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A Universal Investigation of n-representations of n-quivers

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Abstract. We have two goals in this paper. First, we investigate and construct cofree coalgebras over n-representations of quivers, limits and colimits of n-representations of quivers, and limits and colimits of coalgebras in the monoidal categories of n-representations of quivers. Second, for any given quivers Q_1,Q_2,..., Q_n, we construct a new quiver $\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)}$, called an n-quiver, and identify each category $Rep_k(Q_j)$ of representations of a quiver Q_j as a full subcategory of the category $Rep_k(\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)})$ of representations of $\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)}$ for every $j \in \{1, 2, \ldots, n\}$.

1 Introduction

The notions of quiver and their representation can be traced back to 1972 when they were introduced by Gabriel [16]. Since then, it has been studied as a vibrant subject with a strong linkage with many other mathematics areas. This comes from the modern approach that quiver representation theory suggests. Due to its inherent combinatorial flavor, this theory has re-

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cently been largely studied as extremely important theory with connections to many theories, such as associative algebra, combinatorics, algebraic topology, algebraic geometry, quantum groups, Hopf algebras, tensor categories. Further, it bridges the gap between combinatorics and category theory, and this simply comes from the well-known fact that there is a forgetful functor, which has a left adjoint, from the category of small categories to the category of quivers. It turns out that it gives "new techniques, both of combinatorial, geometrical and categorical nature" [11, p. ix].

To study a subject more extensively, one might need to generalize it in a certain way. The notion of n-representations of quivers can be introduced as a generalization of representations of quivers. We start with 2-representations of quivers and inductively define n-representations quivers. Then we mainly concentrate our study on 2-representations of quivers because they roughly give a complete description of n-representations of quivers which can be established analogously. We alternatively and preferably call 2-representations of quivers birepresentations of quivers. As the reader might notice, birepresentations of quivers are fundamentally different from representations of biquivers[1] introduced by Sergeichuk in [27, p. 237].

This paper is mainly devoted to two parallel goals. The first one is to investigate and construct cofree coalgebras, limits and colimits of coalgebras in the categories of n-representations of quivers.

The other goal of this paper is to introduce a generalization for quivers and prove that this can be recast into n-representations of quivers. Accordingly, for any given quivers $Q_1, Q_2, ..., Q_n$, one might build a new quiver $\mathscr{Q}_{(Q_1,Q_2,...,Q_n)}$, called n-quiver, by which we are able to view each category $Rep_k(Q_j)$ of representations of a quiver Q_j as a full subcategory of the category $Rep_k(\mathscr{Q}_{(Q_1,Q_2,...,Q_n)})$ of representations of $\mathscr{Q}_{(Q_1,Q_2,...,Q_n)}$ for every $j \in \{1, 2, \dots, n\}$. It is worth mentioning that for dealing with finite dimensional representations, one could consider the corresponding map between the quiver coalgebra of $Q_1, Q_2, ..., Q_n$ and the quiver coalgebra of $\mathscr{Q}_{(Q_1,Q_2,...,Q_n)}$ [12].

Before formulating our problem, we recall some categorical definitions. Let $\mathfrak{X}$ be a category. A *concrete category* over $\mathfrak{X}$ is a pair $(\mathfrak{A}, \mathfrak{U})$, where $\mathfrak{A}$

[1] A directed graph with usual and dashed arrows will be called a **biquiver**. Its **representation** is given by assigning to each vertex a complex vector space, to each usual arrow a linear mapping, and to each dashed arrow a semilinear mapping [27, p. 237].

is a category and $\mathfrak{U} : \mathfrak{A} \to \mathfrak{X}$ is a faithful functor [2, p. 61]. Let $(\mathfrak{A}, \mathfrak{U})$ be a concrete category over $\mathfrak{X}$. Following [2, pp. 140-143], a *free object* over $\mathfrak{X}$-object X is an $\mathfrak{A}$-object A such that there exists a *universal arrow* (A, u) over X; that is, $u : X \to \mathfrak{U}A$ such that for every arrow $f : X \to \mathfrak{U}B$, there exists a unique morphism $f' : A \to B$ in $\mathfrak{A}$ such that $\mathfrak{U}f'u = f$. We also say that (A, u) is the free object over X. A concrete category $(\mathfrak{A}, \mathfrak{U})$ over $\mathfrak{X}$ is said to *have free objects* provided that for each $\mathfrak{X}$-object X, there exists a universal arrow over X. For example, the category $Vect_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$ has free objects. So do the category **Top** of topological spaces and the category of **Grp** of groups. However, some interesting categories do not have free objects [2, p. 142]).

Dually, *co-universal arrows*, *cofree objects* and categories that *have cofree objects* can be defined. For the basic concepts of concrete categories, free objects, and cofree objects, we refer the reader to [17, pp. 138-155].

It turns out that a concrete $(\mathfrak{A}, \mathfrak{U})$ over $\mathfrak{X}$ has (co)free objects if and only if the functor that constructs (co)free object is a (right) left adjoint to the faithful functor $\mathfrak{U} : \mathfrak{A} \to \mathfrak{X}$. Our problem can be formulated as follows. Let $\mathscr{U}_n : CoAlg(Rep_{(Q_1,Q_2,\ldots,Q_n)}) \to Rep_{(Q_1,Q_2,\ldots,Q_n)}$ be the forgetful functor from the category of coalgebras in the category $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ to the category $Rep_{(Q_1,Q_2,\ldots,Q_n)}$. The question is: does $\mathscr{U}_n$ have a right adjoint?

An expected strategy for the answer of this question is to use the dual of Special Adjoint Functor Theorem (D-SAFT).

We prove that the category of representations of n-quivers is equivalent to the category of usual quiver representations of a quiver (the n-quiver construction we introduce with respect to the n quivers). This shows that this category is abelian, has limits and colimits, is Grothendieck and even hereditary. Our main focus is, however, to give explicit examples and constructions of several categorical elements such as limits, kernels or cokernels.

The sections of this paper can be summarized as follows.

In Section 2, we give some detailed background on quiver representations and few categorical notions that we need for the next sections.

In Section 3, we introduce the notion of n-representations of quivers, and we explicitly give concrete examples of birepresentations of quivers. In addition, we establish the categories of n-representations of quivers.

In Section 4, we show that limits of birepresentations (2-representations of quivers) exist and inductively extend our results to limits of n-representations of quivers. We also construct them in terms of limits of representations of quivers. In similar fashion, we investigate and construct colimits of n-representations of quivers, and then we end the section by showing that the categories of n-representations of quivers are abelian.

In Section 5, we introduce the notion of 2-quivers and inductively define n-quivers. We explicitly give concrete examples of n-quivers and representations of n-quivers. By using the concept of n-quivers, we identify the categories of n-representations of quivers and the categories of representations of n-quivers as equivalent categories. This helps us to have an explicit description for the generators of the categories of n-representations of quivers and characterize some properties of n-representations of quivers.
Finally, we investigate cofree coalgebras in the monoidal categories of n-representations of quivers. We also construct them in terms of colimits and generators and show that cofree coalgebras in these monoidal categories can be obtained from cofree coalgebras in the monoidal categories of quiver representations.

2 Preliminaries

Throughout this paper k is an algebraically closed field, $n \geq 2$, and

$$Q,\ Q',\ Q_1,\ Q_2, ...,\ Q_n$$

are quivers. We also denote kQ the path algebra of Q. Unless otherwise specified, we will consider only finite, connected, and acyclic quivers. Let $\mathcal{A}$ be a (locally small) category and A, B objects in $\mathcal{A}$. We denote by $\mathcal{A}(A, B)$ the set of all morphisms from A to B. Let $\mathcal{A}$, $\mathcal{B}$ be categories. Following [21, p. 74], the **product category** $\mathcal{A} \times \mathcal{B}$ is the category whose objects are all pairs of the form (A, B), where A is an object of $\mathcal{A}$ and B an object of $\mathcal{B}$. An arrow is a pair $(f, g) : (A, B) \to (A', B')$, where $f : A \to A'$ is an arrow of $\mathcal{A}$ and $g : B \to B'$ is an arrow of $\mathcal{B}$. The identity arrow for $\mathcal{A} \times \mathcal{B}$ is (id_A, id_B) and composition is defined component-wise, so $(f, g)(f', g') = (ff', gg')$. There is a projective functor $P_1 : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ defined by $P_1(A, B) = A$ and $P_1(f, g) = f$. Similarly, we have a projective functor $P_2 : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ defined by $P_2(A, B) = B$ and $P_2(f, g) = g$.

The following consequences are important for our investigation.

Theorem 2.1. (D-SAFT) [15, p. 148] *If $\mathfrak{A}$ is cocomplete, co-wellpowered and with a generating set, then every cocontinuous functor from $\mathfrak{A}$ to a locally small category has a right adjoint.*

Proposition 2.2. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, $CoMon(\mathcal{C})$ be the category of comonoids of $\mathcal{C}$ and $U : CoMon(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor.*

(i) *If $\mathcal{C}$ is cocomplete, then $CoMon(\mathcal{C})$ is cocomplete and U preserves colimits.*

(ii) *If furthermore $\mathcal{C}$ is co-wellpowered, then so is $CoMon(\mathcal{C})$.*

For the fundamental concepts of category theory, we refer to [18], [**?**], [5], [24], [2], [9], [15], [23], or [22].

Following [25], a **quiver** $Q = (\mathsf{Q}_0, \mathsf{Q}_1, s, t)$ consists of

- Q_0 a set of vertices,
- Q_1 a set of arrows,
- $s : \mathsf{Q}_1 \to \mathsf{Q}_0$ a map from arrows to vertices, mapping an arrow to its starting point,
- $t : \mathsf{Q}_1 \to \mathsf{Q}_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

We will represent an element $\alpha \in \mathsf{Q}_1$ by drawing an arrow from its starting point $s(\alpha)$ to its endpoint $t(\alpha)$ as: $s(\alpha) \xrightarrow{\alpha} t(\alpha)$.
A **representation** $M = (M_i, \varphi_\alpha)_{i \in \mathsf{Q}_0, \alpha \in \mathsf{Q}_1}$ of a quiver Q is a collection of k-vector spaces M_i one for each vertex $i \in \mathsf{Q}_0$, and a collection of k-linear maps $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ one for each arrow $\alpha \in \mathsf{Q}_1$.

A representation M is called **finite-dimensional** if each vector space M_i is finite-dimensional.

Let Q be a quiver and let $M = (M_i, \varphi_\alpha)$, $M' = (M'_i, \varphi'_\alpha)$ be two representations of Q. A **morphism** of representations $f : M \to M'$ is a collection $(f_i)_{i \in \mathsf{Q}_0}$ of k-linear maps $f_i : M_i \to M'_i$ such that for each arrow $s(\alpha) \xrightarrow{\alpha} t(\alpha)$ in Q_1 the diagram

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{\phi_\alpha} & M_{t(\alpha)} \\ {\scriptstyle f_{s(\alpha)}}\downarrow & & \downarrow{\scriptstyle f_{t(\alpha)}} \\ M'_{s(\alpha)} & \xrightarrow[\phi'_\alpha]{} & M'_{t(\alpha)} \end{array} \tag{2.1}$$

commutes.

A morphism of representations $f = (f_i) : M \to M'$ is an isomorphism if each f_i is bijective. The class of all representations that are isomorphic to a given representation M is called the **isoclass** of M.

The above definition introduces a category $Rep_k(Q)$ of k-linear representations of Q. We denote by $rep_k(Q)$ the full subcategory of $Rep_k(Q)$ consisting of the finite dimensional representations.

Given two representations $M = (M_i, \phi_\alpha)$ and $M' = (M'_i, \phi'_{\alpha'})$ of Q, the representation

$$M \oplus M' = (M_i \oplus M'_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \phi'_\alpha \end{bmatrix}) \tag{2.2}$$

is the **direct sum** of M and M' in $Rep_k(Q)$ [3, p. 71].
A nonzero representation of a quiver Q is said to be **indecomposable** if it is not isomorphic to a direct sum of two nonzero representations [14, p. 21].

Following [25, p. 114], the **path algebra** kQ of a quiver Q is the algebra with basis the set of all paths in the quiver Q and with multiplication defined on two basis elements c, c' by

$$c.c' = \begin{cases} cc', & \text{if } s(c') = t(c) \\ 0, & \text{otherwise.} \end{cases}$$

We will need the following propositions.

Proposition 2.3. [3, p. 70] *Let Q be a finite quiver. Then $Rep_k(Q)$ and $rep_k(Q)$ are k-linear abelian categories.*

Proposition 2.4. [3, p. 74] *Let Q be a finite, connected, and acyclic quiver. There exists an equivalence of categories $Mod\,kQ \simeq Rep_k(Q)$ that restricts to an equivalence $Mod\,kQ \simeq rep_k(Q)$, where kQ is the path algebra of Q, $Mod\,kQ$ denotes the category of right kQ-modules, and $Mod\,kQ$ denotes the full subcategory of $Mod\,kQ$ consisting of the finitely generated right kQ-modules.*

This is a very brief review of the basic concepts involved with our work. For the basic notions of quiver representations theory, we refer the reader to [3], [25], [4], [7], [14], [8], [30].

3 n-representations of quivers: Basic concepts

Let $Q = (\mathsf{Q}_0, \mathsf{Q}_1, s, t)$, $Q' = (\mathsf{Q}'_0, \mathsf{Q}'_1, s', t')$ be quivers.

Definition 3.1. A **2-representation** of (Q, Q') (or a **birepresentation** of (Q, Q')) is a triple $\bar{M} = ((M_i, \phi_\alpha), (M'_{i'}, \phi'_\beta), (\psi^\alpha_\beta))_{i\in\mathsf{Q}_0, i'\in Q'_0, \alpha\in\mathsf{Q}_1, \beta\in\mathsf{Q}'_1}$, where (M_i, ϕ_α), $(M'_{i'}, \phi'_\beta)$ are representations of Q, Q' respectively, and (ψ^α_β) is a collection of k-linear maps $\psi^\alpha_\beta : M_{t(\alpha)} \to M'_{s(\beta)}$, one for each pair of arrows $(\alpha, \beta) \in \mathsf{Q}_1 \times \mathsf{Q}'_1$.

