continuous real functions, using a concept, called “overlap", an attempt has been made to fill the vacuum of the concept of image of continuous real functions in pointfree topology. In this main section, we give a determined version of the image of continuous real functions on a topological space X in the pointfree topology and we show that this is independent of what we see in [13].

Definition 2.1. For every $\alpha \in C(L)$, we define $\mathrm{pim}(\alpha)$, called pre-image of α, as

$$\mathrm{pim}(\alpha) = \bigcap \{w \in O\mathbb{R} : \ \alpha(w) = \mathbf{Top}\}.$$

At below we provide an example in which we demonstrate that $\mathrm{pim}(\alpha)$ is an appropriate model of image of the real-valued functions in pointfree topology.

Example 2.2. Let $C(X)$ be the ring of real-valued continuous functions on a topological space X. We know that for all $f \in C(X)$ we have $\mathcal{O}f \in C(\mathcal{O}X)$ and clearly, we can write

$$Im(f) = f(X) = \bigcap_{f(X)\subseteq w} w = \bigcap\{w \in \mathcal{O}\mathbb{R} : f^{-1}(w) = X\}$$

$$= \bigcap\{w \in \mathcal{O}\mathbb{R} : \mathcal{O}f(w) = \mathbf{Top}\}.$$

Therefore, $Im(f) = \mathrm{pim}(\mathcal{O}f)$.

Hereinafter, by $\mathbb{R}_x$, we mean $\mathbb{R} \setminus \{x\}$.

Proposition 2.3. *For every $\alpha \in C(L)$, the following statements hold:*

(a) $\mathrm{pim}(\alpha) = \bigcap\{\mathbb{R}_x : \ \alpha(\mathbb{R}_x) = \mathbf{Top}\}$.

(b) $x \notin \mathrm{pim}(\alpha)$ *if and only if* $\alpha(\mathbb{R}_x) = \mathbf{Top}$.

Proof. (a): Suppose that $\mathcal{B} = \{\mathbb{R}_x : \ \alpha(\mathbb{R}_x) = \mathbf{Top}\}$. Obviously $\mathrm{pim}(\alpha) \subseteq \bigcap \mathcal{B}$. Now, assuming $x \notin \mathrm{pim}(\alpha)$, there exists $w \in \mathcal{O}\mathbb{R}$ such that $x \notin w$ and $\alpha(w) = \mathbf{Top}$. Hence, $w \subseteq \mathbb{R}_x$, consequently $\alpha(\mathbb{R}_x) = \mathbf{Top}$ and so $x \notin \mathbb{R}_x \in \mathcal{B}$. Therefore, $\bigcap \mathcal{B} \subseteq \mathrm{pim}(\alpha)$ and subsequently $\mathrm{pim}(\alpha) = \bigcap \mathcal{B}$.

(b): According to (a), it is obvious that we can write

$$x \notin \mathrm{pim}(\alpha) \quad \Rightarrow \quad \exists \mathbb{R}_y, \quad \alpha(\mathbb{R}_y) = \mathbf{Top}, \quad x \notin \mathbb{R}_y.$$

Since $x \notin \mathbb{R}_y$, $x = y$ and consequently $\alpha(\mathbb{R}_x) = \mathbf{Top}$. Conversely, assume that $\alpha(\mathbb{R}_x) = \mathbf{Top}$. Thus, $\mathrm{pim}(\alpha) \subseteq \mathbb{R}_x$ and so $x \notin \mathrm{pim}(\alpha)$. □

Estaji and at al. in [8], put

$$R_\alpha = \{r \in \mathbb{R} : \ coz(\alpha - r) \neq \mathbf{Top}\}$$

for every $\alpha \in C(L)$, and they studied some of its properties. By Proposition 2.3, it is evident that $R_\alpha = \mathrm{pim}(\alpha)$.

Recall that $w^* = \mathbb{R} \setminus \overline{w}$ and $\overline{w} = \bigcap_{x \in w^*} \mathbb{R}_x$ for every $w \in \mathcal{O}\mathbb{R}$.

Proposition 2.4. *For every $w \in \mathcal{O}\mathbb{R}$ and every $\alpha \in C(L)$, the following statements hold:*

(a) *If* $\alpha(w^*) = \perp$, *then* $\alpha(\mathbb{R}_x) = \mathbf{Top}$ *for all* $x \in w^*$.

(b) *If* $\alpha(w^*) = \perp$, *then* $\overline{\mathrm{pim}(\alpha)} \subseteq \overline{w}$.

(c) *If* $r \in \overline{\mathrm{pim}(\alpha)}$ *and* $w \in \mathcal{O}_r$, *then* $\alpha(w) \neq \perp$.

Proof. (a): Suppose that $w \in \mathcal{O}\mathbb{R}$ and $\alpha \in C(L)$. Then for every $x \in w^*$, we can write

$$\mathbb{R}_x \cup w^* = \mathbb{R} \quad \Rightarrow \quad \alpha(\mathbb{R}_x) = \alpha(\mathbb{R}_x) \vee \alpha(w^*) = \alpha(\mathbb{R}_x \cup w^*) = \alpha(\mathbb{R}) = \mathbf{Top}.$$

(b): Since $\alpha(w^*) = \perp$, by part (a), for all $x \in w^*$, we have $\alpha(\mathbb{R}_x) = \mathbf{Top}$ and so

$$\operatorname{pim}(\alpha) = \bigcap \{\mathbb{R}_x : \ \alpha(\mathbb{R}_x) = \mathbf{Top}\} \subseteq \bigcap_{x \in w^*} \mathbb{R}_x = \overline{w}.$$

(c): Suppose that $r \in \overline{\operatorname{pim}(\alpha)}$ and $w \in \mathcal{O}_r$. Thus, there exists $y \in w \cap \operatorname{pim}(\alpha)$ and therefore

$$\mathbf{Top} = \alpha(\mathbb{R}) = \alpha(\mathbb{R}_y \vee w) = \alpha(\mathbb{R}_y) \vee \alpha(w).$$

On the other hand, since $y \in \operatorname{pim}(\alpha)$, $\alpha(\mathbb{R}_y) \neq \mathbf{Top}$ and so $\alpha(w) \neq \perp$. □

By Example 2.2, it is easy to see that if $\operatorname{pim}(\mathcal{O}f) \subseteq w \in \mathcal{O}\mathbb{R}$, then $\mathcal{O}f(w) = \mathbf{Top}$. Also, if $\mathcal{O}f(w) \neq \perp$, for every $w \in \mathcal{O}_r$, $r \in \overline{\operatorname{pim}(\alpha)}$. So here are two natural question.

Question 1: Suppose that $\alpha \in C(L)$ and $w \in \mathcal{O}\mathbb{R}$. Can we imply $\alpha(w) = \mathbf{Top}$ from $\operatorname{pim}(\alpha) \subseteq w$?

