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Actions of a separately strict cpo-monoid on pointed directed complete posets

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Abstract. In the present article, we study some categorical properties of the category $\mathbf{Cpo_{Sep}}$-S of all separately strict S-cpo's; cpo's equipped with a compatible right action of a separately strict cpo-monoid S which is strict continuous in each component. In particular, we show that this category is reflective and coreflective in the category of S-cpo's, find the free and cofree functors, characterize products and coproducts. Furthermore, epimorphisms and monomorphisms in $\mathbf{Cpo_{Sep}}$-S are studied, and show that $\mathbf{Cpo_{Sep}}$-S is not cartesian closed.

1 Introduction

The category **Dcpo** of directed complete partially ordered sets plays an important role in Theoretical Computer Science, specially in Domain Theory (see [1, 9, 11]). This category is complete and cocomplete. The completeness of **Dcpo** has been proved, in a constructive way, by Achim Jung [1] but it is stated there that to describe the colimits is quite difficult. In [8], Fiech characterizes and describes colimits in **Dcpo**, but his construction is rather complicated. The cartesian closedness of **Dcpo** has also been proved by Achim Jung (see [11]). It is also shown that the category **Cpo** of directed complete partially ordered sets, each with a bottom

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(smallest) element, and continuous maps, preserving bottom elements, between them is monoidal closed, complete, and cocomplete (see [1, 11]). The free dcpo over a poset, and free dcpo algebras have been studied in [12].

The action of a monoid on sets is also an important algebraic structure in mathematics as well as in computer science. For example, computer scientists use the notion of a projection algebra (sets with an action of the monoid $(\mathbb{N}^\infty, min)$) as convenient means of the algebraic specification of process algebras (see [7] and its references).

Combining the notions of a poset and an act, many algebraic and categorical properties of the category of actions of a pomonoid on a poset have been studied (see for example [2, 3, 6]).

In this paper, considering the actions of a separately strict cpo-monoid S on a cpo, we study the properties of the category so obtained. Showing the existence of limits and colimits, we see that this category is both complete and cocomplete. We also find the free and cofree objects over cpo's, and show that this category is not cartesian closed. Also, monomorphisms and epimorphisms in this category are studied. We should also mention that, the objects we consider and call separately strict S-cpo's are neither a kind of dcpo algebras considered in [12], nor a kind of modules over a monoid (or algebras for a monad), so their categorical properties studied here are not known.

2 Preliminaries

In this section, we recall some preliminary notions about the actions of a monoid on a set and on a poset. For more information see [3, 5, 13].

2.1 The category of S-acts

In this subsection, we briefly recall the preliminary notions about the action of a monoid on a set. For more information see [5, 13].

Definition 2.1. Let S be a monoid with 1 as its identity. An *S-act* (also called *S-set, S-polygon, S-system, S-transition system, S-automata*) is a set A equipped with an *action* $A \times S \to A, (a, s) \rightsquigarrow as$, such that $a1 = a$ and $a(st) = (as)t$.

Remark 2.2. (a) Let A be an S-act with the action $A \times S \to A$. Then, we have the following right and left translations:
(*Right translation*) For each $s \in S$, $R_s : A \to A,\ R_s(a) = as$.
(*Left translation*) For each $a \in A$, $L_a : S \to A,\ L_a(t) = at$.

(b) Let S be a monoid, A be a set, and $A \times S \to A$ be a function. Then, A is an S-act if and only if $(A, (R_s)_{s\in S})$ is a *unary algebra* with $R_s \circ R_t = R_{ts}$ and $R_1 = \mathrm{id}_A$.

(c) Each monoid S can be clearly considered as an S-act with the action given by its binary operation $S \times S \to S$. Note that, the unary algebra related to this S-act is $(S; (R_s)_{s\in S})$, where $R_s : S \to S$ is defined by $R_s(t) = ts$.

Definition 2.3. An *S-map* $f : A \to B$ between S-acts is an action-preserving map, that is, $f(as) = f(a)s$ for each $a \in A, s \in S$.

The category of all S-acts and S-maps between them is denoted by **Act**-S.

An element a of an S-act A is said to be a *zero* (or a *fixed*) element of A if $as = a$, for all $s \in S$. Note that if S is a monoid with a zero element z, then for each S-act A and $a \in A$, az is a zero element of A.

We recall that limits (products, equalizers, pullbacks) in **Act**-S are computed as in sets endowed with a natural action on them (for more information see [5, 13]), the same is true for coproducts.

2.2 The category of S-posets

Here we recall the definition, and give some categorical ingredients, of **Pos**-S needed in the sequel (for more information see [3]).

Note that, for a monoid S we have the functions: the binary operation $S \times S \to S$, the right and the left translations $R_s, L_s : S \to S$. Using these functions we have the following ordered monoids (borrowing some terms from topological semigroups).

Definition 2.4. Let S be a monoid with a partial order $\leq$. Then:

(a) $(S, \leq)$ is a *pomonoid* if the partial order $\leq$ is compatible with the monoid operation; that is, for $s, t, s', t' \in S$, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$. In other words,

$$(s, s') \leq (t, t') \;\Rightarrow\; ss' \leq tt'.$$

(b) $(S, \leq)$ is a *separately strict pomonoid* if for each $s \in S$, both the right and the left translations $R_s, L_s : S \to S$ are order-preserving.

(c) $(S, \leq)$ is a *right* (*left*) *weak separately strict pomonoid* if for each $s \in S$, the right (left) translation R_s (L_s) is order-preserving.

Note that $(S, \leq)$ is a pomonoid if and only if it is a *separately strict pomonoid.* This remark also shows that a weak separately strict pomonoid is not necessarily a pomonoid.

Definition 2.5. Let S be a pomonoid, A be an S-act, and $\leq$ be a partial order on A. Then:

(a) $(A, \leq)$ is an *S-poset* if the action on A is order-preserving; that is,

$$(a, s) \leq (b, t) \;\Rightarrow\; as \leq bt.$$

(b) $(A, \leq)$ is a *separately strict S-poset* if for each $s \in S$ and $a \in A$, both $R_s : A \to A,\ L_a : S \to A$ are order-preserving.

(c) $(A, \leq)$ is a *weak separately strict S-poset* if for each $s \in S$, the right translation $R_s : A \to A$ is order-preserving.

Remark 2.6. (a) Let S be a pomonoid, A be an S-act, and $\leq$ be a partial order on A. Then A is an S-poset if and only if it is a separately strict S-poset.

(b) Let S be a pomonoid, A be an S-act, and $\leq$ be a partial order on A. Then, A is a weak separately strict S-poset if and only if $(A, (R_s)_{s\in S})$ is a *unary algebra* in the category **Pos** of posets, satisfying $R_s \circ R_t = R_{ts}$ and $R_1 = \mathrm{id}_A$.

(c) Note that if $(S, \leq)$ is a left or right separately strict pomonoid or $(A, \leq)$ has some extra properties, then some of the above remarks may change accordingly. For example, as in this paper, each S-act A may have a smallest element $\perp_A$, in which case the action may preserve the smallest element $(\perp_S, \perp_A)$ but the right translations may not preserve $\perp_A$, and vice versa (see Example 3.2).

Definition 2.7. An *S-poset map* $f : A \to B$ between S-posets is an action-preserving monotone map.

The category of all S-posets with action-preserving monotone maps between them is denoted by **Pos**-S.

Products, terminal object, equalizers, pullbacks, and coproducts of S-posets are as in **Act**-S with the obvious order.

2.3 The category of directed complete posets

In the following, we recall the category **Cpo** of cpo's (see [11]).

Recall that a non-empty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$, and P is called a *directed complete poset*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P. A dcpo which has a bottom element $\perp_P$ is said to be a *cpo*.

Also, recall that a *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. By a *cpo map* between cpo's, we mean a continuous map which is *strict*; that is, preserves the bottom element. We denote the category of all cpo's with cpo maps between them by **Cpo**.

Recall from [1] that the product of a family of cpo's is their cartesian product, with componentwise order and ordinary projection maps. In particular, the terminal object of **Cpo** is the one element poset. Also, the coproduct of a family of cpo's is their *coalesced sum*. Recall that *the coalesced sum* of the family $\{A_i \mid i \in I\}$ of cpo's is defined to be

$$\biguplus_{i\in I} A_i = \bot \oplus \dot{\bigcup}_{i\in I}(A_i \setminus \{\bot_{A_i}\}).$$

In particular, the initial object of **Cpo** is the singleton poset $\{\theta\}$.

