Categories and
General Algebraic Structures
with Applications
Volume 6, Special issue on the occasion of Banaschewski's 90th Birthday, January 2017, 51-66.

C-connected frame congruences

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. We discuss the congruences θ that are connected as elements of the (totally disconnected) congruence frame $\mathfrak{C}L$, and show that they are in a one-to-one correspondence with the completely prime elements of L, giving an explicit formula. Then we investigate those frames L with enough connected congruences to cover the whole of $\mathfrak{C}L$. They are, among others, shown to be T_D-spatial; characteristics for some special cases (Boolean, linear, scattered and Noetherian) are presented.

Introduction

The aim of this paper is a study of connectedness of frame congruences. There is the standard representation of a frame congruence θ as the join

Support from the projects P202/12/G061 (Grant Agency of the Czech Republic) and MTM2015-63608-P (Ministry of Economy and Competitiveness of Spain) is gratefully acknowledged. We also gratefully acknowledge the grant (No. 93107) from the NRF (South Africa) under the Blue Skies Research Programme.
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Keywords: Frame, frame congruence, congruence and sublocale lattice, connectedness, T_D-spatiality.
Mathematics Subject Classification [2010]: 06A06, 06B10, 06D20, 06D22.
Received: 1 June 2016, Accepted: 28 September 2016
ISSN Print: 2345-5853 Online: 2345-5861

$\bigvee\{\Delta_a \cap \nabla_b \mid (a,b) \in \theta\}$ with open congruences $\Delta_a = \{(x,y) \mid a \wedge x = a \wedge y\}$ and closed congruences $\nabla_b = \{(x,y) \mid b \vee x = b \vee y\}$. Since the congruences $\Delta_a \cap \nabla_b$ are complemented in the frame $\mathfrak{C}L$ of congruences in L, this shows that $\mathfrak{C}L$ is zero-dimensional, in other words, totally disconnected. Hence one can expect that an individual frame congruence θ (as an element of $\mathfrak{C}L$, not to be confused with the connectedness of the resulting quotient frames L/θ in Chen [6]) is seldom connected.

This is indeed the case, but such congruences seem to be of more interest than meets the eye, particularly if one asks about the associated *local C-connectedness* (short for *local congruence connectedness*), that is, about the frames for which $\mathfrak{C}L$ is covered by connected congruences.

The connected congruences are the $\Delta_a \cap \nabla_b$ with a immediately preceding b. Their geometric characteristics may be of interest: they are precisely the congruences the associated sublocales of which are the complements of the T_D-points in the spectrum of L. This, and more detailed characteristics of the connected θ, is presented in Section 3 (after preliminary Sections 1 and 2).

In Section 4 we present a general characterization of a locally C-connected frame; in particular we show that such a frame is T_D-spatial.

Section 5 is devoted to some special cases in which the characteristics of local C-connectedness becomes fairly explicit. In the Boolean case we obtain precisely the discrete locales, in the linear case we have precisely the well-ordered complete posets, in the scattered case the local C-connectedness is equivalent with T_D-spatiality. Finally we show that a Noetherian frame is locally C-connected if and only if for each $a < b$ one has a sequence $a = a_1, a_2, \ldots, a_n = b$ with a_i immediately preceding a_{i+1}.

1 Preliminaries I: Basics

1.1 For an element a of a poset $(X, \leq)$ we write, as usual, ${\downarrow}a = \{x \mid x \leq a\}$ and ${\uparrow}a = \{x \mid x \geq a\}$. The suprema (joins) of subsets $A \subseteq X$ are denoted by $\bigvee A$, and $a \vee b = \bigvee\{a,b\}$; similarly we write $\bigwedge A$ and $a \wedge b = \bigwedge\{a,b\}$ for infima (meets).

The opposite (dual) poset of $(X, \leq)$ (that is, $(X, \leq_1)$ with the order $x \leq_1 y$ if and only if $y \leq x$) will be denoted by $(X, \leq)^{\mathrm{op}}$.

1.2 Recall that a *frame* is a complete lattice L satisfying the distributivity rule

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{frm}$$

for all $A \subseteq L$ and $b \in L$, and that a *frame homomorphism* $h\colon L \to M$ preserves all joins and all finite meets. The resulting category is denoted by **Frm**.

If L^{op} is a frame we say that L is a co-frame.

