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Applications of the Kleisli and Eilenberg-Moore 2-adjunctions

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Abstract. In 2010, J. Climent Vidal and J. Soliveres Tur developed, among other things, a pair of 2-adjunctions between the 2-category of adjunctions and the 2-category of monads. One is related to the Kleisli adjunction and the other to the Eilenberg-Moore adjunction for a given monad.

Since any 2-adjunction induces certain natural isomorphisms of categories, these can be used to classify bijections and isomorphisms for certain structures in monad theory. In particular, one important example of a structure, lying in the 2-category of adjunctions, where this procedure can be applied to is that of a lifting. Therefore, a lifting can be characterized by the associated monad structure, lying in the 2-category of monads, through the respective 2-adjunction. The same can be said for Kleisli extensions.

Several authors have been discovered this type of bijections and isomorphisms but these pair of 2-adjunctions can collect them all at once with an extra property, that of naturality.

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1 Introduction and preliminaries

Motivated by [2] and [3], the authors apply 2-adjunctions of Kleisli and Eilenberg-Moore in order to get some classical isomorphisms of categories and bijections of structures related to monads.

Among the examples given in this article, there is one of high importance. In [7], I. Moerdijk gave an equivalence between the lifting of a monoidal structure, over a category $\mathcal{C}$, to a monoidal structure on the category of Eilenberg-Moore algebras $\mathcal{C}^F$, for a monad with endofunctor F on the category $\mathcal{C}$, and the colax monad structures on F for the monoidal category $\mathcal{C}$. This equivalence of structures lacks of naturality but using the 2-adjunction of Eilenberg-Moore it can be incorporated.

Analogously, the following case is analysed. The equivalence between extensions of monoidal structure over a category $\mathcal{C}$ to a monoidal structure on the Kleisli category $\mathcal{C}_F$, for a monad with endofunctor F on the category $\mathcal{C}$, and the lax monad structures on F for the monoidal category $\mathcal{C}$, cf. [7] and [10].

The 2-adjunctions of Kleisli and Eilenberg-Moore are generalized to the context of 2-categories that accept the constructions of algebras.

We give the structure of the article.

In Section 2, we give the formal 2-adjunction corresponding to the Kleisli situation.

In Section 3, we give the formal 2-adjunction corresponding to the Eilenberg-Moore case.

In Section 4, we apply the 2-adjunction of EM to the case where the 2-category is ${}_2Cat$.

In Section 5, we prove the theorem of I. Moerdijk on the equivalence of lifted monoidal structures and colax monads.

In Section 6, we use the Kleisli 2-adjunction for the ${}_2Cat$ case.

In Section 7, we apply this 2-adjunction to *extensions* of a monoidal structure on the Kleisli category and relate it with lax monads.

In Section 8, we apply the 2-adjunction of Eilenberg-Moore to the well known case of liftings of functors and commutative diagrams for the forgetful functor, check [1] and [9].

In Section 9, we relate actions of the category $\mathcal{C}$ over its Kleisli category $\mathcal{C}_F$ with strong monads.

In Section 10, we finalize with left and right functor algebras for a monad and relate this to certain liftings and extensions, respectively, for the underlying functors, cf. [4].

We give some remarks on notation. Suppose that we had an adjunction of the form $\mathfrak{L} \dashv \mathfrak{R}$, then the unit and counit for this adjunction will be denoted as $\eta^{\mathfrak{R}\mathfrak{L}}$ and $\varepsilon^{\mathfrak{L}\mathfrak{R}}$, respectively. This notation is complicated but it is clear and prevents the proliferation of several greek letters to denote new units and counits. As the article develops, the reader might see the advantage in the usage of this notation.

We will be working with monoidal categories denoted as $(\mathcal{C}, \otimes, I, a, l, r)$ and also as $(\mathcal{C}, \otimes, I)$, as a contraction, that leaves understood the natural constraint transformations. We will be working with the constant functor $\delta_I : \mathbf{1} \longrightarrow \mathcal{C}$, at I, where $\mathbf{1}$ is the category with only one object 0 and only one arrow 1_0. That is to say, $\delta_I(0) = I$.

On the other hand, it is known that a category with binary products and a terminal object has a canonical (cartesian) monoidal structure. This is the case for the category Cat, of small categories. The natural constraint transformations, taken on components, are functors, for example, for $\mathcal{C}, \mathcal{D}, \mathcal{E}$, $a_{\mathcal{C},\mathcal{D},\mathcal{E}} : (\mathcal{C} \times \mathcal{D}) \times \mathcal{E} \longrightarrow \mathcal{C} \times (\mathcal{D} \times \mathcal{E})$ is the obvious functor. In order to compact the notation, we will agree that in the case that the component be the object $\mathcal{C}, \mathcal{C}, \mathcal{C}$, the asociativity functor will be denoted simply as $a_{\mathcal{C}}$. In turn, the respective constraint functors will be denoted as $l_{\mathcal{C}}$ and $r_{\mathcal{C}}$.

Finally, the horizontal composition in a general 2-category $\mathcal{A}$ will be denoted as $\cdot$ or by juxtaposition, this notation will be used indistinctively. The vertical composition on 2-cells will be given the symbol $\circ$.

2 Formal Kleisli 2-adjunction

In order to construct the Kleisli 2-adjunction, the involved 2-category $\mathcal{A}^{op}$ has to *admit the construction of algebras*, [8].

Definition 2.1. Consider a 2-category $\mathcal{A}$ and its corresponding 2-functor $\mathrm{Inc}_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{Mnd}(\mathcal{A})$ which maps a 0-cell A in $\mathcal{A}$ to the trivial monad $(A, 1_A)$. It is said that the 2-category $\mathcal{A}$ *admits the construction of algebras* iff the 2-functor $\mathrm{Inc}_{\mathcal{A}}$ admits a right adjoint.

If the 2-category $\mathcal{A}^{op}$ admits the construction of algebras then an additional 2-adjunction of the form

$$\mathbf{Mnd}(\mathcal{A}^{op}) \underset{\Psi_K}{\overset{\Phi_K}{\rightleftarrows}} \mathbf{Adj}_R(\mathcal{A}^{op}) \tag{2.1}$$

can be defined.

If we describe the 2-adjunction over $\mathcal{A}$ rather than on the opposite one then the 2-category $\mathbf{Mnd}(\mathcal{A}^{op})$ will be isomorphic to $\mathbf{Mnd}^{\bullet}(\mathcal{A})$ and the 2-category $\mathbf{Adj}_R(\mathcal{A}^{op})$ will be isomorphic to $\mathbf{Adj}_L(\mathcal{A})$. Note that in [8] the category $\mathcal{A}^{op}$ is denoted as $\mathcal{A}^*$.

The description of the 2-category $\mathbf{Mnd}^{\bullet}(\mathcal{A})$ is given as follows:

1. The 0-cells are monads in $\mathcal{A}$, that is, (A, f, μ^f, η^f). The short notation (A, f) will be used for such a monad.

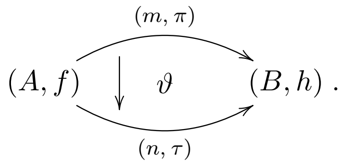
2. The 1-cells, which we call indistinctively as morphisms of monads, are pairs of the form $(m, \pi) : (A, f) \longrightarrow (B, h)$; where $m : A \longrightarrow B$ is a 1-cell in $\mathcal{A}$ and $\pi : mf \longrightarrow hm$ is a 2-cell in $\mathcal{A}$ such that the following diagrams commute:

$$\begin{array}{ccccc} mff & \xrightarrow{\pi f} & hmf & \xrightarrow{h\pi} & hhm \\ {\scriptstyle m\mu^f}\downarrow & & & & \downarrow{\scriptstyle \mu^h m} \\ mf & & \xrightarrow[\pi]{\qquad\qquad} & & hm \end{array} \quad , \quad \begin{array}{ccc} & m & \\ {\scriptstyle m\eta^f}\swarrow & & \searrow{\scriptstyle \eta^h m} \\ mf & \xrightarrow[\pi]{} & hm. \end{array}$$

3. The 2-cells, which we call indistinctively as transformations of monads, have the form $\vartheta : (m, \pi) \longrightarrow (n, \tau) : (A, f) \longrightarrow (B, h)$, such that $\vartheta : m \longrightarrow n : A \longrightarrow B$ is a 2-cell in $\mathcal{A}$ and the following diagram commutes:

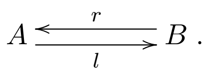
$$\begin{array}{ccc} mf & \xrightarrow{\pi} & hm \\ {\scriptstyle \vartheta f}\downarrow & & \downarrow{\scriptstyle h\vartheta} \\ nf & \xrightarrow[\tau]{} & hn. \end{array}$$

This 2-cell is displayed as follows

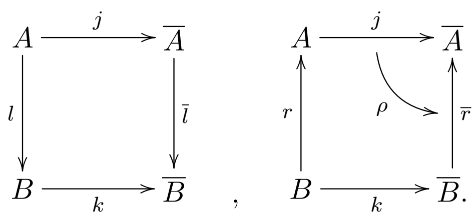


The structure of the 2-category $\mathbf{Adj}_L(\mathcal{A})$ is given as follows:

1. The 0-cells are made of adjunctions



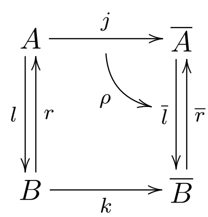
2. The 1-cells are of the form (j, k, ρ) such that the second diagram is the 2-cell mate of the first one that commutes



The mate ρ is described, since the left one commutes, by

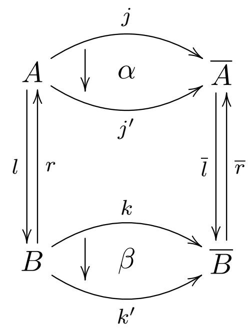
$$\rho = \overline{r}k\varepsilon \circ \overline{\eta}jr. \tag{2.2}$$

This morphism can be represented as



and denoted as $(j, k, \rho) : l \dashv r \longrightarrow \bar{l} \dashv \overline{r}$. Since the diagram corresponding to the left adjoints commutes, the 2-category of adjunctions has the subindex L.

3. The 2-cells are made of a pair of 2-cells in $\mathcal{A}$, (α, β) as in



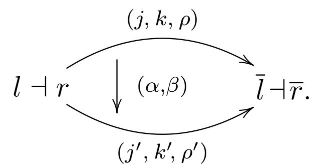
such that they fulfill one of the following equivalent conditions:

(i) $\bar{l}\alpha = \beta l$,

(ii) $\rho' \circ \alpha r = \overline{r}\beta \circ \rho$.

Remark 2.2. Note that the previous conditions can be seen as commutative surface diagrams.

This 2-cell can be displayed as



The cell structure described arrange itself to form a 2-category, that is to say, inherits the 2-category structure of $\mathcal{A}$.

Before going into the details on the construction of the 2-functor Ψ_K, we develop some calculations. These calculations are dual to those made in [8]. Note that we are going to be switching between the 2-categories $\mathcal{A}^{op}$ and $\mathbf{Mnd}(\mathcal{A}^{op})$ to $\mathcal{A}$ and $\mathbf{Mnd}^{\bullet}(\mathcal{A})$, respectively.

Since the 2-category $\mathcal{A}^{op}$ admits the construction of algebras, the functor $\mathrm{Inc}_{\mathcal{A}^{op}} : \mathcal{A}^{op} \longrightarrow \mathbf{Mnd}(\mathcal{A}^{op})$ admits a right adjoint, denoted as $\mathrm{Alg}_{\mathcal{A}^{op}} : \mathbf{Mnd}(\mathcal{A}^{op}) \longrightarrow \mathcal{A}^{op}$. These 2-functors are going to be short denoted as $\mathbf{I}^{\bullet}$ and $\mathrm{A}^{\bullet}$, respectively.

The corresponding counit, on the component (A, f^{op}), is $\varepsilon^{IA^{\bullet}}(A, f^{op}) : \mathrm{Inc}_{\mathcal{A}^{op}}\mathrm{Alg}_{\mathcal{A}^{op}}(A, f^{op}) \longrightarrow (A, f^{op})$. If we define $\mathrm{Alg}_{\mathcal{A}^{op}}(A, f^{op}) = A_f$, the *Kleisli object*, then $\varepsilon^{IA^{\bullet}}(A, f^{op}) = (g_f, \iota_f) : (A, f) \longrightarrow (A_f, 1_{A_f})$. This last 1-cell belongs to $\mathbf{Mnd}^{\bullet}(\mathcal{A})$, where $g_f : A \longrightarrow A_f$ and $\iota_f : g_f f \longrightarrow g_f$.

Following [8], for any monad (A, f^{op}) in $\mathbf{Mnd}(\mathcal{A}^{op})$, there exists an adjunction in $\mathcal{A}$,

$$A \underset{g_f}{\overset{v_f}{\rightleftarrows}} A_f$$

such that it generates the monad (A, f), with unit η^f and counit ε^{gv_f}. It can be checked that $\iota_f = \varepsilon^{gv_f} g_f$. This adjunction is called the *Kleisli adjunction*.

Suppose that there is a morphism of monads $(m^{op}, \pi) : (B, h^{op}) \longrightarrow (A, f^{op})$ in $\mathbf{Mnd}(\mathcal{A}^{op})$, *i.e.* $(m, \pi) : (A, f) \longrightarrow (B, h)$ in $\mathbf{Mnd}^{\bullet}(\mathcal{A})$. Take the following composition of morphisms of monads $(g_h, \iota_h) \cdot (m, \pi) = (g_h m, \iota_h m \circ g_h \pi) : (A, f) \longrightarrow (B_h, 1_{B_h})$.