Unless confusion is possible, we denote a birepresentation simply by $\bar{M} = (M, M', \psi)$. Next, we inductively define n-representations for any integer $n \geq 2$.

For any $m \in \{1, \ldots, n\}$, let $Q_m = (\mathsf{Q}^{(m)}_0, \mathsf{Q}^{(m)}_1, s^{(m)}, t^{(m)})$ be a quiver. An n**-representation** of $(Q_1,\ Q_2,\ \ldots, Q_n)$ is $(2n-1)$-tuple

$$\bar{V} = (V^{(1)}, V^{(2)}, ..., V^{(n)}, \psi_1, \psi_2, ..., \psi_{n-1}),$$

where for every $m \in \{1, 2, \ldots, n\}$, $V^{(m)}$ is a representation of Q_m, and $(\psi^{\gamma^{(m)}}_{m\gamma^{(m-1)}})$ is a collection of k-linear maps

$$\psi^{\gamma^{(m)}}_{m\gamma^{(m-1)}} : V^{(m-1)}_{t^{(m-1)}(\gamma^{(m-1)})} \to V^{(m-1)}_{s^{(m)}(\gamma^{(m)})},$$

one for each pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in \mathsf{Q}^{(m-1)}_1 \times \mathsf{Q}^{(m)}_1$ and $m \in \{2, \ldots, n\}$.

Remark 3.2.

(i) When no confusion is possible, we simply write s and t instead of s' and t', respectively, and for every $m \in \{1, 2, \ldots, n\}$, we write s and t instead of $s^{(m)}$ and $t^{(m)}$, respectively.

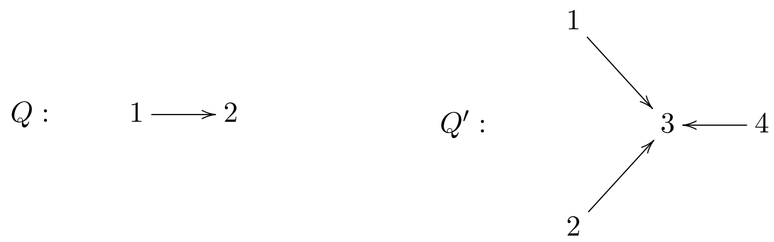
(ii) It is clear that if $(V^{(1)}, V^{(2)}, ..., V^{(n)}, \psi_1, \psi_2, ..., \psi_{n-1})$ is an n-representation of $(Q_1,\ Q_2,\ \ldots, Q_n)$, then

$$(V^{(1)}, V^{(2)}, ..., V^{(n-1)}, \psi_1, \psi_2, ..., \psi_{n-2})$$

is an $(n-1)$-representation of $(Q_1,\ Q_2,\ \ldots, Q_{n-1})$ for every integer $n \geq 2$.

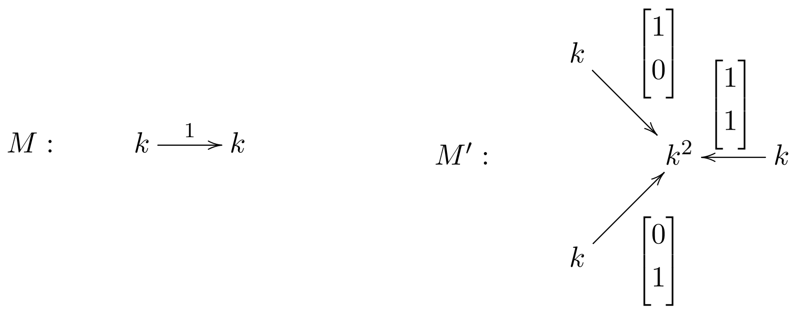
(iii) Part (ii) implies that for any integer $n > 2$, n-representations roughly inherit all the properties and universal constructions that $(n-1)$-representations have. Thus, we mostly focus on studying birepresentations, since they can be regarded as a mirror in which one can see a clear description of n-representations for any integer $n > 2$.

Example 3.3. Let Q and Q' be the following quivers



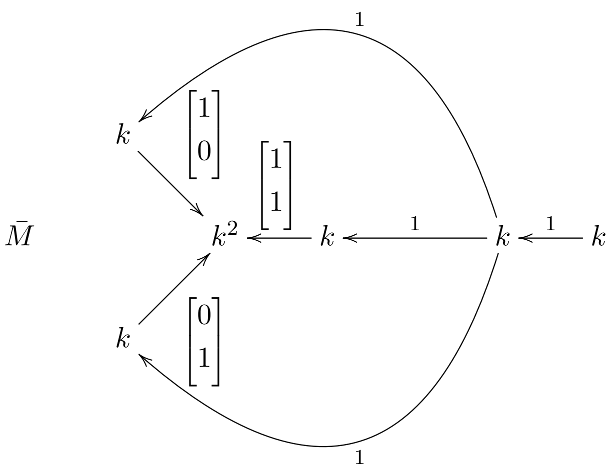
(3.1)

and consider the following:

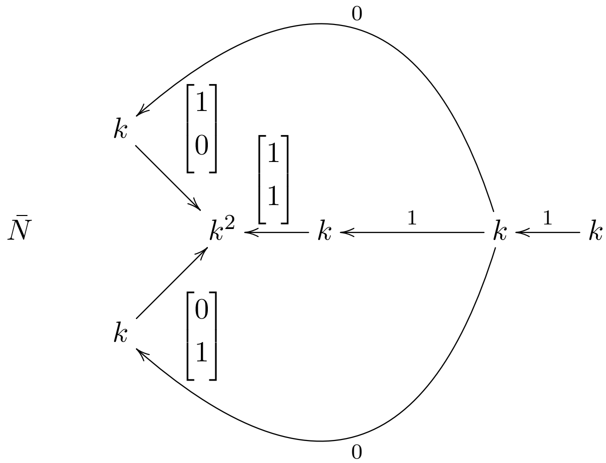


(3.2)

Then M and M' are representations of Q and Q' respectively [25]. The following are birepresentations of (Q, Q').

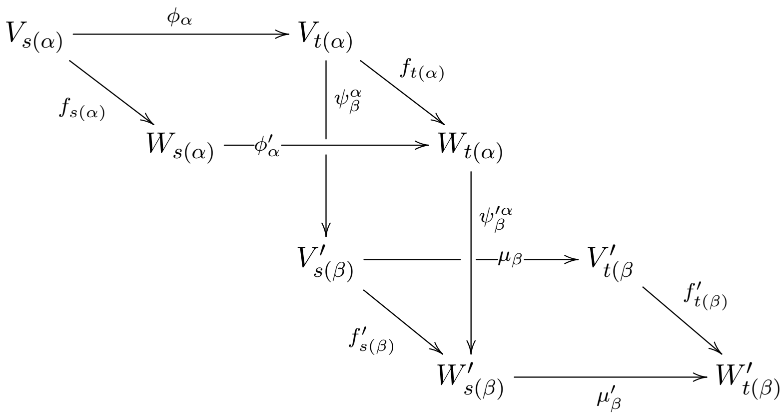


(3.3)



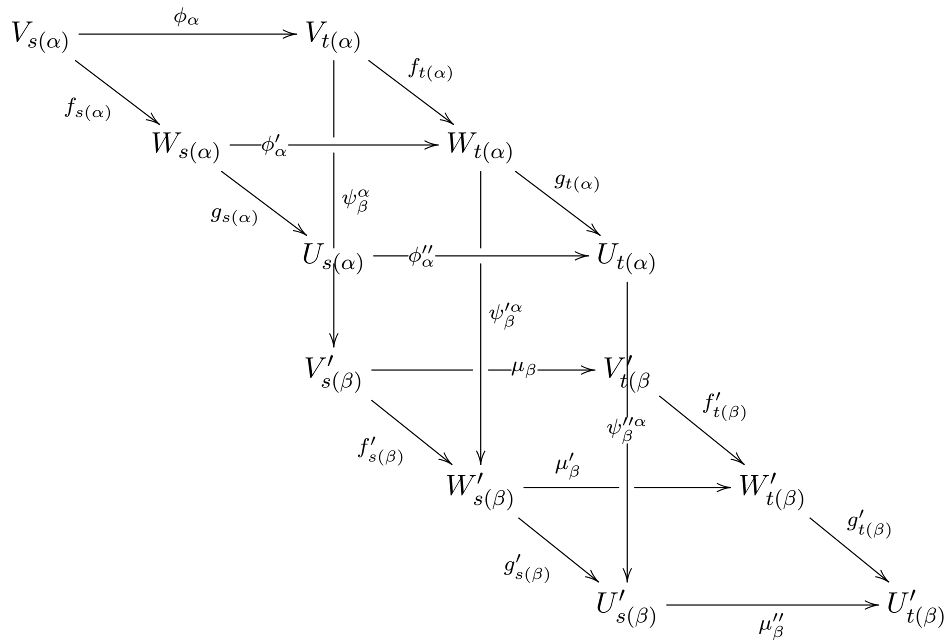
(3.4)

Definition 3.4. Let $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$ be birepresentations of (Q, Q'). Write $V = (V_i, \phi_\alpha)$, $V' = (V'_{i'}, \mu_\beta)$, $W = (W_i, \phi'_\alpha)$, $W' = (W'_{i'}, \mu'_\beta)$. A morphism of birepresentations $\bar{f} : \bar{V} \to \bar{W}$ is a pair $\bar{f} = (f, f')$, where $f = (f_i) : (V_i, \phi_\alpha) \to (W_i, \phi'_\alpha)$, $f' = (f'_{i'}) : (V'_{i'}, \mu_\beta) \to (W'_{i'}, \mu'_\beta)$ are morphisms in $Rep_k(Q)$, $Rep_k(Q')$ respectively such that the following diagram commutes.



(3.5)

The composition of two maps (f, f') and (g, g') can be depicted as the following diagram.



(3.6)

In general, if $\bar{V} = (V^{(1)}, V^{(2)}, ..., V^{(n)}, \psi_1, \psi_2, ..., \psi_{n-1})$ and

$$\bar{W} = (W^{(1)}, W^{(2)}, ..., W^{(n)}, \psi'_1, \psi'_2, ..., \psi'_{n-1})$$

are n-representations of $(Q_1,\ Q_2,\ \dots\ , Q_n)$, then a morphism of n-representations $\bar{f} : \bar{V} \to \bar{W}$ is n-tuple $\bar{f} = (f^{(1)}, f^{(2)}, ..., f^{(n-1)})$, where

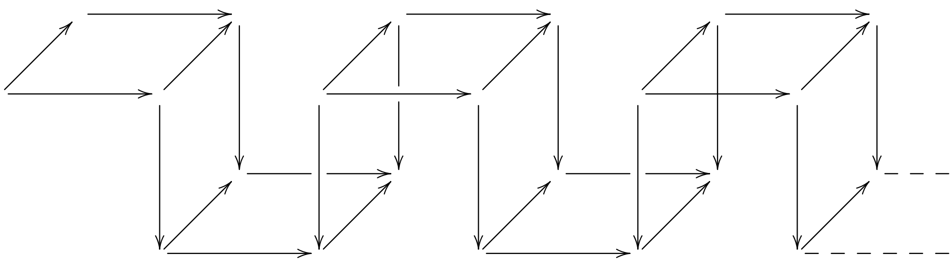
$$f^{(m)} = (f^{(m)}_{i^{(m)}}) : (V_{i^{(m)}}, \phi^{i^{(m)}}_{\gamma^{(m)}}) \to (W_{i^{(m)}}, \mu^{i^{(m)}}_{\gamma^{(m)}}),$$

is a morphism in $Rep_k(Q_m)$ for any $m \in \{2, \dots, n\}$, and for each pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in \mathsf{Q}_1^{(m-1)} \times \mathsf{Q}_1^{(m)}$ the following diagram is commutative.

$$\begin{array}{ccc} V^{(m-1)}_{t^{(m-1)}(\gamma^{(m-1)})} & \xrightarrow{\psi^{\gamma^{(m)}}_{m\gamma^{(m-1)}}} & V^{(m)}_{s^{(m)}(\gamma^{(m)})} \\ \Big\downarrow f^{(m-1)}_{t^{(m-1)}(\gamma^{(m-1)})} & & \Big\downarrow f^{(m)}_{s^{(m)}(\gamma^{(m)})} \\ W^{(m-1)}_{t^{(m-1)}(\gamma^{(m-1)})} & \xrightarrow[\psi'^{\gamma^{(m)}}_{m\gamma^{(m-1)}}]{} & W^{(m)}_{s^{(m)}(\gamma^{(m)})} \end{array} \tag{3.7}$$

for every $m \in \{2, \ldots, n\}$.

A morphism of n-representations can be depicted as:



(3.8)

Remark 3.5.

(i) The above definition provides a category $Rep_{(Q,Q')}$ of k-linear birepresentations of (Q, Q'). We denote by $rep_{(Q,Q')}$ the full subcategory of $Rep_{(Q,Q')}$ consisting of the finite dimensional birepresentations. Similarly, it also creates a category $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ of n-representations. We denote $rep_{(Q_1,Q_2,\ldots,Q_n)}$ the full subcategory of $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ consisting of the finite dimensional n-representations.

(ii) If Q' is the empty quiver, then $Rep_{(Q,Q')}$ is isomorphic to the category $Rep_k(Q)$. Thus, representations of a quiver can be seen as a particular case of n-representations of quivers.