1.2.1. The equality (frm) states, in other words, that for every $b \in L$ the mapping $- \wedge b = (x \mapsto x \wedge b) : L \to L$ preserves all joins (suprema). Hence every $- \wedge b$ has a right Galois adjoint resulting in a *Heyting operation* $\to$ with

$$a \wedge b \leq c \quad \text{if and only if} \quad a \leq b \to c. \tag{$**$}$$

Thus, each frame is a Heyting algebra (note that, however, the frame homomorphisms do not coincide with the Heyting ones so that **Frm** differs from the category of complete Heyting algebras). The operation $\to$ and some of its properties (for example, $a \to b = 1$ if and only if $a \leq b$, $1 \to a = a$, $a \to (\bigwedge b_i) = \bigwedge a \to b_i, a \leq b \to a$, $a \wedge (a \to b) \leq b$, $a \to (b \to c) = (a \wedge b) \to c$ that immediately follow from $(**)$) will often be used in the sequel.

1.2.2. A standard, but perhaps less obvious Heyting identity. *For all a and x,*

$$a = (a \vee x) \wedge (x \to a).$$

(Indeed, trivially $\leq$; on the other hand,

$$(a \vee x) \wedge (x \to a) = (a \wedge (x \to a) \vee (x \wedge (x \to a)) \leq a.)$$

1.3 Subobjects in Frm. Subobjects in the category of frames can be represented in various equivalent ways.

Congruences θ (and surjective homomorphisms $h : L \to L/\theta$): In the inclusion order they constitute a complete lattice $\mathfrak{C}L$. It is an important fact that

$$\mathfrak{C}L \text{ is a frame}$$

and that the infima in $\mathfrak{C}L$ are the intersections (see [10] or [11]; see also Birkhoff [5] VI.4).

Nuclei: A *nucleus* on a frame L is a monotone mapping $\nu : L \to L$ such that $x \leq \nu(x)$, $\nu(\nu(x)) = \nu(x)$ and $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$.

Sublocales: A *sublocale* of a frame L is a subset $S \subseteq L$ such that

1. $M \subseteq S$ implies $\bigwedge M \in S$, and
2. if $a \in L$ and $s \in S$ then $a \to s \in S$.

Sublocales of L ordered by inclusion constitute a co-frame $\mathcal{S}(L)$. The infima in $\mathcal{S}(L)$ are intersections, and the suprema are given by the formula

$$\bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\}.$$

There are natural invertible correspondences between these representations: With a sublocale S we have the associated nucleus

$$\nu(a) = \nu_S(a) = \bigwedge\{s \mid s \in S,\ a \leq s\}$$

and a nucleus ν yields a sublocale $S = \nu[L]$. A congruence θ generates the nucleus $\nu(a) = \bigvee\{x \mid x\theta a\}$ and a nucleus ν yields a congruence $\{(x,y) \mid \nu(x) = \nu(y)\}$. The resulting correspondence between congruences and sublocales is contravariant, that is, the inclusion is inverted, and the joins respectively meets in $\mathfrak{C}L$ correspond to meets respectively joins in $\mathcal{S}(L)$.

It should be noted that sublocales are subobjects in a particularly natural sense. When considering the *category of locales* dual to **Frm** (to obtain a covariant generalization of spaces) with the localic maps represented as right Galois adjoints of frame homomorphisms, then the sublocales are precisely the subsets embedded by extremally monomorphic localic maps.

1.4 Open and closed congruences and sublocales. For an element $a \in L$ we have the following special (frame) congruences

$$\Delta_a = \{(x,y) \mid a \wedge x = a \wedge y\} \quad \text{(the open congruences), and}$$
$$\nabla_a = \{(x,y) \mid a \vee x = a \vee y\} \quad \text{(the closed congruences).}$$

They are complements of each other in the frame $\mathfrak{C}L$, and correspond to the open respectively closed sublocales

$$\mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\} \quad \text{respectively} \quad \mathfrak{c}(a) = {\uparrow}a.$$

Needless to say, they are complements of each other in $\mathcal{S}(L)$, they naturally extend the concepts of open and closed subspace of a space, and correspond to the open and closed parts of Isbell's pioneering paper [9].

We have

$$\mathfrak{o}(0) = \{1\}, \mathfrak{o}(1) = L, \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b) \quad \text{and} \quad \mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i),$$

and if $a \neq b$ then $\mathfrak{o}(a) \neq \mathfrak{o}(b)$ (and $\mathfrak{c}(a) \neq \mathfrak{c}(b)$).

1.4.1. It is a well known fact that for every congruence θ

$$\theta = \bigvee\{\Delta_a \cap \nabla_b \mid a\theta b\} = \bigvee\{\Delta_a \cap \nabla_b \mid a\theta b,\ a < b\}$$

and correspondingly for $S \in \mathcal{S}(L)$

$$S = \bigwedge\{\mathfrak{o}(a) \vee \mathfrak{c}(b) \mid \nu_S(a) = \nu_S(b)\} = \bigwedge\{\mathfrak{o}(a) \vee \mathfrak{c}(b) \mid \nu_S(a) = \nu_S(b),\ a < b\}.$$

1.5 Pseudocomplements and complements. The *pseudocomplement* of an a in a lattice L, that is the (unique, if it exists) b such that $x \wedge a = 0$ if and only if $x \leq b$, will be denoted by a^*. In a frame, $a^* = a \to 0$.

