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On zero divisor graph of unique product monoid rings over Noetherian reversible ring

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Abstract. Let R be an associative ring with identity and $Z^*(R)$ be its set of non-zero zero divisors. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the graph whose vertices are the non-zero zero-divisors of R, and two distinct vertices r and s are adjacent if and only if $rs = 0$ or $sr = 0$. In this paper, we bring some results about undirected zero-divisor graph of a monoid ring over reversible right (or left) Noetherian ring R. We essentially classify the diameter-structure of this graph and show that $0 \leq \text{diam}(\Gamma(R)) \leq \text{diam}(\Gamma(R[M])) \leq 3$. Moreover, we give a characterization for the possible $\text{diam}(\Gamma(R))$ and $\text{diam}(\Gamma(R[M]))$, when R is a reversible Noetherian ring and M is a u.p.-monoid. Also, we study relations between the girth of $\Gamma(R)$ and that of $\Gamma(R[M])$.

1 Introduction and definitions

The concept of a zero-divisor graph of a commutative ring was first introduced by Beck in [8]. In his work all elements of the ring were vertices of the

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graph. Inspired by his study, Anderson and Livingston [4], redifined and studied the zero-divisor graph whose vertices are the non-zero zero-divisors of a ring. Several results concerning zero-divisor graph of a commutative ring R are given in [4].

Recently Redmond, in [30], has extended this concept to arbitrary rings. Let R be an associative ring with identity. The set *zero-divisors* of R, denoted by $Z(R)$, is the set of elements $a \in R$ such that there exists a non-zero element $b \in R$ with $ab = 0$ or $ba = 0$. Set $Z^*(R) = Z(R) \setminus \{0\}$. Redmond, in [30], considered an undirected zero-divisor graph of a non-commutative ring R, the graph $\Gamma(R)$, with vertices in the set $Z^*(R)$ and such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$ or $ba = 0$. Several papers are devoted to studying the relationship between the zero-divisor graph and algebraic properties of rings (cf. [1, 2, 4–6, 8, 24, 30]). Using the notion of a zero-divisor graph, it has been proven in [31] that for any finite ring R, the sum $\sum_{x\in R} |r_R(x) - l_R(x)|$ is even, where $r_R(x)$ and $l_R(x)$ denote the right and left annihilator of the element x in R, respectively.

Recall that a graph is said to be *connected* if for each pair of distinct vertices u and v, there is a finite sequence of distinct vertices $v_1 = u, v_2, \ldots, v_n = v$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a *path* and for two distinct vertices a and b in the simple (undirected) graph Γ, the *distance* between a and b, denoted by $d(a, b)$, is the length of a shortest path connecting a and b, if such a path exists; otherwise we put $d(a, b) = \infty$. Recall that the *diameter* of a graph Γ is defined as follows:

$$\mathrm{diam}(\Gamma) = \sup\{d(a, b)|\ a \text{ and } b \text{ are distinct vertices of } \Gamma\}.$$

The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is *complete*; i.e., each pair of distinct vertices forms an edge. The *girth* of a graph Γ, denoted by $\mathrm{g}r(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise, $\mathrm{g}r(\Gamma) = \infty$.

There is considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. The first such extensions that come to mind are those of polynomial and power series extensions. Axtell, Coykendall and Stickles [7] examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings. Also, Lucas [24] continued the study of the diameter of zero-divisor graphs of polynomial and power series rings over commutative rings. Moreover, Anderson and Mulay [5], studied the girth and diameter of zero-divisor graph of a commutative ring and investigated the girth and diameter of zero-divisor graphs of polynomial and power series rings over commutative rings. Since we are especially interested in non-commutative aspects of the theory, we do not wish to restrict ourselves to commutative cases. In this paper, we extend to non-commutative rings many results known earlier for zero-divisor graphs of commutative rings. Nevertheless, we wish to emphasize that some of our results about zero-divisor graph of a monoid rings over non-commutative ring are new even for commutative rings.

According to Cohn [11], a ring R is called *reversible* if $ab = 0$ implies that $ba = 0$ for each $a, b \in R$. Anderson and Camillo [3], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa and Niewieczerzal [21] took the term C_0 for it. Clearly, *reduced* rings (i.e., rings with no non-zero nilpotent elements) and commutative rings are reversible. Kim and Lee [20], studied extensions of reversible rings and showed that polynomial rings over reversible rings need not be reversible in general.