(iii) For any $m \in \{1, \ldots, n\}$, let $Q_m = (\mathsf{Q}_0^{(m)}, \mathsf{Q}_1^{(m)}, s^{(m)}, t^{(m)})$ be a quiver and fix $j \in \{1, \ldots, n\}$. Let $\Upsilon_{Rep_k(Q_j)}$ be the subcategory of $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ whose objects are $(2n-1)$-tuples

$$\bar{X} = (0, 0, ..., V^{(j)}, 0, ..., 0, \psi_1, \psi_2, ..., \psi_{n-1}),$$

where $V^{(j)}$ is a representation of Q_j, and $\psi^{\gamma^{(m)}}_{m\gamma^{(m-1)}} = 0$ for every pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in \mathsf{Q}_1^{(m-1)} \times \mathsf{Q}_1^{(m)}$ and $m \in \{2, \ldots, n\}$. Then $\Upsilon_{Rep_k(Q_j)}$ is clearly a full subcategory of $Rep_{(Q_1,Q_2,\ldots,Q_n)}$. Notably, we have an equivalence of categories $\Upsilon_{Rep_k(Q_j)} \simeq Rep_k(Q_j)$, and thus by by Proposition (2.4), we have $Rep_k(Q_j) \simeq \Upsilon_{Rep_k(Q_j)} \simeq Mod\, kQ_j$. It turns out that the category $Rep_k(Q_j)$ and $Mod\, kQ_j$ can be identified as full subcategories of $Rep_{(Q_1,Q_2,\ldots,Q_n)}$.
The category $\Upsilon_{Rep_k(Q_j)}$ has a full subcategory $\Upsilon_{rep_k(Q_j)}$ when we restrict the objects on the finite dimensional representations. Therefore, we also have $rep_k(Q_j) \simeq \Upsilon_{rep_k(Q_j)} \simeq mod\, kQ_j$.

Remark 3.6. Let $\mathcal{B}_0$ be the class of all quivers.

(i) One might consider the class $\mathcal{B}_0$ and full subcategories of the categories of birepresentations of quivers to build a bicategory. Indeed, there is a bicategory $\mathcal{B}$ consists of

- the objects or the 0-cells of $\mathcal{B}$ are simply the elements of $\mathcal{B}_0$,
- for each $Q, Q' \in \mathcal{B}_0$, we have $\mathcal{B}(Q, Q') = Rep_k(Q) \times Rep_k(Q')$, whose objects are the 1-cells of $\mathcal{B}$, and whose morphisms are the 2-cells of $\mathcal{B}$,
- for each $Q, Q', Q'' \in \mathcal{B}_0$, a composition functor

$$\mathcal{F} : Rep_k(Q') \times Rep_k(Q'') \times Rep_k(Q) \times Rep_k(Q') \to Rep_k(Q) \times Rep_k(Q'')$$

  defined by:

$$\mathcal{F}((N', N''), (M, M')) = (M, N''), \quad \mathcal{F}((g', g''), (f, f')) = (f, g'') \text{ on 1-cells } (M, M'), (N', N'') \text{ and 2-cells } (f, f'), (g', g''),$$

- for any $Q \in \mathcal{B}_0$ and for each $(M, M') \in \mathcal{B}(Q, Q)$, we have

  $\mathcal{F}((M, M'), (M, M)) = (M, M')$ *and* $\mathcal{F}((M', M'), (M, M')) = (M, M')$.

  Also, for any 2-cell (f, f'), we have $\mathcal{F}((f', f'), (f, f')) = (f, f')$ and $\mathcal{F}((f, f'), (f, f)) = (f, f')$. Thus, the identity and the unit coherence axioms hold.

The rest of bicategories axioms are obviously satisfied. For each $Q, Q' \in \mathcal{B}_0$, let $\beth_{(Q,Q')}$ be the full subcategory of $Rep_{(Q,Q')}$ whose objects are the triples (X, X', Ψ), where $(X, X') \in Rep_k(Q) \times Rep_k(Q')$ and $\Psi^\alpha_\beta = 0$ for every pair of arrows $(\alpha, \beta) \in \mathsf{Q}_1 \times \mathsf{Q}'_1$, and whose morphisms are usual morphisms of birepresentations between them. Clearly, $\beth_{(Q,Q')} \cong Rep_k(Q) \times Rep_k(Q')$ for any $Q, Q' \in \mathcal{B}_0$. Thus, by considering the class $\mathcal{B}_0$ and these full subcategories (described above) of the birepresentations categories of quivers, we can always build a bicategory as above.
Obviously, the discussion above implies that for each $Q, Q' \in \mathcal{B}_0$, the product category $Rep_k(Q) \times Rep_k(Q')$ can be viewed as a full subcategory of $Rep_{(Q,Q')}$. Further, it implies that the product category $Rep_k(Q_1) \times Rep_k(Q_2) \times ... \times Rep_k(Q_n)$ can be viewed as a full subcategory of $Rep_{(Q_1,Q_2,...,Q_n)}$, where $Q_1, Q_2, ..., Q_n \in \mathcal{B}_0$ and $n \geq 2$.
We also have the same analogue if we replace $Rep_{(Q_1,Q_2,...,Q_n)}$, by $rep_{(Q_1,Q_2,...,Q_n)}$, and $Rep_k(Q_1)$, $Rep_k(Q_2)$, ... , $Rep_k(Q_n)$ by $rep_k(Q_1), rep_k(Q_2)$, ... , $rep_k(Q_n)$, respectively.
For the basic notions of bicategories, we refer the reader to [19].

(ii) For any $Q, Q', Q'' \in \mathcal{B}_0$, let $\mathbb{H}_{(Q,Q',Q'')}$ be the full subcategory of $Rep_{(Q,Q')} \times Rep_{(Q',Q'')}$ whose objects are pairs of triples of the type $((X, X', \psi), (X', X'', \psi'))$, where $(X, X', \psi) \in Rep_{(Q,Q')}$ and $(X', X'', \psi') \in Rep_{(Q',Q'')}$, and whose morphisms are usual morphisms (in $Rep_{(Q,Q')} \times Rep_{(Q',Q'')}$) between them.

Let $\Theta_{(Q,Q',Q'')} : \mathbb{H}_{(Q,Q',Q'')} \to Rep_{(Q,Q'')}$ be a map defined by:

$$\Theta_{(Q,Q',Q'')}((M, N, \psi), (N, L, \psi')) = (M, L, \Psi),$$

$$\Theta_{(Q,Q',Q'')}((f, f'), (g, g')) = (f, g') : (M, L, \Psi) \to (M', L', \Psi')$$

for any objects $((M, N, \psi), (N, L, \psi')), ((M', N', \psi), (N', L', \psi'))$ in $\mathbb{H}_{(Q,Q',Q'')}$, and for any morphism

$$(f, f') : (M, M', \psi) \to (N, N', \psi'), (g, g') : (N, N', \psi') \to (L, L', \psi'')$$

where $M = (M_i, \phi_\alpha)$, $M' = (M'_i, \phi'_\alpha)$, $N = (N_{i'}, \mu_\beta)$, $N' = (N'_{i'}, \mu'_\beta)$, $L = (L_{i''}, \nu_\gamma)$, $L' = (L'_{i''}, \nu'_\gamma)$, $\psi = (\psi^\alpha_\beta)$, $\psi' = (\psi'^\alpha_\beta)$, $\tilde{\psi} = (\tilde{\psi}^\beta_\gamma)$, $\tilde{\psi}' = (\tilde{\psi}'^\beta_\gamma)$, $\Psi^\alpha_\gamma = \sum_\beta \tilde{\psi}^\beta_\gamma \, \mu_\beta \, \psi^\alpha_\beta$ and $\Psi'^\alpha_\gamma = \sum_\beta \tilde{\psi}'^\beta_\gamma \, \mu'_\beta \, \psi'^\alpha_\beta$.

Note that $\Theta_{(Q,Q',Q'')} : ((f,f'),(g,g')) = (f,g') : (M,L,\Psi) \to (M',L',\Psi')$ is a morphism in $Rep_{(Q,Q'')}$. To show this, consider the following diagram:

$$\begin{array}{ccccccccccc} M_{s(\alpha)} & \xrightarrow{\phi_\alpha} & M_{t(\alpha)} & \xrightarrow{\psi^\alpha_\beta} & N_{s(\beta)} & \xrightarrow{\mu_\beta} & N_{t(\beta)} & \xrightarrow{\tilde{\psi}^\beta_\gamma} & L_{s(\gamma)} & \xrightarrow{\nu_\gamma} & L_{t(\gamma)} \\ \downarrow{\scriptstyle f_{s(\alpha)}} & & \downarrow{\scriptstyle f_{t(\alpha)}} & & {\scriptstyle f'_{s(\beta)}}\downarrow{\scriptstyle g_{s(\beta)}} & & {\scriptstyle f'_{t(\beta)}}\downarrow{\scriptstyle g_{t(\beta)}} & & \downarrow{\scriptstyle g'_{s(\gamma)}} & & \downarrow{\scriptstyle g'_{t(\gamma)}} \\ M'_{s(\alpha)} & \xrightarrow[\phi'_\alpha]{} & M'_{t(\alpha)} & \xrightarrow[\psi'^\alpha_\beta]{} & N'_{s(\beta)} & \xrightarrow[\mu'_\beta]{} & N'_{t(\beta)} & \xrightarrow[\tilde{\psi}'^\beta_\gamma]{} & L'_{s(\gamma)} & \xrightarrow[\nu'_\gamma]{} & L'_{t(\gamma)} \end{array} \tag{3.9}$$

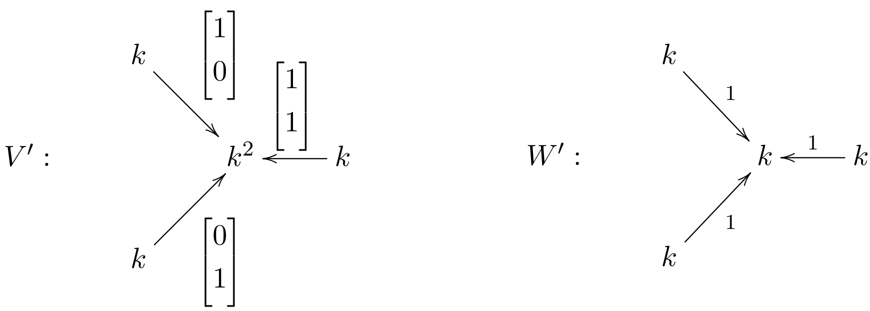
For any $\alpha \in \mathsf{Q}_1$, $\beta \in \mathsf{Q}'_1$, $\gamma \in \mathsf{Q}''_1$, we have:

$$\begin{aligned} g'_{s(\gamma)}\ \Psi^\alpha_\gamma &= g'_{s(\gamma)} \sum_\beta \tilde{\psi}^\beta_\gamma\ \mu_\beta\ \psi^\alpha_\beta = \sum_\beta g'_{s(\gamma)}\ \tilde{\psi}^\beta_\gamma\ \mu_\beta\ \psi^\alpha_\beta \\ &= \sum_\beta \tilde{\psi}'^\beta_\gamma\ g_{t(\beta)}\ \mu_\beta\ \psi^\alpha_\beta = \sum_\beta \tilde{\psi}'^\beta_\gamma\ f'_{t(\beta)}\ \mu_\beta\ \psi^\alpha_\beta \\ &= \sum_\beta \tilde{\psi}'^\beta_\gamma\ \mu'_\beta\ f'_{s(\beta)}\ \psi^\alpha_\beta = \sum_\beta \tilde{\psi}'^\beta_\gamma\ \mu'_\beta\ \psi'^\alpha_\beta\ f_{t(\beta)} \\ &= \Psi'^\alpha_\gamma\ f_{t(\beta)} \end{aligned}$$

Therefore, $\Theta_{(Q,Q',Q'')}$ is a functor for any $Q, Q', Q'' \in \mathcal{B}_0$. Indeed, for any $Q, Q', Q'' \in \mathcal{B}_0$, $\Theta_{(Q,Q',Q'')}$ has a left adjoint functor.

Example 3.7. Let Q, Q' be the quivers defined in Example 3.3 and consider the following:

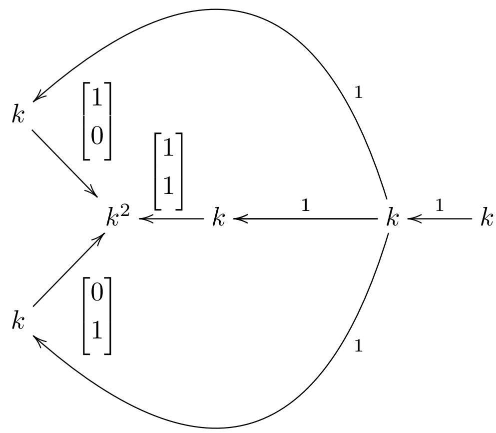
$$V: \quad k \xrightarrow{1} k \qquad\qquad W: \quad k \xrightarrow{0} k \tag{3.10}$$



(3.11)

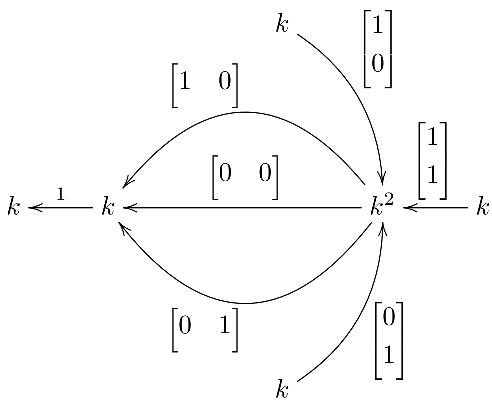
Then V and W (respectively, V' and W') are representations of Q (respectively, Q') [25]. It is also straightforward to verify that $Rep_k(Q)(V,W) \cong k$ and $Rep_k(Q')(V',W') \cong k^2$. We refer the reader to [25] for more details. Consider the following.