The final reminder is the following lemma which is used frequently in this paper.

Lemma 2.8. [11] *Let P, Q, and R be dcpo's, and $f : P\times Q \to R$ be a function of two variables. Then f is continuous if and only if f is continuous in each variable; which means that for all $a \in P$, $b \in Q$, $f_a : Q \to R$ ($b \mapsto f(a,b)$) and $f_b : P \to R$ ($a \mapsto f(a,b)$) are continuous.*

3 The category of separately strict S-cpo's

In this section, after introducing the category of separately strict S-cpo's, we study the reflection and coreflection of $\mathbf{Cpo_{Sep}}$-S in **Cpo**-S.

Definition 3.1. (a) A *dcpo* (*cpo*)*-monoid* is a monoid which is also a dcpo (cpo) whose binary operation is a (strict) continuous map.

(b) Let S be a (cpo) dcpo-monoid. By an *S-dcpo* (*S-cpo*) we mean a dcpo (cpo) A which is also an S-act whose action $A\times S \to A$ is (strict) continuous. The category of all S-dcpo's (cpo's) with action-preserving (strict) continuous maps, namely S-dcpo (cpo) maps, between them is denoted by **Dcpo**-S (**Cpo**-S).

(c) A *separately strict cpo-monoid* is a monoid which is also a cpo whose binary operation is strict continuous map in each component; that is, each R_s, L_s is strict continuous.

(d) Let S be a separately strict cpo-monoid. By a *separately strict S-cpo* we mean a cpo A which is also an S-act whose action $A \times S \to A$ is strict continuous in each component; that is, each R_s, L_a is strict continuous.

The category of all separately strict S-cpo's with action-preserving strict continuous maps between them is denoted by $\mathbf{Cpo_{Sep}}$-S.

In the following example we see that a cpo-monoid is not necessarily a separately strict cpo-monoid.

Example 3.2. Consider the pomonoid $\{0 < 1\}$ with the binary operation max. It is clear that max is strict continuous. So $\{0 < 1\}$ is a cpo-monoid. But $\{0 < 1\}$ is not a separately strict cpo-monoid because the continuous map $R_1 : S \to S$ is not strict, in fact $\max\{1, 0\} = 1 \neq 0$.

Remark 3.3. (1) Considering a dcpo-monoid S, if we define the notion of a separately strict S-dcpo similar to a separately strict S-cpo, then applying Lemma 2.8 we get that the notions of separately strict S-dcpo and S-dcpo coincide, which is studied in [14].

(2) Applying Lemma 2.8, it is clear that for a cpo-monoid S, a separately strict S-cpo is an S-cpo. In fact the category of separately strict S-cpo is a subcategory of the category of S-cpo's.

(3) Again by Lemma 2.8, for a separately strict cpo-monoid S, an S-cpo P is a separately strict S-cpo if and only if $p\bot_S = \bot_P$ and $\bot_P s = \bot_P$ for all $p \in P$ and $s \in S$.

Lemma 3.4. *For a separately strict cpo-monoid S, every separately strict S-cpo A has exactly one zero element, $\bot_A$.*

Proof. Assuming the contrary, let $p \in A$ be a zero different from $\bot_A$. Then the continuous map $L_p : S \to A$ is not strict ($ps = p \neq \bot_A$), which is a contradiction. □

In the following, we show that the category $\mathbf{Cpo_{Sep}}$-S is both a reflective and a coreflective subcategory of the category $\mathbf{Cpo}$-S.

Theorem 3.5. *The category $\mathbf{Cpo_{Sep}}$-S is a coreflective subcategory of the category $\mathbf{Cpo}$-S.*

Proof. We show that for every S-cpo there exists a coreflection. To see this, for an S-cpo P, let $P^{[S]}$ be the set of all S-cpo maps from S to P with the pointwise order and the action defined by $(fs)(t) = f(st)$, for $s, t \in S$ and $f \in P^{[S]}$. Then, we show that $P^{[S]}$ is a separately strict S-cpo and the map $\sigma : P^{[S]} \to P$ defined by $\sigma(f) = f(1)$ is a coreflection.

First, we show that the map $f_\bot : S \to P$, $s \mapsto \bot_P$, is an S-cpo map. It is clearly a cpo map, also for $s, t \in S$, we have $f_\bot(st) = \bot_P = \bot_P t = f_\bot(s)t$, where the second equality is because $\bot_S$ is a zero element, and so is $\bot_P$. In fact, $f_\bot$ is the bottom element of $P^{[S]}$. Now, let $F \subseteq^d P^{[S]}$. Then $\bigvee^d F$ exists, which is defined by $(\bigvee^d F)(s) = \bigvee^d_{f\in F} f(s)$. First notice that $\bigvee^d F$ is a cpo map (See [11]). To show that $\bigvee^d F$ is action-preserving, let $s, t \in S$. Then

$$(\bigvee^d F)(st) = \bigvee^d_{f\in F} f(st) = \bigvee^d_{f\in F} (f(s)t) = (\bigvee^d_{f\in F} f(s))t = ((\bigvee^d F)(s))t.$$

Therefore, $P^{[S]}$ is a cpo. Now, we show that the action is well-defined. Take $f \in P^{[S]}$, $s \in S$. We have $(fs)(\bot_S) = f(s\bot_S) = f(\bot_S) = \bot_P$. To prove continuity of fs, let $T \subseteq^d S$. Then

$$(fs)(\bigvee^d T) = f(s(\bigvee^d T)) = f(\bigvee^d_{t\in T} st) = \bigvee^d_{t\in T} f(st) = \bigvee^d_{t\in T} (fs)(t).$$

To show that fs is action-preserving, let $t, t' \in S$. Then

$$(fs)(tt') = f(s(tt')) = f((st)t') = f(st)t' = (fs)(t)t'.$$

Hence the action is well-defined. Recall that this action is continuous (see [15]), and so by Lemma 2.8 it is continuous in each component. Now, we show that it is strict in each component. This is because $(f_\perp s)(t) = f_\perp(st) = \perp_P = f_\perp(t)$ for all $s, t \in S$ and $(f\perp_S)(t) = f(\perp_S t) = f(\perp_S) = \perp_P = f_\perp(t)$, for all $f \in P^{[S]}$ and $t \in S$. Consequently $P^{[S]}$ is a separately strict S-cpo. Moreover, the map $\sigma : P^{[S]} \to P$ defined by $\sigma(f) = f(1)$ is continuous. This is because directed suprema of functions are pointwise. Also, it is strict and action-preserving, since $\sigma(f_\perp) = f_\perp(1) = \perp_P$, and $\sigma(fs) = fs(1) = f(s1) = f(s) = f(1s) = f(1)s = (\sigma(f))s$, for all $s \in S$ and $f \in P^{[S]}$. Finally, it is a coreflection, because for a given S-cpo map $\alpha : A \to P$ from a separately strict S-cpo A, the map $\overline{\alpha} : A \to P^{[S]}$, defined by $\overline{\alpha}(a)(s) = \alpha(as)$, is a unique S-cpo map satisfying $\sigma \circ \overline{\alpha} = \alpha$. We also have that $\overline{\alpha}$ is strict, because for $s \in S$, $\overline{\alpha}(\perp_A)(s) = \alpha(\perp_A s) = \alpha(\perp_A) = \perp_P = f_\perp(s)$. Also, $\overline{\alpha}$ is continuous, since taking $D \subseteq^d A$ and $s \in S$, we get

$$\begin{aligned}\overline{\alpha}(\textstyle\bigvee^d D)(s) &= \alpha((\textstyle\bigvee^d D)s) = \alpha(\textstyle\bigvee^d_{x\in D} xs)\\ &= \textstyle\bigvee^d_{x\in D} \alpha(xs) = \bigvee^d_{x\in D} \overline{\alpha}(x)(s) = (\bigvee^d_{x\in D} \overline{\alpha}(x))(s).\end{aligned}$$

Further, $\overline{\alpha}$ is action-preserving, because for $s, t \in S$ and $a \in A$ we have

$$\overline{\alpha}(as)(t) = \alpha((as)t) = \alpha(a(st)) = \overline{\alpha}(a)(st) = (\overline{\alpha}(a)s)(t).$$

Finally, the uniqueness of $\overline{\alpha}$ is because taking an S-cpo map $h : A \to P^{[S]}$ with $\sigma \circ h = \alpha$, it follows that

$$\begin{aligned}h(a)(s) &= h(a)(s1) = (h(a)s)(1) = \sigma(h(a)s)\\ &= \sigma(h(as)) = \alpha(as) = \overline{\alpha}(a)(s)\end{aligned}$$

for $a \in A$ and $s \in S$. □

Before giving the left adjoint to the above inclusion functor, we need some lemmas and remarks.