Since the counit is universal from $\mathrm{Inc}_{\mathcal{A}^{op}}$ to (A, f^{op}), there exists a 1-cell $m_\pi : A_f \longrightarrow B_h$, in $\mathcal{A}$, such that the following diagram commute:

$$\begin{array}{ccc} & (A, f) & \\ {}^{(g_f, \iota_f)}\swarrow & & \searrow^{(g_h m, \iota_h m \circ g_h \pi)} \\ (A_f, 1_{A_f}) & \xrightarrow[(m_\pi, 1_{m_\pi})]{} & (B_h, 1_{B_h}). \end{array}$$

In particular, $g_h m = m_\pi g_f$ and $\iota_h m \circ g_h \pi = m_\pi \iota_f$. Note that the associated mate to the first equality is $\rho_\pi = v_h m_\pi \varepsilon^{gv_h} \circ \eta^h m v_f$ and that $\rho_\pi g_f = \pi$.

Consider a 2-cell of monads $\vartheta : (m, \pi) \longrightarrow (n, \tau) : (A, f) \longrightarrow (B, h)$ in $\mathbf{Mnd}^{\bullet}(\mathcal{A})$. Due to the construction of algebras for $\mathcal{A}^{op}$, the 2-adjunction $\mathrm{Alg}_{\mathcal{A}^{op}} \dashv \mathrm{Inc}_{\mathcal{A}^{op}}$ provides an isomorphism of categories, for (A, f^{op}) in $\mathbf{Mnd}(\mathcal{A}^{op})$ and B in $\mathcal{A}^{op}$, of the form

$$Hom_{\mathbf{Mnd}(\mathcal{A}^{op})}\big((A, f^{op}), \mathrm{Inc}_{\mathcal{A}^{op}}(B)\big) \cong Hom_{\mathcal{A}^{op}}\big(\mathrm{Alg}_{\mathcal{A}^{op}}(A, f^{op}), B\big),$$

which translates, in the non-opposite case, into the following assignment:

$$A_f \underset{b}{\overset{a}{\Downarrow \alpha}} B \quad \longmapsto \quad (A, f) \underset{(b g_f, b\iota_f)}{\overset{(a g_f, a\iota_f)}{\Downarrow \alpha g_f}} (B, 1_B). \qquad (2.3)$$

On the other hand, we have an equality of 2-cells

$$(A,f) \underset{(g_h n,\, \iota_h n \circ g_h \tau)}{\overset{(g_h m,\, \iota_h m \circ g_h \pi)}{\Downarrow g_h\vartheta}} (B_h, 1_{B_h}) \quad = \quad (A,f) \underset{(n_\tau g_f,\, n_\tau \iota_f)}{\overset{(m_\pi g_f,\, m_\pi \iota_f)}{\Downarrow g_h\vartheta}} (B_h, 1_{B_h}).$$

Therefore, to the 2-cell $g_h\vartheta$ there corresponds, through the asignment (2.3), a 2-cell $\beta_\vartheta = \mathrm{Alg}_{\mathcal{A}^{op}}(g_h\vartheta) \cdot \eta^{IA^\bullet}(B_h)$, such that $g_h\vartheta = \beta_\vartheta g_f$, where $\beta_\vartheta : m_\pi \to n_\tau$. We change, at this point, the notation as $\beta_\vartheta = \widetilde{\vartheta}$.

Without any further ado, we provide the description of the 2-functor Ψ_K.

1. For the monad (A, f, μ^f, η^f) in $\mathbf{Mnd}^\bullet(\mathcal{A})$, $\Psi_K(A,f) = g_f \dashv v_f$, that is, the Kleisli adjunction.

2. For the morphism $(m,\pi) : (A,f) \longrightarrow (B,h)$, $\Psi_K(m,\pi) = (m, m_\pi, \rho_\pi)$.

3. For the transformation $\vartheta : (m,\pi) \longrightarrow (n,\tau) : (A,f) \longrightarrow (B,g)$, $\Psi_K(\vartheta) = (\vartheta, \widetilde{\vartheta})$, where $\widetilde{\vartheta}$ is given as above.

The description of the 2-functor Φ_K is given as follows.

1. For the adjunction $l \dashv r$, $\Phi_K(l \dashv r) = (A, rl)$.

2. For the morphism of adjunctions $(j,k,\rho) : (l \dashv r) \longrightarrow (\bar{l} \dashv \bar{r})$, $\Phi_K(j,k,\rho) = (j, \pi_\rho)$. Where $\pi_\rho = \rho l$.

3. For the transformation of adjunctions $(\alpha,\beta) : (j,k,\rho) \longrightarrow (j',k',\rho') : l \dashv r \longrightarrow \bar{l} \dashv \bar{r}$, $\Phi_K(\alpha,\beta) = \vartheta_{(\alpha,\beta)} = \alpha$.

Yet again, following [8], it can be shown that for the adjunction $l \dashv r$, there exists a *dual comparison 1-cell* $k_{rl} : A_{rl} \longrightarrow B$, such that $l = k_{rl} g_{rl}$, $v_{rl} = r k_{rl}$ and $\varepsilon^{rl} l = k_{rl} \iota_{rl}$.

The unit of the 2-adjunction in (2.1), $\eta^{\Phi\Psi_K} : 1_{\mathbf{Mnd}^\bullet(\mathcal{A})} \longrightarrow \Phi_K \Psi_K$ is defined, in the component (A,f), as

$$\eta^{\Phi\Psi_K}(A,f) := (1_A, 1_f) : (A,f) \longrightarrow (A,f) \quad \mathit{in} \quad \mathbf{Mnd}^\bullet(\mathcal{A}).$$

In turn, the counit $\varepsilon^{\Psi\Phi_K} : \Psi_K\Phi_K \longrightarrow 1_{\mathbf{Adj}_l(\mathcal{A})}$ is defined, in the component $l \dashv r$, as

$$\varepsilon^{\Psi\Phi_K}(l \dashv r) := (1_A, k_{rl}, 1_{v_{rl}}) : g_{rl} \dashv v_{rl} \longrightarrow l \dashv r \quad in \quad \mathbf{Adj}_L(\mathcal{A}).$$

Theorem 2.3. *There exists a* 2*-adjunction* $\Psi_K \dashv \Phi_K$*.*

Proof. We prove only one of the triangular identities, that is, $\Phi_K\varepsilon^{\Psi\Phi_K} \circ \eta^{\Phi\Psi_K}\Phi_K = 1_{\Phi_K}$,

$$\begin{aligned}\big(\Phi_K\varepsilon^{\Psi\Phi_K} \circ \eta^{\Phi\Psi_K}\Phi_K\big)(l \dashv r) &= \Phi_K\varepsilon^{\Psi\Phi_K}(l \dashv r) \cdot \eta^{\Phi\Psi_K}\Phi_K(l \dashv r)\\ &= \Phi_K(1_A, k_{rl}, 1_{v_{rl}}) \cdot \eta^{\Phi\Psi_K}(A, rl)\\ &= (1_A, 1_{v_{rl}}g_{rl}) \cdot (1_A, 1_{rl}) = (1_A, 1_{rl}) = 1_{(A,rl)}\\ &= 1_{\Phi_K(l\dashv r)} = 1_{\Phi_K}(l \dashv r).\end{aligned}$$

□

Since the left 2-adjoint Ψ_K assigns the Kleisli adjunction to a monad, the 2-adjunction is called Kleisli 2-*adjunction.*

3 Formal Eilenberg-Moore 2-adjunction

Consider a 2-category $\mathcal{A}$ which admits the construction of algebras. With this property of $\mathcal{A}$, we will construct a 2-adjunction of the form

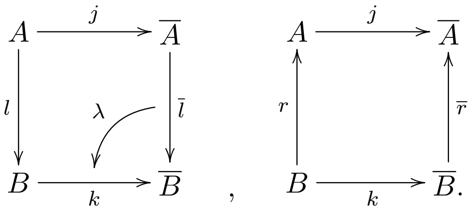
$$\mathbf{Adj}_R(\mathcal{A}) \underset{\Phi_E}{\overset{\Psi_E}{\rightleftarrows}} \mathbf{Mnd}(\mathcal{A}).$$

The 2-category $\mathbf{Adj}_R(\mathcal{A})$ is described as follows.

1. The 0-cells are made of adjunctions

$$A \underset{l}{\overset{r}{\leftrightarrows}} B.$$

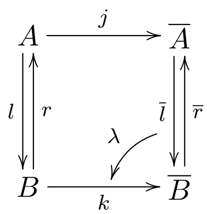
2. The 1-cells are pairs, of 1-cells in $\mathcal{A}$, (j, k) such that the first diagram is the 2-cell mate to the second commutative one



The mate is described by

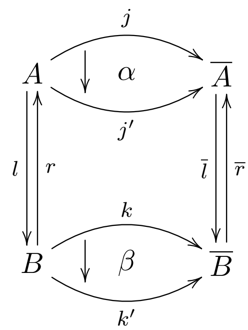
$$\lambda = \overline{\varepsilon} kl \circ \bar{l} j\eta. \tag{3.1}$$

This morphism can be represented as



and denoted as $(j, k, \lambda) : l \dashv r \longrightarrow \bar{l} \dashv \overline{r}$.

3. The 2-cells are made of a pair of 2-cells in $\mathcal{A}$, (α, β) as in



such that they fulfill one of the following equivalent conditions:

(i) $\lambda' \circ \bar{l}\alpha = \beta l \circ \lambda$,

(ii) $\alpha r = \overline{r}\beta$.

Remark 3.1. Note that the previous conditions can be seen as commutative surface diagrams.

This 2-cell can be displayed as follows

$$l \dashv r \;\underset{(j',k',\lambda')}{\overset{(j,k,\lambda)}{\rightrightarrows}}\; \bar{l} \dashv \bar{r}. \qquad \Big\Downarrow (\alpha,\beta)$$

The described cell structure arrange itself to form a 2-category.
The 2-category $\mathbf{Mnd}(\mathcal{A})$ is formed as follows.

1. The 0-cells are monads in $\mathcal{A}$, (A, f, μ^f, η^f). The short notation (A, f) will be used for such a monad.

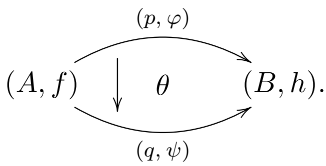
2. The 1-cells are *morphisms of monads* $(p, \varphi) : (A, f) \longrightarrow (B, h)$, which consist of a 1-cell $p : A \longrightarrow B$ and a 2-cell $\varphi : hp \longrightarrow pf$, both in $\mathcal{A}$, such that the following diagrams commutes:

$$\begin{array}{ccccc} hhp & \xrightarrow{h\varphi} & hpf & \xrightarrow{\varphi f} & pff \\ {\scriptstyle \mu^h p}\big\downarrow & & & & \big\downarrow{\scriptstyle p\mu^f} \\ hp & & \xrightarrow{\quad\varphi\quad} & & pf \end{array} \quad , \quad \begin{array}{ccc} & p & \\ {\scriptstyle \eta^h p}\swarrow & & \searrow{\scriptstyle p\eta^f} \\ hp & \xrightarrow{\quad\varphi\quad} & pf. \end{array}$$

3. The 2-cells, or *transformations of monads*, $\theta : (p, \varphi) \longrightarrow (q, \psi) : (A, f) \longrightarrow (B, h)$ consist of a 2-cell $\theta : p \longrightarrow q$ in $\mathcal{A}$ and fulfills the commutativity of the diagram

$$\begin{array}{ccc} hp & \xrightarrow{\varphi} & pf \\ {\scriptstyle h\theta}\big\downarrow & & \big\downarrow{\scriptstyle \theta f} \\ hq & \xrightarrow[\psi]{} & qf. \end{array}$$

This 2-cell is displayed as



The description of the 2-functor Φ_E is given as follows.

1. On 0-cells, $\Phi_E(l \dashv r) = (A, rl, r\varepsilon l, \eta)$, that is, the induced monad by an adjunction.
2. On 1-cells, $(j, k, \lambda) : (A, rl) \longrightarrow (\overline{A}, \overline{r}\overline{l})$, $\Phi_E(j, k, \lambda) = (j, \overline{r}\lambda) : (A, rl) \longrightarrow (\overline{A}, \overline{r}\overline{l})$.
3. On 2-cells, $(\alpha, \beta) : (j, k, \lambda) \longrightarrow (j', k', \lambda')$, $\Phi_E(\alpha, \beta) = \alpha : (j, \overline{r}\lambda) \longrightarrow (j', \overline{r}\lambda')$.

Before the description of the 2-functor Ψ_E, we realize some calculations.

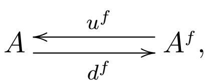
Since the 2-category $\mathcal{A}$ admits the construction of algebras, the 2-functor $\text{Inc}_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{Mnd}(\mathcal{A})$ admits a right adjoint, denoted as $\text{Alg}_{\mathcal{A}} : \mathbf{Mnd}(\mathcal{A}) \longrightarrow \mathcal{A}$.