$\bar{V} = (V, V', \psi)$



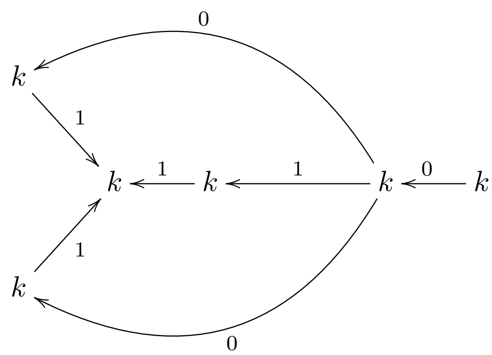
(3.12)

$\underline{\bar{V}} = (V', V, \psi')$



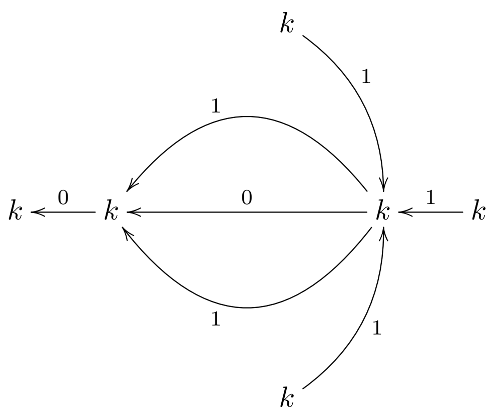
(3.13)

$$\bar{W} = (W, W', \psi')$$



(3.14)

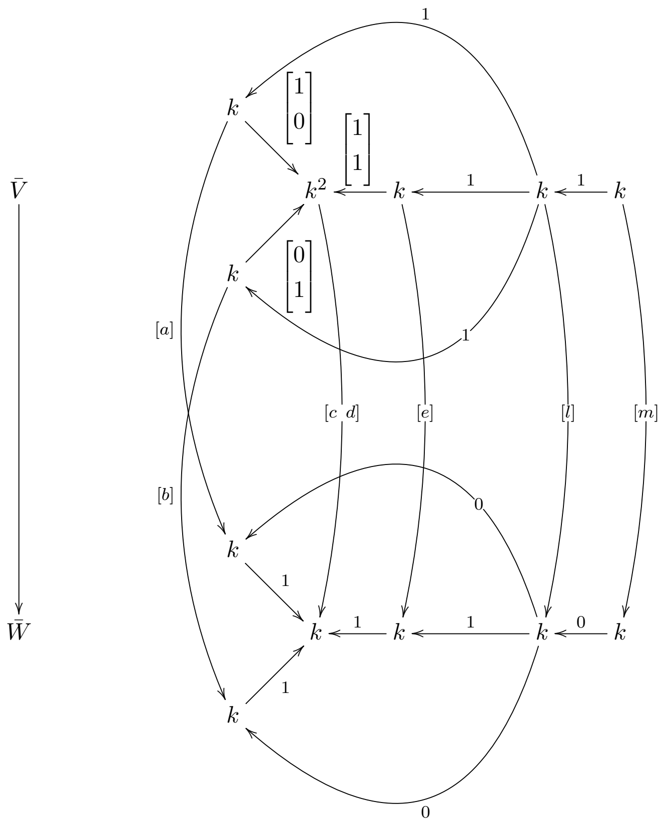
$$\underline{\bar{W}} = (W', W, \underline{\psi'})$$



(3.15)

Then $\bar{V}$ and $\bar{W}$ are birepresentations of (Q, Q'), and $\underline{\bar{V}}$ and $\underline{\bar{W}}$ are birepresentations of (Q', Q).

To compute $Rep_{(Q,Q')}(\bar{V}, \bar{W})$, consider the following diagram.



(3.16)

The commuting squares give the relations

$$a = 0, \quad b = 0, \quad c = a = 0, \quad d = b = 0, \quad e = c + d = 0, \quad l = e = 0. \quad (3.17)$$

Hence, we obtain $Rep_{(Q,Q')}(\bar{V}, \bar{W}) \cong \{(0,0,0,0,0,0,m) : m \in k\} \cong k$.

We leave it to the reader to compute $Rep_{(Q,Q')}(\bar{W}, \bar{V})$, $Rep_{(Q',Q)}(\underline{\bar{V}}, \underline{\bar{W}})$ and $Rep_{(Q',Q)}(\underline{\bar{W}}, \underline{\bar{V}})$.

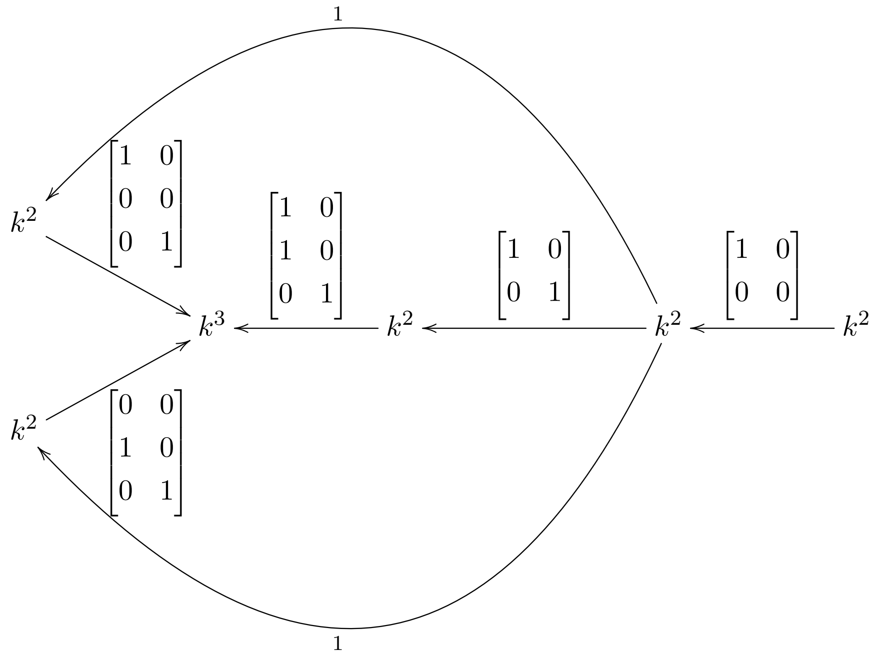
Definition 3.8. Let $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$ be birepresentations of (Q, Q'). Write $V = (V_i, \phi_\alpha)$, $V' = (V'_{i'}, \phi'_\beta)$, $W = ((W_i, \mu_\alpha)$, $W' = (W'_{i'}, \mu'_\beta)$. Then

$$\bar{V} \oplus \bar{W} = ((V_i \oplus W_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \mu_\alpha \end{bmatrix}), (V'_{i'} \oplus W'_{i'}, \begin{bmatrix} \phi'_\alpha & 0 \\ 0 & \mu'_\alpha \end{bmatrix}), \begin{bmatrix} \psi^\alpha_\beta & 0 \\ 0 & \psi'^\alpha_\beta \end{bmatrix}), \quad (3.18)$$

where $(V_i \oplus W_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \mu_\alpha \end{bmatrix})$, $(V'_{i'} \oplus W'_{i'}, \begin{bmatrix} \phi'_\alpha & 0 \\ 0 & \mu'_\alpha \end{bmatrix})$ are the direct sums of $(V_i, \phi_\alpha), (W_i, \mu_\alpha)$ and $(V'_{i'}, \phi'_\beta), (W'_{i'}, \mu'_\beta)$ in $Rep_k(Q)$, $Rep_k(Q')$ respectively, is a birepresentation of (Q, Q') called the **direct sum** of $\bar{V}, \bar{W}$ (in $Rep_{(Q,Q')}$).

Similarly, direct sums in $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ can be defined.

Example 3.9. Consider the birepresentations in Example 3.7. Then the direct sum $\bar{V} \oplus \bar{W}$ is the birepresentation



(3.19)

Definition 3.10. A birepresentation $\bar{V} \in Rep_{(Q,Q')}$ is called **indecomposable** if $\bar{M} \neq 0$ and $\bar{M}$ cannot be written as a direct sum of two nonzero birepresentations, that is, whenever $\bar{M} \cong \bar{L} \oplus \bar{N}$ with $\bar{L}, \bar{N} \in Rep_{(Q,Q')}$, then $\bar{L} = 0$ or $\bar{N} = 0$.

Example 3.11. Consider the birepresentations in Example 3.7. The birepresentation $\bar{V}$ is indecomposable, but the birepresentation $\bar{W}$ is not.

The above example also shows that if $\bar{W} = ((W_i, \phi_\alpha), (W'_{i'}, \phi'_\beta), (\psi^\alpha_\beta))$ is birepresentation of (Q, Q') such that the representations (W_i, ϕ_α) and

$(W'_{i'}, \phi'_\beta)$ are indecomposable in $Rep_k(Q)$, $Rep_k(Q')$ respectively, then $\bar{W}$ need not be indecomposable. The proof of the following proposition is straightforward.

Proposition 3.12. *Let $\bar{V} = ((V_i, \phi_\alpha), (V'_{i'}, \phi'_\beta), (\psi^\alpha_\beta)) \in Rep_{(Q,Q')}$ be an indecomposable birepresentation, then the representations (W_i, ϕ_α) and $(W'_{i'}, \phi'_\beta)$ are indecomposable in $Rep_k(Q)$ and $Rep_k(Q')$ respectively.*

4 Completeness, cocompleteness and canonical decomposition of morphisms in the categories of n-representations

Recall that a category $\mathfrak{C}$ is cocomplete when every functor $\mathfrak{F} : \mathfrak{D} \to \mathfrak{C}$, with $\mathfrak{D}$ a small category has a colimit [9]. For the basic notions of cocomplete categories and examples, we refer to [2], [9], or [26]. A functor is *cocontinuous* if it preserves all small colimits [15, p. 142].

Proposition 4.1. *The category $Rep_{(Q,Q')}$ is complete and the forgetful functor*

$$\mathcal{U} : Rep_{(Q,Q')} \to Rep_k(Q) \times Rep_k(Q')$$

is continuous. In addition, the limit of objects in $Rep_{(Q,Q')}$ can be obtained by the corresponding construction for objects in $Rep_k(Q) \times Rep_k(Q')$.

Proof. Let $\mathscr{D}$ be a small category, and let $\mathscr{F} : \mathscr{D} \to Rep_{(Q,Q')}$ be a functor, and consider the composition of the following functors.

$$\mathscr{D} \xrightarrow{\mathscr{F}} Rep_{(Q,Q')} \xrightarrow{\mathcal{U}} Rep_k(Q) \times Rep_k(Q') \xrightarrow{\mathcal{P}_1} Rep_k(Q) \tag{4.1}$$

$$\mathscr{D} \xrightarrow{\mathscr{F}} Rep_{(Q,Q')} \xrightarrow{\mathcal{U}'} Rep_k(Q) \times Rep_k(Q') \xrightarrow{\mathcal{P}_2} Rep_k(Q') \tag{4.2}$$

where $\mathcal{U}, \mathcal{U}'$ are the obvious forgetful functors, and $\mathcal{P}_1$, $\mathcal{P}_2$ are the projection functors.

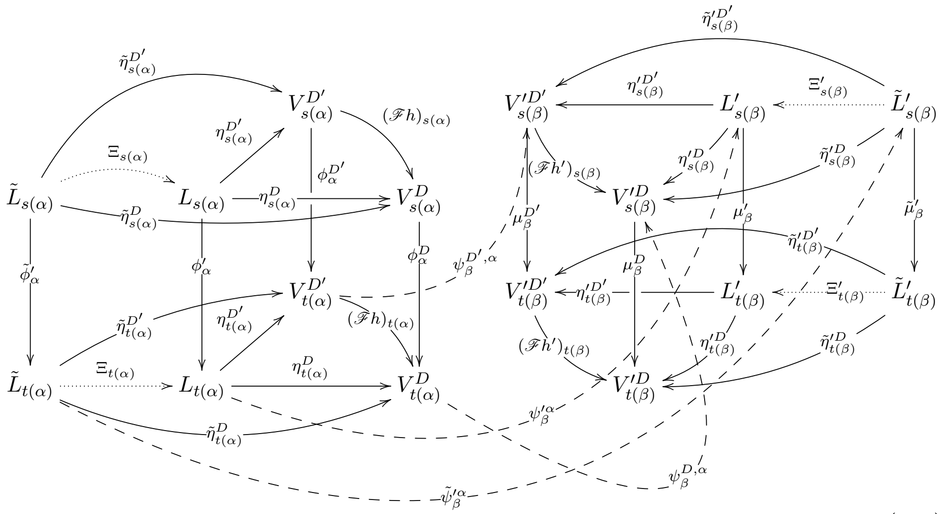
For all $D \in \mathscr{D}$, let $\mathscr{F}D = \bar{V} = (V, V', \psi^{D,})$. Since $Rep_k(Q)$, $Rep_k(Q')$ are complete categories, the functors $\mathcal{P}_1\mathcal{U}\mathscr{F}$, $\mathcal{P}_2\mathcal{U}\mathscr{F}$ have limits. Let $(L, (\eta^D)_{D\in\mathscr{D}})$,

$(L', (\eta'^{D})_{D\in\mathscr{D}})$ be limits of $\mathcal{P}_1\mathcal{U}\mathscr{F}$, $\mathcal{P}_2\mathcal{U}\mathscr{F}$ respectively, where η^{D}, η'^{D} are the morphisms

$$L \xrightarrow{\eta^D} V^D, \qquad L' \xrightarrow{\eta'^D} V'^D \quad , \tag{4.3}$$

for every $D \in \mathscr{D}$. Write $L = (L_i, \phi'_\alpha)$, $L' = (L'_{i'}, \mu'_\beta)$, $V^D = (V_i^D, \phi_\alpha^D)$, $V^{D'} = (V_i^{D'}, \phi_\alpha^{D'})$ $V'^{D'} = (V'^{D'}_{i'}, \mu_\beta^{D'})$, $V'^D = (V'^D_i, \mu_\beta^D)$.

Let $h : D \to D'$ be a morphism in $\mathscr{D}$ and consider the following diagram 4.4.