Question 2: Suppose that $\alpha(w) \neq \perp$, for every $w \in \mathcal{O}_r$. Can we conclude that $r \in \overline{\operatorname{pim}(\alpha)}$?

Example 2.8 shows that the answer to these two questions is generally negative (in the first question, even if w is an unbounded interval in $\mathbb{R}$). But, in the following proposition, we will find that the answer to the first question is positive under some conditions.

Proposition 2.5. *Let $\alpha \in C(L)$, $w \in \mathcal{O}\mathbb{R}$ and $\operatorname{pim}(\alpha) \subseteq w$, then the following statements hold:*

(a) *If w is dense in $\mathbb{R}$ and the boundary of w is finite, then $\alpha(w) = \mathbf{Top}$.*

(b) *Let $\mathcal{U} \subseteq \mathcal{O}\mathbb{R}$ be such that one of these families is bounded, $\operatorname{pim}(\alpha) \subseteq \bigcap \mathcal{U}$ and $\alpha(u) = \mathbf{Top}$ for every $u \in \mathcal{U}$. If $\bigcap_{u \in \mathcal{U}} \overline{u} \subseteq w$, then it follows that $\alpha(w) = \mathbf{Top}$.*

Proof. (a): It is clear.

(b): Without loss of generality, we can suppose that $\overline{u}$ is compact for all $u \in \mathcal{U}$. Now, it is easy to see that there exist $u_1, \cdots, u_n \in \mathcal{U}$ such that $\bigcap_{i=1}^{n} \overline{u_i} \subseteq w$. Therefore,

$$\mathbf{Top} = \bigwedge_{i=1}^{n} \alpha(u_i) = \alpha\Big(\bigcap_{i=1}^{n} u_i\Big) \leqslant \alpha(w) \quad \Rightarrow \quad \alpha(w) = \mathbf{Top}.$$

□

Suppose that $\alpha \in C(L)$ and $S \subseteq \mathbb{R}$. We recall from [13] that α is an overlap of S, denoted by $\alpha \blacktriangleleft S$, whenever $i(u) \subseteq i(v)$ implies $\alpha(u) \leqslant \alpha(v)$; that is, $u \cap S \subseteq v \cap S$ implies $\alpha(u) \leqslant \alpha(v)$. In the following propositions and example, we will see that although this concept and $\operatorname{pim}(\alpha)$ are closely related, but they are different from each other.

Proposition 2.6. *Suppose that* $\alpha \in C(L)$ *and* $OV(\alpha) = \{S \subseteq \mathbb{R} : \ \alpha \blacktriangleleft S\}$. *Then* $\operatorname{pim}(\alpha) = \bigcap_{S \in OV(\alpha)} S$.

Proof. Let $S \in OV(\alpha)$ and $x \notin S$. Thus, $\mathbb{R}_x \cap S = \mathbb{R} \cap S$ and so $\mathbf{Top} = \alpha(\mathbb{R}) = \alpha(\mathbb{R}_x)$; that , $x \notin \operatorname{pim}(\alpha)$. Therefore, $\operatorname{pim}(\alpha) \subseteq \bigcap_{S \in OV(\alpha)} S$. Conversely, suppose $x \notin \operatorname{pim}(\alpha)$; it suffices to show that $\mathbb{R}_x \in OV(\alpha)$. To see this, for every $u, v \in \mathcal{O}\mathbb{R}$, we can write

$$u \cap \mathbb{R}_x \subseteq v \cap \mathbb{R}_x \quad \Rightarrow \quad \alpha(u) = \alpha(u) \wedge \mathbf{Top} = \alpha(u) \wedge \alpha(\mathbb{R}_x)$$

$$= \alpha(u \cap \mathbb{R}_x) \leqslant \alpha(v \cap \mathbb{R}) = \alpha(v).$$

□

Proposition 2.7. *Suppose that* $\alpha \in C(L)$, $w \in \mathcal{O}\mathbb{R}$ *and* $\alpha(w) = \mathbf{Top}$, *then* $\alpha \blacktriangleleft w$.

Proof. Let $u, v \in \mathcal{O}\mathbb{R}$ and $u \cap w \subseteq v \cap w$. Hence

$$\alpha(u) = \alpha(u) \wedge \mathbf{Top} = \alpha(u) \wedge \alpha(w)$$

$$= \alpha(u \cap w) \leqslant \alpha(v \cap w) = \alpha(v) \wedge \alpha(w) = \alpha(v) \wedge \mathbf{Top} = \alpha(v).$$

□

In this way, it turns out that the following equality is in place, too.

$$\mathrm{pim}(\alpha)=\bigcap\{w\in\mathcal{O}\mathbb{R}:\ \alpha\blacktriangleleft w\}.$$

Example 2.8. There is a frame L and $\beta\in C(L)$ such that $\beta\not\blacktriangleleft\mathrm{pim}(\beta)$. To see this, let L, β and the family $\{S_c\}_{c\in\mathcal{I}}$ be same as in [13, Example 3.18]. Then, $\mathrm{pim}(\beta)\subseteq\bigcap_{c\in\mathcal{I}}S_c=\emptyset$. Thus, $\beta\blacktriangleleft\mathrm{pim}(\beta)$ does not hold. Furthermore, since $\beta(\emptyset)=\perp$, there exists $w\in\mathcal{O}\mathbb{R}$ such that $\beta(w)\neq\mathbf{Top}$. Clearly, $\mathrm{pim}(\beta)=\emptyset\subseteq w$ whereas $\beta(w)\neq\mathbf{Top}$. Thus, the answer to Question 1 is negative. Also, since $\beta(\mathbf{Top})=\mathbf{Top}$, there exists an element $r\in\mathbb{R}$ such that for every $w\in\mathcal{O}_r$ we have $\beta(w)\neq\perp$, whereas $r\notin\emptyset=\overline{\mathrm{pim}(\beta)}$. Therefore, the answer to Question 2 is also negative.

Now, we want to find the relationship between $\mathrm{pim}(|\alpha|)$ and $\mathrm{pim}(\alpha)$.

Lemma 2.9. *For every $\alpha\in C(L)$ and every $x\in\mathbb{R}$, we have*

$$|\alpha|(\mathbb{R}_x)=\Big(\alpha(x,+\infty)\vee\alpha(-\infty,|x|)\Big)\wedge\Big(\alpha(-|x|,+\infty)\vee\alpha(-\infty,-x)\Big).$$

Proof. By Proposition 1.4, the proof is straightforward. □

The following corollary is followed from the above lemma immediately.

Corollary 2.10. *Assume that $\alpha\in C(L)$ and $x\in\mathbb{R}$. Then the following statements hold:*

(a) *If $x<0$, then $|\alpha|(\mathbb{R}_x)=\mathbf{Top}$.*
(b) *If $x\geqslant 0$, then $|\alpha|(\mathbb{R}_x)=\alpha(\mathbb{R}_x)\wedge\alpha(\mathbb{R}_-x)$.*
(c) $\mathrm{pim}(|\alpha|)\subseteq\mathbb{R}^+$.