Remark 3.6. (1) Recall that for a dcpo P, a subset $A \subseteq P$ is said to be a Scott-closed subset of P if it is a down-closed subset of P and if $D \subseteq^d A$ then $\bigvee^d D \in A$.

(2) For an S-cpo P, take Z to be the set of all zero elements of P and $\overline{Z}$ be the smallest Scott-closed subact of P containing Z. We know from Lemma 3.4 that any separately strict S-cpo B has only one zero element, $\perp_B$, and so every S-cpo map from P to a separately strict S-cpo B, takes Z to $\perp_B$ (this is because every action-preserving map preserves the zero elements and B has just one zero element).

In the following lemma, we show that every S-cpo map from an S-cpo P to a separately strict S-cpo B takes $\overline{Z}$ to $\perp_B$.

Lemma 3.7. *Let P be an S-cpo and $f : P \to B$ be an S-cpo map to a separately strict S-cpo B. Then $f(\overline{Z}) = \perp_B$ where Z is the set of all zero elements of P.*

Proof. We show that $f(\overline{Z}) = \perp_B$. We know $\downarrow \perp_B = \{\perp_B\}$ is a Scott-closed subset of B. Then $f^{-1}(\perp_B)$ is a Scott-closed subset of A containing Z. Also, for all $a \in f^{-1}(\perp_B)$ and $s \in S$, $as \in f^{-1}(\perp_B)$. This is because $\perp_B$ is a zero element. Hence $f^{-1}(\perp_B)$ is a Scott-closed subact of A containing Z, and so $\overline{Z} \subseteq f^{-1}(\perp_B)$ as required. $\square$

Lemma 3.8. *Let B be an S-cpo and I a Scott-closed subset of B which is also a subact of B. Then $B^* = (B \setminus I) \cup \{\perp_B\}$ is an S-cpo.*

Proof. First we show that B^* with the order of B is a cpo. To show that the supremum of every directed subset in B^* exists, take $D \subseteq^d B^*$. If $D = \{\perp_B\}$, then $\bigvee^d D = \perp_B \in B^*$. If $D \neq \{\perp_B\}$, then D is a directed subset of B and so $\bigvee^d D$ exists in B. Also $\bigvee^d D \notin I$, since D is not a subset of I, and so $\bigvee^d D \in B^*$ as required. Hence B^* is a cpo. Now, we show that B^* with the action defined by

$$a \cdot s = \begin{cases} as & \text{if} \quad as \notin I \\ \perp_B & \text{if} \quad as \in I \end{cases}$$

for all $a \in B^*$ and $s \in S$, is an S-cpo. First we show $a \cdot (st) = (a \cdot s) \cdot t$. We consider two cases:

Case (1): $a(st) \in I$. In this case, $a \cdot (st) = \perp_B$ and if $as \in I$, then $(a \cdot s) \cdot t = \perp_B \cdot t = \perp_B$; also if $as \notin I$, then $a \cdot s = as$ and $(a \cdot s) \cdot t = (as) \cdot t = \perp_B$ (since $(as)t = a(st) \in I$).

Case (2): $a(st) \notin I$. In this case, $a \cdot (st) = a(st) = (as)t = (a \cdot s) \cdot t$ ($as \notin I$ otherwise $(as)t = a(st) \in I$ which is a contradiction).

Now we show that the action is strict continuous. First notice that it is strict. This is true because $\perp_B \cdot \perp_S = \perp_B$, by the definition. The action is also continuous. Applying Lemma 2.8, we show that it is continuous in each component. First we show that the action is continuous in the first component. To see this, let $D \subseteq^d B^*$

and $s \in S$. Then $(\bigvee^d D) \cdot s = \bigvee^d_{y \in D} y \cdot s$. To see this, take $\bigvee^d D = x$ and consider two cases:

Case (1): $xs \in I$. In this case, $x \cdot s = \perp_B$. Also $xs \in I$ and since the action is order-preserving, $ys \in I$ for all $y \in D$. Then $x \cdot s = \perp_B = \bigvee^d_{y \in D} y \cdot s$, as required.

Case (2): $xs \notin I$. In this case, $x \cdot s = xs$. But for all $y \in D$, xs is an upper bound for the set $\{y \cdot s \mid \ y \in D\}$. This is because $y \cdot s = ys$ or $y \cdot s = \perp_B$. Also, if $b \in B^*$ is an upper bound of the mentioned set, then we show that $xs \leq b$. Let $K = \{y \in D \mid \ ys \notin I\}$. Then:

(1) $K \neq \emptyset$, because if $K = \emptyset$, then $ys \in I$, for all $y \in D$ and so $\bigvee^d_{y \in D} ys \in I$. This gives $xs = (\bigvee^d D)s = \bigvee^d_{y \in D} ys \in I$, which is a contradiction.

(2) For every $y \notin K$, there exists $y' \in K$ with $y \leq y'$. This is because for $y \notin K$ and $y_0 \in K$ (such element exists because $K \neq \emptyset$) there exists an element $y' \in K$ such that $y, y_0 \leq y'$ since D is directed. But then, $y_0 s \leq y's$, and hence $y' \in K$, since $y \in K$.

Therefore, for all $y \in K$, $ys = y \cdot s \leq b$, and for every $y \notin K$ by (2) there exists $y' \in K$ such that $ys \leq y's = y' \cdot s \leq b$. Thus, since $xs = \bigvee^d_{y \in D} ys$, we get $xs \leq b$ as required.

The continuity in the second component is proved similarly. Consequently, B^* with the action and the order defined above is an S-cpo. □

Remark 3.9. Let B be a separately strict S-cpo and I a Scott-closed subset of B which is also a subact of B. Then $B^* = (B \setminus I) \cup \{\perp_B\}$ with the action and order defined in the above lemma is a separately strict S-cpo. In fact, by the above lemma, B^* is an S-cpo. Furthermore, by the definition of the action, we have $\perp_B \cdot s = \perp_B$, for all $s \in S$. Also $a\perp_S = \perp_B \in I$ (B is a separately strict S-cpo and then $a\perp_S = \perp_B$, for all $a \in B$) gives $a \cdot \perp_S = \perp_B$ for all $a \in B^*$. Consequently, by part (3) of Remark 3.3, $B^* = (B \setminus I) \cup \{\perp_B\}$ is a separately strict S-cpo.

Lemma 3.10. *Let B be a separately strict S-cpo and I be a Scott-closed subset of B which is also a subact of B, and B^* be as in the above remark. Then, the mappings $h : B \to B^*$ defined by $h(b) = \perp_B$ for all $b \in B$, and $\gamma : B \to B^*$ defined by $\gamma(x) = \perp_B$ for all $x \in I$ and $\gamma(x) = x$ for all $x \notin I$ are S-cpo maps.*

Proof. It is clear that the mapping h is an S-cpo map. For γ, it is strict by its definition. For continuity, let $D \subseteq^d B$. We consider two cases:
Case (1): $\bigvee^d D \in I$. Then $\gamma(\bigvee^d D) = \perp_B = \bigvee^d \gamma(D)$. The last equality is because $D \subseteq I$.
Case (2): $\bigvee^d D \notin I$. Then $\gamma(\bigvee^d D) = \bigvee^d D$ and the set $K = \{y \in D \mid \ y \notin I\}$ is non-empty and for each $y \notin K$ there exists $y' \in K$ such that $y \leq y'$. Now, we show $\bigvee^d_{y \in D} \gamma(y) = \bigvee^d D$. It is clear that $\bigvee^d D$ is an upper bound of the set $\{\gamma(y) \mid \ y \in D\}$ (since $\gamma(y) = y$ or $\gamma(y) = \perp_B$). Let $b \in B^*$

be any upper bound for the set $\{\gamma(y) \mid\ y \in D\}$. Then, for all $y \in K$, $y = \gamma(y) \leq b$ and also for every $y \notin K$, $y \leq y' \leq b$ where $y' \in K$. Hence $y \leq b$ for all $y \in D$ and so $\bigvee^d D \leq b$, as required. To show that γ is action-preserving, let $b \in B$ and $s \in S$. We consider two cases:
Case (1): $bs \in I$. Then $\gamma(bs) = \perp_B$. If $b \notin I$, then $\gamma(b) \cdot s = b \cdot s = \perp_B$. Also if, $b \in I$, then $\gamma(b) \cdot s = \perp_B \cdot s = \perp_B$. Hence $\gamma(bs) = \gamma(b) \cdot s$ for all $b \in B$ and $s \in S$.
Case (2): $bs \notin I$. This gives $b \notin I$ and $\gamma(bs) = bs$. Also $\gamma(b) \cdot s = b \cdot s = bs$. Hence $\gamma(bs) = \gamma(b) \cdot s$ for all $b \in B$ and $s \in S$. Therefor γ is an S-cpo map. □

Theorem 3.11. *The category* $\mathbf{Cpo_{Sep}}$-S *is a reflective subcategory of the category* $\mathbf{Cpo}$-S.