(4.4)

Clearly, $(L'_{s(\beta)}, (\eta'^{D}_{s(\beta)})_{D\in\mathscr{D}})$ can be viewed as a limit of a functor $\rho : \mathscr{D} \to Vect_k$. Also, for any $D, D' \in \mathscr{D}$, we have

$$\begin{aligned}
(\mathscr{F}h')_{s(\beta)}\ \psi_\beta^{D',\alpha}\ \eta^{D'}_{t(\alpha)} &= \psi_\beta^{D,\alpha}\ (\mathscr{F}h)_{t(\alpha)}\ \eta^{D'}_{t(\alpha)} \\
&\text{(since } \bar{\mathscr{F}}h = (\mathscr{F}h, \mathscr{F}h') \text{ is a morphism in } Rep_{(Q,Q')}) \\
&= \psi_\beta^{D,\alpha}\ \eta^D_{t(\alpha)} \\
&\text{(since } (L, (\eta^{D})_{D\in\mathscr{D}}) \text{ is a limit of } \mathcal{P}_1\mathcal{U}\mathscr{F})
\end{aligned}$$

Thus, $(L_{t(\alpha)}, (\psi^{D,}\ \eta^D_{t(\alpha)})_{D\in\mathscr{D}})$ is a cone on ρ. Since Vec_k is complete, there exists a unique k-linear map $\psi'^\alpha_\beta : L_{t(\alpha)} \to L'_{s(\beta)}$ with $\eta'^{D}_{s(\beta)}\ \psi'^\alpha_\beta = \psi_\beta^{D,\alpha}\ \eta^D_{t(\alpha)}$ for every $D \in \mathscr{D}$ and for each pair of arrows $(\alpha, \beta) \in Q_1 \times \in Q'_1$. Hence, $\bar{\eta}^D$ is a morphism in $Rep_{(Q,Q')}$ for any $D \in \mathscr{D}$.

Let $\bar{L} = (L, L', \psi')$, $\bar{\eta}^D = (\eta^D, \eta'^D)$. We claim that $(\bar{L}, (\bar{\eta}^{D})_{D\in\mathscr{D}})$ is a limit of $\mathscr{F}$. Then obviously all we need is to show that for any cone

$(\bar{\tilde{L}}, (\bar{\tilde{\eta}}^D)_{D\in\mathscr{D}})$, there exists a unique morphism $\bar{\Xi} = (\Xi, \Xi')$ in $Rep_{(Q,Q')}$ with $\bar{\eta}^D\ \bar{\Xi} = \bar{\tilde{\eta}}^D$ for every $D \in \mathscr{D}$. Let $(\bar{\tilde{L}}, (\bar{\tilde{\eta}}^D)_{D\in\mathscr{D}})$ be a cone on $\mathscr{F}$ and write $\bar{\tilde{L}} = (\tilde{L}, \tilde{L}', \tilde{\psi}')$. Since $Rep_k(Q)$, $Rep_k(Q')$ are complete categories, there exist unique morphisms Ξ, Ξ' in $Rep_k(Q)$, $Rep_k(Q')$ respectively such that $\eta^D\ \Xi = \tilde{\eta}^D$, $\eta'^D\ \Xi' = \tilde{\eta}'^D$ for every $D \in \mathscr{D}$. It remains to show that $\bar{\Xi} = (\Xi, \Xi')$ is a morphism in $Rep_{(Q,Q')}$. For any $D \in \mathscr{D}$, we have

$$\begin{aligned}
\eta'^D_{s(\beta)}\ \Xi'_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta &= \tilde{\eta}'^D_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta \\
&\quad (\text{since } \eta'^D\ \Xi' = \tilde{\eta}'^D) \\
&= \psi^{D,\alpha}_\beta\ \tilde{\eta}^D_{t(\alpha)} \\
&\quad (\text{since } \bar{\tilde{\eta}}'^D \text{ is a morphism in } Rep_{(Q,Q')}) \\
&= \psi^{D,\alpha}_\beta\ \eta^D_{t(\alpha)}\ \Xi_{t(\alpha)} \\
&\quad (\text{since } \eta^D\ \Xi = \tilde{\eta}^D) \\
&= \eta'^D_{s(\beta)} \psi'^\alpha_\beta\ \Xi_{t(\alpha)} \\
&\quad (\text{since } \bar{\eta}^D \text{ is a morphism in } Rep_{(Q,Q')})
\end{aligned}$$

Notably, $(\tilde{L}_{t(\alpha)}, (\tilde{\eta}'^D_{s(\beta)}\ \tilde{\psi}')_{D\in\mathscr{D}})$ can be viewed as a limit of a functor $\rho' : \mathscr{D} \to Vect_k$. In addition, $(\tilde{L}_{t(\alpha)}, (\tilde{\eta}'^D_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta)_{D\in\mathscr{D}})$ is a cone on ρ' since

$$\begin{aligned}
(\mathscr{F}h')_{s(\beta)}\ \tilde{\eta}'^{D'}_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta &= \mathscr{F}h'_{s(\beta)}\ \eta'^{D'}_{s(\beta)}\ \Xi'_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta \\
&\quad (\text{since } \bar{\tilde{\eta}}^{D'} = \bar{\eta}^{D'}\ \bar{\Xi}) \\
&= \eta'^D_{s(\beta)}\ \Xi'_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta \\
&\quad (\text{since } \bar{\mathscr{F}}h\ \bar{\eta}^{D'} = \bar{\eta}^D) \\
&= \tilde{\eta}'^D_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta \quad .
\end{aligned}$$

Since Vec_k is complete, it follows from the universal property of the limit that $\Xi'_{s(\beta)}\ \tilde{\psi}'^\alpha_\beta = \psi'^\alpha_\beta\ \Xi_{t(\alpha)}$. Thus, $\bar{\Xi}$ is a morphism in $Rep_{(Q,Q')}$. Thus, $(\bar{L}, (\bar{\eta}^D)_{D\in\mathscr{D}})$ is a limit of $\mathscr{F}$, as desired. □

From Proposition 4.1 and Remark 3.2, we obtain.

Proposition 4.2. *The category $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ is complete and the forgetful functor*

$$\mathcal{U}_n : Rep_{(Q_1,Q_2,\ldots,Q_n)} \to Rep_k(Q_1) \times Rep_k(Q_2) \times \ldots \times Rep_k(Q_n)$$

is continuous. Further, the limit of objects in $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ can be obtained by the corresponding construction for objects in $Rep_k(Q_1) \times Rep_k(Q_2) \times \ldots \times Rep_k(Q_n)$.

Since a category is complete if and only if it has products and equalizers [2], the following is an immediate consequence of Proposition 4.2.

Corollary 4.3. *The category* $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ *has kernels.*

Proposition 4.4. *The category* $Rep_{(Q,Q')}$ *is cocomplete and the forgetful functor*

$$\mathcal{U} : Rep_{(Q,Q')} \to Rep_k(Q) \times Rep_k(Q')$$

is cocontinuous. Moreover, the colimit of objects in $Rep_{(Q,Q')}$ *can be obtained by the corresponding construction for objects in* $Rep_k(Q) \times Rep_k(Q')$*.*

Proof. Let $\mathscr{D}$ be a small category, and let $\mathscr{F} : \mathscr{D} \to Rep_{(Q,Q')}$ be a functor, and consider the composition of the following functors.

$$\mathscr{D} \xrightarrow{\mathscr{F}} Rep_{(Q,Q')} \xrightarrow{\mathcal{U}} Rep_k(Q) \times Rep_k(Q') \xrightarrow{\mathcal{P}_1} Rep_k(Q) \tag{4.5}$$

$$\mathscr{D} \xrightarrow{\mathscr{F}} Rep_{(Q,Q')} \xrightarrow{\mathcal{U}'} Rep_k(Q) \times Rep_k(Q') \xrightarrow{\mathcal{P}_2} Rep_k(Q') \tag{4.6}$$

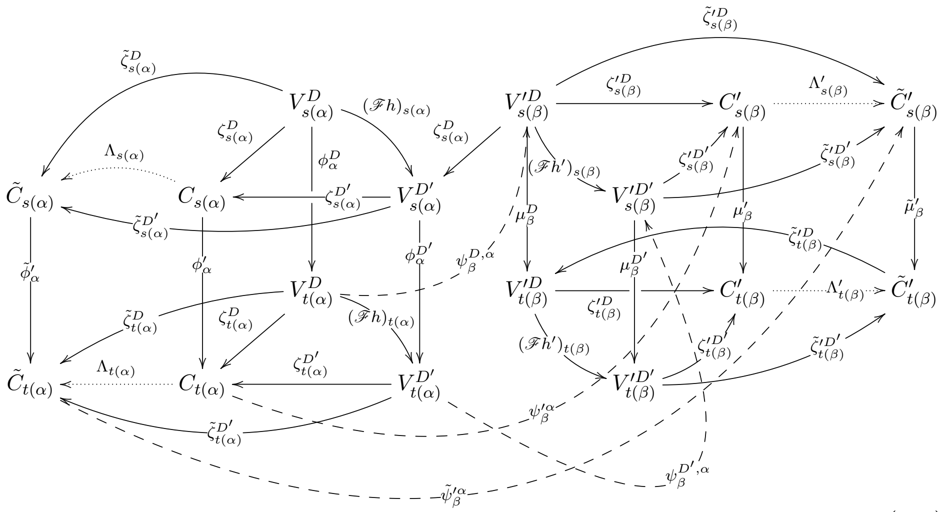
where $\mathcal{U}, \mathcal{U}'$ are the obvious forgetful functors, and $\mathcal{P}_1, \mathcal{P}_2$ are the projection functors.

For all $D \in \mathscr{D}$, let $\mathscr{F}D = \bar{V} = (V, V', \psi^{D,})$. Since $Rep_k(Q)$, $Rep_k(Q')$ are cocomplete categories, the functors $\mathcal{P}_1\mathcal{U}\mathscr{F}$, $\mathcal{P}_2\mathcal{U}\mathscr{F}$ have colimits. Let $(C, (\zeta^D)_{D\in\mathscr{D}})$, $(C', (\zeta'^D)_{D\in\mathscr{D}})$ be colimits of $\mathcal{P}_1\mathcal{U}\mathscr{F}$, $\mathcal{P}_2\mathcal{U}\mathscr{F}$ respectively, where ζ^D, ζ'^D are the morphisms

$$V^D \xrightarrow{\zeta^D} C, \qquad V'^D \xrightarrow{\zeta'^D} C' \quad , \tag{4.7}$$

for every $D \in \mathscr{D}$. Write $C = (C_i, \phi'_\alpha)$, $C' = (C'_{i'}, \mu'_\beta)$, $V^D = (V_i^D, \phi_\alpha^D)$, $V^{D'} = (V_i^{D'}, \phi_\alpha^{D'})$ $V'^{D'} = (V_{i'}'^{D'}, \mu_\beta^{D'})$, $V'^D = (V_i'^D, \mu_\beta^D)$.

Let $h : D \to D'$ be a morphism in $\mathscr{D}$ and consider the following diagram.



(4.8)

It is clear that $(C_{t(\alpha)}, (\zeta^D_{t(\alpha)})_{D\in\mathscr{D}})$ can be viewed as a colimit of a functor $\upsilon : \mathscr{D} \to Vect_k$. Moreover, we have for any $D, D' \in \mathscr{D}$,

$$\begin{aligned}
\zeta'^{D'}_{s(\beta)}\ \psi^{D',\alpha}_{\beta}\ (\mathscr{F}h')_{t(\alpha)} &= \zeta'^{D'}_{s(\beta)}\ (\mathscr{F}h)_{s(\beta)}\ \psi^{D,\alpha}_{\beta} \\
&\text{(since } \bar{\mathscr{F}}h = (\mathscr{F}h, \mathscr{F}h') \text{ is a morphism in } Rep_{(Q,Q')}) \\
&= \zeta'^{D}_{s(\beta)}\ \psi^{D,\alpha}_{\beta} \\
&\text{(since } (C', (\zeta^{D})_{D\in\mathscr{D}}) \text{ is a colimit of } \mathcal{P}_2\mathcal{U}\mathscr{F}).
\end{aligned}$$

Therefore, $(C'_{s(\beta)}, (\zeta'^{D}_{s(\beta)}\ \psi^{D,})_{D\in\mathscr{D}})$ is a cocone on υ. Since Vec_k is cocomplete, there exists a unique k-linear map $\psi'^{\alpha}_{\beta} : C_{t(\alpha)} \to C'_{s(\beta)}$ with $\psi'^{\alpha}_{\beta}\ \zeta^{D}_{t(\alpha)} = \zeta'^{D}_{s(\beta)}\ \psi^{D,\alpha}_{\beta}$ for every $D \in \mathscr{D}$ and for each pair of arrows $(\alpha, \beta) \in Q_1 \times \in Q'_1$.