A *complement* of a is a b such that $a \vee b = 1$ and $a \wedge b = 0$. In a distributive lattice each complement is a pseudocomplement and hence it is uniquely determined, and we can use the symbol a^* again. A complement of a does not have to exist; if it does we speak of a *complemented* element a.

Provided all the symbols make sense we have the DeMorgan formula

$$(\bigvee a_i)^* = \bigwedge a_i^*. \tag{1.5.1}$$

In $\mathcal{S}(L)$ we will also use the symbol S^* for the complement of (a complemented) S. But we have to keep in mind that here it is not a special case of a pseudocomplement (what we said above holds in a co-frame for the *supplement*, the smallest b such that $a \vee b = 1$). But a complement is both a pseudocomplement and a supplement and hence for *complemented* sublocales we have

$$(\bigcap S_i)^* = \bigvee S_i^* \tag{1.5.2}$$

while $(\bigvee S_i)^* = \bigcap S_i^*$ does not hold generally.

For more about frames see, for example, [10] or [11].

2 Preliminaries II: Prime, completely prime, and spatialities

2.1 In a poset we write $a \triangleleft b$ if $a < b$ and $a \le x \le b$ implies that either $a = x$ or $x = b$.

In the sequel, the posets L will always be complete distributive lattices.

2.2 An element p of L is *prime* (or, *meet-irreducible*) if

$$a \wedge b = p \quad \Rightarrow \quad a = p \text{ or } b = p.$$

It is *completely prime* if for any $A \subseteq L$,

$$\bigwedge A = p \quad \Rightarrow \quad \exists a \in A,\ p = a.$$

It is *maximal* if $p < 1$ and if $p < x$ only for $x = 1$.

Obviously,

$$\text{maximal} \quad \Rightarrow \quad \text{completely prime} \quad \Rightarrow \quad \text{prime}$$

and none of the implications can be reversed.

2.3 Proposition. *The following statements are equivalent*:

(1) *p is completely prime,*

(2) *there is precisely one q such that $p \triangleleft q$,*

(3) *p is prime and there exists q such that $p \triangleleft q$.*

Proof. (1)⇒(2): Consider $q = \bigwedge\{x \mid p < x\}$. Then by the complete primeness $p < q$.

(2)⇒(1): If $a > p$ for all $a \in A$ then $\bigwedge A \ge q > p$.

Now evidently (1)≡(2) implies (3). Finally,

(3)⇒(2): If $p \triangleleft q_1, q_2$ and p is prime then $p \le q_1 \wedge q_2 \le q_i$ and since by primeness $p \ne q_1 \wedge q_2$ we have $q_1 \wedge q_2 = q_i$. □

2.3.1. For a completely prime p, the uniquely determined q with $p \triangleleft q$ will be denoted by

$$p^+.$$

2.3.2. A completely prime p is not necessarily maximal, but we have the following (certainly well known) observation.

Fact. *We have* $(p^+)^* = 0$ (*and hence* $(p^+)^{**} = 1$).

Proof. $p^+ \leq p \vee (p^+)^*$ would make $p^+ = p^+ \wedge p^+ = (p \wedge p^+) \vee ((p^+)^* \wedge p^+) = p$. Hence necessarily $p = p \vee (p^+)^*$ so that $(p^+)^* \leq p < p^+$ and $(p^+)^* \wedge (p^+)^* \leq p^+ \wedge (p^+)^* = 0$. □

2.4 One-point sublocales. Each sublocale contains $1 = \bigwedge \emptyset$. Thus, the empty subspace is modelled by the smallest sublocale $\mathbb{O} = \{1\}$ ("the empty sublocale") and the next smallest are the *one-point sublocales* (briefly, *points*) $\{p, 1\}$ with $p \neq 1$ (necessarily) prime.