Proof. Let P be an S-cpo, Z be the set of all zero elements of P, and $\overline{Z}$ be the smallest Scott-closed subact of P. Then by Lemma 3.8, $P^* = (P \setminus \overline{Z}) \cup \{\perp_P\}$ is an S-cpo. Now we show that P^* is a separately strict S-cpo. By part (3) of Remark 3.3, it is sufficient to show that the action is strict in each of its components. Note that it is strict in the first component by its definition ($\perp_P \cdot s = \perp_P$, for all $s \in S$). Also it is strict in the second component. In fact, $p\perp_S$ is a zero element of P (notice that S is a separately strict cpo-monoid and so $\perp_S$ is a zero element of S). This gives $p\perp_S \in \overline{Z}$ and so $p \cdot \perp_S = \perp_P$ for all $p \in P^*$, as required. Consequently P^* is a separately strict S-cpo. Define the reflection map $\tau\colon P \to P^*$ by

$$\tau(p) = \begin{cases} p & \text{if} \quad p \notin \overline{Z} \\ \perp_P & \text{if} \quad p \in \overline{Z} \end{cases}$$

We show that τ is an S-cpo map. First notice that τ is strict, since $\perp_P$ is a zero element. To show that it is continuous, let $D \subseteq^d P$. We consider two cases:
Case (1): $\bigvee^d D \notin \overline{Z}$, then $\tau(\bigvee^d D) = \bigvee^d D$. We show $\bigvee^d_{x \in D} \tau(x) = \bigvee^d D$. Let $K = \{x \in D \mid\ x \notin \overline{Z}\}$. The set K is non-empty (If $K = \emptyset$, then $D \subseteq^d \overline{Z}$ and so $\bigvee^d D \in \overline{Z}$). For all $x \in K$, $\tau(x) = x \leq \bigvee^d D$ and also for all $x \notin K$, $\tau(x) = \perp_P \leq \bigvee^d D$. Let $b \in P^*$ be any upper bound for the set $\{\tau(x) \mid x \in D\}$. For $x \in K$, $x = \tau(x) \leq b$. Also for $x \notin K$ and $x' \in K$ (because $K \neq \emptyset$) there exists $x'' \in K$ such that $x < x''$ and $x' \leq x''$ (since D is directed, such x'' exists. Also, since $\overline{Z}$ is a lower set and $x' \notin \overline{Z}$, we have $x'' \notin \overline{Z}$ and so $x'' \in K$). Hence $x < x'' = \tau(x'') \leq b$. Consequently $\bigvee^d D \leq b$, as required.
Case (2): $\bigvee^d D \in \overline{Z}$; then $D \subseteq \overline{Z}$ (because $\overline{Z}$ is a lower set). Hence $\tau(\bigvee^d D) = \perp_P = \bigvee^d_{x \in D} \tau(x)$.

To see that τ is action-preserving, let $p \in P$ and $s \in S$. Then, consider two cases:
Case (1): $bs \in \overline{Z}$, then $\tau(bs) = \perp_P$. If $b \in \overline{Z}$, then $\tau(b) \cdot s = \perp_B \cdot s = \perp_P$. If $b \notin \overline{Z}$, then $\tau(b) \cdot s = b \cdot s = \perp_B$ (the last equality is because $bs \in \overline{Z}$).
Case (2): $bs \notin \overline{Z}$, then $b \notin \overline{Z}$ and so $\tau(bs) = bs = b \cdot s = \tau(b) \cdot s$. Hence τ is an

S-cpo map. Further, given an S-cpo map $\alpha\colon P \to B$ to a separately strict S-cpo B, the map $\overline{\alpha}\colon P^* \to B$, given by $\overline{\alpha}(p) = \alpha(p)$, for all $p \in P^*$, is a unique S-cpo map satisfying $\overline{\alpha} \circ \tau = \alpha$. Notice that $\overline{\alpha}$ is strict. This is because α is strict. To prove continuity, let $D \subseteq^d P^*$. Then

$$\overline{\alpha}(\bigvee^d D) = \alpha(\bigvee^d D) = \bigvee^d_{x\in D} \alpha(x) = \bigvee^d_{x\in D} \overline{\alpha}(x).$$

The second equality is because D is also a directed subset of P and α is continuous.

To show that $\overline{\alpha}$ is action-preserving, let $p \neq \bot_P \in P^*$ and $s \in S$. We consider two cases:
Case (1): If $ps \in \overline{Z}$, then

$$\overline{\alpha}(p \cdot s) = \overline{\alpha}(\bot_P) = \bot_B = \alpha(ps) = \alpha(p)s = \overline{\alpha}(p)s$$

where the third equality holds by Lemma 3.7.
Case (2): If $ps \notin \overline{Z}$, then $p \notin \overline{Z}$ and so

$$\overline{\alpha}(p \cdot s) = \overline{\alpha}(ps) = \alpha(ps) = \alpha(p)s = \overline{\alpha}(p)s$$

Hence $\overline{\alpha}$ is an S-cpo map.

Now, we show $\overline{\alpha} \circ \tau = \alpha$. For $p \in P$ we consider two cases:
Case (1): $p \in \overline{Z}$, then

$$(\overline{\alpha} \circ \tau)(p) = \overline{\alpha}(\tau(p)) = \overline{\alpha}(\bot_P) = \bot_B = \alpha(p)$$

The last equality holds by Lemma 3.7.
Case (2): $p \notin \overline{Z}$, then

$$(\overline{\alpha} \circ \tau)(p) = \overline{\alpha}(\tau(p)) = \overline{\alpha}(p) = \alpha(p)$$

To establish the uniqueness of $\overline{\alpha}$, suppose that $h : P^* \to B$ is also an S-cpo map such that $h \circ \tau = \alpha$. Notice that $h(\bot_P) = \bot_B = \overline{\alpha}(\bot_P)$. Also for all $p \neq \bot_P \in P^*$, $h(p) = h(\tau(p)) = \alpha(p) = \overline{\alpha}(p)$, where the second equality is because $p \in P^*$ and so $\tau(p) = p$ $(p \notin \overline{Z})$. □

4 Free and Cofree separately strict S-cpo's

In this section we give a description of free and cofree separately strict S-cpo's on a cpo. The following two lemmas are frequently used in this section.

Lemma 4.1. [4, 11] *Let $\{A_i \mid i \in I\}$ be a family of dcpo's. Then the directed join of a directed subset $D \subseteq^d \prod_{i\in I} A_i$ is calculated as $\bigvee^d D = (\bigvee^d D_i)_{i\in I}$ where, for each $i \in I$,*

$$D_i = \{a \in A_i \mid \exists d = (d_k)_{k\in I} \in D, a = d_i\}.$$

Lemma 4.2. *Let A be a dcpo. Then $D \subseteq A_\perp = \perp \oplus A$ is directed if and only if $D \subseteq^d A$, $D = \{\perp\}$, or $D = \{\perp\} \cup D'$ where $D' \subseteq^d A$.*

Proof. It is clear that the directed subsets of A, and subsets $D \subseteq A$ which are of the forms $\{\perp\}$ or $\{\perp\}\cup D'$, where $D' \subseteq^d A$, are directed subsets of $A_\perp$. Conversely, let $D \subseteq^d A_\perp$. Then in the case that $\perp \notin D$ we have $D \subseteq^d A$, and in the case where $\perp \in D$ we have $D = \{\perp\} \cup D'$ and $D' \subseteq^d A$, because for $x, y \in D'$ there exists $z \in D$ such that $x, y \leq z$, and this gives $z \neq \perp$. □

Free separately strict S-cpo on a cpo P. By a *free separately strict S-cpo on a cpo P* we mean a separately strict S-cpo F together with a strict continuous map $\tau : P \to F$ with the universal property that given any separately strict S-cpo A and a strict continuous map $f : P \to A$ there exists a unique S-cpo map $\overline{f} : F \to A$ such that $\overline{f} \circ \tau = f$.