It turns out that $\bar{\zeta}^D$ is a morphism in $Rep_{(Q,Q')}$ for any $D \in \mathscr{D}$. Let $\bar{C} = (C, C', \psi')$, $\bar{\zeta}^D = (\zeta^D, \zeta'^D)$. We claim that $(\bar{C}, (\bar{\zeta}^D)_{D\in\mathscr{D}})$ is a colimit of $\mathscr{F}$. To substantiate this claim, we need to show that for any cocone $(\bar{\tilde{C}}, (\bar{\tilde{\zeta}}^D)_{D\in\mathscr{D}})$, there exists a unique morphism $\bar{\Lambda} = (\Lambda, \Lambda')$ in $Rep_{(Q,Q')}$ with $\bar{\Lambda}\ \bar{\zeta}^D = \bar{\tilde{\zeta}}^D$ for every $D \in \mathscr{D}$. Let $(\bar{\tilde{C}}, (\bar{\tilde{\zeta}}^D)_{D\in\mathscr{D}})$ be a cocone on $\mathscr{F}$ and write $\bar{\tilde{C}} = (\tilde{C}, \tilde{C}', \tilde{\psi}')$. Since $Rep_k(Q)$, $Rep_k(Q')$ are cocomplete categories, exists unique morphisms Λ, Λ' in $Rep_k(Q)$, $Rep_k(Q')$, respectively, such that $\Lambda\ \zeta^D = \tilde{\zeta}^D$, $\Lambda'\ \zeta'^D = \tilde{\zeta}'^D$ for every $D \in \mathscr{D}$. It remains to show that $\bar{\Lambda} = (\Lambda, \Lambda')$ is a morphism in $Rep_{(Q,Q')}$. For any $D \in \mathscr{D}$, we have

$$\begin{aligned}
\tilde{\psi}'^{\alpha}_{\beta}\ \Lambda_{t(\alpha)}\ \zeta^{D}_{t(\alpha)} &= \tilde{\psi}'^{\alpha}_{\beta}\ \tilde{\zeta}^{D}_{t(\alpha)} \\
&\text{(since } \bar{\Lambda}\ \bar{\zeta}^{D} = \bar{\bar{\zeta}}^{D}) \\
&= \tilde{\zeta}'^{D}_{s(\beta)}\ \psi^{D,\alpha}_{\beta} \\
&\text{(since } \bar{\bar{\zeta}}^{D} \text{ is a morphism in } Rep_{(Q,Q')}) \\
&= \Lambda'_{s(\beta)}\ \zeta'^{D}_{s(\beta)}\ \psi^{D,\alpha}_{\beta} \\
&\text{(since } \bar{\Lambda}\ \bar{\zeta}^{D} = \bar{\bar{\zeta}}^{D}) \\
&= \Lambda'_{s(\beta)}\ \psi'^{\alpha}_{\beta}\ \zeta^{D}_{t(\alpha)} \\
&\text{(since } \bar{\zeta}^{D} \text{ is a morphism in } Rep_{(Q,Q')})
\end{aligned}$$

Notably, $(C_{t(\alpha)}, (\zeta^{D}_{t(\alpha)})_{D\in\mathscr{D}})$ can be viewed as a colimit of a functor $\upsilon' : \mathscr{D} \to Vect_k$. Moreover, $(\tilde{C}'_{s(\beta)}, (\tilde{\psi}'^{\alpha}_{\beta}\ \tilde{\zeta}^{D}_{t(\alpha)})_{D\in\mathscr{D}})$ is a cocone on υ' since for any $D, D' \in \mathscr{D}$, we have

$$\begin{aligned}
\tilde{\psi}'^{\alpha}_{\beta}\ \tilde{\zeta}^{D'}_{t(\alpha)}\ (\mathscr{F}h)_{t(\alpha)} &= \tilde{\psi}'^{\alpha}_{\beta}\ \Lambda_{t(\alpha)}\ \zeta^{D'}_{t(\alpha)}\ (\mathscr{F}h)_{t(\alpha)} \\
&\text{(since } \bar{\Lambda}\ \bar{\zeta}^{D} = \bar{\bar{\zeta}}^{D}) \\
&= \tilde{\psi}'^{\alpha}_{\beta}\ \Lambda_{t(\alpha)}\ \zeta^{D}_{t(\alpha)} \\
&\text{(since } (C, (\zeta^{D})_{D\in\mathscr{D}}) \text{ is a colimit of } \mathcal{P}_1\mathcal{U}\mathscr{F}) \\
&= \tilde{\psi}'^{\alpha}_{\beta}\ \tilde{\zeta}^{D}_{s(\beta)} \\
&\text{(since } \bar{\Lambda}\ \bar{\zeta}^{D} = \bar{\bar{\zeta}}^{D})
\end{aligned}$$

Since Vec_k is cocomplete, it follows from the universal property of the colimit that $\Lambda'_{s(\beta)}\ \tilde{\psi}'^{\alpha}_{\beta} = \psi'^{\alpha}_{\beta}\ \Lambda_{t(\alpha)}$. Consequently, $\bar{\Lambda}$ is a morphism in $Rep_{(Q,Q')}$, which completes the proof. □

From Proposition 4.4 and Remark 3.2, we obtain.

Proposition 4.5. *The category $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ is cocomplete and the forgetful functor*

$$\mathcal{U}_n : Rep_{(Q_1,Q_2,\ldots,Q_n)} \to Rep_k(Q_1) \times Rep_k(Q_2) \times \ldots \times Rep_k(Q_n)$$

is cocontinuous. Furthermore, the colimit of objects in $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ can be obtained by the corresponding construction for objects in $Rep_k(Q_1) \times Rep_k(Q_2) \times \ldots \times Rep_k(Q_n)$.

Since a category is cocomplete if and only if it has coproducts and coequalizers [2], we have the following immediate consequence.

Corollary 4.6. *The category $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ has cokernels.*

Next, we aim to show that the Categories of n-representations are abelian.

Following [13, p. 2], an **additive** category is a category $\mathcal{C}$ satisfying the following axioms:

(i) Every set $\mathcal{C}(X,Y)$ is equipped with a structure of an abelian group (written additively) such that composition of morphisms is biadditive with respect to this structure.

(ii) There exists a zero object $0 \in \mathcal{C}$ such that $\mathcal{C}(0,0) = 0$.

(iii) (Existence of direct sums.) For any objects $X, X' \in \mathcal{C}$, the direct sum $X \oplus X' \in \mathcal{C}$.

Let k be a field. An additive category $\mathcal{C}$ is said to be k-**linear** if for any objects $X, Y \in \mathcal{C}$, $\mathcal{C}(X,Y)$ is equipped with a structure of a vector space over k, such that composition of morphisms is k-linear.
An **abelian** category is an additive category $\mathcal{C}$ in which for every morphism $f : X \to Y$ there exists a sequence

$$K \xrightarrow{k} X \xrightarrow{\iota} I \xrightarrow{j} Y \xrightarrow{c} C \tag{4.9}$$

with the following properties:

(i) $ji = f$,

(ii) $(K,k) = Ker(f)$, $(C,c) = Coker(f)$,

(iii) $(I,\iota) = Coker(k)$, $(I,j) = Ker(c)$.

A sequence (4.11) is called a **canonical decomposition** of f.

Let $\bar{f} = (f,f') : \bar{X} \to \bar{Y}$ be a morphism in $Rep_{(Q,Q')}$. It follows that $f : X \to Y$, $f' : X' \to Y'$ are morphisms in $Rep_k(Q)$, $Rep_k(Q')$ respectively. From Proposition 2.3, $f : X \to Y$, $f' : X' \to Y'$ have the canonical decompositions

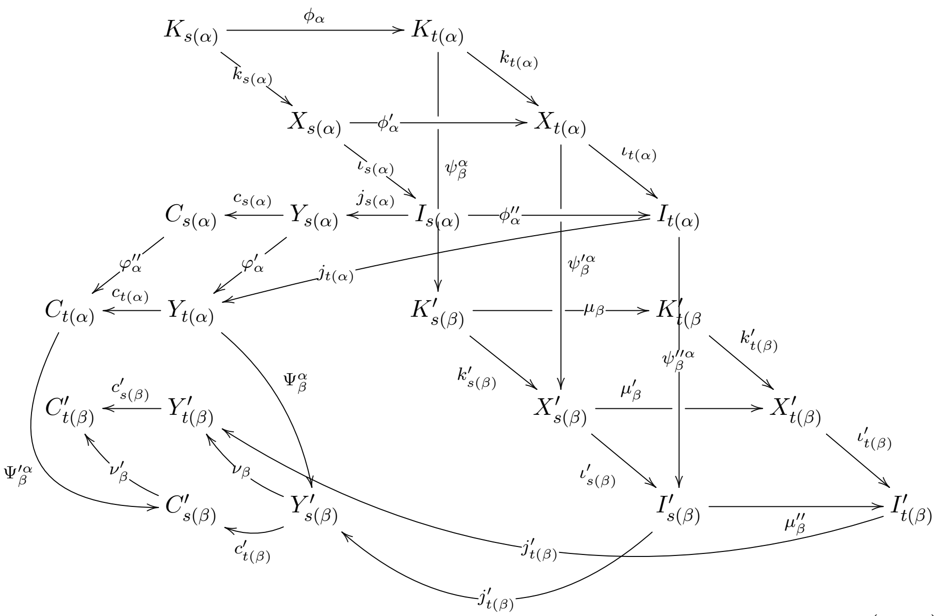
$$K \xrightarrow{k} X \xrightarrow{\iota} I \xrightarrow{j} Y \xrightarrow{c} C \tag{4.10}$$

$$K' \xrightarrow{k'} X' \xrightarrow{\iota'} I' \xrightarrow{j'} Y' \xrightarrow{c'} C'$$

in $Rep_k(Q)$, $Rep_k(Q')$ respectively. From the above results, kernels and cokernels exist in $Rep_{(Q,Q')}$. It turns out that $\bar{f}$ has a canonical decomposition

$$\bar{K} \xrightarrow{\bar{k}} \bar{X} \xrightarrow{\bar{\iota}} \bar{I} \xrightarrow{\bar{j}} \bar{Y} \xrightarrow{\bar{c}} \bar{C} \tag{4.11}$$

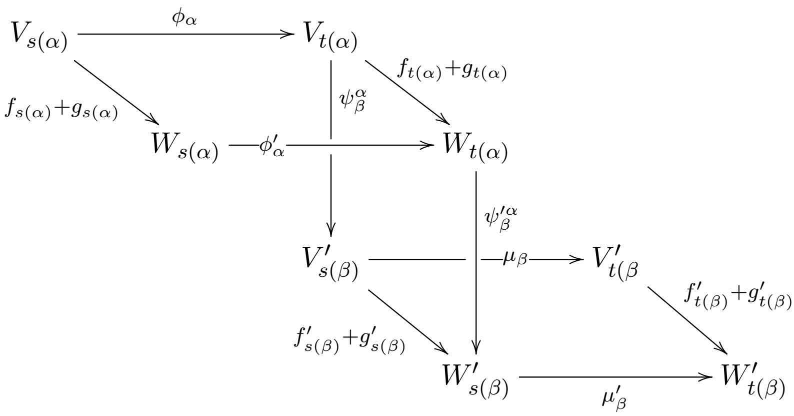
in $Rep_{(Q,Q')}$, and this decomposition can explicitly be seen in the following commutative diagram.



(4.12)

This implies that any morphism $\underline{f} : \bar{V} \to \bar{W}$ of n-representations has a canonical decomposition in $Rep_{(Q_1,Q_2,\ldots,Q_n)}$.

Remark 4.7. Let $\bar{f}$, $\bar{g} : \bar{V} \to \bar{W}$ be morphisms in $Rep_{(Q,Q')}$. Write $\bar{f} = (f, f')$, $\bar{g} = (g, g')$, $f = (f_i)$, $g = (g_i)$, $f' = (f'_{i'})$, $g' = (g'_{i'})$, $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$. Define $\bar{f} + \bar{g} = (f + g, f' + g') = ((f_i + g_i), (f'_{i'} + g'_{i'}))$. Since $Rep_k(Q)$ and $Rep_k(Q')$ are abelian, the sets $Rep_k(Q)(V, W)$, $Rep_k(Q)(V', W')$ are equipped with a structure of an abelian group such that composition of morphisms is biadditive with respect to this structure [3, p. 70]. Since $\bar{f}$, $\bar{g}$ are morphisms in $Rep_{(Q,Q')}$ and since the category Vec_k is abelian, we have the following commutative diagram:



(4.13)

Thus, the set $Rep_{(Q,Q')}(\bar{V},\bar{W})$ is equipped with a structure of an abelian group such that composition of morphisms is biadditive with respect to the above structure.

We end this section with the following result.

Theorem 4.8. *The category $Rep_{(Q,Q')}$ is a k-linear abelian category. More generally, the category $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ is a k-linear abelian category for any integer $n \geq 2$.*

5 n-quivers and n-representations

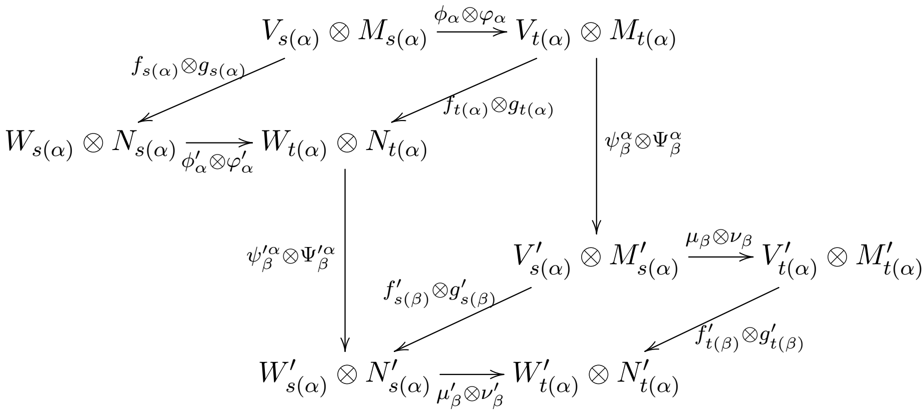
Let $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$ be birepresentations of (Q, Q'). For simplicity, we suppress k from the tensor product $\otimes_k$ and use $\otimes$ instead. Define

$$\bar{V} \otimes \bar{W} = (V \otimes W, V' \otimes W', \psi \otimes \psi'). \tag{5.1}$$

Then $\bar{V} \otimes \bar{W}$ is clearly a birepresentation of (Q, Q'). Let $\bar{f} : \bar{V} \to \bar{W}$, $\bar{g} : \bar{M} \to \bar{N}$ be morphisms in $Rep_{(Q,Q')}$ and write $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$, $\bar{M} = (M, M', \Psi)$, $\bar{N} = (N, N', \Psi')$. Define

$$\bar{f} \otimes \bar{g} : \bar{V} \otimes \bar{M} \to \bar{W} \otimes \bar{N}. \tag{5.2}$$

Then it is clear that $\bar{f}\otimes\bar{g}$ is a morphism in $Rep_{(Q,Q')}$, and hence the following diagram is commutative.



(5.3)

Thus, the category $Rep_{(Q,Q')}$ is a monoidal category, and hence by Remark 3.2, $Rep_{(Q_1,Q_2,...,Q_n)}$ is a monoidal category for any $n \geq 2$. For the basic notions of monoidal categories, we refer the reader to [13], [28], [6], and [10, Chapter 6].

Definition 5.1. Let $Q = (Q_0, Q_1, s, t)$, $Q' = (Q'_0, Q'_1, s', t')$ be quivers, and let $\{\varrho^\alpha_\beta : \alpha \in Q_1,\ \beta \in Q'_1\}$ be a collection of of arrows $t(\alpha) \xrightarrow{\varrho^\alpha_\beta} s'(\beta)$ one for each pair of arrows $(\alpha, \beta) \in Q_1 \times Q'_1$.