If p is completely prime respectively maximal we speak of $\{p, 1\}$ as of a T_D*-point* respectively T_1*-point.*

2.4.1. Proposition. *If a one-point sublocale* $\{p, 1\}$ *is complemented then* p *is completely prime.*

Proof. Let S be the complement. We have $S \cap \{p, 1\} = \{1\}$ and hence $p \notin S$. Since $S \vee \{p, 1\} = L$ we have $M = \{x \mid x > p\} \subseteq S$ (because no $x > p$ can be obtained as $y \wedge p$) and hence $\bigwedge M \in S$, so that $\bigwedge M \neq p$ and p^+ exists. □

2.4.2. Note. The reverse implication also holds, that is, $\{p, 1\}$ with completely prime p is always complemented. Recall the (somewhat surprising) formula for supplement $S^\#$ in $\mathcal{S}(L)$, the smallest T such that $T \vee S = L$,

$$S^\# = \bigvee \{T \mid T \cap S = \mathbb{O}\}$$

(see [11], VI,4.5.2). In general it is not (also) a pseudocomplement, because in the co-frame $\mathcal{S}(L)$ the meet $\cap$ does not generally distribute over $\bigvee$. But in this concrete case it is: $\bigvee \{T \mid T \cap \{p, 1\} = \mathbb{O}\} \cap \{p, 1\} = \mathbb{O}$ since if we had $p = \bigwedge \{x_T \mid x_T \in T\}$ we would have to have $p = x_T$ for some T.

2.5 If X is a topological space then $\Omega(X) = \{U \mid U \subseteq X,\ U \text{ open}\}$ is a frame. A frame L is *spatial* if there is a space X such that $L \cong \Omega(X)$.

X is a T_D space if for each $x \in X$ there is an open $U \ni x$ such that $U \smallsetminus \{x\}$ is open ([1]). A frame L is T_D*-spatial* respectively T_1*-spatial* if $L \cong \Omega(X)$ with a T_D- respectively T_1-space X.

2.5.1. Proposition. *A frame L is spatial respectively T_D-spatial respectively T_1-spatial if and only if for all $a \in L$, $a = \bigwedge\{p \mid a \le p,\ p$ prime$\}$ respectively $a = \bigwedge\{p \mid a \le p,\ p$ completely prime$\}$ respectively $a = \bigwedge\{p \mid a \le p,\ p$ maximal$\}$.*

(See [4, 10, 12] and also [9].)

2.5.2. From the formula for the join in $\mathcal{S}(L)$ we now obtain

Corollary. L is spatial, T_D-spatial, T_1-spatial, respectively, if and only if

$$L = \bigvee\{P \mid P \text{ a point, } T_D\text{-point, } T_1\text{-point, respectively, in } L\}.$$

2.6 Corollary. (see also [11] VI.3) *Each complemented sublocale of a spatial (T_D-spatial, T_1-spatial) frame L, is spatial (T_D-spatial, T_1-spatial).*

Proof. Let $L = \bigvee \mathcal{P}$ for a set of points $\mathcal{P}$, and let sublocales $S, T \subseteq L$ be complements of each other. Since $S \vee T = L$ we have for each $\{p, 1\} \in \mathcal{P}$ $p = s \wedge t$ with $s \in S$ and $t \in T$, and since p is prime, we have either $p = s \in S$ or $p = t \in T$. Set

$$\mathcal{P}_S = \{\{p,1\} \in \mathcal{P} \mid p \in S\}, \quad \mathcal{P}_T = \{\{p,1\} \in \mathcal{P} \mid p \in T\}.$$

Then $\bigvee \mathcal{P}_S \vee \bigvee \mathcal{P}_T = \bigvee \mathcal{P} = L$, $\bigvee \mathcal{P}_S \subseteq S$, $\bigvee \mathcal{P}_T \subseteq T$, and $S \cap T = \{1\}$. We conclude that $\bigvee \mathcal{P}_S = S$ and $\bigvee \mathcal{P}_T = T$. $\square$

3 C-connected congruences

3.1 A congruence θ on a a frame L is said to be *connected* (Chen [6]) if the associated sublocale S of L is a connected frame. We will be, however, interested in the connectedness of the congruences as elements of the congruence frame $\mathfrak{C}L$. Thus, to avoid confusion with the terminology of [6], we will speak of *C-connected* (short for "connected as an element of the congruence lattice") θ if

$$\theta = \theta_1 \vee \theta_2 \ \text{ and } \ \theta_1 \cap \theta_2 = \mathsf{O} \ \text{ implies that either } \ \theta_1 = \mathsf{O} \ \text{ or } \ \theta_2 = \mathsf{O}.$$

The following result will be markedly refined in Theorem 3.6.