Recall that the *smash product* of the cpo's A and B is the cpo $A \otimes B = \perp \oplus ((A \setminus \{\perp_A\}) \times (B \setminus \{\perp_B\}))$.

Theorem 4.3. *Let S be a separately strict cpo-monoid. Then for a given cpo P, the free separately strict S-cpo on P is $F = P \otimes S$.*

Proof. The action on $P \otimes S$ is defined by $(p, t) \cdot s = (p, ts)$ if $ts \neq \perp_S$, $(p, t) \cdot s = \perp$ if $ts = \perp_S$, and $\perp \cdot s = \perp$, for all $s, t \in S$, $(p, t) \in (P \setminus \{\perp_P\}) \times (S \setminus \{\perp_S\})$. First we check the properties of the action. To show $(p, t) \cdot (ss') = ((p, t) \cdot s) \cdot s'$, we consider two cases:

Case (1): If $t(ss') = \perp_S$, then $(p, t) \cdot (ss') = \perp$. Now, if $ts = \perp_S$ then $(p, t) \cdot s = \perp$ and so $((p, t) \cdot s) \cdot s' = \perp \cdot s' = \perp$. If $ts \neq \perp_S$ then $(p, t) \cdot s = (p, ts)$ and so $((p, t) \cdot s) \cdot s' = (p, ts) \cdot s' = \perp$; the last equality is because $(ts)s' = t(ss') = \perp_S$.

Case (2): If $t(ss') \neq \perp_S$, then $(p, t) \cdot (ss') = (p, tss') = (p, ts) \cdot s' = ((p, t) \cdot s) \cdot s'$. The second equality is because $ts \neq \perp_S$, otherwise $t(ss') = (ts)s' = \perp_S s = \perp_S$ which is a contradiction, and the last equality is because of the definition of the action. Hence $P \otimes S$ with this action is an S-act. Now, we show that it is a separately strict S-cpo. We know from [1] that $P \otimes S$ is a cpo. Then we show that the mappings $R_s : P \otimes S \to P \otimes S$, $x \rightsquigarrow x \cdot s$, and $L_x : S \to P \otimes S$, $s \rightsquigarrow x \cdot s$, are strict continuous, for all $s \in S$ and $x \in P \otimes S$. Since $\perp \cdot s = \perp$, R_s is strict for all $s \in S$ and since S is a separately strict cpo-monoid, $s\perp_S = \perp_S$ and so $(p, s) \cdot \perp_S = \perp$. Therefore $L_x : S \to P \otimes S$ is strict for all $x \in P \otimes S$. Now, we show that $R_s : P \otimes S \to P \otimes S$ is continuous for all $s \in S$. To prove continuity, let $D \subseteq^d A \otimes B$. Applying Lemma 4.2 we consider two cases:

Case (1): Let $D \subseteq^d (P \setminus \{\perp_P\}) \times (S \setminus \{\perp_S\})$. In this case, by Lemma 4.1, $\bigvee^d D = (\bigvee^d D_1, \bigvee^d D_2)$ where $D_1 = domD$ and $D_2 = codomD$. Now we consider two subcases:

Subcase (1a): $(\bigvee^d D_2)s \neq \perp_S$. In this subcase, we have

$$(\bigvee^d D) \cdot s = (\bigvee^d D_1, \bigvee^d D_2) \cdot s = (\bigvee^d D_1, (\bigvee^d D_2)s) = (\bigvee^d D_1, \bigvee^d_{y \in D_2} ys).$$

Then we claim

$$\bigvee^d_{(x,y)\in D} (x, y) \cdot s = (\bigvee^d D_1, \bigvee^d_{y \in D_2} ys) \quad (*).$$

Let $K = \{(a, b) \in D \mid \ bs \neq \perp_B\}$. Then K satisfies:
(1) $K \neq \emptyset$, because otherwise $(\bigvee^d D_2)s = \perp_S$ which is a contradiction.
(2) For all $(a, b) \in K$, $(a, b) \cdot s = (a, bs)$, by the definition of the action on $P \otimes S$.
(3) For all $(a, b) \in K$ and $(a', b') \notin K$, there exists $(a'', b'') \in K$ with $(a, b) \leq (a'', b'')$ and $(a', b') \leq (a'', b'')$, since D is directed. But, then $bs \leq b''s$, and hence $b's \neq \perp_S$ and so $(a'', b'') \in K$.

Now to prove $(*)$, first we see that $(\bigvee^d D_1, \bigvee^d_{y\in D_2} ys)$ is an upper bound of the set $\{(a, b) \cdot s \mid (a, b) \in D\}$. Also for all $(x, y) \in K$, $(x, y) \cdot s = (x, ys) \leq (\bigvee^d D_1, \bigvee^d_{y\in D_2} ys)$. For $(a, b) \notin K$, $(a, b) \cdot s = \perp \leq (\bigvee^d D_1, \bigvee^d_{y\in D_2} ys)$, as required. Secondly, if (a', b') is an upper bound of the set $\{(a, b) \cdot s \mid (a, b) \in D\}$, let $x \in D_1$. Then there exists $y \in D_2$ such that $(x, y) \in D$. If $(x, y) \in K$, then $(x, ys) \leq (a', b')$ and so $x \leq a'$. If $(x, y) \notin K$, then by (3) there exists $(x'', y'') \in K$ such that $(x, y) < (x'', y'')$. This gives $x \leq x''$ and $(x'', y'') \cdot s = (x'', y''s) \leq (a', b')$. Then $x \leq x'' \leq a'$. So for all $x \in D_1$, $x \leq a'$ and so $\bigvee^d D_1 \leq a'$ $(**)$. Also let $y \in D_2$. Then there exists $x \in D_1$ such that $(x, y) \in D$. If $(x, y) \in K$ then $(x, ys) \leq (a', b')$ and so $ys \leq b'$. If $(x, y) \notin K$ then $ys = \perp_S$ and so $ys \leq b'$. Hence $\bigvee^d_{y\in D_2} ys \leq b'$ $(***)$. Then, by $(**)$ and $(***)$, we get $(\bigvee^d D_1, \bigvee^d_{y\in D_2} ys) \leq (a', b')$, as required.

Subcase (1b): $(\bigvee^d D_2)s = \perp_S$. In this subcase, we have

$$(\bigvee^d D) \cdot s = (\bigvee^d D_1, \bigvee^d D_2) \cdot s = \perp = \bigvee^d_{(x,y)\in D} (x, y) \cdot s$$

The last equality is because for all $(x, y) \in D$ we have $y \in D_2$ and so $ys \leq (\bigvee^d D_2)s = \perp_S$. This gives $ys = \perp_S$ and so by the definition of the action, $(x, y) \cdot s = \perp$ for all $(x, y) \in D$.