The corresponding counit, on the component (A, f), is

$$\varepsilon^{IA}(A, f) : \text{Inc}_{\mathcal{A}}\text{Alg}_{\mathcal{A}}(A, f) \longrightarrow (A, f).$$

If we define $\text{Alg}_{\mathcal{A}}(A, f) = A^f$, the *Eilenberg-Moore object* for (A, f), then $\varepsilon^{IA}(A, f) := (u^f, \chi^f) : (A^f, 1_{A^f}) \longrightarrow (A, f)$, where $u^f : A^f \longrightarrow A$ and $\chi^f : u^f d^f u^f \longrightarrow u^f$.

In Theorem 2, at [8], the author proved that if $\mathcal{A}$ admits the construction of algebras then for any monad (A, f) in $\mathbf{Mnd}(\mathcal{A})$, there exists an adjunction in $\mathcal{A}$



such that it generates the monad (A, f), with unit η^f and counit ε^{du^f}. It can be checked that $\chi^f = u^f \varepsilon^{du^f}$. This adjunction is called the *The Eilenberg-Moore adjunction.*

Suppose there is a morphism of monads $(p, \varphi) : (A, f) \longrightarrow (B, h)$. Take the composition of morphisms of monads $(p, \varphi) \cdot (u^f, \chi^f) = (pu^f, p\chi^f \circ \varphi u^f) : \mathrm{Inc}_{\mathcal{A}}(A^f) = (A^f, 1_{A^f}) \longrightarrow (B, h)$.

The previous counit, ε^{IA}, is universal from the functor $\mathrm{Inc}_{\mathcal{A}}$, in particular, for the 1-cell $(pu^f, p\chi^f \circ \varphi u^f) : \mathrm{Inc}_{\mathcal{A}}(A^f) \longrightarrow (A, f)$ exists a unique 1-cell in $\mathcal{A}$ of the form $p^\varphi : A^f \longrightarrow \mathrm{Alg}_{\mathcal{A}}(B, h) = B^h$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Inc}_{\mathcal{A}}(A^f) & \xrightarrow{\mathrm{Inc}_{\mathcal{A}}(p^\varphi)} & \mathrm{Inc}_{\mathcal{A}}(B^h) \\ & {\scriptstyle (pu^f, p\chi^f \circ \varphi u^f)} \searrow \quad \swarrow {\scriptstyle (u^h, \chi^h)} & \\ & (A, f). & \end{array}$$

In particular, $pu^f = u^h p^\varphi$ and $p\chi^f \circ \varphi u^f = \chi^h p^\varphi$. Observe that the associated mate, to the first equality, is $\lambda = \varepsilon^h p^\varphi d^f \circ d^h p \eta^f$ and that $u^h \lambda = \varphi$.

Consider a 2-cell of monads, $\theta : (p, \varphi) \longrightarrow (q, \psi) : (A, f) \longrightarrow (B, h)$. Because of the construction of algebras for $\mathcal{A}$, the 2-adjunction provides an isomorphism of categories, for A in $\mathcal{A}$ and (X, f) in $\mathbf{Mnd}(\mathcal{A})$,

$$Hom_{\mathcal{A}}(A, \mathrm{Alg}_{\mathcal{A}}(X, f)) \cong Hom_{\mathbf{Mnd}(\mathcal{A})}(\mathrm{Inc}_{\mathcal{A}}(A), (X, f))$$

given by the following assignment

$$\left(A \underset{b}{\overset{a}{\Downarrow \alpha}} X^f \right) \longmapsto \left((A, 1_A) \underset{(u^f b, \chi^f b)}{\overset{(u^f a, \chi^f a)}{\Downarrow u^f \alpha}} (X, f) \right) \tag{3.2}$$

cf. [8]. On the other hand, we have an equality of 2-cells

$$(A^f, 1_{A^f}) \underset{(qu^f, q\chi^f \circ \psi u^f)}{\overset{(pu^f, p\chi^f \circ \varphi u^f)}{\Downarrow \theta u^f}} (B, h) \quad = \quad (A^f, 1_{A^f}) \underset{(u^h q^\psi, \chi^h q^\psi)}{\overset{(u^h p^\varphi, \chi^h p^\varphi)}{\Downarrow \theta u^f}} (B, h).$$

Therefore, to the 2-cell θu^f there corresponds, through the assignment (3.2), a 2-cell $\mathrm{Alg}_{\mathcal{A}}(\theta u^f)\eta^{AI}(A^f) := \beta^\theta$, where $\beta^\theta : p^\varphi \longrightarrow q^\psi$ and such that $u^h \beta^\theta = \theta u^f$. We change the notation as follows $\beta^\theta = \widehat{\theta}$.

With these calculations at hand, we define the 2-functor Ψ_E.

1. On 0-cells, (A, f) , $\Psi_E(A, f) = d^f \dashv u^f$, that is, the Eilenberg-Moore adjunction.

2. On 1-cells, $(p, \varphi) : (A, f) \longrightarrow (B, h)$, $\Psi_E(p, \varphi) = (p, p^\varphi) : d^f \dashv u^f \longrightarrow d^h \dashv u^h$.

3. On 2-cells, $\theta : (p, \varphi) \longrightarrow (q, \psi) : (A, f) \longrightarrow (B, h)$, $\Psi_E(\theta) = (\theta, \widehat{\theta}) : (p, p^\varphi) \longrightarrow (q, q^\psi) : d^f \dashv u^f \longrightarrow d^h \dashv u^h$.

The unit and the counit for this 2-adjunction are given as follows. The component of the unit, at $l \dashv r$, is $\eta^{\Psi\Phi_E}(l \dashv r) : l \dashv r \longrightarrow \Psi_E\Phi_E(l \dashv r)$, where $\Psi_E\Phi_E(l \dashv r) = d^{rl} \dashv u^{rl}$.

In [8], Theorem 3, the author proved the existence of a *comparison* 1-cell $k^{rl} : B \longrightarrow A^{rl}$, such that $u^{rl}k^{rl} = r$ and $d^{rl} = k^{rl}l$. Therefore, we can make the definition $\eta^{\Psi\Phi_E}(l \dashv r) = (1_A, k^{rl}, 1_{d^{rl}}) : l \dashv r \longrightarrow d^{rl} \dashv u^{rl}$.

In turn, the component of the counit, at (A, f), is

$$\varepsilon^{\Phi\Psi_E}(A, f) : \Phi_E\Psi_E(A, f) \longrightarrow (A, f),$$

where $\Phi_E\Psi_E(A, f) = (A, f)$. In this case, the counit is defined as $\varepsilon^{\Phi\Psi_E}(A, f) = (1_A, 1_f) : (A, f) \longrightarrow (A, f)$.

Theorem 3.2. *There exists a 2-adjunction* $\Phi_E \dashv \Psi_E$.

Proof. We prove only one of the triangular identities and the other one is left to the reader. Using the definition of the unit and counit for this 2-adjunction, the triangular identity $\varepsilon^{\Phi\Psi_E}\Phi_E \circ \Phi_E\eta^{\Psi\Phi_E} = 1_{\Phi_E}$ is proved as

$$\begin{aligned}
(\varepsilon^{\Phi\Psi_E}\Phi_E \circ \Phi_E\eta^{\Psi\Phi_E})(l \dashv r) &= \varepsilon^{\Phi\Psi_E}\Phi_E(l \dashv r) \cdot \Phi_E\eta^{\Psi\Phi_E}(l \dashv r) \\
&= \varepsilon^{\Phi\Psi_E}(A, rl) \cdot \Phi_E(1_A, k^{rl}, 1_{d^{rl}}) \\
&= (1_A, 1_{rl}) \cdot (1_A, u^{rl}1_{d^{rl}}) \\
&= (1_A, 1_{rl}) = 1_{(A,rl)} = 1_{\Phi_E(l \dashv r)} \\
&= 1_{\Phi_E}(l \dashv r).
\end{aligned}$$

□

Since the right 2-adjoint assigns the Eilenberg-Moore adjunction to a monad (A, f), this 2-adjunction is called the *Eilenberg-Moore 2-adjunction.*

4 Eilenberg-Moore 2-adjunction

In this section, we apply the results of Section 3 to the 2-category ${}_2Cat$, the 2-category of small categories and functors, due to the fact that this 2-category admits the construction of algebras. Therefore, we have a 2-adjunction

$$\mathbf{Adj}_R({}_2Cat) \underset{\Phi_E}{\overset{\Psi_E}{\rightleftarrows}} \mathbf{Mnd}({}_2Cat).$$

Since the complete description, for a general $\mathcal{A}$, has been given above, we only give some remarks on the derived properties for this particular 2-category.

The description of the 2-functor Ψ_E, for this particular 2-category, is given by the entries

1. On 0-cells, $\Psi_E(\mathcal{C}, F) = D^F \dashv U^F$, that is, the Eilenberg-Moore adjunction.

2. On 1-cells, $(P, \varphi) : (\mathcal{C}, F) \longrightarrow (\mathcal{D}, H)$, $\Psi_E(P, \varphi) = (P, P^\varphi, \lambda^\varphi)$. The action of the functor $P^\varphi : \mathcal{C}^F \longrightarrow \mathcal{D}^H$ is the following

   (i) On objects, (M, χ_M) in $\mathcal{C}^F$, $P^\varphi(M, \chi_M) = (PM, P\chi_M \cdot \varphi_M)$.

   (ii) On morphisms, p, $P^\varphi(p) = Pp$.

   (iii) The natural transformation λ^φ is the mate of the identity $U^H P^\varphi = PU^F$. Using (3.1), we get the component of λ^φ at A, in $\mathcal{C}$,

$$\begin{aligned} \lambda^\varphi A &= \left(\varepsilon^{DU^H} P^\varphi D^F \circ D^H P \eta^{UD^F}\right)(A) \\ &= P\mu^F A \cdot \varphi F A \cdot H P \eta^F A \\ &= \varphi A. \end{aligned}$$

3. On 2-cells, $\theta : (P, \varphi) \longrightarrow (Q, \psi)$, we have

$$\Psi_E(\theta) = (\alpha^\theta, \beta^\theta) = (\theta, \hat{\theta}).$$

   The induced natural tranformation $\hat{\theta} : P^\varphi \longrightarrow Q^\psi : \mathcal{C}^F \longrightarrow \mathcal{D}^H$ is defined through its components, using the condition $\theta U^F = U^H \hat{\theta}$, as

$$\hat{\theta}(M, \chi_M) = \theta M.$$

Since we have a 2-adjunction, the following isomorphism of categories takes place, natural for $L \dashv R$ in $\mathbf{Adj}_R({}_2Cat)$ and $(\mathcal{X}, H)$ in $\mathbf{Mnd}({}_2Cat)$:

$$Hom_{\mathbf{Adj}_R({}_2Cat)}(L \dashv R, \Psi_E(\mathcal{X}, H)) \cong Hom_{\mathbf{Mnd}({}_2Cat)}(\Phi_E(L \dashv R), (\mathcal{X}, H)). \tag{4.1}$$

5 Monoidal liftings (Eilenberg-Moore type)

In this section, we relate monoidal liftings to colax monad structures. In order to do so, we give the definition for this last concept.

Definition 5.1. A colax monad $\big((F, \xi, \gamma), \mu^F, \eta^F\big)$ over the monoidal category $(\mathcal{C}, \otimes, I)$ consists of the following

1. (F, μ^F, η^F) is a monad on $\mathcal{C}$.
2. $(F, \xi, \gamma) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{C}, \otimes, I)$ is a colax monoidal functor. That is to say, the natural transformations $\xi : F \cdot \otimes \longrightarrow \otimes \cdot (F \times F)$ and $\gamma : F \cdot \delta_I \longrightarrow \delta_I$ fulfills the commutativity of the following diagrams:

$$\begin{array}{ccccc} F((A \otimes B) \otimes C) & \xrightarrow{\xi_{A \otimes B, C}} & F(A \otimes B) \otimes FC & \xrightarrow{\xi_{A,B} \otimes FC} & (FA \otimes FB) \otimes FC \\ \downarrow{\scriptstyle Fa_{A,B,C}} & & & & \downarrow{\scriptstyle a_{FA,FB,FC}} \\ F(A \otimes (B \otimes C)) & \xrightarrow[\xi_{A, B \otimes C}]{} & FA \otimes F(B \otimes C) & \xrightarrow[FA \otimes \xi_{B,C}]{} & FA \otimes (FB \otimes FC) \end{array} \tag{5.1}$$

$$\begin{array}{ccccc} F(I \otimes A) & \xrightarrow{\xi_{I,A}} & FI \otimes FA & \xrightarrow{\gamma \otimes FA} & I \otimes FA \\ & {\scriptstyle Fl_A} \searrow & & \swarrow {\scriptstyle l_{FA}} & \\ & & FA & & \end{array} \tag{5.2}$$

$$\begin{array}{ccccc} FA \otimes I & \xleftarrow{FA \otimes \gamma} & FA \otimes FI & \xleftarrow{\xi_{A,I}} & F(A \otimes I) \\ & {\scriptstyle r_{FA}} \searrow & & \swarrow {\scriptstyle Fr_A} & \\ & & FA. & & \end{array} \tag{5.3}$$

3. $\mu^F : (F,\xi,\gamma)\cdot(F,\xi,\gamma) \longrightarrow (F,\xi,\gamma)$ and $\eta^F : (1_{\mathcal{C}}, 1_{\otimes}, 1_{\delta_I}) \longrightarrow (F,\xi,\gamma)$ are colax natural transformations, that is, apart from the fact that they are natural transformations, they have to fulfill the following commutative diagrams:

$$\begin{array}{ccccc} FF\otimes & \xrightarrow{F\xi} & F\otimes(F\times F) & \xrightarrow{\xi(F\times F)} & \otimes(FF\times FF) \\ \downarrow{\scriptstyle \mu^F\otimes} & & & & \downarrow{\scriptstyle \otimes(\mu^F\times\mu^F)} \\ F\otimes & & \xrightarrow{\quad\quad\xi\quad\quad} & & \otimes(F\times F) \end{array} \tag{5.4}$$

$$\begin{array}{ccccc} FF\delta_I & \xrightarrow{F\gamma} & F\delta_I & \xrightarrow{\gamma} & \delta_I \\ \downarrow{\scriptstyle \mu^F\delta_I} & & & \nearrow{\scriptstyle \gamma} & \\ F\delta_I & & & & \end{array} \tag{5.5}$$

$$\begin{array}{ccc} \otimes & \xrightarrow{1_{\otimes}} & \otimes \\ \downarrow{\scriptstyle \eta^F\otimes} & & \downarrow{\scriptstyle \otimes(\eta^F\times\eta^F)} \\ F\otimes & \xrightarrow{\xi} & \otimes(F\times F) \end{array} \quad , \quad \begin{array}{ccc} \delta_I & \xrightarrow{1_{\delta_I}} & \delta_I \\ \downarrow{\scriptstyle \eta^F\delta_I} & \nearrow{\scriptstyle \gamma} & \\ F\delta_I . & & \end{array} \tag{5.6}$$

Since the natural transformation γ has only one component, at 0 in $\mathbf{1}$, then this natural transformation and its component will be denoted indistinctly as γ.