In this paper, first we prove some results about zero divisors of a reversible ring R. Then for a u.p.-monoid M, we show that $0 \leq \operatorname{diam}(\Gamma(R)) \leq \operatorname{diam}(\Gamma(R[M])) \leq 3$, when R is a reversible and right or left Noetherian ring. Also we give a characterization for the possible $\operatorname{diam}(\Gamma(R))$ and $\operatorname{diam}(\Gamma(R[M]))$, when R is a Noetherian reversible ring and M is a u.p.-monoid. In closing, we give some relations between the girth of $\Gamma(R)$ and that of $\Gamma(R[M])$.

Throughout, R denotes an associative ring with identity, unless otherwise indicated. For $X \subseteq R$, the ideal generated by X is denoted by $\langle X \rangle$. For an element $a \in R$, let $\ell_R(a) = \{b \in R \mid ba = 0\}$ and $r_R(a) = \{b \in R \mid ab = 0\}$. Note that if R is a reversible ring and $a \in R$, then $\ell_R(a) = r_R(a)$ is an ideal of R, and we denote it by $ann(a)$. We write $Z_\ell(R)$, $Z_r(R)$ and $Z(R)$ for the set of all left zero-divisors of R, the set of all right zero-divisors of R and the set $Z_\ell(R) \cup Z_r(R)$, respectively. For a non-empty subset S of R, $\ell_R(S)$ and $r_R(S)$ denote the left annihilator and the right annihilator of S in R, respectively. Note that if R is a reversible ring, then $Z_\ell(R) = Z_r(R) = Z(R)$.

2 Zero divisors of reversible rings

In this section, we study the structure of zero divisors of reversible rings. Recall that a ring R is *abelian* if every idempotent of R is central. Since reversible rings are abelian, hence by a similar way as used in the proof of [4, Theorem 2.5], one can prove the following result.

Remark 2.1. Let R be a reversible ring. Then there exists a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times D$ where D is a domain or $Z(R)$ is an annihilator ideal.

By using Remark 2.1 and a similar way as used in the proof of [4, Theorem 2.8], one can prove the following result.

Remark 2.2. Let R be a reversible ring. Then $\Gamma(R)$ is complete if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.

Recall that an ideal $\mathcal{P}$ of R is *completely prime* if $ab \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$ for $a, b \in R$.

Proposition 2.3. *Let R be a reversible ring and $\mathfrak{A} = \{ann(a) \mid 0 \neq a \in R\}$. If $\mathcal{P}$ is a maximal element of $\mathfrak{A}$, then $\mathcal{P}$ is a completely prime ideal of R.*

Proof. Let $xy \in \mathcal{P} = ann(a)$ and $x \notin \mathcal{P}$. Then $xa \neq 0$, and hence $ann(ax) \in \mathfrak{A}$. Since $\mathcal{P} \subseteq ann(xa)$ and $\mathcal{P}$ is a maximal element of $\mathfrak{A}$, so $ann(a) = \mathcal{P} = ann(ax)$. Since $axy = 0$, we have $ay = 0$, which implies that $y \in \mathcal{P}$. Therefore $\mathcal{P}$ is a completely prime ideal of R. □

Proposition 2.4. *Let R be a reversible ring. Then $\Gamma(R)$ is a connected graph with $diam(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $\mathrm{g}r(\Gamma(R)) \leq 4$.*

Proof. By a similar way as used in the proof of [4, Theorem 2.3], one can show that $\mathrm{diam}(\Gamma(R)) \leq 3$. Also, by [30, Theorem 2.4], $\mathrm{g}r(\Gamma(R)) \leq 4$. □

By a similar argument as used in the proof of [4, Theorem 2.2] one can prove the following theorem.

Theorem 2.5. *Let R be a reversible ring. Then $\Gamma(R)$ is finite if and only if either R is finite or a domain.*

Lemma 2.6. [14, Lemma 3.1] *Let R be a reversible ring. Then $Z(R)$ is an union of prime ideals.*

Hence the collection of zero-divisors of a reversible ring R is the set-theoretic union of prime ideals. We write $Z(R) = \cup_{i\in\Lambda}\mathcal{P}_i$ with each $\mathcal{P}_i$ prime. We will also assume that these primes are maximal with respect to being contained in $Z(R)$.

For a reversible ring R, $r_R(a)$ is an ideal of R for each $a \in R$. Hence by a similar way as used in the proof of [19, Theorem 80] one can prove the following result.

Remark 2.7. Let R be a reversible and right or left Noetherian ring. Then we have $Z(R) = \cup_{i\in\Lambda}\mathcal{P}_i$, where Λ is a finite set and each $\mathcal{P}_i$ is an annihilator of a non-zero element in $Z(R)$.