Case (2): $D = D' \cup \{\perp\}$ where $D' \subseteq (P \setminus \{\perp_P\}) \times (S \setminus \{\perp_S\})$ is directed. Then by Lemma 4.1, $\bigvee^d D' = (\bigvee^d D_1, \bigvee^d D_2)$ where $D_1 = domD'$ and $D_2 = codomD'$. Then by case (1), $(\bigvee^d D') \cdot s = \bigvee^d_{(x,y) \in D'} (x, y) \cdot s$ and so

$$\begin{aligned}(\bigvee^d D) \cdot s \ &= (\bigvee^d D') \cdot s = ((\bigvee^d D') \cdot s) \vee \perp \\ &= (\bigvee^d_{(x,y)\in D'} (x, y) \cdot s) \vee \perp \cdot s = \bigvee^d_{(x,y)\in D} (x, y) \cdot s\end{aligned}$$

as required. Now, we show that $L_x : S \to P \otimes S$ is continuous for all $x \in P \otimes S$. For this, let $T \subseteq^d S$ and $x \in P \otimes S$. If $x = \perp$, then $\perp \cdot (\bigvee^d T) = \perp = \bigvee^d_{t\in T} \perp \cdot t$. If $x = (p, s)$ then we consider two cases:
Case (1): $s(\bigvee^d T) = \perp_S$, then $(p, s)\cdot(\bigvee^d T) = \perp = \bigvee^d_{t\in T}((p, s)\cdot t)$, the last equality is because $s(\bigvee^d T) = \bigvee^d_{t\in T} st = \perp_S$ and so $st = \perp_S$ for all $t \in T$.
Case (2): $s(\bigvee^d T) \neq \perp_S$, then

$$(p, s) \cdot (\bigvee^d T) = (p, s(\bigvee^d T)) = (p, \bigvee^d_{t\in T} st) = \bigvee^d_{t\in T} (p, s) \cdot t$$

where the second equality is because the action on S is continuous in each component. To prove the last equality, let $K' = \{t \in T \mid st \neq \perp_S\}$. Similar to the proof in the above for K, we have:
(1) K' is non-empty (otherwise, $\bigvee^d_{t\in T} st = s(\bigvee^d T) = \perp_B$ which is a contradiction.)
(2) For $t \notin K'$ and $t' \in K'$, there exists $t'' \in K'$ with $t \leq t''$ and $t' \leq t''$, since T is directed. But, then $st' \leq st''$ and hence $st'' \neq \perp_S$, so $t'' \in K'$.
First notice that $(p, \bigvee^d_{t\in T} st)$ is an upper bound for the set $\{(p, s) \cdot t \mid t \in T\}$. In fact, for every $(p, s)\cdot t$, if $st = \perp_S$, then $(p, s)\cdot t = \perp \leq (p, \bigvee^d_{t\in T} st)$. Also if $st \neq \perp_S$, then $(p, s) \cdot t = (p, st) \leq (p, \bigvee^d_{t\in T} st)$. Let (q, s') be any upper bound of the set $\{(p, s) \cdot t \mid t \in T\}$. Since K' is non-empty then for $t_0 \in K'$, $(p, st_0) = (p, s) \cdot t_0 \leq (q, s')$. This gives $p \leq q$. Also for $t \in K'$, $(p, st) = (p, s) \cdot t \leq (q, s')$ and so $st \leq s'$. For $t \notin K'$, by (2), there exists $t' \in K'$ such that $t \leq t'$. This gives $st \leq st' \leq s'$. Hence for all $t \in T$, $st \leq s'$ and so $\bigvee^d_{t\in T} st \leq s'$. Thus $(p, \bigvee^d_{t\in T} st) \leq (q, s')$, as required.

Now, we show that the map $\tau : P \to P \otimes S$ defined by $\tau(p) = (p, 1)$ and $\tau(\perp_P) = \perp$ is a universal strict continuous map. It is strict by its definition. To prove continuity, let $D \subseteq^d P$ and consider two cases:
Case (1): $\perp_P \in D$, then

$$\tau(\bigvee^d D) = (\bigvee^d D, 1) = \bigvee^d_{x\in D\setminus\{\perp_P\}} (x, 1) = (\bigvee^d_{x\in D\setminus\{\perp_P\}} (x, 1)) \vee \perp = \bigvee^d_{x\in D} \tau(x)$$

Case (2): $\perp_P \notin D$, then

$$\tau(\bigvee^{d} D) = (\bigvee^{d} D, 1) = \bigvee^{d}_{x\in D} (x,1) = \bigvee^{d}_{x\in D} (x,1) = \bigvee^{d}_{x\in D} \tau(x).$$

Finally, to prove the universal property of $\tau : P \to P \otimes S$, take a strict continuous map $f : P \to B$ to a separately strict S-cpo B. Then the map $\overline{f} : P \otimes S \to B$ defined by $\overline{f}(p,s) = f(p)s$ and $f(\perp) = \perp_B$, for all $(p,s) \in (P \setminus \{\perp_P\}) \times (S \setminus \{\perp_S\})$, is the unique separately strict S-cpo satisfying $\overline{f} \circ \tau = f$. We show that $\overline{f}$ is strict continuous and action-preserving. First notice that it is strict by its definition. To prove continuity, let $D \subseteq^d P \otimes S$. Applying Lemma 4.2, we consider two cases:
Case (1): $D \subseteq^d (P \setminus \{\perp_P\}) \times (S \setminus \{\perp_S\})$. In this case by Lemma 4.1, $\bigvee^d D = (\bigvee^d D_1, \bigvee^d D_2)$ where $D_1 = domD$ and $D_2 = codomD$. Then

$$\begin{aligned}\overline{f}(\bigvee^d D) &= \overline{f}((\bigvee^d D_1, \bigvee^d D_2)) = f(\bigvee^d D_1)(\bigvee^d D_2)\\ &= (\bigvee^d_{x\in D_1} f(x))(\bigvee^d D_2) = \bigvee^d_{x\in D_1}(f(x)(\bigvee^d D_2))\\ &= \bigvee^d_{x\in D_1}\bigvee^d_{t\in D_2}(f(x)t) = \bigvee^d_{(x,s)\in D}\overline{f}((x,s)).\end{aligned}$$

Case (2): $D = D' \cup \{\perp\}$, where $D' \subseteq^d (P\setminus\{\perp_P\})\times(S\setminus\{\perp_S\})$. Then $\bigvee^d D = \bigvee^d D'$ and by Case (1), $\overline{f}(\bigvee^d D') = \bigvee^d_{(x,s)\in D'} \overline{f}((x,s))$. Hence

$$\begin{aligned}\overline{f}(\bigvee^d D) &= \overline{f}(\bigvee^d D') = (\overline{f}(\bigvee^d D')) \vee \perp_B\\ &= (\bigvee^d_{(x,s)\in D'} \overline{f}((x,s))) \vee \overline{f}(\perp) = \bigvee^d \overline{f}(D).\end{aligned}$$

Now, we show that the mapping $\overline{f}$ is action-preserving. First notice that $\overline{f}(\perp \cdot s) = \overline{f}(\perp) = \perp_B = \perp_B s = \overline{f}(\perp)s$. Secondly for $(p,t) \in (P \setminus \{\perp_P\}) \times (S \setminus \{\perp_S\})$ and $s \in S$, we consider two cases:
Case (1): If $ts \neq \perp_S$, then

$$\overline{f}((p,t)\cdot s) = \overline{f}((p,ts)) = f(p)(ts) = (f(p)t)s = \overline{f}((p,t))s.$$

Case (2): If $ts = \perp_S$, then

$$\overline{f}((p,t)\cdot s) = \overline{f}(\perp) = \perp_B = f(p)\perp_S = f(p)(ts) = (f(p)t)s = (\overline{f}((p,t)))s.$$

To establish the uniqueness of $\overline{f}$, suppose that $h : P\otimes S \to B$ is also a separately strict S-cpo map such that $h\circ\tau = f$. Then for all $(p,t) \in (P\setminus\{\perp_P\})\times(S\setminus\{\perp_S\})$,

$$h((p,t)) = h((p,1)\cdot t) = h((p,1))t = ((h\circ\tau)(p))t = f(p)t = \overline{f}((p,t))$$

as required. □

Corollary 4.4. *The forgetful functor from* $\mathbf{Cpo_{Sep}}$-S *to* $\mathbf{Cpo}$ *has a left adjoint.*

Cofree separately strict S-cpo over a cpo. By a *cofree separately strict S-cpo on a cpo P* we mean a separately strict S-cpo K together with a strict continuous map $\sigma : K \to P$ with the universal property that given any separately strict S-cpo A and a strict continuous map $g : A \to P$ there exists a unique S-cpo map $\overline{g} : A \to K$ such that $\sigma \circ \overline{g} = g$.