Using the isomorphism (4.1), the following bijection can be obtained, cf. [7]

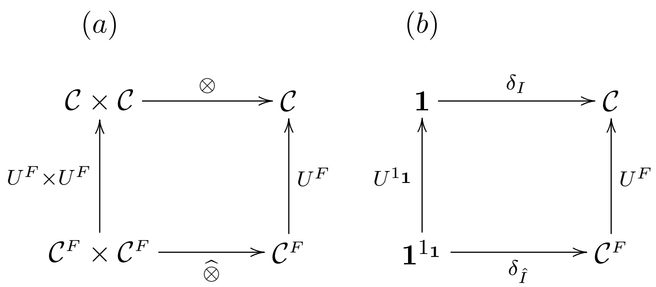
Theorem 5.2. *There is a bijective correspondance between the following structures*

(1) *Colax monads* $\big((F,\xi,\gamma),\mu^F,\eta^F\big)$, *for the monoidal structure* $(\mathcal{C},\otimes,I,a,l,r)$.

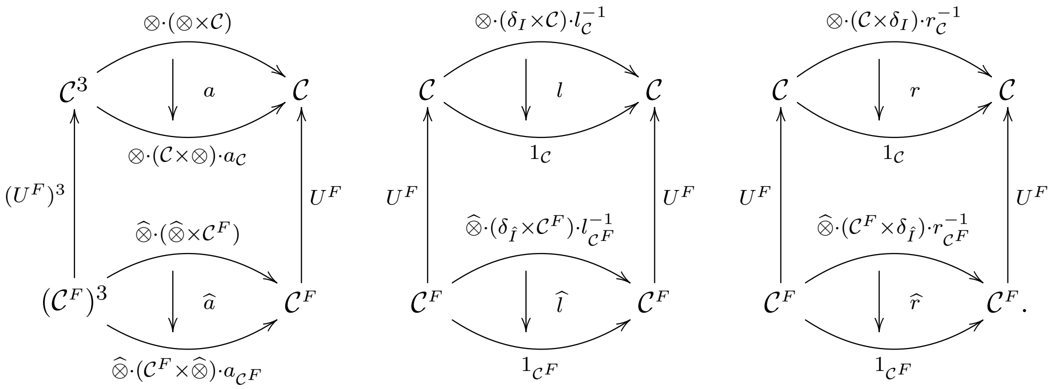
(2) *Morphisms and natural transformations of monads of the form*

$$\begin{aligned}
(\otimes,\xi) &: (\mathcal{C}\times\mathcal{C}, F\times F) \longrightarrow (\mathcal{C},F),\\
(\delta_I,\gamma) &: (\mathbf{1}\,,1_{\mathbf{1}}\,) \longrightarrow (\mathcal{C},F)\\
a &: (\otimes\cdot(\otimes\times\mathcal{C}), \otimes(\xi\times F)\circ\xi(\otimes\times\mathcal{C})) \longrightarrow\\
&\quad (\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}}, \otimes(F\times\xi)a_{\mathcal{C}}\circ\xi(\mathcal{C}\times\otimes)a_{\mathcal{C}}):\\
&\qquad\qquad ((\mathcal{C}\times\mathcal{C})\times\mathcal{C},(F\times F)\times F) \longrightarrow (\mathcal{C},F),\\
l &: (\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1}, \otimes(\gamma\times F)\,l_{\mathcal{C}}^{-1}\circ\xi(\delta_I\times\mathcal{C})\,l_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}},1_F):\\
&\qquad\qquad (\mathcal{C},F)\longrightarrow(\mathcal{C},F),\\
r &: (\otimes\cdot(\mathcal{C}\times\delta_I)\cdot r_{\mathcal{C}}^{-1}, \otimes(F\times\gamma)\,r_{\mathcal{C}}^{-1}\circ\xi(\mathcal{C}\times\delta_I)\,r_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}},1_F):\\
&\qquad\qquad (\mathcal{C},F)\longrightarrow(\mathcal{C},F).
\end{aligned}$$

(3) *Monoidal structures for the Eilenberg-Moore category,* $(\mathcal{C}^F,\widehat{\otimes},\hat{I},\hat{a},\hat{l},\hat{r})$ *such that the following diagram of arrows and surfaces commutes:*



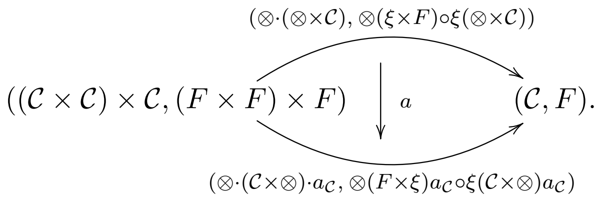
(5.7)



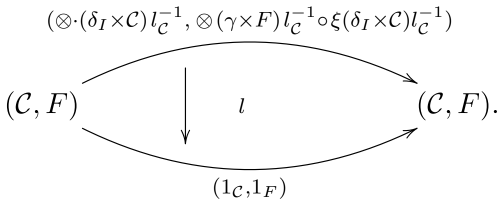
Proof. (1)⇒(2) Consider a colax monad $\big((F,\xi,\gamma),\mu^F,\eta^F\big)$, for the monoidal structure $(\mathcal{C},\otimes,I)$. In particular, the multiplication and the unit of the monad are colax natural transformations then (5.4) and the first diagram in (5.6) commute. Therefore, we have a monad morphism $(\otimes,\xi):(\mathcal{C}\times\mathcal{C},F\times F)\longrightarrow(\mathcal{C},F)$.

Likewise, the commutativity of (5.5) and the second diagram in (5.6) implies that $(\delta_I,\gamma):(\mathbf{1},1_{\mathbf{1}})\longrightarrow(\mathcal{C},F)$ is a morphism of monads. Note that the requirement (δ_I,γ) is a monad morphism is equivalent to the statement (I,γ) is an Eilenberg-Moore algebra.

Since $(\otimes,\xi)$ is a morphism of monads, the following are also morphisms of monads $\big(\otimes\cdot(\otimes\times\mathcal{C}),\otimes(\xi\times F)\circ\xi(\otimes\times\mathcal{C})\big)$ and $(\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}},\otimes(F\times\xi)a_{\mathcal{C}}\circ\xi(\mathcal{C}\times\otimes)a_{\mathcal{C}})$ from $((\mathcal{C}\times\mathcal{C})\times\mathcal{C},(F\times F)\times F)$ to $(\mathcal{C},F)$. Furthermore, due to the commutativity of the diagram (5.1), the following is a 2-cell in $\mathbf{Mnd}({}_2Cat)$



Likewise, because $(\otimes,\xi)$ and (δ_I,γ) are monad morphisms, $(\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1},\otimes(\gamma\times F)\,l_{\mathcal{C}}^{-1}\circ\xi(\delta_I\times\mathcal{C})\,l_{\mathcal{C}}^{-1})$ is also a monad morphism. Using the commutativity of the diagram (5.2), we can consider the monad 2-cell



In a similar way, the following is a monad transformation, $r:(\otimes\cdot(\mathcal{C}\times\delta_I)\cdot r_{\mathcal{C}}^{-1},\otimes(F\times\gamma)\,r_{\mathcal{C}}^{-1}\circ\xi(\mathcal{C}\times\delta_I)\,r_{\mathcal{C}}^{-1})\longrightarrow(1_{\mathcal{C}},1_F):(\mathcal{C},F)\longrightarrow(\mathcal{C},F)$.

(2)⇒(1) Note that the aforementioned claims can be reverted.

(2)⇒(3) Take the monad morphism $(\otimes,\xi):(\mathcal{C}\times\mathcal{C},F\times F)\longrightarrow(\mathcal{C},F)$. In order to use the isomorphism (4.1), we make $L\dashv R=D^F\times D^F\dashv U^F\times U^F$ and $(\mathcal{X},H,\mu^H,\eta^H)=(\mathcal{C},F,\mu^F,\eta^F)$. Therefore, to this monad morphism corresponds a morphism of adjunctions of the form $(\otimes, \otimes^{\xi}) : D^F \times D^F \dashv U^F \times U^F \longrightarrow D^F \dashv U^F$ such that a diagram like (5.7a) commutes. According to the definition of Ψ^E, the functor $\otimes^{\xi}$ acts, on objects, as

$$\otimes^{\xi}\big((M, \chi_M), (N, \chi_N)\big) = \big(\otimes(M, N), \otimes(\chi_M, \chi_N) \cdot \xi_{M,N}\big).$$

The previous action is defined at the beginning of the proof of Theorem 7.1, [7]. On morphisms, we have

$$\otimes^{\xi}(p, q) = \otimes(p, q).$$

We change the notation from $\otimes^{\xi}$ to $\widehat{\otimes}$.

If in the isomorphism (4.1), we make $L \dashv R = 1_{\mathbf{1}} \dashv 1_{\mathbf{1}}$ and $(\mathcal{X}, H, \mu^H, \eta^H) = (\mathcal{C}, F, \mu^F, \eta^F)$, the monad morphism (δ_I, γ) has an associated morphism of adjunctions of the form $(\delta_I, {\delta_I}^{\gamma}) : (1_{\mathbf{1}} \dashv 1_{\mathbf{1}}) \longrightarrow D^F \dashv U^F$ such that a diagram like (5.7b) commutes. According to the definition of Ψ^E, the functor ${\delta_I}^{\gamma}$ acts as

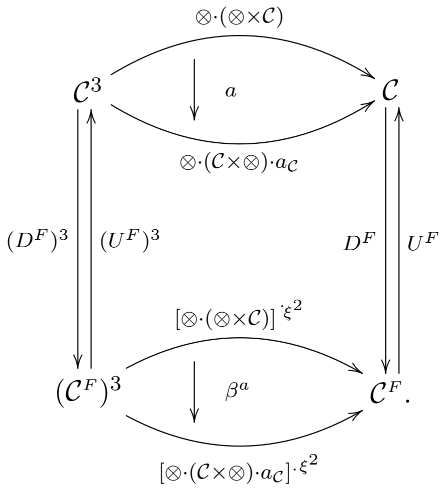
$${\delta_I}^{\gamma}(0, 1_0) = (\delta_I(0), \delta_I(1_0) \cdot \gamma) = (I, \gamma).$$

On morphisms,

$${\delta_I}^{\gamma}(1_0) = \delta_I(1_0) = 1_I = 1_{(I,\gamma)}.$$

If we make the definition $\hat{I} = (I, \gamma)$, then ${\delta_I}^{\gamma} := \delta_{\hat{I}}$. The algebra (I, γ) is the unit of the monoidal structure on $\mathcal{C}^F$.