Proposition 2.11. $\mathrm{pim}(|\alpha|)=\{|x|:\ x\in\mathrm{pim}(\alpha)\}$ *for every $\alpha\in C(L)$.*

Proof. Supposing $A=\{|x|:\ x\in\mathrm{pim}(\alpha)\}$, clearly, $A=\{x\in\mathbb{R}^+:\ x\in\mathrm{pim}(\alpha)\text{ or }-\mathrm{x}\in\mathrm{pim}(\alpha)\}$. Accordingly to Lemma 2.9, for every $x\geqslant 0$, we can write

$$x\notin A\ \Leftrightarrow\ x,-x\notin\mathrm{pim}(\alpha)\ \Leftrightarrow\ \alpha(\mathbb{R}_x)=\alpha(\mathbb{R}_{-x})=\mathbf{Top}\ \Leftrightarrow\ |\alpha|(\mathbb{R}_x)$$
$$=\mathbf{Top}\ \Leftrightarrow\ x\notin\mathrm{pim}(|\alpha|).$$

□

Proposition 2.12. *The following relations are true for each $\alpha \in C(L)$ and each $r \in \mathbb{R}$:*

(a) $\operatorname{pim}(\mathbf{r}) = \{r\}$.

(b) $\operatorname{pim}(\mathbf{r}\alpha) = r \operatorname{pim}(\alpha)$.

(c) $\operatorname{pim}(\mathbf{r} + \alpha) = r + \operatorname{pim}(\alpha)$.

Proof. (a): Clearly, for every $r \in \mathbb{R}$, we can write

$$\mathbf{r}(\mathbb{R}_x) = \mathbf{Top} \quad \Leftrightarrow \quad x \neq r. \qquad \therefore \quad \operatorname{pim}(\mathbf{r}) = \bigcap_{x \neq r} \mathbb{R}_x = \{r\}.$$

(b): For every $r \in \mathbb{R}$, we can write (without loss of generality, assume that $r \neq 0$)

$$\begin{aligned} \operatorname{pim}(\mathbf{r}\alpha) \subseteq \mathbb{R}_x \quad &\Leftrightarrow \quad (\mathbf{r}\alpha)(\mathbb{R}_x) = \mathbf{Top} \quad \Leftrightarrow \quad \alpha(\frac{1}{r}\,\mathbb{R}_x) = \alpha(\mathbb{R}_{\frac{x}{r}}) = \mathbf{Top} \\ &\Leftrightarrow \quad \operatorname{pim}(\alpha) \subseteq \mathbb{R}_{\frac{x}{r}} \quad \Leftrightarrow \quad r.\operatorname{pim}(\alpha) \subseteq \mathbb{R}_x \\ &\Leftrightarrow \quad \operatorname{pim}(\mathbf{r}).\operatorname{pim}(\alpha) \subseteq \mathbb{R}_x. \end{aligned}$$

(c): For every $r \in \mathbb{R}$, we can write

$$\begin{aligned} \operatorname{pim}(\mathbf{r} + \alpha) \subseteq \mathbb{R}_x \quad &\Leftrightarrow \quad (\mathbf{r} + \alpha)(\mathbb{R}_x) = \mathbf{Top} \quad \Leftrightarrow \quad \alpha(-r + \mathbb{R}_x) = \alpha(\mathbb{R}_{x-r}) = \mathbf{Top} \\ &\Leftrightarrow \quad \operatorname{pim}(\alpha) \subseteq \mathbb{R}_{x-r} = -r + \mathbb{R}_x \quad \Leftrightarrow \quad r + \operatorname{pim}(\alpha) \subseteq \mathbb{R}_x \\ &\Leftrightarrow \quad \operatorname{pim}(\mathbf{r}) + \operatorname{pim}(\alpha) \subseteq \mathbb{R}_x. \end{aligned}$$

□

Now, we state the relation between $\operatorname{pim}(\alpha)$, $\operatorname{pim}(\alpha^+)$, and $\operatorname{pim}(\alpha^-)$ in the following.

Proposition 2.13. *For every $\alpha \in C(L)$, the following relations hold:*

(a) $\operatorname{pim}(\alpha) \cap (0, +\infty) = \operatorname{pim}(\alpha^+) \setminus \{0\}$.

(b) $\operatorname{pim}(\alpha) \cap (-\infty, 0) = \operatorname{pim}(-\alpha^-) \setminus \{0\}$.

(c) $\operatorname{pim}(\alpha) \setminus \{0\} = ((\operatorname{pim}(\alpha^+) \cup \operatorname{pim}(-\alpha^-)) \setminus \{0\}$.

Proof. (a): For every $x > 0$, by Proposition 1.4, we have

$$\alpha^+(-\infty, x) = (\alpha \vee \mathbf{0})(-\infty, x) = \alpha(-\infty, x) \wedge \mathbf{0}(-\infty, x) = \alpha(-\infty, x)$$

and similarly,

$$\alpha^{+}(x,\,+\infty) = (\alpha \vee \mathbf{0})(x,\,+\infty) = \alpha(x,\,+\infty) \vee \mathbf{0}(x,\,+\infty) = \alpha(x,\,+\infty).$$

Therefore, for every $x > 0$, we can deduce that

$$\alpha(\mathbb{R}_x) = \alpha(-\infty,\,x) \vee \alpha(x,\,+\infty) = \alpha^{+}(-\infty,\,x) \vee \alpha^{+}(x,\,+\infty) = \alpha^{+}(\mathbb{R}_x).$$

Hence, $(0,\,+\infty) \cap \mathrm{pim}(\alpha) = \mathrm{pim}(\alpha^{+}) \setminus \{0\}$.

(b): For every $x < 0$, by part (a), we can write

$$\begin{aligned} -\alpha^{-}(\mathbb{R}_x) &= -\alpha^{-}[(-\infty,\,x) \vee (x,\,+\infty)] \\ &= -\alpha^{-}(-\infty,\,x) \vee -\alpha^{-}(x,\,+\infty) \\ &= \alpha^{-}(-x,+\infty) \vee \alpha^{-}(-\infty,-x) \\ &= (-\alpha)^{+}(-x,+\infty) \vee (-\alpha)^{+}(-\infty,-x) \\ &= -\alpha(-x,+\infty) \vee -\alpha(-\infty,-x) \\ &= \alpha(-\infty,\,x) \vee \alpha(x,\,+\infty) = \alpha(\mathbb{R}_x). \end{aligned}$$

Therefore, $(-\infty,0) \cap \mathrm{pim}(\alpha) = \mathrm{pim}(-\alpha^{-}) \setminus \{0\}$.