Theorem 4.5. *For a given cpo P and separately strict cpo-monoid S, the cofree separately strict S-cpo on P is the set $K = [S \to P]$, of all strict continuous maps from S to P, with pointwise order and the action given by $(fs)(t) = f(st)$, for $s, t \in S$ and $f \in [S \to P]$.*

Proof. We know from [11] that $[S \to P]$ is a cpo. Also the action on $[S \to P]$ is continuous (see [15]). Then by Lemma 2.8, the mappings $R_s : [S \to P] \to [S \to P]$ defined by $R_s(f) = fs$ and $L_f : S \to [S \to P]$ define by $L_f(s) = fs$, for all $s \in S$ and $f \in [S \to P]$, are continuous. Also, it is an easy computation to show $f_\perp s = f_\perp$ and $f\perp_S = f_\perp$, for all $s \in S$ and $f \in [S \to P]$, where $f_\perp$ is the bottom element of $[S \to P]$. Hence $[S \to P]$ is a separately strict S-cpo. Now, take the cofree map $\sigma : [S \to P] \to P$ defined by $\sigma(f) = f(1)$. It is continuous (see [15]). Now, we show that σ is strict. We have $\sigma(f_\perp) = f_\perp(1) = \perp_P$. Then σ is an S-cpo map. Further, given a strict continuous map $\alpha : A \to P$ from a separately strict S-cpo A, the map $\overline{\alpha} : A \to [S \to P]$, given by $\overline{\alpha}(a)(s) = \alpha(as)$, is the unique separately strict S-cpo map satisfying $\sigma \circ \overline{\alpha} = \alpha$. First notice that $\overline{\alpha}$ is a continuous action-preserving map (see [15]). Now, we show that it is strict. In fact,

$$\overline{\alpha}(\perp_A)(s) = \alpha(\perp_A s) = \alpha(\perp_A) = \perp_P = f_\perp(s).$$

Then $\overline{\alpha}$ is an S-cpo map. To establish the uniqueness of $\overline{\alpha}$, suppose that $h : A \to [S \to P]$ is also an S-cpo map such that $\sigma \circ h = \alpha$. Then for all $a \in A$ and $s \in S$,

$$\begin{aligned} h(a)(s) &= h(a)(s1) = (h(a)s)(1) = \sigma(h(a)s) \\ &= \sigma(h(as)) = \alpha(as) = \overline{\alpha}(a)(s). \end{aligned}$$

□

5 Some categorical properties of $\mathbf{Cpo_{sep}}$-S

In this section we show that the category $\mathbf{Cpo_{Sep}}$--S is complete and cocomplete, and also give a description of products and coproducts in this category. Further, we study the monomorphisms and epimorphisms. Finally, we show that this category is not cartesian closed.

5.1 Limits and coproducts in $\mathbf{Cpo_{Sep}}$-S

First recall from [1, 14] that the product of a family of dcpo's, cpo's, S-dcpo's, and S-cpo's is their cartesian product with the natural order and action. In particular, the terminal object in these categories and also in the category $\mathbf{Cpo_{Sep}}$-S is the singleton $\{\theta\}$.

Remark 5.1. (1) Since the category of separately strict S-cpo's is a reflective subcategory of the category of S-cpo's, by Theorem 3.11, the category $\mathbf{Cpo_{Sep}}$-S is complete and all limits in this category are calculated the same as in the category of S-cpo's. Thus, the product of a family of separately strict S-cpo's is their cartesian product, and the equalizer of S-cpo maps $f, g : A \to B$ between separately strict S-cpo's is $E = \{x \in A \mid f(x) = g(x)\}$.

(2) Since the inclusion functor from $\mathbf{Cpo_{Sep}}$-S to $\mathbf{Cpo}$-S has a right adjoint by Theorem 3.5, it preserves colimits. So, by [14], the initial object in $\mathbf{Cpo_{Sep}}$-S is the singleton and the coproduct of the family $\{A_i \mid \ i \in I\}$ of separately strict S-cpo's is $A = \biguplus_{i\in I} A_i$.

Following Theorem 23.14 [10], to prove that $\mathbf{Cpo_{Sep}}$-S is cocomplete, having the above remark (1) it is enough to show that it is well-powered and has a coseparator.

Recall that an object C of a category $\mathcal{C}$ is called a *coseparator* if the functor $hom(-, C) : \mathcal{C}^{op} \to \mathbf{Set}$ is faithful. In other words, for each pair of distinct arrows $f, g : A \to B$ there exists an arrow $h : B \to C$ such that $h \circ f \neq h \circ g$.

Theorem 5.2. *The category* $\mathbf{Cpo_{Sep}}$-S *has a coseparator.*

Proof. We show that for each cpo P with $|P| \geq 2$, the cofree object $[S \to P]$ described in Theorem 4.5 is a coseparator. Let $f, g : A \to B$ be separately strict S-cpo maps with $f \neq g$. To give a separately strict S-cpo map $h : B \to [S \to P]$ with $h \circ f \neq h \circ g$, first we define a cpo map $k : B \to P$ such that $k \circ f \neq k \circ g$.

Since $f \neq g$, there exists $a \in A$ with $f(a) \neq g(a)$. We consider three cases

$$(1)\ \ f(a) < g(a) \qquad (2)\ \ g(a) < f(a) \qquad (3)\ \ f(a) \parallel g(a)$$

Let $f(a) < g(a)$. Take $B' = \{b \in B \mid \ b \leq f(a)\}$. Define $k : B \to P$ by

$$k(b) = \begin{cases} \perp_P & \text{if } b \in B' \\ y & \text{otherwise} \end{cases}$$

where $y \in P$ and $y \neq \perp_P$ (such y exists since $|P| \geq 2$). First we show that k is order-preserving, and hence it takes directed subsets to directed ones. Let $b_1, b_2 \in B$ with $b_1 \leq b_2$. If $b_1 \in B'$, then for the case where $b_2 \in B'$, $\perp_P = k(b_1) = k(b_2)$; and for the case where $b_2 \notin B'$, $\perp_P = k(b_1) < y = k(b_2)$. Also, if $b_1 \notin B'$, then

$b_2 \notin B'$ and so $k(b_1) = k(b_2) = y$. To prove the continuity of k, let $D \subseteq^d B$. Notice that $\bigvee^d D \in B' \Leftrightarrow D \subseteq B'$. Now, if $\bigvee^d D \in B'$, then $D \subseteq B'$ and so $k(\bigvee^d D) = \perp_P = \bigvee^d_{z \in D} k(z)$. Also, if $\bigvee^d D \notin B'$, then $k(\bigvee^d D) = y$ and $D \not\subseteq B'$. Thus $D \setminus B' \neq \emptyset$, and

$$\bigvee_{z\in D}^{d} k(z) = \bigvee_{z \in (D \setminus B') \cup (B' \cap D)}^{d} k(z) = y \vee \perp_P = y$$

as required. Finally, since $[S \to P]$ is the cofree separately strict S-cpo on P, there exists a unique separately strict S-cpo map $h : B \to [S \to P]$ such that $\sigma \circ h = k$, where σ is the cofree map defined in Theorem 4.5. This gives $h \circ f \neq h \circ g$, and so $[S \to P]$ is a coseparator.

The case (2) is proved similarly. And for case (3), take $B' = \{b \in B \mid \ b \leq f(a)\}$ or $B' = \{b \in B \mid \ b \leq g(a)\}$ in the proof of case (1). □

Theorem 5.3. *The category* $\mathbf{Cpo_{Sep}}$-S *is well-powered.*

Proof. We should prove that the class of isomorphic subobjects of any separately strict S-cpo is a set. This is true since, by Lemma 5.5, monomorphisms are one-one. □

Proposition 5.4. *The category* $\mathbf{Cpo_{Sep}}$-S *is cocomplete.*

Proof. The result follows by completeness of $\mathbf{Cpo_{Sep}}$-S and by Theorem 23.14 [10] and Theorems 5.2, 5.3. □

5.2 Monomorphisms and Epimorphisms

In this subsection, we study the relation between epimorphisms and onto maps and between monomorphisms and one-one maps in the category of separately strict S-cpo's. First the following less surprising theorem.

Theorem 5.5. *A morphism in* $\mathbf{Cpo_{Sep}}$-S *is a monomorphism if and only if it is one-one.*

Proof. Let $h : A \to B$ be a monomorphism in $\mathbf{Cpo_{Sep}}$-S, and $h(a) = h(a')$. Consider the S-cpo maps $f, g : S \to A$ given by $f(s) = as$ and $g(s) = a's$, for $a, a' \in A$. Then, $h \circ f = h \circ g$, and so we conclude that $f = g$. Thus, $a = a'$. □

In the following example, we see that an epimorphism is not necessarily an onto map.