Suppose that we have a natural transformation of the form $a : (\otimes \cdot (\otimes \times \mathcal{C}), \otimes(\xi \times F) \circ \xi(\otimes \times \mathcal{C})) \longrightarrow (\otimes \cdot (\mathcal{C} \times \otimes) \cdot a_{\mathcal{C}}, \otimes(F \times \xi)a_{\mathcal{C}} \circ \xi(\mathcal{C} \times \otimes)a_{\mathcal{C}}) : ((\mathcal{C} \times \mathcal{C}) \times \mathcal{C}, (F \times F) \times F) \longrightarrow (\mathcal{C}, F)$ then we can make $L \dashv R = (D^F \times D^F) \times D^F \dashv (U^F \times U^F) \times U^F$ and $(\mathcal{X}, H, \mu^H, \eta^H) = (\mathcal{C}, F, \mu^F, \eta^F)$. Therefore, to the previous 2-cell of monads, we can associate a 2-cell of adjunctions of the form



In order to reduce expressions, we used and will be using the notation

$$\begin{aligned} {}^{\cdot}\xi^2 &:= \otimes(\xi\times F)\circ\xi(\otimes\times\mathcal{C}),\\ {}_{\cdot}\xi^2 &:= \otimes(F\times\xi)a_{\mathcal{C}}\circ\xi(\mathcal{C}\times\otimes)a_{\mathcal{C}},\\ (\cdot)^3 &:= (\cdot\times\cdot)\times\cdot. \end{aligned}$$

Since Ψ_E is a 2-functor, we have

$$\begin{aligned} [\otimes\cdot(\otimes\times\mathcal{C})]^{{}^{\cdot}\xi^2} &= \Psi_E(\otimes\cdot(\otimes\times\mathcal{C}),\otimes(\xi\times F)\circ\xi(\otimes\times\mathcal{C}))\\ &= \Psi_E\big((\otimes,\xi)\cdot(\otimes\times\mathcal{C},\xi\times F)\big)\\ &= \Psi_E(\otimes,\xi)\cdot\Psi_E(\otimes\times\mathcal{C},\xi\times F)\\ &= \otimes^{\xi}\cdot(\otimes\times\mathcal{C})^{\xi\times F} = \widehat{\otimes}\cdot(\widehat{\otimes}\times\mathcal{C}^F). \end{aligned}$$

In the same way, we can check that $[\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}}]^{{}_{\cdot}\xi^2} = \widehat{\otimes}\cdot(\mathcal{C}^F\times\widehat{\otimes})\cdot a_{\mathcal{C}^F}$. We change the notation β^a for $\widehat{a}$ and we get a natural transformation

$$\widehat{a}:\widehat{\otimes}(\widehat{\otimes}\times\mathcal{C}^F)\longrightarrow\widehat{\otimes}(\mathcal{C}^F\times\widehat{\otimes})\cdot a_{\mathcal{C}^F}:(\mathcal{C}^F\times\mathcal{C}^F)\times\mathcal{C}^F\longrightarrow\mathcal{C}^F.$$

Using the definition of the functor Ψ_E on the 2-cell a, we get the components as

$$\widehat{a}\big(\big((M,\chi_M),(N,\chi_N)\big),(M',\chi_{M'})\big) = a(M,N,M').$$

Suppose we have a 2-cell in $\mathbf{Mnd}({}_2Cat)$ of the form $l : (\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1}, \otimes(\gamma\times F)\, l_{\mathcal{C}}^{-1}\circ\xi(\delta_I\times\mathcal{C})\, l_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}}, 1_F) : (\mathcal{C}, F) \longrightarrow (\mathcal{C}, F)$. If in the isomorphism (4.1), we make $L \dashv R = D^F \dashv U^F$ and $(\mathcal{X}, H, \mu^H, \eta^H) = (\mathcal{C}, F, \mu^F, \eta^F)$, it can be obtained a 2-cell in the 2-category $\mathbf{Adj}_R({}_2Cat)$ of the form $(l, \beta^l) : (\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1}, 1_{\mathcal{C}}) \longrightarrow ([\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1}]^{\gamma\circ\xi}, [1_{\mathcal{C}}]^{1_F}) : D^F \dashv U^F \longrightarrow D^F \dashv U^F$, where we used the notation $\gamma\circ\xi = \otimes(\gamma\times F)\, l_{\mathcal{C}}^{-1}\circ\xi(\delta_I\times\mathcal{C})\, l_{\mathcal{C}}^{-1}$. We change the notation from β^l to $\hat{l}$.

In the same way as before, it can be proved that $[\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1}]^{\gamma\circ\xi} = \widehat{\otimes}(\delta_{\hat{I}}\times\mathcal{C}^F)\, l_{\mathcal{C}^F}^{-1}$ and $[1_{\mathcal{C}}]^{1_F} = 1_{\mathcal{C}^F}$. Therefore, we obtain a natural transformation $\hat{l} : \widehat{\otimes}(\delta_{\hat{I}}\times\mathcal{C}^F)\, l_{\mathcal{C}^F}^{-1} \longrightarrow 1_{\mathcal{C}^F} : \mathcal{C}^F \longrightarrow \mathcal{C}^F$. Using the definition of the 2-functor Ψ_E on the 2-cell l, the component of the natural transformation $\hat{l}$ on (M, χ_M) is

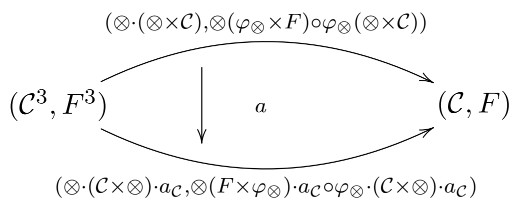
$$\hat{l}(M, \chi_M) = l_M.$$

Similarly to the 2-cell $r : (\otimes\cdot(\mathcal{C}\times\delta_I)\cdot r_{\mathcal{C}}^{-1}, \otimes(F\times\gamma)\, r_{\mathcal{C}}^{-1}\circ\xi\,(\mathcal{C}\times\delta_I)\, r_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}}, 1_F) : (\mathcal{C}, F) \longrightarrow (\mathcal{C}, F)$ there corresponds a natural transformation $\hat{r} : \widehat{\otimes}(\mathcal{C}^F\times\delta_{\hat{I}})\, r_{\mathcal{C}^F}^{-1} \longrightarrow 1_{\mathcal{C}^F} : \mathcal{C}^F \longrightarrow \mathcal{C}^F$. The component of this natural transformation, at (M, χ_M), is

$$\hat{r}(M, \chi_M) = r_M. \tag{5.8}$$

Since the natural transformations a, l and r fulfill the coherence conditions for a monoidal struture and U^F is faithful then $\hat{a}, \hat{l}$ and $\hat{r}$ fulfill the pentagon and the triangle coherence conditions. Therefore, $(\mathcal{C}^F, \widehat{\otimes}, \hat{I}, \hat{a}, \hat{l}, \hat{r})$ is a monoidal structure over $\mathcal{C}^F$.

(3)⇒(2) Note that the aforementioned statements can be reverted. For example, take the morphism of adjunctions $(a, \widehat{a}) : (\otimes\cdot(\otimes\times\mathcal{C}), \widehat{\otimes}\cdot(\widehat{\otimes}\times\mathcal{C}^F)) \longrightarrow (\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}}, \widehat{\otimes}\cdot(\mathcal{C}^F\times\widehat{\otimes})\cdot a_{\mathcal{C}^F}) : (U^F\times U^F)\times U^F \dashv (D^F\times D^F)\times D^F \longrightarrow U^F \dashv D^F$. The image of this 2-cell, under Φ_E, is $a : (\otimes, \varphi_\otimes)\cdot(\otimes\times\mathcal{C}, \varphi_\otimes\times F) \longrightarrow (\otimes, \varphi_\otimes)\cdot(\mathcal{C}\times\otimes, F\times\varphi_\otimes)\cdot(a_{\mathcal{C}}, 1_{F\times(F\times F)\cdot a_{\mathcal{C}}}) : ((\mathcal{C}\times\mathcal{C})\times\mathcal{C}, (F\times F)\times F) \longrightarrow (\mathcal{C}, F)$, that is,



is a 2-cell in $\mathbf{Mnd}({}_2Cat)$.

Everytime we used the isomorphism (4.1), the monad $(\mathcal{C}, F, \mu^F, \eta^F)$ was always taken fixed, therefore the implication $(2 \Rightarrow 3)$ is natural in the monad $(\mathcal{C}, F, \mu^F, \eta^F)$. $\square$

6 Kleisli 2-adjunction

Based on either [2] or [3], the following 2-adjunction takes place

$$\mathbf{Mnd}^{\bullet}({}_2Cat) \underset{\Psi_K}{\overset{\Phi_K}{\leftrightarrows}} \mathbf{Adj}_L({}_2Cat)$$

which can also be deduced from the general 2-adjunction given by (2.1). In this sense, we provide only a few remarks on the structure for the several objects that build this 2-adjunction.

The description of 2-functor, Ψ_K, is given completely in order to provide the necessary notation. The structure of such 2-functor goes as

1. On 0-cells, $\Psi_K(\mathcal{C}, F) = G_F \dashv V_F$, that is, the Kleisli adjunction.

2. On 1-cells, $(P, \pi) : (\mathcal{C}, F) \longrightarrow (\mathcal{D}, H)$, $\Psi_K(P, \pi) = (P, P_\pi, \rho_\pi)$. In the definition of the functor $P_\pi : \mathcal{C}_F \longrightarrow \mathcal{D}_H$, we use the notation $(\cdot)^\sharp$ given for a morphism in $\mathcal{C}_F$ and $(\cdot)^\flat$ for a morphism in $\mathcal{D}_H$. This notation is used in [5] and [9].

   (i) On objects, X in $\mathcal{C}_F$, $P_\pi X = PX$.

   (ii) On morphisms, $x^\sharp : X \longrightarrow Y$ in $\mathcal{C}_F$, $P_\pi x^\sharp = (\pi C_{x^\sharp} \cdot Px)^\flat$, where $C_{x^\sharp}$ is the notation for the codomain of the morphism $x^\sharp$ as in C_F, which in this case is Y.

   (iii) In order to define ρ_π we have to prove that the following equality of functors takes place, $G_H P = P_\pi G_F$. On objects and morphisms $f : A \longrightarrow B$ in $\mathcal{C}$,

$$\begin{aligned} G_H PA &= PA = P_\pi A = P_\pi G_F A, \\ G_H Pf &= (HPf \cdot \eta^H PA)^\flat = (HPf \cdot \pi A \cdot P\eta^F A)^\flat \\ &= (\pi B \cdot PFf \cdot P\eta^F A)^\flat = P_\pi(Ff \cdot \eta^F A)^\sharp = P_\pi G_F f, \end{aligned}$$

where the second equality takes place because of the unitality condition on π and the third one is due to the naturality on π. Using (2.2), we get the mate for this identity

$$\rho_\pi = V_H P_\pi \varepsilon^{DU^F} \circ \eta^H P V_F,$$

whose component, at X in $\mathcal{C}_F$, is $\rho_\pi X = \mu^H PX \cdot H\pi X \cdot \eta^H PFX = \pi X$.

3. On 2-cells, $\vartheta : (P, \pi) \longrightarrow (Q, \tau)$, we have

$$\Psi_K(\vartheta) = (\alpha_\vartheta, \beta_\vartheta)$$

where $\alpha_\vartheta := \vartheta$ and we rename β_ϑ as $\tilde{\vartheta}$. The induced natural tranformation $\tilde{\vartheta} : P_\pi \longrightarrow Q_\tau : \mathcal{C}_F \longrightarrow \mathcal{D}_H$ is defined through its component on X, using the condition $G_H \vartheta = \tilde{\vartheta} G_F$, as

$$\tilde{\vartheta} X = (\eta^H QX \cdot \vartheta X)^\flat. \tag{6.1}$$

Since we have a 2-adjunction, the following isomorphism of categories takes place, natural in $(\mathcal{X}, H)$ and $L \dashv R$

$$Hom_{\mathbf{Mnd}^\bullet(_2Cat)}\big((\mathcal{X}, H), \Phi_K(L \dashv R)\big) \cong Hom_{\mathbf{Adj}_L(_2Cat)}\big(\Psi_K(\mathcal{X}, H), L \dashv R\big). \tag{6.2}$$

7 Monoidal extensions (Kleisli type)

In this section, we give the dual version of the monoidal liftings, therefore the definition of a *lax monad* is provided.

Theorem 7.1. *A lax monad $((F, \zeta, \omega), \mu^F, \eta^F)$ over a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ consists of*

(1) *(F, μ^F, η^F) is a monad on $\mathcal{C}$.*

(2) *$(F, \zeta, \omega) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{C}, \otimes, I)$ is a lax monoidal functor. This means that the natural transformations $\zeta : \otimes \cdot (F \times F) \longrightarrow F \cdot \otimes$ and*

$\omega : \delta_I \longrightarrow F \cdot \delta_I$, *fulfills the commutativity on the diagrams*

$$\begin{array}{ccccc}
(FA \otimes FB) \otimes FC & \xrightarrow{\zeta_{A,B} \otimes FC} & F(A \otimes B) \otimes FC & \xrightarrow{\zeta_{A \otimes B, C}} & F((A \otimes B) \otimes C) \\
\Big\downarrow{\scriptstyle a_{FA,FB,FC}} & & & & \Big\downarrow{\scriptstyle Fa_{A,B,C}} \\
FA \otimes (FB \otimes FC) & \xrightarrow[FA \otimes \zeta_{B,C}]{} & FA \otimes F(B \otimes C) & \xrightarrow[\zeta_{A, B \otimes C}]{} & F(A \otimes (B \otimes C))
\end{array} \tag{7.1}$$

$$\begin{array}{ccccc}
I \otimes FA & \xrightarrow{\omega \otimes FA} & FI \otimes FA & \xrightarrow{\zeta_{I,A}} & F(I \otimes A) \\
& {\scriptstyle l_{FA}} \searrow & & \swarrow {\scriptstyle Fl_A} & \\
& & FA & &
\end{array} \tag{7.2}$$

$$\begin{array}{ccccc}
F(A \otimes I) & \xleftarrow{\zeta_{A,I}} & FA \otimes FI & \xleftarrow{FA \otimes \omega} & FA \otimes I \\
& {\scriptstyle Fr_A} \searrow & & \swarrow {\scriptstyle r_{FA}} & \\
& & FA. & &
\end{array} \tag{7.3}$$

$\mu^F : (F, \zeta, \omega) \cdot (F, \zeta, \omega) \longrightarrow (F, \zeta, \omega)$ *and* $\eta^F : (1_{\mathcal{C}}, 1_{\otimes}, 1_{\delta_I}) \longrightarrow (F, \zeta, \omega)$ *are lax natural transformations, the adjective lax adds, to the naturality, the commutative diagrams*

$$\begin{array}{ccccc}
\otimes(FF \times FF) & \xrightarrow{\zeta(F \times F)} & F \otimes (F \times F) & \xrightarrow{F\zeta} & FF\otimes \\
\Big\downarrow{\scriptstyle \otimes(\mu^F \times \mu^F)} & & & & \Big\downarrow{\scriptstyle \mu^F \otimes} \\
\otimes(F \times F) & & \xrightarrow[\zeta]{} & & F\otimes
\end{array} \tag{7.4}$$

$$\begin{array}{ccccc}
\delta_I & \xrightarrow{\omega} & F\delta_I & \xrightarrow{F\omega} & FF\delta_I \\
& & {\scriptstyle \omega} \searrow & & \Big\downarrow{\scriptstyle \mu^F \delta_I} \\
& & & & F\delta_I
\end{array} \tag{7.5}$$

$$\begin{array}{ccc} \otimes & \xrightarrow{\;1_{\otimes}\;} & \otimes \\ {\scriptstyle \otimes(\eta^F\times\eta^F)}\downarrow & & \downarrow{\scriptstyle \eta^F\otimes} \\ \otimes(F\times F) & \xrightarrow[\;\zeta\;]{} & F\otimes \end{array} \qquad \begin{array}{ccc} \delta_I & \xrightarrow{\;1_{\delta_I}\;} & \delta_I \\ & {\scriptstyle \omega}\searrow & \downarrow{\scriptstyle \eta^F\delta_I} \\ & & F\delta_I. \end{array} \tag{7.6}$$

Note 7.2. Necessarily $\omega(0) = \eta^F I$.