3.2 Proposition. *A non-trivial C-connected congruence θ is of the form*

$$\Delta_a \cap \nabla_b \quad \textit{with} \quad a < b.$$

Proof. We have (recall 1.4.1) $\theta = \bigvee\{\Delta_a \cap \nabla_b \mid (a,b) \in \theta, a < b\}$. Suppose that for some individual $C = \Delta_{a_0} \cap \nabla_{b_0}$ we have $D = \bigvee\{\Delta_a \cap \nabla_b \mid (a_0, b_0) \neq (a,b) \in \theta, a < b\}$ non-trivial. C is complemented and hence we obtain from $\theta = C \vee D$ a disjoint $\theta = C \vee (C^* \cup D)$. $\square$

3.3 Lemma. *Let θ be C-connected. Then $\theta = \Delta_a \cap \nabla_b$ for every $(a,b) \in \theta$ such that $a < b$.*

Proof. Consider an arbitrary pair $(a,b) \in \theta$ with $a < b$. We have $\theta = ((\Delta_a \cap \nabla_b) \cap \theta) \vee ((\Delta_a \cap \nabla_b)^* \cap \theta)$ and since θ is C-connected and $\Delta_a \cap \nabla_b \subseteq \theta$ we conclude that the second summand is trivial, and hence $\Delta_a \cap \nabla_b = \theta$. $\square$

3.4 Proposition. *A congruence class of a C-connected congruence θ contains at most two elements.*

Consequently, if a non-trivial C-connected θ is represented as $\Delta_a \cap \nabla_b$ then $a \lhd b$, and by Lemma 3.3 we have $u \lhd v$ for any $(u,v) \in \theta$, $u < v$.

Proof. Let C be a θ-congruence class with at least three elements. Set $c = \bigvee C$. There exist $a, b \in C$, $a \neq b$, $a, b < c$. Hence we have (in the frame $\mathfrak{C}L$) $\Delta_c \subseteq \Delta_a, \Delta_b$ and, by Lemma 3.3, $\theta = \Delta_a \cap \nabla_c = \Delta_b \cap \nabla_c$. Then

$$\begin{aligned}\Delta_a &= \Delta_a \cap (\Delta_c \vee \nabla_c) = (\Delta_a \cap \Delta_c) \vee (\Delta_a \cap \nabla_c) = \Delta_c \vee (\Delta_b \cap \nabla_c) = \\ &= (\Delta_b \cap \Delta_c) \vee (\Delta_b \cap \nabla_c) = \Delta_b \cap (\Delta_c \vee \nabla_c) = \Delta_b\end{aligned}$$

so that $a = b$, a contradiction.

As for the second statement: if $a \leq x, y \leq b$ we have $a \wedge x = a = a \wedge y$ and $b \vee x = b = b \vee y$ and hence $x\theta y$. $\square$

From now on we will extensively use the sublocales associated with the congruences in question.

3.5 Proposition. *Let $a \lhd b$. Then the complement S^* of $S = \mathfrak{o}(a) \vee \mathfrak{c}(b)$ is the one-point sublocale*

$$P = S^* = \{b \to a, 1\}.$$

In particular $p = b \to a$ is completely prime. Furthermore we have $p^+ \to p = p$ and, since $\{p, 1\} = S^*$, a canonical representation

$$S = \mathfrak{o}(p) \vee \mathfrak{c}(p^+)$$

with a uniquely determined completely prime p.

Proof. $S^* = \mathfrak{c}(a) \wedge \mathfrak{o}(b)$ and hence $x \in S^*$ if and only if $x \geq a$ and $b \to x = x$. Consequently
- either $x \geq b$ and then $x \in \mathfrak{c}(b) \wedge \mathfrak{o}(b) = \mathbb{O} = \{1\}$
- or $x \ngeq b$.

In the latter case, $a \leq x, b$ and $a \leq x \wedge b < b$ and hence $a = x \wedge b$. Thus,

$$b \to a = b \to (x \wedge b) = (b \to x) \wedge (b \to b) = x \wedge 1 = x$$

so that $S^* = \{b \to a, 1\}$, a one-point sublocale. Now (recall 2.4), one-point sublocales are precisely the $\{p, 1\}$ with p prime, and this one is complemented and hence p is completely prime (see Proposition 2.4.1.).

For the last statement it suffices to prove that for $p = b \to a$, $p = p^+ \to p$. But this is immediate:

$$x \leq p^+ \to p \quad \text{if and only if} \quad x \wedge p^+ \leq p \quad \text{if and only if} \quad x \leq p$$

since p is prime. $\square$

3.5.1. Remark. Note that the requirement that $P = \{p, 1\}$ being complemented is essential. The fact that p is prime and that p can be written non-trivially as $b \to a$ does not make the point automatically complemented. See the unit interval $L = \langle 0, 1\rangle$. Here every p with $0 < p < 1$ is prime, and for any b with $p < b < 1$ we have $p = b \to p$, while none of such $\{p, 1\}$ is complemented.