Example 5.6. (1) We show that $(\mathbb{N}^\infty, min, \leq)$ in which the order is the natural order, is a cpo-monoid. First notice that it is a cpo. In fact for every subset X, we have $\bigvee^d X = max\{X\}$ if X is a finite set and $\bigvee^d X = \infty$ if X is an infinite set. Now, we show that the operation min is strict continuous. Since, $min(1,1) = 1$, the operation min is strict. To prove continuity, we show that it is continuous separately strict in each component. Since the operation min is commutative, it is enough to show that $min(\bigvee^d X, n) = \bigvee^d_{x\in X} min(x,n)$ for all $X \subseteq \mathbb{N}^\infty$ and $n \in \mathbb{N}^\infty$. Let $X \subseteq \mathbb{N}^\infty$ and $n \in \mathbb{N}^\infty$, then we consider two cases:

Case (1): X is a finite set. Then $\bigvee^d X = max\{X\}$. Now we consider two subcases:
Subcase (1a): $\bigvee^d X \leq n$, then $min(\bigvee^d X, n) = \bigvee^d X = \bigvee^d_{x\in X} min(x,n)$ (the last equality is because $x \leq \bigvee^d X \leq n$ for all $x \in X$).
Subcase (1b): $\bigvee^d X \geq n$, then $min(\bigvee^d X, n) = n$ and

$$\begin{aligned}\textstyle\bigvee^d_{x\in X} min(x,n) &= (\textstyle\bigvee^d\{min(x,n)|\ x \leq n\}) \vee (\textstyle\bigvee^d\{min(x,n)|\ x \leq n\}) \\ &= \{x, n | x \leq n\} \vee \{n\} = n\end{aligned}$$

Case (2): X is an infinite set. Then $min(\bigvee^d X, n) = min(\infty, n) = n$ and

$$\bigvee^d_{x\in X} min(x,n) = (\bigvee_{x\leq n} min(x,n)) \vee (\bigvee^d_{x>n} min(x,n)) = (\bigvee^d_{x\leq n}\{x\}) \vee \{n\} = n$$

(notice that $\{x|\ x > n\} \neq \emptyset$, because X is an infinite set). Hence $(\mathbb{N}^\infty, min, \leq)$ is a cpo-monoid.
(2) Consider the cpo-monoid $S = (\mathbb{N}^\infty, min, \leq)$. Since $\perp_{\mathbb{N}^\infty} = 1$ and $min(1,n) = min(n,1) = 1$ for all $n \in \mathbb{N}^\infty$, by the part (3) of Remark 3.3, it is a separately strict S-cpo. Now, consider the cpo $\mathbb{N}$ with the order defined by $n \leq m$ if and only if $n = 1$ or $n = m$, for all $n, m \in \mathbb{N}$. Then $\mathbb{N}$ with the action $\lambda : \mathbb{N} \times \mathbb{N}^\infty \to \mathbb{N}$, $(n,m) \rightsquigarrow min(n,m)$, is a separately strict $\mathbb{N}^\infty$-cpo. Since $min(1,m) = 1$, each $R_m : \mathbb{N} \to \mathbb{N}$, $n \rightsquigarrow min(n,m)$, is strict. To prove continuity, let $D \subseteq^d \mathbb{N}$. Then $D = \{n\}$ or $D = \{1, n\}$ for $n \in \mathbb{N}$. For $D = \{1, n\}$, $R_m(\bigvee^d D) = R_m(n) = min(n,m)$ and $\bigvee^d\{R_m(1), R_m(n)\} = \bigvee^d\{1, min(n,m)\} = min(n,m)$. Also for $D = \{n\}$, $R_m(\bigvee^d D) = R_m(n) = \bigvee^d\{R_m(n)\}$. Hence R_m is strict continuous for all $m \in \mathbb{N}^\infty$.

Also $L_n : \mathbb{N}^\infty \to \mathbb{N}$, $m \rightsquigarrow min(n,m)$, is strict, because $L_n(1) = min(n,1) = 1$. To prove continuity, let $D \subseteq^d \mathbb{N}^\infty$. Then consider two cases:

Case (1): $D \subseteq^d \mathbb{N}^\infty$ is infinite. Then $\bigvee^d D = \infty$ and so $L_n(\bigvee^d D) = L_n(\infty) = min(n, \infty) = n$ and

$$\begin{aligned}\textstyle\bigvee^d L_n(D) &= (\textstyle\bigvee^d\{L_n(x) \mid x \leq n\}) \vee (\textstyle\bigvee^d\{L_n(x) \mid n < x\}) \\ &= (\textstyle\bigvee^d\{x \mid x \leq n\}) \vee (\textstyle\bigvee^d\{n \mid n < x\}) = n\end{aligned}$$

as required.

Case (2): $D \subseteq^d \mathbb{N}^\infty$ is finite. Then $\bigvee^d D = maxD$. If $n \leq maxD$, then $L_n(\bigvee^d D) = L_n(maxD) = min(n, maxD) = n$ and $\bigvee^d L_n(D) = \bigvee^d \{L_n(x) \mid x \in D\} = \bigvee^d \{min(n,x) \mid x \in D\} = \bigvee^d \{n\} = n$. If $maxD < n$, then $L_n(\bigvee^d D) = L_n(maxD) = min(n, maxD) = maxD$ and

$$\begin{aligned}\bigvee^d L_n(D) &= \bigvee^d \{L_n(x) \mid x \in D\} = \bigvee^d \{min(n,x) \mid x \in D\}\\ &= \bigvee^d \{x \mid x \in D\} = \bigvee^d D = maxD\end{aligned}$$

as required. Hence $\mathbb{N}$ is a separately strict $\mathbb{N}^\infty$-monoid. Now consider the inclusion $h : \mathbb{N} \to \mathbb{N}^\infty$. Then h is clearly action-preserving. In fact $h(\lambda(n,m)) = h(min(n,m)) = min(n,m) = \lambda(n,m) = \lambda(h(n), m)$, for all $n \in \mathbb{N}$ and $m \in \mathbb{N}^\infty$. Also, h is strict continuous. To see this, let $D \subseteq^d \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that $D = \{1, n\}$ or $D = \{n\}$. For $D = \{1, n\}$, we have

$$h(\bigvee^d D) = h(n) = n = \bigvee^d \{1, n\} = \bigvee^d \{h(1), h(n)\} = \bigvee^d h(D).$$

The other case is clear. Thus h is a separately strict $\mathbb{N}^\infty$-cpo map which is not surjective (∞ is not in the image of h). We show that h is an epimorphism. Let $f_1, f_2 : \mathbb{N}^\infty \to P$ be separately strict $\mathbb{N}^\infty$-cpo maps with $f_1 \circ h = f_2 \circ h$. Then, we have

$$f_1(1) = f_1(h(1)) = (f_1 \circ h)(1) = (f_2 \circ h)(1) = f_2(h(1)) = f_2(1),$$

$$f_1(n) = f_1(h(n)) = (f_1 \circ h)(n) = (f_2 \circ h)(n) = f_2(h(n)) = f_2(n)$$

for all $n \in \mathbb{N}$, and

$$f_1(\infty) = f_1(\bigvee^d \mathbb{N}) = \bigvee^d_{n \in \mathbb{N}} f_1(n) = \bigvee^d_{n \in \mathbb{N}} f_2(n) = f(\bigvee^d \mathbb{N}) = f_2(\infty).$$

Therefore, $f_1 = f_2$, and so h is an epimorphism.

Theorem 5.7. *Let $f : A \to B$ be an epimorphism in $\mathbf{Cpo_{Sep}}$-S. Then f is surjective if and only if its image is a Scott-closed subset of B.*

Proof. Let $f(A)$ be a Scott-closed subset of B. We consider the mappings $h, \gamma : B \to B^*$ in Lemma 3.10, where B^* is a separately strict S-cpo of Remark 3.9, for $I = f(A)$. Then $h \circ f = \gamma \circ f$. Since f is an epimorphism, $h = \gamma$ and so $f(A) = B$, as required. The converse is true because $f(A) = B$ is a Scott-closed subset of B. $\square$

We close the paper with the following result.

Theorem 5.8. *The category* $\mathbf{Cpo_{Sep}}$-S *is not cartesian closed.*

Proof. For every separately strict S-cpo A, the functor $A \times -$ does not preserve the initial object and so does not have a right adjoint. Note that the initial object in the category $\mathbf{Cpo_{Sep}}$-S is the singleton poset. Hence the category $\mathbf{Cpo_{Sep}}$-S is not cartesian closed. $\square$

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