The natural transformation ω has only one component at 0 in $\mathbf{1}$, then both are going to be denoted by ω, that is, $\omega = \omega(0) = \eta^F I$.

We are going to make use of the isomorphism (6.2). The result we want to obtain using this isomorphism is the following.

Theorem 7.3. *There is a bijective correspondence between the following structures:*

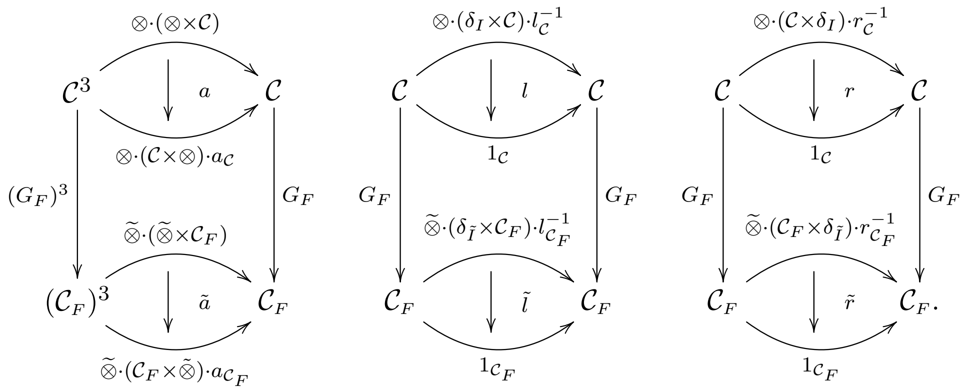
(1) *Lax monads* $((F, \zeta, \omega), \mu^F, \eta^F)$*, for the monoidal structure* $(\mathcal{C}, \otimes, I, a, l, r)$*.*

(2) *Morphims and transformations of monads of the form*

$$\begin{array}{rcl} (\otimes, \zeta) & : & (\mathcal{C}\times\mathcal{C}, F\times F) \longrightarrow (\mathcal{C}, F), \\ (\delta_I, \omega) & : & (\mathbf{1}\,, 1_{\mathbf{1}}\,) \longrightarrow (\mathcal{C}, F) \\ a & : & (\otimes\cdot(\otimes\times\mathcal{C}), \zeta(\otimes\times\mathcal{C})\circ\otimes(\zeta\times F)) \longrightarrow \\ & & (\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}}, \zeta(\mathcal{C}\times\otimes)a_{\mathcal{C}}\circ\otimes(F\times\zeta)a_{\mathcal{C}}) : \\ & & \qquad ((\mathcal{C}\times\mathcal{C})\times\mathcal{C}, (F\times F)\times F) \longrightarrow (\mathcal{C}, F), \\ l & : & (\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1}, \zeta\,(\delta_I\times\mathcal{C})\,l_{\mathcal{C}}^{-1}\circ\otimes(\omega\times F)\,l_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}}, 1_F) : \\ & & \qquad (\mathcal{C}, F) \longrightarrow (\mathcal{C}, F), \\ r & : & (\otimes\cdot(\mathcal{C}\times\delta_I)\cdot r_{\mathcal{C}}^{-1}, \zeta\,(\mathcal{C}\times\delta_I)\,r_{\mathcal{C}}^{-1}\circ\otimes(F\times\omega)\,r_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}}, 1_F) : \\ & & \qquad (\mathcal{C}, F) \longrightarrow (\mathcal{C}, F). \end{array}$$

Monoidal structures for the Kleisli category $(\mathcal{C}_F, \widetilde{\otimes}, \tilde{I})$ *such that the following* *diagrams of arrows and surfaces commute:*

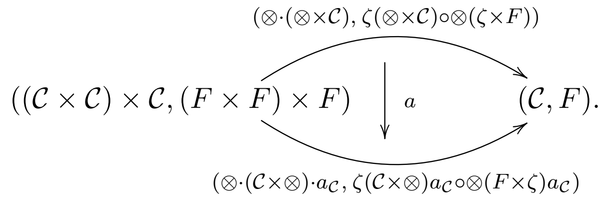
$$
\begin{array}{ccc}
(a) & & (b) \\
\begin{array}{ccc}
\mathcal{C}\times\mathcal{C} & \xrightarrow{\ \otimes\ } & \mathcal{C} \\
{\scriptstyle G_F\times G_F}\downarrow & & \downarrow{\scriptstyle G_F} \\
\mathcal{C}_F\times\mathcal{C}_F & \xrightarrow[\ \widetilde{\otimes}\ ]{} & \mathcal{C}_F
\end{array}
& \qquad &
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\ \delta_I\ } & \mathcal{C} \\
{\scriptstyle G_{1_{\mathbf{1}}}}\downarrow & & \downarrow{\scriptstyle G^F} \\
\mathbf{1}_{1_{\mathbf{1}}} & \xrightarrow[\ \delta_{\tilde I}\ ]{} & \mathcal{C}^F
\end{array}
\end{array}
\tag{7.7}
$$



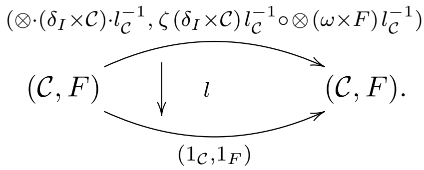
Proof. (1)⇒(2) Consider a lax monad $((F,\zeta,\omega),\mu^F,\eta^F)$ for the monoidal category $(\mathcal{C},\otimes,I)$. In particular, μ^F and η^F are natural lax monoidal transformations. Therefore, the commutativity of (7.4) and the first diagram in (7.6) is equivalent to the condition that $(\otimes,\zeta):(\mathcal{C}\times\mathcal{C},F\times F)\longrightarrow(\mathcal{C},F)$ be a monad morphism.

The commutativity conditions in (7.5) and the second on (7.6) is equivalent to the condition for the following to be a monad morphism (δ_I,ω) : $(\mathbf{1}\,,1_{\mathbf{1}}\,)\longrightarrow(\mathcal{C},F)$.

Since $(\otimes,\zeta)$ is a morphism of monads so are $(\otimes\cdot(\otimes\times\mathcal{C}),\ \zeta(\otimes\times\mathcal{C})\circ\otimes(\zeta\times F))$ and $(\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}},\ \zeta(\mathcal{C}\times\otimes)a_{\mathcal{C}}\circ\otimes(F\times\zeta)a_{\mathcal{C}})$. Yet again, since $((F,\zeta),\mu^F,\eta^F)$ is a lax monad over the monoidal category $(\mathcal{C},\otimes,I,a,l,r)$, then a commutative diagram like (7.1) takes place. Therefore the following is a 2-cell in $\mathbf{Mnd}^\bullet({}_2Cat)$.



Since $(\otimes, \zeta)$ and (δ_I, ω) are monad morphisms so is $(\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1}, \zeta\,(\delta_I\times\mathcal{C})\,l_{\mathcal{C}}^{-1}\circ\otimes(\omega\times F)\,l_{\mathcal{C}}^{-1})$ and taking into account the commutativity of the diagram (7.2a), we can state that the following is a 2-cell in $\mathbf{Mnd}^\bullet({}_2Cat)$



In the very same way, $r : (\otimes\cdot(\mathcal{C}\times\delta_I)\cdot r_{\mathcal{C}}^{-1}, \zeta(\mathcal{C}\times\delta_I)\,r_{\mathcal{C}}^{-1}\circ\otimes(F\times\omega)\,r_{\mathcal{C}}^{-1})$ is a 2-cell of monads.

(2)⇒(1) The previous assertions can be reverted.

(2)⇒(3) Suppose we have a monad morphism $(\otimes, \zeta)$. Use the isomorphism (6.2), with $(\mathcal{D}, H, \mu^H, \eta^H) = (\mathcal{C}\times\mathcal{C}, F\times F, \mu^F\times\mu^F, \eta^F\times\eta^F)$ and $L\dashv R = G_F\dashv V_F$ to get an associated morphism of adjunctions $(\otimes, \otimes_\zeta) : G_F\times G_F\dashv V_F\times V_F \longrightarrow G_F\dashv V_F$, such that a diagram like (7.7a) commutes. According to the definition of Ψ_K, the functor $\otimes_\zeta$ acts as follows. On objects,

$$\otimes_\zeta(X, Y) = \otimes(X, Y) = X\otimes Y,$$

and on morphisms,

$$\otimes_\zeta(x^\sharp, y^\sharp) = (\zeta_{C_{x^\sharp}, C_{y^\sharp}}\cdot(x\otimes y))^\sharp,$$

where $C_{x^\sharp}$ is codomain of the morphism $x^\sharp$ for example. We rename $\otimes_\zeta$ as $\widetilde{\otimes}$.

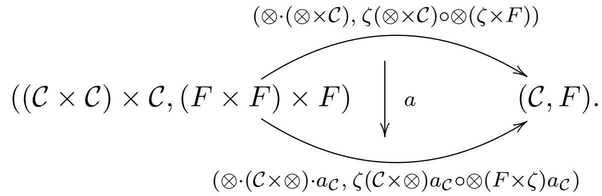
For the monad morphism, $(\delta_I, \omega) : (\mathbf{1}, 1_{\mathbf{1}}) \longrightarrow (\mathcal{C}, F)$, use the mentioned isomorphism with $(\mathcal{D}, H, \mu^H, \eta^H) = (\mathbf{1}\,, 1_{\mathbf{1}}, 1_{1_{\mathbf{1}}}, 1_{1_{\mathbf{1}}})$, that is, the trivial monad on the category $\mathbf{1}$, and $L\dashv R = G_F\dashv V_F$. Therefore, there exists an adjunction morphism $(\delta_I, [\delta_I]_\omega) : G_{1_\mathbf{1}} \dashv V_{1_\mathbf{1}} \longrightarrow G_F \dashv V_F$. According to the 2-functor Ψ_K, the functor $[\delta_I]_\omega : \mathbf{1} \longrightarrow \mathcal{C}_F$, acts as

$$[\delta_I]_\omega(0) = \delta_I(0) = I,$$

$$[\delta_I]_\omega(1_0) = (\omega(0) \cdot \delta_I(1_0))^\sharp = (\eta^F I)^\sharp.$$

That is to say $[\delta_I]_\omega = \delta_{\tilde{I}} : \mathbf{1}_{1_\mathbf{1}} \longrightarrow \mathcal{C}_F$, where $\tilde{I} = I$.