(c): Straightforward from (a) and (b), it is concluded that

$$\mathrm{pim}(\alpha) \setminus \{0\} = \big((\mathrm{pim}(\alpha^{+})) \cup \mathrm{pim}(-\alpha^{-})\big) \setminus \{0\}.$$

□

Question 3: Now, this question arises whether the following relations, similar to what we have for real functions on topological spaces, hold.

$$\mathrm{pim}(\alpha \vee \beta) \subseteq \mathrm{pim}(\alpha) \cup \mathrm{pim}(\beta) \quad , \quad \mathrm{pim}(\alpha \wedge \beta) \subseteq \mathrm{pim}(\alpha) \cap \mathrm{pim}(\beta)$$

$$\mathrm{pim}(\alpha + \beta) \subseteq \mathrm{pim}(\alpha) + \mathrm{pim}(\beta) \quad , \quad \mathrm{pim}(\alpha\beta) \subseteq \mathrm{pim}(\alpha)\mathrm{pim}(\beta).$$

We show that under some achievable conditions, the answer is positive. But first we need some preparations.

Definition 2.14. An ideal I in a frame L is called $\vee$-complete (countably $\vee$-complete) if from $D \subseteq I$ (countable set $D \subseteq I$), it follows that $\bigvee D \in I$.

Example 2.15. (a) Every principal ideal is $\vee$-complete.

(b) Suppose that ω_1 is the first uncountable ordinal and $L = \downarrow \omega_1$. Clearly L is a frame and if we put $P = L \setminus \{\mathbf{Top}\}$, then P is a countably $\vee$-complete ideal whereas it is not a $\vee$-complete ideal.

Definition 2.16. For every $P \in \mathrm{Spec}(L)$, we define $A_P(\alpha) = \{x \in \mathbb{R} : \ \alpha(x, +\infty) \in P\}$ and $B_P(\alpha) = \{x \in \mathbb{R} : \ \alpha(-\infty, x) \in P\}$.

Because these two sets $A_P(\alpha)$ and $B_P(\alpha)$ are important in our work, we discuss them briefly.

Lemma 2.17. *Let $P \in Spec(L)$ and $\alpha \in C(L)$. Then*

(a) $A_P(\alpha) \cup B_P(\alpha) = \mathbb{R}$.

(b) *Any element of $A_P(\alpha)$ is an upper bound of $B_P(\alpha)$ and any element of $B_P(\alpha)$ is a lower bound of $A_P(\alpha)$.*

(c) $\uparrow A_P(\alpha) = A_P(\alpha)$ *and* $\downarrow B_P(\alpha) = B_P(\alpha)$.

Proof. (a): Assuming $x \notin A_P(\alpha)$, it follows that $\alpha(x, +\infty) \notin P$. Since P is prime and $\alpha(x, +\infty) \wedge \alpha(-\infty, x) = \perp \in P$, we deduce that $\alpha(-\infty, x) \in P$. Hence $x \in B_P(\alpha)$.

(b): Assume that $x \in A_P(\alpha)$ and, on the contrary, there exists an element $c \in B_P(\alpha)$ such that $x < c$. Therefore, $\mathbf{Top} = \alpha(\mathbb{R}) = \alpha(-\infty, c) \vee \alpha(x, +\infty) \in P$ and this is a contradiction. Similarly, any element of $B_P(\alpha)$ is a lower bound of $A_P(\alpha)$.

(c): Supposing $x \in \uparrow A_P(\alpha)$, there exists an element $a \in A_P(\alpha)$ such that $a \leqslant x$. Thus, $\alpha(x, +\infty) \leqslant \alpha(a, +\infty) \in P$ and consequently $x \in A_P(\alpha)$. $\square$

Corollary 2.18. *Let $P \in Spec(L)$ and $\alpha \in C(L)$. Then the following statements are equivalent:*

(a) $\inf A_P(\alpha) \in \mathbb{R}$

(b) $A_P(\alpha) \neq \emptyset \neq B_P(\alpha)$.

(c) $\sup B_P(\alpha) \in \mathbb{R}$

(d) *There exists an element $x \in \mathbb{R}$ such that*

$$(x, +\infty) \subseteq (\inf A_P(\alpha), +\infty) \subseteq [x, +\infty) \text{ and}$$

$$(-\infty, x) \subseteq (-\infty, \sup B_P(\alpha)) \subseteq (-\infty, x].$$

(e) $\inf A_P(\alpha) = \sup B_P(\alpha) \in \mathbb{R}$.

Proof. (a) $\Rightarrow$ (b): By hypothesis, clearly, $A_P(\alpha) \neq \emptyset$ and there exists an element $x \in \mathbb{R}$ such that $x \notin A_P(\alpha)$. By Lemma 2.17, $x \in B_P(\alpha)$. Thus, $B_P(\alpha)$ is also non-empty.

(b) $\Rightarrow$ (c): By Lemma 2.17, it is clear.

(c) $\Rightarrow$ (d): Similar to (a) $\Rightarrow$ (b), it follows that $A_P(\alpha) \neq \emptyset \neq B_P(\alpha)$. Hence, by part (b) of Lemma 2.17, $A_P(\alpha)$ (respectively, $B_P(\alpha)$) is non-empty and bounded below (respectively, bounded above). Hence, $\inf A_P(\alpha)$ and $\sup B_P(\alpha)$ exist. It is easy, by using Lemma 2.17, once again, to see that $\inf A_P(\alpha) = x = \sup B_P(\alpha)$ and in addition, we have $(x, +\infty) \subseteq (\inf A_P(\alpha), +\infty) \subseteq [x, +\infty)$ and $(-\infty, x) \subseteq (-\infty, \sup B_P(\alpha)) \subseteq (-\infty, x]$.

The implications (d) $\Rightarrow$ (e) $\Rightarrow$ (a) are obvious. □

Definition 2.19. $P \in \mathrm{Spec}(L)$ is said to be real with respect to $\alpha \in C(L)$ if $A_P(\alpha)$ and $B_P(\alpha)$ are non-empty closed subsets in $\mathbb{R}$. If P is real with respect to every $\alpha \in C(L)$, then we say P is real.

Lemma 2.20. *Assume that $P \in Spec(L)$ and $\alpha \in C(L)$. Then, the following statements are equivalent:*

(a) *P is real with respect to α.*

(b) $\inf A_P(\alpha) \in A_P(\alpha)$ *and* $\sup B_P(\alpha) \in B_P(\alpha)$.

(c) *There is an element $x \in \mathbb{R}$ such that $A_P(\alpha) \cap B_P(\alpha) = \{x\}$.*

(d) *There exists an element $x \in \mathbb{R}$ such that $\alpha(\mathbb{R}_x) \in P$.*

Proof. By Corollary 2.18, it is clear. □

Lemma 2.21. *Let $P \in Spec(L)$ be countably $\vee$-complete. Then P is real.*

Proof. Suppose that $\alpha \in C(L)$. Since P is countably $\vee$-complete, it follows that $\inf A_P(\alpha) \in \mathbb{R}$ and so, by Corollary 2.18, there exists an element $x \in \mathbb{R}$ such that

$$(x, +\infty) \subseteq (\inf A_P(\alpha),\, +\infty) \subseteq [x, +\infty)$$

and

$$(-\infty, x) \subseteq (-\infty,\ \sup B_P(\alpha)) \subseteq (-\infty, x].$$

By Lemma 2.20, it is enough to show that $x \in A_P(\alpha) \cap B_P(\alpha)$. This is obvious, since P is countably $\vee$-complete and $\mathbb{Q}$ is dense in $\mathbb{R}$. □

By the above lemma, $\downarrow p$ is real for each $p \in \mathrm{Sp}L$.