3.6 Theorem. *The non-trivial C-connected congruences θ are precisely the $\Delta_a \cap \nabla_b$ with $a \triangleleft b$.*

Proof. After Proposition 3.4 it remains to be proved that for any $a \triangleleft b$ in L the congruence $\theta = \Delta_a \cap \nabla_b$ is C-connected. Using the sublocale reasoning: we have the sublocale $S = \mathfrak{o}(a) \vee \mathfrak{c}(b)$ (with $a \triangleleft b$) associated with θ and want to prove that

$$\text{if } \ S = T \cap U \ \text{ and } \ T \vee U = L \ \text{ then either } \ T = S \ \text{ or } \ U = S. \qquad (*)$$

First realize that all the S, T, U are complemented ($T \vee (U \cap S^*) = (T \vee U) \cap (T \vee S^*) = L$ and $T \cap U \cap S^* = S \cap S^* = \mathbb{O}$, so that $T^* = U \cap S^*$ and similarly $U^* = T \cap S^*$) and hence the complement transforms the formulas in $(*)$ to

$$S^* = T^* \vee U^* \quad \text{and} \quad T^* \cap U^* = \mathbb{O} = \{1\}.$$

By Proposition 3.5, $S^* = \{p, 1\}$. Since $S^* \supseteq T^*, U^*$, and $T^* \cap U^* = \{1\}$, one of these sublocales, say T^*, is $\{p, 1\}$ and the other, say U^*, is $\mathbb{O}$ and then $U = L$. $\square$

3.7.1. Lemma. *For every non-trivial C-connected congruence θ there is a completely prime p such that*

$$(a < b \textit{ and } a\,\theta\, b) \quad \textit{if and only if} \quad (a \neq b,\ a \leq p,\ b \leq p^+ \textit{ and } a = b \wedge p).$$

Proof. By Theorem 3.6, $\theta = \Delta_a \cap \nabla_b$ for some $a \triangleleft b$, and by Proposition 3.5, $\Delta_a \cap \nabla_b = \Delta_p \cap \nabla_{p^+}$ with $p = b \to a$, and p is completely prime.

Let $a < b$ and $a\theta b$. Then by Lemma 3.3 and Proposition 3.4 we can apply Proposition 3.5 on $\Delta_a \cap \nabla_b$ for this particular pair a, b. Now $a \leq b \to a = p$ and since $(a, b) \in \Delta_p \cap \nabla_{p^+}$ we have in particular $p^+ = p^+ \vee a = p^+ \vee b$, and hence $b \leq p^+$, $a = p \wedge a = p \wedge b$.

On the other hand, let $a \leq p,\ b \leq p^+$ and $a = b \wedge p$. Then $a < b$ (since $a \neq b$). We have $p\,\theta\, p^+$ and hence $a = (b \wedge p)\,\theta\,(b \wedge p^+) = b$. $\square$

3.7.2. Theorem. *The assignment*

$$p \quad \mapsto \quad \Delta_p \cap \nabla_{p^+}$$

constitutes a one-to-one correspondence between completely prime elements in L and non-trivial C-connected congruences on L. The formula for the congruence associated with p is explicitly given by

$$\{(a, b) \mid a = b \textit{ or } (a \neq b,\ a, b \leq p^+ \textit{ and either } a = p \wedge b \textit{ or } b = p \wedge a)\}.$$

Proof. By Proposition 3.5 and Theorem 3.6 the correspondence is onto and one-to-one.

The formula follows from Lemma 3.7.1 because if $a \neq b$ are congruent we have a, b and $a \wedge b$ congruent and since a congruence class cannot have three distinct elements we have either $a < b$ or $b < a$. $\square$

4 Locally C-connected frames

4.1 A frame is *locally C-connected* if each congruence on L is a join of C-connected congruences. Using 1.4.1 and Proposition 3.5 we immediately obtain, in the language of sublocales, the following translation.

Proposition. *A frame L is locally C-connected if and only if for every $a < b$ in L*

$$\mathfrak{o}(a) \vee \mathfrak{c}(b) = \bigcap_{i \in J} \mathfrak{o}(p_i) \vee \mathfrak{c}(p_i^+)$$

for some set $(p_i)_{i \in J}$ of completely prime elements.