Kaplansky [19, Theorem 81] proved that if R is a commutative ring, $J_1, \ldots, J_n$ a finite number of ideals in R, S a subring of R that is contained in the set-theoretic union $J_1 \cup \cdots \cup J_n$, and at least $n-2$ of the J_k's are prime, then S is contained in some J_k. Here we have the following theorem, which is proved in [14, Theorem 3.3].

Theorem 2.8. *Let R be a reversible ring and $Z(R) = \cup_{i\in\Lambda}\mathcal{P}_i$. If Λ is a finite set (in particular if R is left or right Noetherian), and I an ideal of R that is contained in $Z(R)$, then $I \subseteq \mathcal{P}_k$, for some k.*

Note that Remark 2.7 shows that any left or right Noetherian ring satisfies the hypothesis of the theorem.

Corollary 2.9. *Let R be a reversible and left or right Noetherian ring. Let $\mathcal{P}$ be a prime ideal of R which is maximal with respect to being contained in $Z(R)$. Then $\mathcal{P}$ is completely prime and $\mathcal{P} = ann(a)$ for some $a \in R$.*

Proof. It follows from Remark 2.7 and Theorem 2.8. $\square$

Now, by using Theorem 2.8 in conjunction with a method similar to that used in the proof of [7, Corollary 3.5], one can prove the following result.

Corollary 2.10. *Let R be a reversible ring with diam*$(\Gamma(R)) \leq 2$ *and* $Z(R) = \cup_{i\in\Lambda}\mathcal{P}_i$. *If Λ is a finite set (in particular if R is left or right Noetherian), then $|\Lambda| \leq 2$.*

Proposition 2.11. *Let R be a reversible ring with diam*$(\Gamma(R)) = 2$. *If $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ with $\mathcal{P}_1$ and $\mathcal{P}_2$ distinct maximal primes in $Z(R)$, then:*

(1) $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$ *(in particular, for all $x \in \mathcal{P}_1$ and $y \in \mathcal{P}_2$, $xy = 0$).*

(2) *$\mathcal{P}_1$ and $\mathcal{P}_2$ are completely prime ideals of R.*

Proof. (1) By a similar way as used in the proof of [7, Proposition 3.6] one can prove it.

(2) Since $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$, hence $\mathcal{P}_1 = ann(x)$ and $\mathcal{P}_2 = ann(y)$, for each $0 \neq x \in \mathcal{P}_2$ and $0 \neq y \in \mathcal{P}_1$. Let $ab \in \mathcal{P}_1$ and $a \notin \mathcal{P}_1$. Then $xa \neq 0$ for some $0 \neq x \in \mathcal{P}_2$. Hence $b \in ann(xa) = ann(x) = \mathcal{P}_1$. □

3 Relations between diameters of $\Gamma(R)$ and $\Gamma(R[M])$

In this section, we assume R is a reversible ring with $Z(R) = \cup_{i=1}^{n}\mathcal{P}_i$ such that these primes are maximal with respect to being contained in $Z(R)$ and study relation between diam$(\Gamma(R))$ and n. Note that if R is reversible and left or right Noetherian, then $Z(R) = \cup_{i=1}^{n}\mathcal{P}_i$, by Remark 2.7.

Proposition 3.1. *Let R be a reversible ring and $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ with $\mathcal{P}_1 \cap \mathcal{P}_2 \neq 0$. Then diam*$(\Gamma(R)) = 3$.

Proof. We claim that if $ab = 0$ for all $a \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $b \in \mathcal{P}_2 \setminus \mathcal{P}_1$, then $\mathcal{P}_1\mathcal{P}_2 = 0$. Let $x \in \mathcal{P}_1 \cap \mathcal{P}_2$, $a \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $b \in \mathcal{P}_2 \setminus \mathcal{P}_1$. Then $a + x \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $b + x \in \mathcal{P}_2 \setminus \mathcal{P}_1$. Hence $a(b + x) = 0$ and $b(a + x) = 0$, which implies that $bx = 0 = ax$. Now, let $x, y \in \mathcal{P}_1 \cap \mathcal{P}_2$. Then $a + x \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $b + y \in \mathcal{P}_2 \setminus \mathcal{P}_1$, which implies that $0 = (a+x)(b+y) = ab+ay+xb+xy = xy$. Thus $\mathcal{P}_1\mathcal{P}_2 = 0$.