We need the following lemma for the next theorem.

Lemma 2.22. *Let P be prime ideal in a frame L and $\alpha \in C(L)$. The following statements hold:*
(a) $A_P(-\alpha) = -B_P(\alpha)$ *and* $B_P(-\alpha) = -A_P(\alpha)$.
(b) $B_P(\alpha^+) = (-\infty, 0) \cup B_P(\alpha)$.
(c) $A_P(\alpha^+) = (0, +\infty) \cap A_P(\alpha)$.
(d) $B_P(\alpha^-) = (-\infty, 0) \cup -A_P(\alpha)$.
(e) $A_P(\alpha^-) = (0, +\infty) \cap -B_P(\alpha)$.

If, in addition, $\hat{P}(\alpha) = \inf A_P(\alpha) \in \mathbb{R}$, then
(f) $\hat{P}(\alpha^+) = (\hat{P}(\alpha))^+$;
(g) $\hat{P}(\alpha^-) = (\hat{P}(\alpha))^-$.

Proof. (a): It is clear that

$$\begin{aligned} A_P(-\alpha) &= \{x \in \mathbb{R} : \ -\alpha(x, +\infty) \in P\} = \{x \in \mathbb{R} : \ \alpha(-\infty, -x) \in P\} \\ &= -\{y \in \mathbb{R} : \ \alpha(-\infty, y) \in P\} = -B_P(\alpha). \end{aligned}$$

Similarly, we conclude that $B_P(-\alpha) = -A_P(\alpha)$.

(b): We can write

$$\begin{aligned} B_P(\alpha^+) &= \{x \in \mathbb{R} : \ \alpha^+(-\infty, x) \in P\} = \{x \in \mathbb{R} : \ \mathbf{0}(-\infty, x) \wedge \alpha(-\infty, x) \in P\} \\ &= \{x \in \mathbb{R} : \ \mathbf{0}(-\infty, x) \in P\} \cup \{x \in \mathbb{R} : \ \alpha(-\infty, x) \in P\} = (-\infty, 0) \cup B_P(\alpha). \end{aligned}$$

(c): We can write

$$\begin{aligned} A_P(\alpha^+) &= \{x \in \mathbb{R} : \ \alpha^+(x, +\infty) \in P\} = \{x \in \mathbb{R} : \ \mathbf{0}(x, +\infty) \vee \alpha(x, +\infty) \in P\} \\ &= \{x \in \mathbb{R} : \ \mathbf{0}(x, +\infty) \in P\} \cap \{x \in \mathbb{R} : \ \alpha(x, +\infty) \in P\} = (0, +\infty) \cap A_P(\alpha). \end{aligned}$$

(d): By parts (a) and (b), it follows that

$$B_P(\alpha^-) = B_P((-\alpha)^+) = (-\infty, 0) \cup B_P(-\alpha) = (-\infty, 0) \cup -A_P(\alpha).$$

(e): Using (a) and (c), we do similar to (d).
(f): By part (b) and Corollary 2.18, we can write

$$(\hat{P}(\alpha))^+ = 0 \vee \hat{P}(\alpha) = \sup(-\infty,\, 0) \vee \sup B_P(\alpha) = \sup B_P(\alpha^+) = \hat{P}(\alpha^+).$$

(g): By part (d) and Corollary 2.18, we can write

$$(\hat{P}(\alpha))^- = 0 \vee -\hat{P}(\alpha) = \sup((-\infty,\, 0) \cup -A_P(\alpha)) = \sup B_P(\alpha^-) = \hat{P}(\alpha^-).$$

□

The following theorem is an improvement of [6, Proposition 2.3] (also, see [7, Proposition 3.9] and [9, Proposition 2.3]).

Theorem 2.23. *Assume that $P \in Spec(L)$ and is countably $\vee$-complete in L. We define*

$$\hat{P} \,:\, C(L) \to\, \mathbb{R}\;, \quad \hat{P}(\alpha) = \inf\, A_P(\alpha).$$

Then $\hat{P}$ is an f-algebra homomorphism; that is,

(a) $\hat{P}(\alpha+\beta) = \hat{P}(\alpha) + \hat{P}(\beta)$ *for every* $\alpha, \beta \in C(L)$.
(b) $\hat{P}(\alpha\beta) = \hat{P}(\alpha)\hat{P}(\beta)$ *for every* $\alpha, \beta \in C(L)$.
(c) $\hat{P}(r\alpha) = r\hat{P}(\alpha)$ *for every* $r \in \mathbb{R}$ *and every* $\alpha \in C(L)$.
(d) $\hat{P}(\alpha \vee \beta) = \hat{P}(\alpha) \vee \hat{P}(\beta)$ *for every* $\alpha, \beta \in C(L)$.
(e) $\hat{P}(\alpha \wedge \beta) = \hat{P}(\alpha) \wedge \hat{P}(\beta)$ *for every* $\alpha, \beta \in C(L)$.

Proof. (a): Let $x = \hat{P}(\alpha+\beta)$. Since P is countably $\vee$-complete, we have $(\alpha+\beta)(x, +\infty) \in P$. Therefore,

$$\begin{aligned}(\alpha+\beta)(x,+\infty) &= \bigvee \{\alpha(r,s) \wedge \beta(t,u) :\; (r,s)+(t,u) \subseteq (x,+\infty)\}\\ &= \bigvee \{\alpha(r,s) \wedge \beta(t,u) :\; r+t \geqslant x\}\\ &= \bigvee \{\alpha(r,+\infty) \wedge \beta(t,+\infty) :\; r+t \geqslant x\}\\ &= \bigvee \{\alpha(r,+\infty) \wedge \beta(x-r,+\infty) :\; r \in \mathbb{R}\} \in P.\end{aligned}$$

Hence

$$\bigvee \left\{\alpha(r,+\infty) \wedge \beta(x-r,+\infty) :\; r < \hat{P}(\alpha),\; r \in \mathbb{Q}\right\} \in P.$$

Since $\alpha(r,+\infty) \notin P$ for every $r < \hat{P}(\alpha)$, it follows that $\beta(x-r,+\infty) \in P$ for every rational $r < \hat{P}(\alpha)$ and so, by countably $\vee$-completeness of P, we can write

$$\beta(x-\hat{P}(\alpha),+\infty) = \bigvee \left\{\beta(x-r,+\infty) :\; r < \hat{P}(\alpha),\; r \in \mathbb{Q}\right\} \in P.$$