A **2-quiver induced by** (Q, Q') is a quiver $\mathscr{Q}_{(Q,Q')} = (\tilde{Q}_0, \tilde{Q}_1, s'', t'')$, where

- $\tilde{Q}_0 = Q_0 \sqcup Q'_0$,
- $\tilde{Q}_1 = Q_1 \sqcup Q'_1 \sqcup \{\varrho^\alpha_\beta : \alpha \in Q_1,\ \beta \in Q'_1\}$,
- $s'' : \tilde{Q}_1 \to \tilde{Q}_0$ a map from arrows to vertices, mapping an arrow to its starting point,
- $t'' : \tilde{Q}_1 \to \tilde{Q}_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

The notation $\sqcup$ above denotes the disjoint union.

The above definition turns out that one can inductively define n-quivers for any integer $n \geq 2$.
For any $m \in \{1, \ldots, n\}$, let $Q_m = (\mathrm{Q}_0^{(m)}, \mathrm{Q}_1^{(m)}, s^{(m)}, t^{(m)})$ be a quiver, and let $\{\varrho^{\gamma^{(m)}}_{{}_m\gamma^{(m-1)}} : \gamma^{(m)} \in \mathrm{Q}_1^{(m)},\ m \in \{2, \ldots, n\}\}$ be a collection of of arrows

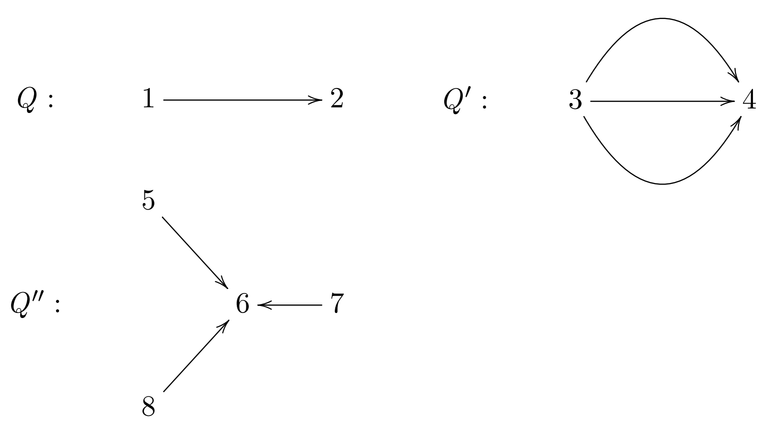
$$t^{(m-1)} \xrightarrow{\varrho^{\gamma^{(m)}}_{{}_m\gamma^{(m-1)}}} s^{(m)}$$

one for each pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in \mathrm{Q}_1^{(m-1)} \times \mathrm{Q}_1^{(m)}$. An **$n$-quiver induced by** $(Q_1,\ Q_2, \ldots, Q_n)$ is a quiver $\mathscr{Q}_{(Q_1, Q_2, \ldots, Q_n)} = (\hat{\mathrm{Q}}_0, \hat{\mathrm{Q}}_1, \hat{s}, \hat{t})$, where

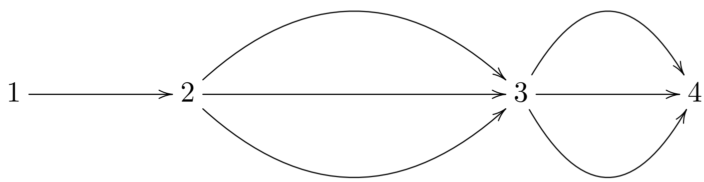
- $\hat{\mathrm{Q}}_0 = \sqcup_{m=1}^{n} \mathrm{Q}_0^{(m)}$,
- $\hat{\mathrm{Q}}_1 = \sqcup_{m=1}^{n} \mathrm{Q}_1^{(m)} \ \sqcup \ \{\varrho^{\gamma^{(m)}}_{{}_m\gamma^{(m-1)}} : \gamma^{(m)} \in \mathrm{Q}_1^{(m)},\ m \in \{2, \ldots, n\}\}$,
- $\hat{s} : \hat{\mathrm{Q}}_1 \to \hat{\mathrm{Q}}_0$ a map from arrows to vertices, mapping an arrow to its starting point,
- $\hat{t} : \hat{\mathrm{Q}}_1 \to \hat{\mathrm{Q}}_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

Example 5.2.

(5.4)

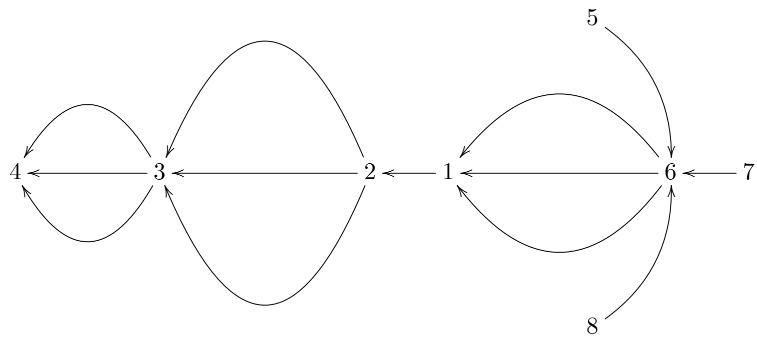


$\mathscr{Q}_{(Q,Q')}$:



(5.5)

$\mathscr{Q}_{(Q'',Q,Q')}$:



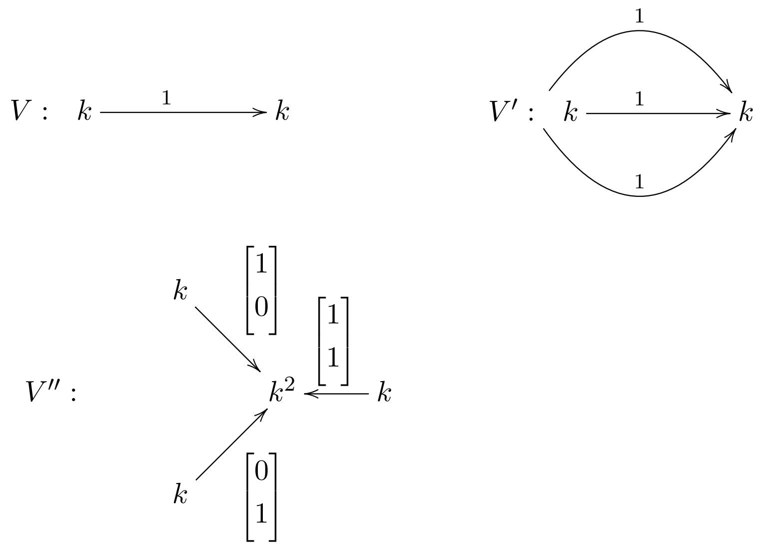
(5.6)

We have $k(Q) \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, $k(Q') \cong \begin{bmatrix} k & k^3 \\ 0 & k \end{bmatrix}$, $k(\mathscr{Q}_{(Q,Q')}) \cong \begin{bmatrix} k & k & k^3 & k^9 \\ 0 & k & k^3 & k^9 \\ 0 & 0 & k & k^3 \\ 0 & 0 & 0 & k \end{bmatrix}$,

and

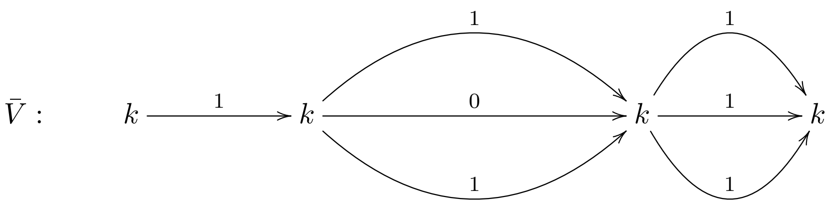
$$k(\mathscr{Q}_{(Q'',Q,Q')}) \cong \begin{bmatrix} k & k & k^3 & k^9 & 0 & 0 & 0 & 0 \\ 0 & k & k^3 & k^9 & 0 & 0 & 0 & 0 \\ 0 & 0 & k & k^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 & 0 & 0 \\ k^3 & k^3 & k^9 & k^{27} & k & k & 0 & 0 \\ k^3 & k^3 & k^9 & k^{27} & 0 & k & 0 & 0 \\ k^3 & k^3 & k^9 & k^{27} & 0 & k & k & 0 \\ k^3 & k^3 & k^9 & k^{27} & 0 & k & 0 & k \end{bmatrix}.$$

Consider the following:

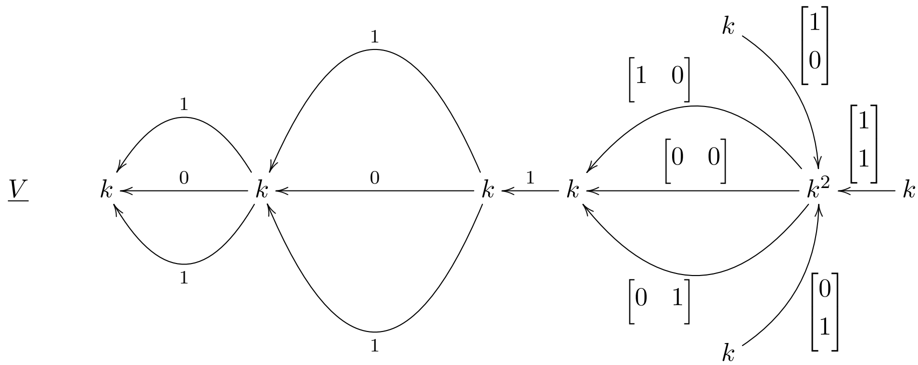


(5.7)

Then V, V', V'' are clearly representations of Q, Q', Q'' respectively. Now, consider the following:



(5.8)



(5.9)

Then $\bar{V}$ is a representations of $\mathscr{Q}_{(Q,Q')}$ and $\underline{V}$ is a representation of $\mathscr{Q}_{(Q'',Q,Q')}$.

Remark 5.3.

(i) In general, if $\Omega_n = \{\varrho^{\gamma^{(n)}}_{n\gamma^{(n-1)}} \ : \ \gamma^{(n-1)} \in Q_1^{(n-1)}, \ \gamma^{(n)} \in Q_1^{(n)}\}$, then we have

$$k\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)} \cong \begin{bmatrix} k\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_{n-1})} & k\Omega_n \\ 0 & kQ_n \end{bmatrix}$$

or

$$k\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)} \cong \begin{bmatrix} k\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_{n-1})} & 0 \\ k\Omega_n & kQ_n \end{bmatrix}$$

where $k\Omega$ is the subspace of $k\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)}$ generated by the set Ω_n. To avoid confusion, we identify $k\mathscr{Q}_{(Q_1)}$ as $k\mathcal{Q}_1$.

(ii) We note that $\bar{V}$ in the previous example can be identified as a birepresentations of (Q, Q'). Indeed, every representation of $\mathscr{Q}_{(Q,Q')}$ can be identified as a birepresentation of (Q, Q'). Conversely, every birepresentation of (Q, Q') can be viewed as a representation of $\mathscr{Q}_{(Q,Q')}$. Similarly, $\underline{V}$ can be identified as a 3-representation of $\mathscr{Q}_{(Q'',Q,Q')}$. This can be explicitly stated as the following.

Proposition 5.4. *There exists an equivalence of categories* $Rep_{(Q,Q')} \simeq Rep_k(\mathscr{Q}_{(Q,Q')})$ *that restricts to an equivalence* $rep_{(Q,Q')} \simeq rep_k(\mathscr{Q}_{(Q,Q')})$.

Proof. This follows from the construction of $\mathscr{Q}_{(Q,Q')}$ in Definition 5.1. In fact, there is a combining functor $\mathcal{F} : Rep_{(Q,Q')} \to Rep_k(\mathscr{Q}_{(Q,Q')})$ that views any birepresentation of (Q, Q') as one piece, a representation of $\mathscr{Q}_{(Q,Q')}$. Its inverse is the decomposing functor $\mathcal{G} : Rep_k(\mathscr{Q}_{(Q,Q')}) \to Rep_{(Q,Q')}$, which breaks each representation of $\mathscr{Q}_{(Q,Q')}$ into three parts (two representations of Q, Q' and a collection of compatibility maps) to form a birepresentation of (Q, Q'). □

Using Induction and Remark 3.2, we have the following.

Proposition 5.5. *For any* $n \geq 2$, *there exists an equivalence of categories* $Rep_{(Q_1,Q_2,\ldots,Q_n)} \simeq Rep_k(\mathscr{Q}_{(Q_1,\ Q_2,\ldots,Q_n)})$ *that restricts to an equivalence* $rep_{(Q_1,Q_2,\ldots,\ Q_n)} \simeq rep_k(\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)})$.

The following is an immediate consequence of Proposition 5.4 and Proposition 2.4.

Proposition 5.6. *There exists an equivalence of categories* $Mod\, k\mathscr{Q}_{(Q,Q')} \simeq Rep_{(Q,Q')}$ *that restricts to an equivalence* $mod\, k\mathscr{Q}_{(Q,Q')} \simeq rep_{(Q,Q')}$.

By Remark 3.2 and Propositions 5.5, 2.4, we have the following.

Proposition 5.7. *For any* $n \geq 2$*, there exists an equivalence of categories* $Mod\, k\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)} \simeq Rep_{(Q_1,Q_2,\ldots,Q_n)}$ *that restricts to an equivalence* $mod\, k\mathscr{Q}_{(Q_1,Q_2,\ldots,Q_n)} \simeq rep_{(Q_1,Q_2,\ldots,Q_n)}$.

We recall the definitions of a co-wellpowered category and a generating set for a category. Let $\mathfrak{E}$ be a class of all epimorphisms of a category $\mathfrak{A}$. Then $\mathfrak{A}$ is called *co-wellpowered* provided that no $\mathfrak{A}$-object has a proper class of pairwise non-isomorphic quotients [2, p. 125]. In other words, for every object the quotients form a set [26, p. 92, 95]. We refer the reader to [2] basics on quotients and co-wellpowered categories.