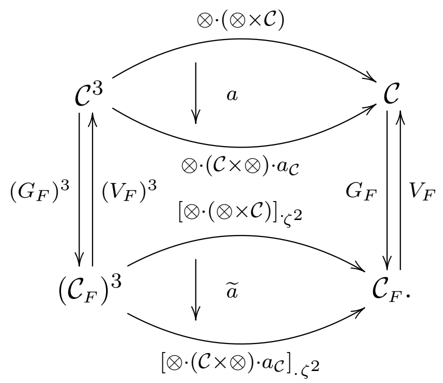
Suppose that we have the following 2-cell in $\mathbf{Mnd}^\bullet({}_2Cat)$,



In order to continue with the calculations, we use the following notation, for the sake of simplification

$$\begin{aligned} {}^{\cdot}\zeta^2 &:= \zeta(\otimes \times \mathcal{C}) \circ \otimes(\zeta \times F), \\ {}_{\cdot}\zeta^2 &:= \zeta(\mathcal{C} \times \otimes)a_\mathcal{C} \circ \otimes(F \times \zeta)a_\mathcal{C}, \\ (\cdot)^3 &:= (\cdot \times \cdot) \times \cdot\,. \end{aligned}$$

According to the isomorphism of categories given by (6.2), to the previous 2-cell in $\mathbf{Mnd}^\bullet({}_2Cat)$ corresponds a 2-cell, (α_a, β_a) in $\mathbf{Adj}_L({}_2Cat)$, where $\alpha_a = a$ and we rename $\beta_a = \tilde{a}$ and such that



It can be show that

$$\begin{aligned}[\otimes\cdot(\otimes\times\mathcal{C})]_{\cdot\zeta^2} &= \widetilde{\otimes}\cdot(\widetilde{\otimes}\times\mathcal{C}_F)\\ [\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}}]_{.\zeta^2} &= \widetilde{\otimes}\cdot(\mathcal{C}_F\times\widetilde{\otimes})\cdot a_{\mathcal{C}_F}.\end{aligned}$$

Therefore, we have a natural transformation $\widetilde{a}:\widetilde{\otimes}\cdot(\widetilde{\otimes}\times\mathcal{C}_F)\longrightarrow\widetilde{\otimes}\cdot(\mathcal{C}_F\times\widetilde{\otimes})\cdot a_{\mathcal{C}_F}$ that will be part of a monoidal structure on $\mathcal{C}_F$. According to the 2-functor Ψ_K, the component of $\widetilde{a}$ at $((X,Y),Z)$ is

$$\widetilde{a}_{X,Y,Z}=(\eta^F(X\otimes(Y\otimes Z))\cdot a_{X,Y,Z})^\sharp.$$

Suppose that we have a 2-cell in $\mathbf{Mnd}^\bullet({}_2Cat)$ of the form $l:\big(\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1},\zeta(\delta_I\times\mathcal{C})\,l_{\mathcal{C}}^{-1}\circ\otimes(\omega\times F)\,l_{\mathcal{C}}^{-1}\big)\longrightarrow(1_{\mathcal{C}},1_F):(\mathcal{C},F)\longrightarrow(\mathcal{C},F)$. Therefore, we obtain a natural transformation $\tilde{l}:\widetilde{\otimes}\cdot(\delta_{\tilde{I}}\times\mathcal{C}_F)\cdot l_{\mathcal{C}_F}^{-1}\longrightarrow 1_{\mathcal{C}_F}$. Using the definition of the functor Ψ_K on the 2-cell l, the component of $\widetilde{l}$, on the object X in $\mathcal{C}_F$, is

$$\tilde{l}X=(\eta^F X\cdot lX)^\sharp. \tag{7.8}$$

Similarly, for the monad morphism $r:(\otimes\cdot(\mathcal{C}\times\delta_I)\cdot r_{\mathcal{C}}^{-1},\zeta(\mathcal{C}\times\delta_I)r_{\mathcal{C}}^{-1}\circ\otimes(F\times\omega)r_{\mathcal{C}}^{-1})\longrightarrow(1_{\mathcal{C}},1_F):(\mathcal{C},F)\longrightarrow(\mathcal{C},F)$, we obtain a natural transformation $\tilde{r}:\otimes_\zeta\cdot(\mathcal{C}_F\times\delta_{\tilde{I}})\cdot r_{\mathcal{C}_F}^{-1}\longrightarrow 1_{\mathcal{C}_F}:\mathcal{C}_F\longrightarrow\mathcal{C}_F$.

The proof of the coherence conditions are left to the reader.

In summary, $(\mathcal{C}_F,\widetilde{\otimes},\tilde{I},\tilde{a},\tilde{l},\tilde{r})$ has a monoidal structure on $\mathcal{C}_F$.

(3)⇒(2) Using the isomorphism, given by (6.2), we get the return of the proof. For example, the image, under Φ_K, for the 2-cell of adjunctions $(a,\tilde{a}):(\otimes\cdot(\otimes\times\mathcal{C}),\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}})\longrightarrow(\widetilde{\otimes}\cdot(\widetilde{\otimes}\times\mathcal{C}_F),\widetilde{\otimes}\cdot(\mathcal{C}_F\times\widetilde{\otimes})\cdot a_{\mathcal{C}_F}):(G_F\times G_F)\times G_F\dashv(V_F\times V_F)\times V_F$ is

$$\begin{aligned}\Phi_K&((a,\tilde{a}))\\ &= a:(\otimes,\pi_\otimes)(\otimes\times\mathcal{C},\pi_{\otimes\times\mathcal{C}})\longrightarrow(\otimes,\pi_\otimes)(\mathcal{C}\times\otimes,\pi_{\mathcal{C}\times\otimes})(a_{\mathcal{C}},\pi_{a_{\mathcal{C}}})\\ &\qquad :(\mathcal{C}^3,F^3)\longrightarrow(\mathcal{C},F)\\ &= a:(\otimes,\pi_\otimes)(\otimes\times\mathcal{C},\pi_\otimes\times F)\longrightarrow\\ &\qquad (\otimes,\pi_\otimes)(\mathcal{C}\times\otimes,F\times\pi_\otimes)(a_{\mathcal{C}},1_{F\times(F\times F)\cdot a_{\mathcal{C}}})\\ &\qquad :(\mathcal{C}^3,F^3)\longrightarrow(\mathcal{C},F)\\ &= a:\big(\otimes\cdot(\otimes\times\mathcal{C}),\pi_\otimes(\otimes\times\mathcal{C})\circ\otimes(\pi_\otimes\times F)\big)\longrightarrow\\ &\qquad \big(\otimes\cdot(\mathcal{C}\times\mathcal{C})\cdot a_{\mathcal{C}},\pi_\otimes(\mathcal{C}\times\otimes)a_{\mathcal{C}}\circ\otimes(F\times\pi_\otimes)a_{\mathcal{C}}\big)\\ &\qquad :(\mathcal{C}^3,F^3)\longrightarrow(\mathcal{C},F).\end{aligned}$$

We used the fact that $a_{\mathcal{C}}$ is a morphism of adjunctions. □

8 Liftings to the Eilenberg-Moore algebras & Extensions to the Kleisli categories

This is probably the most explored section in this article, a few examples of the detailed proofs for the following statements are found in [1] and [9]. In this section, we treated these statements only as direct consequences of the isomorphisms of the categories given by (4.1) and (6.2).

Theorem 8.1. *There is a bijective correspondence, natural in* $(\mathcal{C}, F, \mu^F, \eta^F)$ *and* $(\mathcal{D}, H, \mu^H, \eta^H)$*, between the structures*

(1) *Liftings to the Eilenberg-Moore algebras, for the functor* $P : \mathcal{C} \longrightarrow \mathcal{D}$. *That is to say, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}^F & \xrightarrow{\;Q\;} & \mathcal{D}^H \\ {\scriptstyle U^F}\downarrow & & \downarrow{\scriptstyle U^H} \\ \mathcal{C} & \xrightarrow[\;P\;]{} & \mathcal{D}. \end{array}$$

(2) *Morphisms of monads* $(P, \varphi) : (\mathcal{C}, F) \longrightarrow (\mathcal{D}, H)$. *That is to say, a natural transformation* $\varphi : HP \longrightarrow PF$*, such that the following diagrams commute:*

$$\begin{array}{ccccc} HHP & \xrightarrow{H\varphi} & HPF & \xrightarrow{\varphi F} & PFF \\ {\scriptstyle \mu^H P}\downarrow & & & & \downarrow{\scriptstyle P\mu^F} \\ HP & & \xrightarrow[\varphi]{} & & PF \end{array} \qquad \begin{array}{ccc} & P & \\ {\scriptstyle \eta^H P}\swarrow & & \searrow{\scriptstyle P\eta^F} \\ HP & \xrightarrow[\varphi]{} & PF. \end{array}$$

Theorem 8.2. *There exists a bijective correspondence, natural in* $(\mathcal{C}, F, \mu^F, \eta^F)$ *and* $(\mathcal{D}, H, \mu^H, \eta^H)$*, between the structures*

(1) *Extensions to the Kleisli categories, for the functor* $P : \mathcal{C} \longrightarrow \mathcal{D}$*. That is to say, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{P} & \mathcal{D} \\ {\scriptstyle G_F}\big\downarrow & & \big\downarrow{\scriptstyle G_H} \\ \mathcal{C}_F & \xrightarrow[Q]{} & \mathcal{D}_H . \end{array}$$

(2) *Morphisms of monads* $(P, \varphi) : (\mathcal{C}, F) \longrightarrow (\mathcal{D}, H)$*. That is to say, a natural transformation* $\varphi : PF \longrightarrow HP$ *such that the following diagrams commute*

$$\begin{array}{ccccc} PFF & \xrightarrow{\varphi F} & HPF & \xrightarrow{H\varphi} & HHP \\ {\scriptstyle P\mu^F}\big\downarrow & & & & \big\downarrow{\scriptstyle \mu^H P} \\ PF & & \xrightarrow[\varphi]{\qquad\qquad} & & HP \end{array} \qquad \begin{array}{ccc} & P & \\ {\scriptstyle P\eta^F}\swarrow & & \searrow{\scriptstyle \eta^H P} \\ PF & \xrightarrow[\varphi]{\qquad} & HP. \end{array}$$

9 Actions on the Kleisli category

In this section, we relate actions on the Kleisli category and strong monads through the isomorphism given by the corresponding 2-adjunction. In order to do so, the following definitions have to be stated.

Definition 9.1. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A left $\mathcal{C}$-action on the category $\mathcal{B}$ is a functor $\boxtimes : \mathcal{C} \times \mathcal{B} \longrightarrow \mathcal{B}$ together with natural transformations $\nu : \boxtimes(\otimes \times \mathcal{B}) \longrightarrow \boxtimes(\mathcal{C} \times \otimes)$, $a_* : (\mathcal{C} \times \mathcal{C}) \times \mathcal{B} \longrightarrow \mathcal{B}$ and $j : \boxtimes(\delta_I \times \mathcal{B}) l_{\mathcal{C}}^{-1} \longrightarrow 1_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ such that they fulfill the following commutative diagrams, for objects C, C', C'' in $\mathcal{C}$ and B in $\mathcal{B}$,

$$\begin{array}{ccc}
[(C\otimes C')\otimes C'']\boxtimes B & \xrightarrow{\nu_{C\otimes C',C'',B}} & (C\otimes C')\boxtimes(C''\boxtimes B)\\
\Big\downarrow{\scriptstyle a_{C,C',C''}\boxtimes B} & & \\
[C\otimes(C'\otimes C'')]\boxtimes B & & \Big\downarrow{\scriptstyle \nu_{C,C',C''\boxtimes B}}\\
\Big\downarrow{\scriptstyle \nu_{C,C'\otimes C'',B}} & & \\
C\boxtimes[(C'\otimes C'')\boxtimes B] & \xrightarrow[C\boxtimes\nu_{C',C'',B}]{} & C\boxtimes[C'\boxtimes(C''\boxtimes B)]
\end{array}$$

and

$$\begin{array}{ccc}
(C\otimes I)\boxtimes B & \xrightarrow{\nu_{C,I,B}} & C\boxtimes(I\boxtimes B)\\
{\scriptstyle r_C\boxtimes B}\searrow & & \swarrow{\scriptstyle C\boxtimes j_B}\\
 & C\boxtimes B &
\end{array}
\qquad
\begin{array}{ccc}
(I\otimes C)\boxtimes B & \xrightarrow{\nu_{I,C,B}} & I\boxtimes(C\boxtimes B)\\
{\scriptstyle l_C\boxtimes B}\searrow & & \swarrow{\scriptstyle j_{C\boxtimes B}}\\
 & C\boxtimes B. &
\end{array}$$

Definition 9.2. A *right strong monad* $((F,\sigma^r),\mu^F,\eta^F)$, on the monoidal category $(\mathcal{C},\otimes,I)$, is a usual monad (F,μ^F,η^F), on $\mathcal{C}$, with a natural transformation $\sigma^r : A\otimes FB \longrightarrow F(A\otimes B)$ such that the following diagrams commute:

$$(a)\qquad\begin{array}{ccccc}
A\otimes FFB & \xrightarrow{\sigma^r_{A,FB}} & F(A\otimes FB) & \xrightarrow{F\sigma^r_{A,B}} & FF(A\otimes B)\\
\Big\downarrow{\scriptstyle A\otimes\mu^F B} & & & & \Big\downarrow{\scriptstyle \mu^F(A\otimes B)}\\
A\otimes FB & & \xrightarrow[\sigma^r]{} & & F(A\otimes B)
\end{array}\tag{9.1}$$

$$(b)\qquad\begin{array}{ccc}
 & A\otimes B & \\
{\scriptstyle A\otimes\eta^F B}\swarrow & & \searrow{\scriptstyle \eta^F(A\otimes B)}\\
A\otimes FB & \xrightarrow[\sigma^r]{} & F(A\otimes B)
\end{array}$$

and

$$(a)\quad \begin{array}{ccccc} (A\otimes B)\otimes FC & & \xrightarrow{\sigma^r_{A\otimes B,C}} & & F((A\otimes B)\otimes C) \\ \Big\downarrow{\scriptstyle a_{A,B,FC}} & & & & \Big\downarrow{\scriptstyle Fa_{A,B,C}} \\ A\otimes(B\otimes FC) & \xrightarrow[A\otimes\sigma^r_{B,C}]{} & A\otimes F(B\otimes C) & \xrightarrow[\sigma^r_{A,B\otimes C}]{} & F(A\otimes(B\otimes C)) \end{array} \tag{9.2}$$

$$(b)\quad \begin{array}{ccc} I\otimes FA & \xrightarrow{\sigma^r_{I,A}} & F(I\otimes A) \\ & {\scriptstyle l_{FA}}\searrow \quad \swarrow{\scriptstyle Fl_A} & \\ & FA. & \end{array}$$

Definition 9.3. A *left strong monad* $((F,\sigma^l),\mu^F,\eta^F)$ on a monoidal category $(\mathcal{C},\otimes,I)$, is a usual monad (F,μ^F,η^F) on $\mathcal{C}$, together with a natural transformation $\sigma^l_{A,B} : FA\otimes B \longrightarrow F(A\otimes B)$ such that fulfills the commutativity of dual diagrams like (9.1) and (9.2).