Thus,

$$\hat{P}(\beta) \leqslant x - \hat{P}(\alpha) \quad \Rightarrow \quad \hat{P}(\alpha) + \hat{P}(\beta) \leqslant x. \qquad (1)$$

On the other hand, it is clear that for every $s > \sup B_P(\alpha) = \hat{P}(\alpha)$, we have $\alpha(-\infty, s) \notin P$. Therefore, similar to the above, it conclude that $\beta(-\infty, x - s) \in P$ for every $s > \hat{P}(\alpha)$. Consequently,

$$\beta(-\infty, x - \hat{P}(\alpha)) = \bigvee \Big\{\beta(-\infty, x - s) : \ s > \hat{P}(\alpha), \ s \in \mathbb{Q}\Big\} \in P.$$

Hence, we can write

$$x - \hat{P}(\alpha) \leqslant \hat{P}(\beta) \quad \Rightarrow \quad x \leqslant \hat{P}(\alpha) + \hat{P}(\beta). \qquad (2)$$

The desired equality follows from (1) and (2).

(b): Case (1): $\alpha, \beta \geqslant 0$ and $\hat{P}(\alpha\beta) = 0$. In this case, we show that $\hat{P}(\alpha) = 0$ or $\hat{P}(\beta) = 0$. Since $\hat{P}(\alpha\beta) = 0$, $(\alpha\beta)(0, +\infty) \in P$ and since $\alpha(-\infty, 0) = 0$, $\beta(-\infty, 0) = 0$, we can write

$$\begin{aligned}(\alpha\beta)(0, +\infty) &= \bigvee \{\alpha(r, s) \wedge \beta(t, u) \ : \ (r, s)(t, u) \in (0, +\infty)\} \\ &= \bigvee \{\alpha(r, s) \wedge \beta(t, u) : \ r, t \geqslant 0\} \\ &= \bigvee \{\alpha(r, +\infty) \wedge \beta(t, +\infty) : \ r, t \geqslant 0\} \\ &= \alpha(0, +\infty) \wedge \beta(0, +\infty) \in P.\end{aligned}$$

Therefore, $\beta(\mathbb{R}_0) = \beta(0, +\infty) \in P$ or $\alpha(\mathbb{R}_0) = \alpha(0, +\infty) \in P$. Thus, $\hat{P}(\alpha) = 0$ or $\hat{P}(\beta) = 0$.

Case (2): $\alpha, \beta \geqslant 0$ and $\hat{P}(\alpha\beta) = x > 0$. In this case

$$\alpha\beta(x, +\infty) \in P \quad \Rightarrow \quad \alpha\beta(x, +\infty) = \bigvee_{r>0} \Big(\alpha(r, +\infty) \wedge \beta(\frac{x}{r}, +\infty)\Big) \in P.$$

Since $\alpha(r, +\infty) \notin P$ for every $0 < r < \hat{P}(\alpha)$, it follows that $\beta(\frac{x}{r}, +\infty) \in P$ for every $0 < r < \hat{P}(\alpha)$. Therefore, for every $0 < r < \hat{P}(\alpha)$, we have $\frac{x}{r} \geqslant \hat{P}(\beta)$ and so $\frac{x}{\hat{P}(\alpha)} \geqslant \hat{P}(\beta)$. This implies that

$$x \geqslant \hat{P}(\alpha)\hat{P}(\beta). \qquad (3)$$

Since $\alpha(-\infty, s) \notin P$ for every $s > \hat{P}(\alpha)$, similar to above, we conclude that $\beta(-\infty, \frac{x}{s}) \in P$ for every $s > \hat{P}(\alpha)$. Thus, $\frac{x}{s} \leqslant \hat{P}(\beta)$ for every $s > \hat{P}(\alpha)$ and consequently, $\frac{x}{\hat{P}(\alpha)} \leqslant \hat{P}(\beta)$. Hence,

$$x \leqslant \hat{P}(\alpha)\hat{P}(\beta). \qquad (4)$$

From (3) and (4), it follows that $\hat{P}(\alpha\beta) = \hat{P}(\alpha)\hat{P}(\beta)$.

Final case: Let $\alpha, \beta \in C(L)$ be arbitrary. By previous cases, we can write

$$\begin{aligned}\hat{P}(\alpha\beta) &= \hat{P}\left((\alpha^+ - \alpha^-)(\beta^+ - \beta^-)\right)\\ &= \hat{P}(\alpha^+)\hat{P}(\beta^+) - \hat{P}(\alpha^+)\hat{P}(\beta^-) - \hat{P}(\alpha^-)\hat{P}(\beta^+) + \hat{P}(\alpha^-)\hat{P}(\beta^-).\end{aligned}$$

On the other hand, by Lemma 2.22, we have $\hat{P}(\alpha^-) = (\hat{P}(\alpha))^-$ and $\hat{P}(\alpha^+) = (\hat{P}(\alpha))^+$. Therefore

$$\begin{aligned}\hat{P}(\alpha\beta) &= (\hat{P}(\alpha))^+(\hat{P}(\beta))^+ - (\hat{P}(\alpha))^+(\hat{P}(\beta))^- - (\hat{P}(\alpha))^-(\hat{P}(\beta))^+ + (\hat{P}(\alpha))^-(\hat{P}(\beta))^-\\ &= (\hat{P}(\alpha)^+ - \hat{P}(\alpha)^-)(\hat{P}(\beta)^+ - \hat{P}(\beta)^-) = \hat{P}(\alpha)\hat{P}(\beta).\end{aligned}$$

(c): If $r = 0$, the assertion is clear. If $r > 0$, then

$$\begin{aligned}\hat{P}(\mathbf{r}\alpha) &= \inf\{x:\ \mathbf{r}\alpha(x, +\infty) \in P\} = \inf\left\{x:\ \alpha(\frac{x}{r}, +\infty) \in P\right\}\\ &= \inf\{ry:\ \alpha(y, +\infty) \in P\} = r\hat{P}(\alpha).\end{aligned}$$

Finally, if $r < 0$, then

$$\begin{aligned}\hat{P}(\mathbf{r}(\alpha)) &= \inf\{x:\ \mathbf{r}\alpha(x, +\infty) \in P\} = \inf\{x:\ -\mathbf{r}\alpha(-\infty, -x) \in P\}\\ &= \inf\left\{x \in \mathbb{R}:\ \alpha(-\infty, \frac{x}{r}) \in P\right\} = \inf\{ry:\ \alpha(-\infty, y) \in P\}\\ &= r\sup\{y:\ \alpha(-\infty, y) \in P\} = r\hat{P}(\alpha).\end{aligned}$$

Therefore, $\hat{P}(\mathbf{r}\alpha) = r\hat{P}(\alpha)$ for every $r \in \mathbb{R}$.