Following [**?**, p. 127], a set $\mathcal{G}$ of objects of the category $\mathscr{C}$ is said to *generate* $\mathscr{C}$ when any parallel pair $f, g : X \to Y$ of arrows of $\mathscr{C}$, $f \neq g$ implies that there is an $G \in \mathcal{G}$ and an arrow $\alpha : G \to X$ in $\mathscr{C}$ with $f\alpha \neq g\alpha$ (the term "generates" is well established but poorly chosen; "*separates*" would have been better). For the basic concepts of generating sets, we refer to [**?**], [2], or [15].

The following proposition immediately follows from Remark 3.2, Proposition 5.7 and the fact that the categories of modules are co-wellpowered with generating sets.

Proposition 5.8. *For any* $n \geq 2$*, the category* $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ *is co-well powered.*

Proposition 5.9. *For any* $n \geq 2$*, the category* $Rep_{(Q_1,Q_2,\ldots,Q_n)}$ *has a generating set.*

Using Theorem (2.1), we have the following.

Proposition 5.10. *For any* $n \geq 2$*, the obvious forgetful functor* $\mathcal{U}_n : Rep_{(Q_1,Q_2,\ldots,Q_n)} \to Rep_k(Q_1) \times Rep_k(Q_2) \times \ldots \times Rep_k(Q_n)$ *has a right adjoint. Equivalently, for any* $n \geq 2$*, the concrete category* $(Rep_{(Q_1,Q_2,\ldots,Q_n)}, \mathcal{U}_n)$ *has cofree objects.*

Let $CoAlg(Rep_{(Q,Q')})$ be the category of coalgebras in $Rep_{(Q,Q')}$. By [1, p. 30], the left $k\mathcal{Q}_{(Q,Q')}$-module coalgebras which are finitely generated as left $k\mathcal{Q}_{(Q,Q')}$-modules form a system of generators for $CoAlg(Mod\, k\mathcal{Q}_{(Q,Q')})$. Thus, from Proposition 2.2 and Theorem 2.1, we have the following.

Proposition 5.11. *The forgetful functor* $\mathcal{U} : CoAlg(Rep_{(Q,Q')}) \to Rep_{(Q,Q')}$ *has a right adjoint. Equivalently, the concrete category* $(Rep_{(Q,Q')}, \mathcal{U})$ *has cofree objects.*

Remark (3.2) implies the following consequence.

Proposition 5.12. *For any* $n \geq 2$*, the forgetful functor*

$$\mathcal{U}_n : CoAlg(Rep_{(Q_1,Q_2,\ldots,Q_n)}) \to Rep_{(Q_1,Q_2,\ldots,Q_n)}$$

has a right adjoint. Equivalently, for any $n \geq 2$*, the concrete category* $(Rep_{(Q_1,Q_2,\ldots,Q_n)}, \mathcal{U}_n)$ *has cofree objects.*

Remark 5.13. If the forgetful functor $\mathcal{U} : CoAlg(Rep_{(Q,Q')}) \to Rep_{(Q,Q')}$ has a right adjoint $\mathcal{V}$ and $\bar{M} \in Rep_{(Q,Q')}$, then the cofree coalgebra over $\bar{M}$ can be given by

$$\mathcal{V}(\bar{M}) = \varinjlim_{[\bar{f}:\bar{G}\to\bar{M}]\in Rep_{(Q,Q')},\, \bar{G}\in CoAlg(Rep_{(Q,Q')}),\, \bar{G}\in rep_{(Q,Q')}} \bar{G}.$$

Thus, the concrete category $(CoAlg(Rep_{(Q,Q')}), \mathcal{U})$ has cofree objects given in terms of colimits and generators.

It follows that, for any $n \geq 2$, if the forgetful functor

$$\mathcal{U}_n : CoAlg(Rep_{(Q_1,Q_2,\ldots,Q_n)}) \to Rep_{(Q_1,Q_2,\ldots,Q_n)}$$

has a right adjoint $\mathcal{V}_n$ and $\bar{M} \in Rep_{(Q_1,Q_2,\ldots,Q_n)}$, then the cofree coalgebra over $\bar{M}$ can be given by

$$\mathcal{V}_n(\bar{M}) = \varinjlim_{[\bar{f}:\bar{G}\to\bar{M}]\in Rep_{(Q_1,Q_2,\ldots,Q_n)},\, \bar{G}\in CoAlg(Rep_{(Q_1,Q_2,\ldots,Q_n)}),\, \bar{G}\in rep_{(Q_1,Q_2,\ldots,Q_n)}} \bar{G}.$$

The following proposition explicitly describes colimits and cofree objects in the product categories.

Proposition 5.14. *For any* $m \in \{1, 2, 3, \ldots, n\}$*, let* $\mathcal{U}_m : \mathscr{A}_m \to \mathscr{X}_m$ *be a forgetful functor with a right adjoint* $\mathcal{V}_m$*. Then the product functor* $\mathcal{V}_1 \times \mathcal{V}_2 \times ... \times \mathcal{V}_n : \mathscr{X}_1 \times \mathscr{X}_2 \times ... \times \mathscr{X}_n \to \mathscr{A}_1 \times \mathscr{A}_2 \times ... \times \mathscr{A}_n$ *is a right adjoint of the obvious forgetful functor* $\mathcal{U}_1 \times \mathcal{U}_2 \times ... \times \mathcal{U}_n : \mathscr{A}_1 \times \mathscr{A}_2 \times ... \times \mathscr{A}_n \to \mathscr{X}_1 \times \mathscr{X}_2 \times ... \times \mathscr{X}_n$*. Furthermore, for any* $(X_1, X_2, ..., X_n) \in \mathscr{X}_1 \times \mathscr{X}_2 \times ... \times \mathscr{X}_n$*, the cofree object over* $(X_1, X_2, ..., X_n)$ *is exactly* $(\mathcal{V}_1(X_1) \times \mathcal{V}_2(X_2) \times ... \times \mathcal{V}_n(X_n))$.

Proof. The proof is straightforward. □

For any $m \in \{1, 2, \ldots, n\}$, let $\mathcal{U}_m : CoAlg(Rep_k(Q_m)) \to Rep_k(Q_m)$ be the obvious forgetful functor with a right adjoint $\mathcal{V}_m$. Proposition 5.14 and Remark 3.2 immediately imply the following consequence.

Corollary 5.15. *The product functor* $\mathcal{V}_1 \times \mathcal{V}_2 \times ... \times \mathcal{V}_n : Rep_k(Q_1) \times Rep_k(Q_2) \times ... \times Rep_k(Q_n) \to CoAlg(Rep_k(Q_1)) \times CoAlg(Rep_k(Q_2)) \times ... \times CoAlg(Rep_k(Q_n))$ *is a right adjoint of the obvious product forgetful functor* $\mathcal{U}_1 \times \mathcal{U}_2 \times ... \times \mathcal{U}_n : CoAlg(Rep_k(Q_1)) \times CoAlg(Rep_k(Q_2)) \times ... \times CoAlg(Rep_k(Q_n)) \to Rep_k(Q_1) \times Rep_k(Q_2) \times ... \times Rep_k(Q_n)$.
Furthermore, for any $(M_1, M_2, ..., M_n) \in Rep_k(Q_1) \times Rep_k(Q_2) \times ... \times Rep_k(Q_n)$*, the cofree object over* $(M_1, M_2, ..., M_n)$ *is exactly* $(\mathcal{V}_1(M_1), \mathcal{V}_2(M_2), ..., \mathcal{V}_n(M_n))$.

For any $m \in \{1, 2, \ldots, n\}$, let $\mathcal{U}_m : CoAlg(Rep_k(Q_m)) \to Rep_k(Q_m)$ be the obvious forgetful functor with a right adjoint $\mathcal{V}_m$. By Proposition 5.9, for any $n \geq 2$, the category $Rep_{(Q_1, Q_2, \ldots, Q_n)}$ has a generating set $\mathscr{G}$. For any $m \in \{1, 2, \ldots, n\}$, let $\mathcal{G}_m$ be a generating set of $Rep_k(Q_m)$. The definition of n-representations of quivers implies that each element of $\mathscr{G}$ takes the form $(G_1, G_2, ..., G_n, \chi_1, \chi_2, ..., \chi_{n-1})$ where $G_m \in \mathcal{G}_m$ for every $m \in \{1, 2, 3, \ldots, n\}$. Furthermore, Remark 5.13 and Proposition 4.4 imply the following proposition which gives an explicit description for cofree coalgebras in the concrete category $(Rep_{(Q_1, Q_2, \ldots, Q_n)}, \mathscr{U})$, where $\mathscr{U} : CoAlg(Rep_{(Q_1, Q_2, \ldots, Q_n)}) \to Rep_{(Q_1, Q_2, \ldots, Q_n)}$ is the obvious forgetful functor with a right adjoint $\mathscr{V}$.

Proposition 5.16. *Let*

$$\bar{M} = (M^{(1)}, M^{(2)}, ..., M^{(n)}, \psi_1, \psi_2, ..., \psi_{n-1}) \in Rep_{(Q_1, Q_2, \ldots, Q_n)}.$$

Then the cofree object over $\bar{M}$ *is exactly*

$$(\mathcal{V}_1(M^{(1)}), \mathcal{V}_2(M^{(2)}), ..., \mathcal{V}_n(M^{(n)}), \psi'_1, \psi'_2, ..., \psi'_{n-1}),$$

for some unique k-linear maps $\psi'_1, \psi'_2, ..., \psi'_{n-1}$.

Proof. This can be proved by applying Remark 5.13 and Proposition 4.4. Explicitly, if

$$\bar{G} = (G^{(1)}, G^{(2)}, ..., G^{(n)}, \chi_1, \chi_2, ..., \chi_{n-1}), \quad \bar{f} = (f^{(1)}, f^{(2)}, ..., f^{(n)}),$$

we have the following. for some unique k-linear maps $\psi'_1, \psi'_2, ..., \psi'_{n-1}$. □

We end this paper by pointing out that the universal investigation above can be adjusted to study cofree objects in the centralizer and the center categories of $Rep_{(Q_1,Q_2,...,Q_n)}$. Indeed, one can describe them in terms of cofree objects in centralizer and center categories of $Rep_k(Q_j)$, $j \in \{1, 2, \ldots, n\}$.

References

[1] Abdulwahid, A.H. and Iovanov, M.C., *Generators for comonoids and universal constructions*, Arch. Math. 106 (2016), 21-33.

[2] Adamek, J., Herrlich, H., and Strecker, G.E., "Abstract and Concrete Categories: The Joy of Cats", Dover Publication, 2009.

[3] Assem, I., Skowronski, A., and Simson, D., "Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory", London Math. Soc. Student Texts 65, Cambridge University Press, 2006.

[4] Auslander, M., Reiten, I., S.O. Smalø, S.O., "Representation Theory of Artin Algebras", Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, 1995.

[5] Awodey, S., "Category Theory", Oxford University Press, 2010.

[6] Bakalov, B. and Kirillov, A., Jr., "Lectures on Tensor Categories and Modular Functor", University Lecture Series 21, American Math. Soc., 2001.

[7] Barot, M., "Introduction to the Representation Theory of Algebras", Springer, 2015.

[8] Benson, D.J., "Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras", Cambridge Stud. Adv. Math. 30, Cambridge University Press, 1991.

[9] Borceux, F., "Handbook of Categorical Algebra 1: Basic Category Theory", Cambridge University Press, 1994.

[10] Borceux, F., “Handbook of Categorical Algebra 2: Categories and Structures”, Cambridge University Press, 1994.

[11] Buan, A.B., Reiten, I., and Solberg, O., “Algebras, Quivers and Representations”, Springer, 2013.

[12] Dăscălescu, S., Iovanov, M., Năstăsescu, C., *Quiver algebras, path coalgebras and co-reflexivity*, Pacific J. Math. 262 (2013), 49-79.

[13] Etingof, P., Gelaki, S., Nikshych, D., and Ostrik, V., “Tensor Categories”. Mathematical Surveys and Monographs 205, American Math. Soc., 2015.

[14] Etingof, P., Golberg, O., Hensel, S., Liu, T., Schwendner, A., Vaintrob, D., Yudovina, E., “Introduction to Representation Theory”, Student Mathematical Library 59, American Math. Soc., 2011.

[15] Freyd, P.J. and Scedrov, A., “Categories, Allegories”, Elsevier Science Publishing Company, 1990.

[16] Gabriel, P., *Unzerlegbare Darstellungen I*, Manuscripta Math. 6 (1972), 71-103.

[17] Kilp, M., Knauer, U., Mikhalev, A.V., “Monoids, Acts, and Categories: With Applications to Wreath Products and Graphs”, De Gruyter exposition in Math. 29, 2000.

[18] Leinster, T., “Basic Category Theory”, London Math. Soc. Lecture Note Series 298, Cambridge University Press, 2014.

[19] Leinster, T., “Higher Operads, Higher Categories”, Lecture Note Series 298, London Math. Soc., Cambridge University Press, 2004.

[20] Mac Lane, S., “Categories for the Working Mathematician”, Graduate Texts in Math. 5, Springer-Verlag, 1998.

[21] McLarty, C., “Elementary Categories, Elementary Toposes”, Oxford University Press, 2005.

[22] Mitchell, B., “Theory of Categories”, Academic Press, 1965.

[23] Pareigis, B., “Categories and Functors”, Academic Press, 1971.

[24] Rotman, J.J., “An Introduction to Homological Algebra”, Springer, 2009.

[25] Schiffler, R., “Quiver Representations”, CMS Books in Math. Series, Springer International Publishing, 2014.

[26] Schubert, H., “Categories”, Springer-Verlag, 1972.

[27] Sergeichuk, V.V., *Linearization method in classification problems of linear algebra*, São Paulo J. Math. Sci. 1(2) (2007), 219-240.

[28] Street, R., "Quantum Groups: a Path to Current Algebra", Lecture Series 19, Australian Math. Soc., Cambridge University Press, 2007.

[29] Wisbauer, R., "Foundations of Module and Ring Theory: A Handbook for Study and Research", Springer-Verlag, 1991.

[30] Zimmermann, A., "Representation Theory: A Homological Algebra Point of View", Springer-Verlag, 2014.

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