4.2 Note and Example. One might surmise, at the first sight, that the factors on the right hand side cover the open and closed parts on the right hand side separately, or, more generally, that if $a < b$ and $\mathfrak{o}(a) \vee \mathfrak{c}(b) \subseteq \mathfrak{o}(u) \vee \mathfrak{c}(v)$ with $u \triangleleft v$ then $a \leq u \triangleleft v \leq b$. This is not the case, not even in the Boolean L. Consider L the Boolean algebra of all the subsets of $\{0, 1, 2, 3, 4, 5\}$. We can represent the open sublocales $\mathfrak{o}(x)$ by the $x \in L$ (that is, $x \subseteq \{0, 1, 2, 3, 4, 5\}$) themselves, and the closed sublocales $\mathfrak{c}(x)$ by the complements $L \smallsetminus x$. Consider

$$a = \mathfrak{o}(a) = \{0, 1\},\ b = \mathfrak{o}(b) = \{0, 1, 2, 3\} \text{ so that } \mathfrak{c}(b) = \{4, 5\}, \text{ and}$$
$$u = \mathfrak{o}(u) = \{1, 5\},\ v = \mathfrak{o}(v) = \{1, 2, 5\} \text{ so that } \mathfrak{c}(v) = \{0, 3, 4\}$$

in which case $\mathfrak{o}(a) \vee \mathfrak{c}(b) \subseteq \mathfrak{o}(u) \vee \mathfrak{c}(v)$ while neither $a \leq u$ nor $v \leq b$.

4.3 Using the DeMorgan formula and 3.5 we obtain

Proposition. *If a frame L is locally C-connected then for every $a < b$ in L*

$$\mathfrak{c}(a) \cap \mathfrak{o}(b) = \bigvee_{i \in J} \{p_i, 1\}$$

for some set $(p_i)_{i \in J}$ of completely prime elements.

4.4 Proposition. *A locally C-connected frame L is T_D-spatial.*

Proof. This is an immediate consequence of 4.3 and the formula for the join in $\mathcal{S}(L)$. Take an arbitrary $a \neq 1$ and set $b = 1$. Then $a \in \mathfrak{c}(a) \cap \mathfrak{o}(b)$ and hence it can be obtained as $\bigwedge_i p_i$ for some set $(p_i)_{i\in J}$ of completely prime elements. $\square$

4.5 Note. The formula in 4.3 is not in general sufficient. Note that the reverse DeMorgan formula does not hold. The reader may wonder whether it is not, after all, true in this special case where we have the sublocales in question complemented (having in mind the linearity of such elements, see [9], [11] VI.4.4). But we do not know whether $\bigcap_{i\in J} \mathfrak{o}(p_i) \vee \mathfrak{c}(p_i^+)$ is complemented, which is essential. See also 5.3 below.

5 Some special cases

5.1 Scattered frames. A frame L is *scattered* if $\mathcal{S}(L)$ is a frame (which in fact makes $\mathcal{S}(L)$ a Boolean algebra) – see [2, 7, 13]. Thus we have here both DeMorgan rules and immediately obtain from 4.1 the following.

5.1.1. Proposition. *A scattered frame L is locally C-connected if and only if for every $a < b$ in L*

$$\mathfrak{c}(a) \cap \mathfrak{o}(b) = \bigvee_{i\in J} \{p_i, 1\}$$

for some set $(p_i)_{i\in J}$ of completely prime elements.

Since $\mathfrak{c}(a) \cap \mathfrak{o}(b)$ are complemented we now obtain from 2.5.2 and Corollary 2.6

5.1.2. Proposition. *A scattered frame L is locally C-connected if and only if it is T_D-spatial.*

5.2 One of the extremes: the Boolean case.

Proposition. *A Boolean frame is locally C-connected if and only if it is spatial (that is, if and only if it is atomic, or if and only if it is isomorphic to the lattice of all subsets of a set, or if and only if it is T_1-spatial).*

Here the T_1-spatiality is automatic, but just in case this might be of use in a slightly more general case, recall the observation 2.3.2.

5.3 The other extreme: the linear case. If L is linearly ordered then we have

$$x \to y = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{if } x > y. \end{cases}$$

Consequently,

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \mid x < a\} \cup \{1\}$$

and hence for $a < b$

$$\begin{aligned} \mathfrak{o}(a) \vee \mathfrak{c}(b) &= \{x \mid x < a \text{ or } x \ge b\}, \\ \mathfrak{c}(a) \cap \mathfrak{o}(b) &= \{x \mid a \le x < b\} \cup \{1\} = [a,b)_L \cup \{1\}. \end{aligned} \tag{$*$}$$

Note that for the sublocales S from $(*)$ we have

$$S^* = (L \smallsetminus S) \cup \{1\}$$

and hence

$$\mathfrak{o}(a) \vee \mathfrak{c}(b) = \bigcap_{i \in J} \mathfrak{o}(a_i) \vee \mathfrak{c}(b_i) \quad \text{if and only if} \quad [a,b)_L = \bigcup_{i \in J} [a_i, b_i)_L. \tag{$**$}$$