(d): Clearly, we can write

$$\begin{aligned}\hat{P}(\alpha \vee \beta) &= \sup\{x \in \mathbb{R}:\ (\alpha \vee \beta)(-\infty, x) \in P\}\\ &= \sup\{x:\ \alpha(-\infty, x) \wedge \beta(-\infty, x) \in P\}\\ &= \sup\Big(\{x:\ \alpha(-\infty, x) \in P\} \cup \{x:\ \beta(-\infty, x) \in P\}\Big)\\ &= \sup\{x:\ \alpha(-\infty, x) \in P\} \vee \sup\{x:\ \beta(-\infty, x) \in P\}\\ &= \hat{P}(\alpha) \vee \hat{P}(\beta).\end{aligned}$$

(e): It is similar to the proof of the part (d). □

Note that, by Lemma 2.20, we obtain the following result, clearly.

Corollary 2.24. *Suppose that $P \in Spec(L)$ is countably $\vee$-complete. Then $\hat{P}(\alpha) = x$ if and only if $\alpha(\mathbb{R}_x) \in P$.*

Corollary 2.25. *Assume that $p \in \mathrm{Sp}L$ and*

$$\hat{p} : C(L) \to \mathbb{R} , \quad \hat{p}(\alpha) = \inf\{x \in \mathbb{R} : \alpha(x, +\infty) \leqslant p\}.$$

Then $\hat{p}$ is an f-algebra homomorphism.

Proof. It suffices to put $P = \downarrow p$, then, by Theorem 2.23, we are done. □

We are now ready to answer the Question 3 which we raised earlier.

Theorem 2.26. *Suppose that L is a frame in which every maximal ideal is countable $\vee$-complete. Then for every $\alpha, \beta \in C(L)$, we have the following relations:*

(a) $\mathrm{pim}(\alpha + \beta) \subseteq \mathrm{pim}(\alpha) + \mathrm{pim}(\beta)$.
(b) $\mathrm{pim}(\alpha\beta) \subseteq \mathrm{pim}(\alpha)\mathrm{pim}(\beta)$.
(c) $\mathrm{pim}(\alpha \vee \beta) \subseteq \mathrm{pim}(\alpha) \vee \mathrm{pim}(\beta)$.
(d) $\mathrm{pim}(\alpha \wedge \beta) \subseteq \mathrm{pim}(\alpha) \wedge \mathrm{pim}(\beta)$.

Proof. We only prove part (a); other parts are proved by the same manner. Suppose that $x \in \mathrm{pim}(\alpha + \beta)$. Thus, $(\alpha + \beta)(\mathbb{R}_x) \neq$ **Top** and so there exists an element $M \in \mathrm{Max}(L)$ such that $(\alpha + \beta)(\mathbb{R}_x) \in M$. Therefore, by Theorem 2.23 and Corollary 2.24, $x = \hat{M}(\alpha + \beta) = \hat{M}(\alpha) + \hat{M}(\beta)$. Taking $\hat{M}(\alpha) = a$ and $\hat{M}(\beta) = b$, it is sufficient to show that $a \in \mathrm{pim}(\alpha)$ and $b \in \mathrm{pim}(\beta)$. To see this, by Corollary 2.24, $\alpha(\mathbb{R}_a) \in M$ and $\beta(\mathbb{R}_b) \in M$. Hence, $\alpha(\mathbb{R}_a) \neq$ **Top** $\neq \beta(\mathbb{R}_b)$, so $a \in \mathrm{pim}(\alpha)$ and $b \in \mathrm{pim}(\beta)$. Therefore, $\mathrm{pim}(\alpha + \beta) \subseteq \mathrm{pim}(\alpha) + \mathrm{pim}(\beta)$. □

3 Comparing $\mathrm{pim}(\alpha)$ with images of two real functions $\overline{\alpha}$ and $\hat{\alpha}$

In this section, first, for any $\alpha \in C(L)$, we introduce two real functions $\overline{\alpha}$ and $\hat{\alpha}$ induced naturally by α, then we compare $\mathrm{pim}(\alpha)$ with the images of these two functions.

Definition 3.1. Suppose that $\alpha \in C(L)$. By Corollary 2.25, we can define $\overline{\alpha} : \mathrm{Sp}L \to \mathbb{R}$ with $\overline{\alpha}(p) = \hat{p}(\alpha)$. Also, supposing

$$X_\alpha = \{P \in \mathrm{Spec}(L) : \ P \text{ is real with respect to } \alpha\},$$

we can define $\hat{\alpha} : X_\alpha \to \mathbb{R}$ with $\hat{\alpha}(P) = \hat{P}(\alpha)$.

Note that the mapping $p \to \downarrow p$ is an embedding from $\mathrm{Sp}L$ to $\mathrm{Spec}(L)$, where $\mathrm{Spec}(L)$ is equipped with hall-kernel topology (that is, the Zariski topology). Therefore, we can suppose that $\mathrm{Sp}L$ is a subspace of $\mathrm{Spec}(L)$ and so $\hat{\alpha}|_{\mathrm{Sp}L} = \overline{\alpha}$.

Proposition 3.2. *For every $\alpha \in C(L)$, $\hat{\alpha}$ is continuous and so is $\overline{\alpha}$.*

Proof. Assume that (x, y) is an open interval in $\mathbb{R}$. taking $a = \alpha(x, +\infty)$ and $b = \alpha(-\infty, y)$, it suffices to show that $(\hat{\alpha})^{-1}(x, y) = h^c_{X_\alpha}(a) \cap h^c_{X_\alpha}(b)$, where $h^c_{X_\alpha}(a) = X_\alpha \cap h^c(a)$. Too see this, for every $P \in X_\alpha$, we can write

$$\begin{aligned} P \in (\hat{\alpha})^{-1}(x, y) &\Leftrightarrow x < \hat{\alpha}(P) = \hat{P}(\alpha) < y \\ &\Leftrightarrow a = \alpha(x, +\infty) \notin P \ , \ b = \alpha(-\infty, y) \notin P \\ &\Leftrightarrow P \in h^c_{X_\alpha}(a) \cap h^c_{X_\alpha}(b). \end{aligned}$$

□

The following remark shows that $\overline{\alpha}$ is not a new concept .