The following theorem can be stated, note that the proof is just an adaptation for the corresponding lax monoidal case.

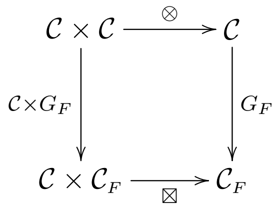
Theorem 9.4. *There exists a bijection between the following structures*

(1) *Right strong monads* $((F,\sigma^r),\mu^F,\eta^F)$ *on the monoidal category* $(\mathcal{C},\otimes,I,a,r,l)$.

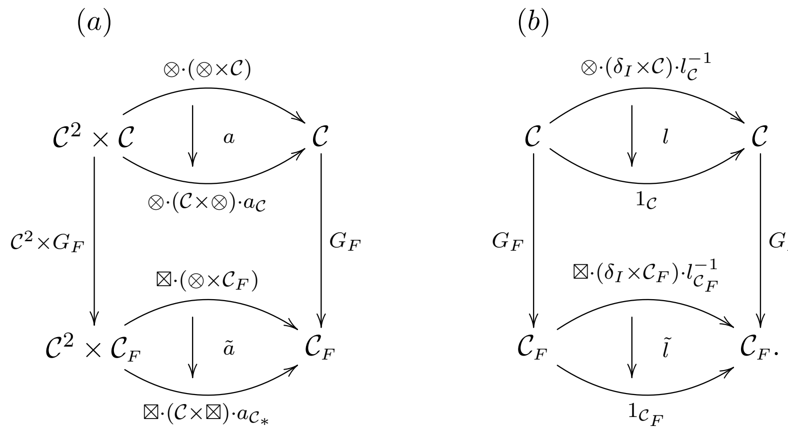
(2) *Morphisms and transformations of monads of the form*

$$\begin{array}{rcl} (\otimes,\sigma^r) & : & (\mathcal{C}\times\mathcal{C},\mathcal{C}\times F) \longrightarrow (\mathcal{C},F) \\ a & : & (\otimes\cdot(\otimes\times\mathcal{C}),\sigma^r(\otimes\times\mathcal{C})) \longrightarrow \\ & & \qquad (\otimes\cdot(\mathcal{C}\times\otimes)\cdot a_{\mathcal{C}},\sigma^r(\mathcal{C}\times\otimes)a_{\mathcal{C}}\circ\otimes(\mathcal{C}\times\sigma^r)a_{\mathcal{C}}) \\ & & \qquad : ((\mathcal{C}\times\mathcal{C})\times\mathcal{C},(\mathcal{C}\times\mathcal{C})\times F) \longrightarrow (\mathcal{C},F) \\ l & : & (\otimes\cdot(\delta_I\times\mathcal{C})\cdot l_{\mathcal{C}}^{-1},\sigma^r(\delta_I\times\mathcal{C})\,l_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}},1_F) : (\mathcal{C},F) \longrightarrow (\mathcal{C},F). \end{array}$$

(3) *Left actions on the Kleisli category,* $\mathcal{C}_F$*,* $\boxtimes : \mathcal{C} \times \mathcal{C}_F \longrightarrow \mathcal{C}_F$ *such that the following diagrams of morphisms and surfaces commute:*



(9.3)



(9.4)

We state the dual theorem as follows.

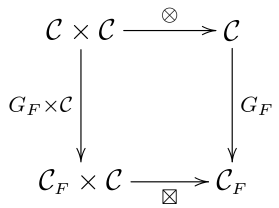
Theorem 9.5. *There exists a bijection between the following structures*

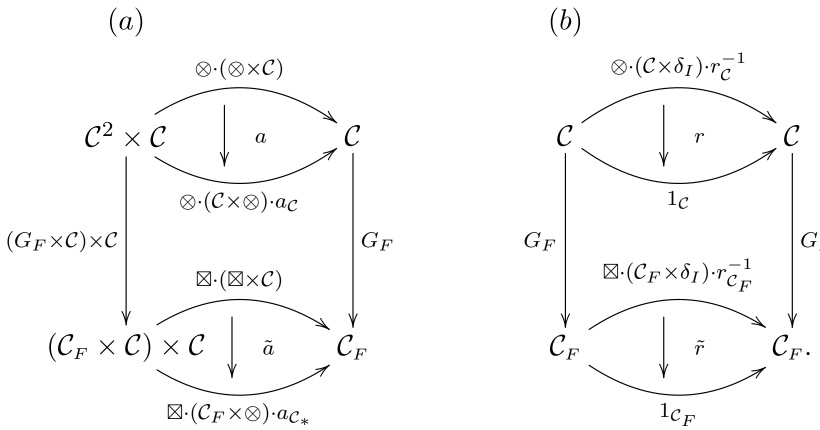
(1) *Left strong monads* $((F, \sigma^l), \mu^F, \eta^F)$ *on the monoidal category* $(\mathcal{C}, \otimes, I, a, r, l)$*.*

(2) *Morphisms and transformations of monads of the form*

$$\begin{gathered}(\otimes, \varphi) : (\mathcal{C} \times \mathcal{C}, F \times \mathcal{C}) \longrightarrow (\mathcal{C}, F) \\ a : (\otimes \cdot (\otimes \times \mathcal{C}), \varphi(\otimes \times \mathcal{C}) \circ \otimes(\varphi \times \mathcal{C})) \longrightarrow \\ (\otimes \cdot (\mathcal{C} \times \otimes) \cdot a_{\mathcal{C}}, \varphi(\mathcal{C} \times \otimes) a_{\mathcal{C}}) \\ : ((\mathcal{C} \times \mathcal{C}) \times \mathcal{C}, (F \times \mathcal{C}) \times \mathcal{C}) \longrightarrow (\mathcal{C}, F) \\ r : (\otimes \cdot (\mathcal{C} \times \delta_I) \cdot r_{\mathcal{C}}^{-1}, \varphi(\mathcal{C} \times \delta_I) r_{\mathcal{C}}^{-1}) \longrightarrow (1_{\mathcal{C}}, 1_F) : (\mathcal{C}, F) \longrightarrow (\mathcal{C}, F).\end{gathered}$$

Right actions on the Kleisli category, $\mathcal{C}_F$, $\boxtimes : \mathcal{C}_F \times \mathcal{C} \longrightarrow \mathcal{C}_F$ *such that the following diagrams of morphisms and surfaces commute:*





We left to the reader the writing of dual statements, that is, the ones that corresponds to the Eilenberg-Moore category, where the direction of the natural transformations are inverted, for example $\widehat{\sigma}^r_{A,B} : F(A \otimes B) \longrightarrow A \otimes FB$.

10 Functor algebras

Check Proposition II.1.1 in [4] and [6] for this section. We define the category of H-left functor algebras for a given monad $(\mathcal{D}, H, \mu^H, \eta^H)$.

Definition 10.1. The category of left H-functor algebras, for the pair $(\mathcal{C}, \mathcal{D})$, denoted as ${}_H\mathcal{F}$ or ${}_H\mathcal{M}$, is defined as follows. The objects are given by (J, λ_J), where $J : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor and $\lambda_J : HJ \longrightarrow J$ is a natural transformation such that the following diagrams commute:

$$
\begin{array}{ccc}
HHJ & \xrightarrow{\mu^H J} & HJ \\
{\scriptstyle H\lambda_J}\downarrow & & \downarrow{\scriptstyle \lambda_J} \\
HJ & \xrightarrow[\lambda_J]{} & J
\end{array}
\qquad
\begin{array}{ccc}
J & \xrightarrow{\eta^H J} & HJ \\
 & {\scriptstyle 1_J}\searrow & \downarrow{\scriptstyle \lambda_J} \\
 & & J.
\end{array}
\tag{10.1}
$$

A *morphism of functor algebras*, $\theta : (J, \lambda_J) \longrightarrow (K, \lambda_K)$, is a natural transformation $\theta : J \longrightarrow K$ such that the following diagram commute:

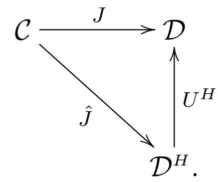
$$
\begin{array}{ccc}
HJ & \xrightarrow{H\theta} & HK \\
{\scriptstyle \lambda_J}\downarrow & & \downarrow{\scriptstyle \lambda_K} \\
J & \xrightarrow[\theta]{} & K.
\end{array}
\tag{10.2}
$$

We realize that the diagrams given by (10.1), for a left H-functor algebra, account for a monad morphism of the form $(J, \lambda_J) : (\mathcal{C}, 1_{\mathcal{C}}) \longrightarrow (\mathcal{D}, H)$. In the same way, the commutative diagram for a morphism of left H-functor algebras, as in (10.2), account for a monad transformation $\theta : (J, \lambda_J) \longrightarrow (K, \lambda_K) : (\mathcal{C}, 1_{\mathcal{C}}) \longrightarrow (\mathcal{D}, H)$.

Using the isomorphism for the Eilenberg-Moore 2-adjunction, given by (4.1), the category ${}_H\mathcal{F}$ is isomorphic to the following category, named possibly as *category of liftings to* $\mathcal{D}^H$, for the pair $(\mathcal{C}, \mathcal{D})$. The objects of such category are functor pairs $(J, \hat{J})$ such that they complete to an adjunction morphism, in $\mathbf{Adj}_R({}_2Cat)$, of the form $(J, \hat{J}) : 1_{\mathcal{C}} \dashv 1_{\mathcal{C}} \longrightarrow D^H \dashv U^H$. That is to say, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{J} & \mathcal{D} \\
{\scriptstyle 1^{1_{\mathcal{C}}}}\uparrow & & \uparrow{\scriptstyle U^H} \\
\mathcal{C}^{1_{\mathcal{C}}} & \xrightarrow[\hat{J}]{} & \mathcal{D}^H
\end{array}
$$

that is,

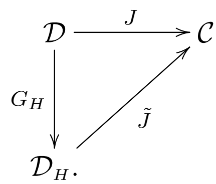


The morphisms of such category are the usual morphisms of adjunctions $(\alpha, \beta) : (J, \hat{J}) \longrightarrow (K, \hat{K}) : 1_{\mathcal{C}} \dashv 1_{\mathcal{C}} \longrightarrow D^H \dashv U^H$. We then proved the following theorem.

Theorem 10.2. *There exists an isomorphism, natural on* $\mathcal{C}$ *and* $(\mathcal{D}, H)$*, between the categories*

(1) *The category of left* H*-functor algebras* ${}_H\mathcal{F}$*.*
(2) *The category of liftings to* $\mathcal{D}^H$*, for the pair* $(\mathcal{C}, \mathcal{D})$*.*

Dually, we have the category of right H-functor algebras, for the monad $(\mathcal{D}, H, \mu^H, \eta^H)$, denoted as $\mathcal{F}_H$ or $\mathcal{M}_H$. The objects are pairs (J, ρ_J), where the natural transformation $\rho_J : JH \longrightarrow J$ is such that it fulfills diagrams dual to those in (10.1). In the same (dual) way as before, this category is the same as the category $Hom_{\mathbf{Mnd}^\bullet({}_2Cat)}((\mathcal{D}, H), (\mathcal{C}, 1_{\mathcal{C}}))$. Therefore, using the isomorphism (6.2), the previous category is isomorphic to the category named as *extensions from* $\mathcal{D}_H$, for the pair $(\mathcal{D}, \mathcal{C})$. The objects of this category are pairs of functors $(J, \tilde{J})$ such that they complete to an adjunction morphism $(J, \tilde{J}) : G_H \dashv V_H \longrightarrow 1_{\mathcal{C}} \dashv 1_{\mathcal{C}}$ in $\mathbf{Adj}_L({}_2Cat)$. In particular, the following diagram commutes:



We also proved the following theorem

Theorem 10.3. *There exists an isomorphism, natural on* $(\mathcal{D}, H)$ *and* $\mathcal{C}$*, between the categories*

(1) *The category of right* H*-functor algebras* $\mathcal{F}_H$*.*
(2) *The category of extensions from* $\mathcal{D}_H$*, for the pair* $(\mathcal{D}, \mathcal{C})$*.*

11 Conclusions and future work

This survey has the objective to show how several situations for the theory of monads are connected in a very simple way, through a 2-adjunction. Any person who has taught a course on monads would agree that this structure, of a 2-adjunction, can be used as an educational purpose in the sense that a simple structure can account for several situations and which can spare the, otherwise cumbersome, details of the proofs.

For future work, we have a few recommendations. The reader may find interesting to extent the part of strong monads and actions over the Kleisli categories to strong symmetrical monads and use the actions for the Eilenberg-Moore case. It would be interesting also to contextualize the case of the monoidal liftings and monoidal extensions according to the formal theory of monoidal monads, and the *standard* objects, given in [10].

The reader may want to find more situations in the monad theory that can use the isomorphism provided by this pair of 2-adjunctions, the authors will certainly pursue this issue.

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