5.3.1. Proposition. *A linearly ordered frame L is locally C-connected if and only if it is well-ordered (which is to say, if and only if each $a \in L$, $a < 1$, is completely prime).*

Proof. $(\Rightarrow)$: If L is locally C-connected then by 4.1, in particular $\{1\} = \mathfrak{o}(0) \vee \mathfrak{c}(1) = \bigcap \{\mathfrak{o}(p) \vee \mathfrak{c}(p^+) \mid p \in \mathcal{P}\}$ for some set $\mathcal{P}$ of completely prime elements, and hence by $(**)$,

$$L \smallsetminus \{1\} = [0,1)_L = \bigcup \{[p,p^+)_L \mid p \in \mathcal{P}\} = \mathcal{P}$$

since $[p,p^+)_L = \{p\}$. Let $M \subseteq L$ be non-empty. Set $a = \bigwedge M$. Then $a \in M$: if $a = 1$ then $M = \{1\}$ (it is not empty), and if $a < 1$ it is completely prime. Hence, a is the least element of M.

$(\Leftarrow)$: On the other hand, if L is well-ordered then each $a \in L$, $a < 1$, is completely prime by Proposition 2.3 (consider the smallest element of $\{x \mid a < x\}$). Then $[a,b)_L = \bigcup \{\{p\} \mid p \in [a,b)_L\} = \bigcup \{[p,p^+)_L \mid p \in [a,b)_L\}$ and L is locally C-connected by $(**)$ and 4.1. $\square$

5.4 The Noetherian case. An element c of a frame is *compact* if $c \leq \bigvee_{i \in J} x_i$ implies that $c \leq \bigvee_{i \in K} x_i$ for a finite $K \subseteq J$. A frame is *Noetherian* if each of its elements is compact.

In the special case that L is Noetherian, the structure of the congruence frame $\mathfrak{C}L$ is much simpler. By [3] one has in particular that the following statements are equivalent:

(N1) *L is Noetherian.*

(N2) *The congruence frame $\mathfrak{C}L$ is compact.*

(N3) *Every lattice congruence on L is a frame congruence.*

(N4) *The complemented elements of $\mathfrak{C}L$ are precisely the compact ones.*

Proposition. *If L is Noetherian then it is locally C-connected if and only if for every $a < b$ in L there is a finite sequence $a = a_1 \triangleleft a_2 \triangleleft \cdots \triangleleft a_n = b$.*

Proof. Here it will be easier to work in the language of congruences than in that of sublocales.

($\Rightarrow$): Take $a < b$. By (N4), $\Delta_a \cap \nabla_b$ is compact and hence it is a finite join $\bigvee_{j=1}^m \theta_j$ of C-connected congruences. Recall from [8] that if θ_1, θ_2 are *lattice* congruences then for $a < b$, $(a, b) \in \theta_1 \vee \theta_2$ if and only if there is a sequence $a = a_1 < a_2 < \cdots < a_n = b$ with $(a_i, a_{i+1}) \in \theta_{k_i}$ for suitable $k_i \in \{1, 2\}$. Using (N3) we can extend this for our join $\bigvee_{j=1}^m \theta_j$ stating that there is a sequence $a = a_1 < a_2 < \cdots < a_n = b$ with $(a_i, a_{i+1}) \in \theta_{k_i}$ for suitable $k_i \in \{1, , \ldots, m\}$. By Proposition 3.4, since $a_i < a_{i+1}$ and θ_j are C-connected, we have $a_i \triangleleft a_{i+1}$.

($\Leftarrow$): Any $\theta \in \mathfrak{C}L$ is a join of congruences $\Delta_a \cap \nabla_b$ with $a < b$. For an individual such $\Delta_a \cap \nabla_b$ take a sequence $a = a_1 \triangleleft a_2 \triangleleft \cdots \triangleleft a_n = b$.

It is easy to see that $\Delta_u \cap \nabla_v$ with $u < v$ is the smallest congruence containing (u, v). Consequently $\bigvee_{i=1}^{n-1} \Delta_{a_i} \cap \nabla_{a_{i+1}}$ is the smallest congruence containing all (a_i, a_{i+1}), $1 \leq i \leq n-1$. Now if a congruence contains (a, b) then it contains all the (x, y) with $a \leq x, y \leq b$; on the other hand, if θ contains all the (a_i, a_{i+1}), $1 \leq i \leq n-1$, we have $a \theta a_2 \theta \cdots \theta a_{n-1} \theta b$. Thus, $\Delta_a \cap \nabla_b = \bigvee_{i=1}^{n-1} \Delta_{a_i} \cap \nabla_{a_{i+1}}$. $\square$

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