Remark 3.3. Recall that $\mathrm{Sp}\mathcal{O}\mathbb{R} = \{\mathbb{R}_x : \ x \in \mathbb{R}\}$ and $g : \mathrm{Sp}\mathcal{O}\mathbb{R} \to \mathbb{R}$ with $g(\mathbb{R}_x) = x$ is a homeomorphism. For every continuous real function $\alpha \in C(L)$, we have $\mathrm{Sp}\alpha : \mathrm{Sp}L \to \mathrm{Sp}\mathcal{O}\mathbb{R}$ with $(\mathrm{Sp}\alpha)(p) = \alpha^*(p) = \bigvee\{w \in \mathcal{O}\mathbb{R} : \ \alpha(w) \leqslant p\}$. Since $\alpha^*(p) \in \mathrm{Sp}\mathcal{O}\mathbb{R}$, there exists a unique $x \in \mathbb{R}$ such that $(\mathrm{Sp}\alpha)(p) = \alpha^*(p) = \mathbb{R}_x$. In fact, $(\mathrm{Sp}\alpha)(p) = \mathbb{R}_x$ if and only if $\alpha(\mathbb{R}_x) \leqslant p$. Therefore, for every $\alpha \in C(L)$, we have a natural function $\overline{\alpha} = g\,\mathrm{Sp}\alpha$ from $\mathrm{Sp}L$ to $\mathbb{R}$ with $\overline{\alpha}(p) = x$ such that $\alpha(\mathbb{R}_x) \leqslant p$. Also, according to this fact, for every $p \in \mathrm{Sp}L$, we can define a function $\hat{p} : C(L) \to \mathbb{R}$ with $\hat{p}(\alpha) = \overline{\alpha}(p)$.

Proposition 3.4. *Assume that* $\alpha \in C(L)$. *Then* $Im(\overline{\alpha}) \subseteq Im(\hat{\alpha}) \subseteq \text{pim}(\alpha)$.

Proof. Clearly, $Im(\overline{\alpha}) \subseteq Im(\hat{\alpha})$. Now, suppose that $x \in Im(\hat{\alpha})$. Thus, there exists a $P \in \text{Spec}(L)$ such that $\hat{\alpha}(P) = x$. Hence, $\hat{P}(\alpha) = x$ and by Corollary 2.24, it follows that $\alpha(\mathbb{R}_x) \in P$. Therefore, $\alpha(\mathbb{R}_x) \neq \mathbf{Top}$ and consequently $x \in \text{pim}(\alpha)$. □

The first inclusion in the above proposition may be strict. To see this, we need the following lemma.

Lemma 3.5. *Suppose that* L *has no non-trivial complemented element. Then for every* $\alpha \in C(L)$, *there exists an element* $x \in \mathbb{R}$ *such that* $\alpha(\mathbb{R}_x) \neq$ **Top**.

Proof. Let $\alpha \in C(L)$ and, on the contrary, for every $x \in \mathbb{R}$, we have $\alpha(\mathbb{R}_x) = \mathbf{Top}$. By hypothesis, for every $x \in \mathbb{R}$, we $\alpha(-\infty, x) = \mathbf{Top}$ and $\alpha(x, +\infty) = \perp$ or $\alpha(-\infty, x) = \perp$ and $\alpha(x, +\infty) = \mathbf{Top}$. It is easy to see that there exists an element $c \in \mathbb{R}$ such that $\alpha(c, +\infty) = \perp$ and so $x_0 = \inf\{x \in \mathbb{R} : \ \alpha(x, +\infty) = \perp\}$ exists. Thus, $\alpha(x_0, +\infty) = \perp$ and $\alpha(t, +\infty) = \mathbf{Top}$ for every $t < x_0$ and so $\alpha(-\infty, t) = \perp$ for every $t < x_0$. Therefore, $\alpha(-\infty, x_0) = \bigvee\{\alpha(-\infty, t) : \ t < x_0\} = \perp$. Hence, $\alpha(\mathbb{R}_{x_o}) = \perp$ and this is a contradiction. □

In the following example we introduce a frame L such that $Im(\overline{\alpha}) \subsetneq \text{pim}(\hat{\alpha})$ for every $\alpha \in C(L)$.

Example 3.6. Suppose $L = [0, 1) \times [0, 1) \oplus \mathbf{Top}$. Clearly, L is a frame, **Top** is a $\vee$-prime element of L and $\text{Sp}L = \emptyset$. Therefore, L does not have any non-trivial complemented element and so, by Lemma 3.5, for every $\alpha \in C(L)$ we have $\alpha(\mathbb{R}_x) \neq \mathbf{Top}$ for some $x \in \mathbb{R}$. We show that $C(L) = \{\mathbf{r} : \ r \in \mathbb{R}\}$. To see this, assume that $\alpha \in C(L)$. Thus, there exists an element $r \in \mathbb{R}$ such that $\alpha(\mathbb{R}_r) \neq \mathbf{Top}$. Now, for every $w \in \mathcal{O}_r$, since **Top** is $\vee$-prime, we can write

$$\mathbf{Top} = \alpha(\mathbb{R}) = \alpha(w \cup \mathbb{R}_r) = \alpha(w) \vee \alpha(\mathbb{R}_r) \quad \Rightarrow \quad \alpha(w) = \mathbf{Top}.$$

This conclude that $\alpha = \mathbf{r}$. On the other hand, it is clear that $Im(\overline{\mathbf{r}}) = \emptyset$, whereas

$$\begin{aligned} x \in Im(\hat{\mathbf{r}}) &\Leftrightarrow \exists P \in \mathrm{Spec}(L), \quad \hat{\mathbf{r}}(P) = x \\ &\Leftrightarrow \exists P \in \mathrm{Spec}(L), \quad \hat{P}(\mathbf{r}) = x \\ &\Leftrightarrow \exists P \in \mathrm{Spec}(L), \quad \mathbf{r}(\mathbb{R}_x) \in P \\ &\Leftrightarrow r = x. \end{aligned}$$

Therefore, $Im(\hat{\mathbf{r}}) = \{r\}$.

Proposition 3.7. *Assume that $\alpha \in C(L)$. Then the following statements hold:*

(a) *If* $\mathrm{Sp}L$ *is cofinal in* $L \setminus \{\mathbf{Top}\}$, *then* $Im(\overline{\alpha}) = Im(\hat{\alpha}) = \mathrm{pim}(\alpha)$.

(b) *If* $\bigcup X_\alpha = L \setminus \{\mathbf{Top}\}$, *then* $Im(\hat{\alpha}) = \mathrm{pim}(\alpha)$.

Proof. (a): It is enough to prove that $\mathrm{pim}(\alpha) \subseteq Im(\overline{\alpha})$. Suppose that $x \in \mathrm{pim}(\alpha)$. Thus, $\alpha(\mathbb{R}_x) \neq \mathbf{Top}$ and by hypothesis, there exists an element $p \in \mathrm{Sp}L$ such that $\alpha(\mathbb{R}_x) \leqslant p$ and this is equivalent to $\overline{\alpha}(p) = \hat{p}(\alpha) = x$. Therefore, $x \in Im(\overline{\alpha})$.

(b): Suppose that $x \in \mathrm{pim}(\alpha)$. Thus, $\alpha(\mathbb{R}_x) \neq \mathbf{Top}$ and by hypothesis, there exists an element $P \in X_\alpha$ such that $\alpha(\mathbb{R}_x) \in P$ and this is equivalent to $\hat{\alpha}(P) = \hat{P}(\alpha) = x$. Therefore, $x \in Im(\hat{\alpha})$. □

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