Now let $x \in \mathcal{P}_1 \cap \mathcal{P}_2$. Then $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2 \subseteq ann(x)$, which is a contradiction. Thus there exist $a \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $b \in \mathcal{P}_2 \setminus \mathcal{P}_1$ such that $ab \neq 0$. If t is a mutual non-zero annihilator for a and b, then $a, b \in ann(t)$. Since $ann(t) \subseteq Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$, we have $ann(t) \subseteq \mathcal{P}_1$ or $ann(t) \subseteq \mathcal{P}_2$. Thus $a, b \in \mathcal{P}_1$ or $a, b \in \mathcal{P}_2$, which is a contradiction. Thus $d(a, b) = 3$ and so diam$(\Gamma(R)) = 3$. □

Proposition 3.2. *Let R be a reversible ring and $Z(R) = \cup_{i=1}^{n} \mathcal{P}_i$. If $n \geq 3$, then diam*$(\Gamma(R)) = 3$.

Proof. Let $x \in \mathcal{P}_1 \setminus \cup_{i=1}^{n} \mathcal{P}_i$ and $y \in \mathcal{P}_2 \setminus \cup_{i=1}^{n} \mathcal{P}_i$. We claim that $xy \neq 0$ and x, y don't have mutual non-zero annihilator. If $xy = 0$, then $xRy = 0 \subseteq \mathcal{P}_3$, and so $x \in \mathcal{P}_3$ or $y \in \mathcal{P}_3$, which is a contradiction. If a is a mutual non-zero annihilator for x, y, then $x, y \in ann(a)$. Since $ann(a) \subseteq Z(R) = \cup_{i=1}^{n} \mathcal{P}_i$, we have $ann(a) \subseteq \mathcal{P}_k$, for some k, by Theorem 2.8. Then $x, y \in \mathcal{P}_k$, which is a contradiction. Therefore $d(x, y) = 3$, and so $\mathrm{diam}(\Gamma(R)) = 3$. $\square$

Proposition 3.3. *Let R be a reversible ring and $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$. Then we have diam*$(\Gamma(R)) = 2$ *if and only if* $|Z(R)| \geq 4$ *and* $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$.

Proof. One can prove the forward direction by a similar way as used in the proof [7, Proposition 3.6]. For the backward direction, let $0 \neq a \in \mathcal{P}_1$ and $0 \neq b \in \mathcal{P}_2$. Then $ab \in \mathcal{P}_1 \cap \mathcal{P}_2 = 0$. Since $|Z(R)| \geq 4$, hence $|\mathcal{P}_1| \geq 3$ or $|\mathcal{P}_2| \geq 3$. We can assume that $|\mathcal{P}_1| \geq 3$. If for all $a, b \in \mathcal{P}_1$, $ab = 0$, then $(\mathcal{P}_1 \cup \mathcal{P}_2) \subseteq ann(x)$, for each $0 \neq x \in \mathcal{P}_1$, which is a contradiction. Thus there are $a, b \in \mathcal{P}_1$ with $ab \neq 0$. Clearly, each non-zero element of $\mathcal{P}_2$ is a mutual annihilator for a, b. Thus $d(a, b) = 2$, and so $\mathrm{diam}(\Gamma(R)) = 2$, as wanted. $\square$

Proposition 3.4. *Let R be a reversible ring with diam*$(\Gamma(R)) = 2$. *If $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ is the union of precisely two maximal primes in $Z(R)$, then $\mathcal{P}_1$ and $\mathcal{P}_2$ are completely prime ideals of R.*

Proof. By Proposition 3.3, $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$. Then $\mathcal{P}_1 = ann(x)$ and $\mathcal{P}_2 = ann(y)$, for each $0 \neq x \in \mathcal{P}_2$ and $0 \neq y \in \mathcal{P}_1$. Let $ab \in \mathcal{P}_1$ and $a \notin \mathcal{P}_1$. Then $xa \neq 0$ for some $0 \neq x \in \mathcal{P}_2$. Then $b \in ann(xa) = ann(x) = \mathcal{P}_1$. $\square$

Theorem 3.5. *Let R be a reversible ring and $Z(R) = \cup_{i=1}^{n} \mathcal{P}_i$. Then*

(1) *diam*$(\Gamma(R)) = 0$ *if and only if* $n = 1$ *and* $|Z(R)| = 2$.

(2) *diam*$(\Gamma(R)) = 1$ *if and only if* $n = 2$ *and* $|Z(R)| = 3$ *or* $n = 1$ *and* $(Z(R))^2 = 0$ *and* $|Z(R)| \geq 3$.

(3) *diam*$(\Gamma(R)) = 2$ *if and only if* $|Z(R)| \geq 4$ *and (i)* $n = 2$ *and* $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$ *or (ii)* $n = 1$ *and* $(Z(R))^2 \neq 0$ *and for each pair of distinct non-zero zero-divisors* a, b, *if* $ab \neq 0$, *then* a, b *have a mutual non-zero annihilator.*

(4) *diam*$(\Gamma(R)) = 3$ *if and only if* $n \geq 3$ *or* $n = 2$ *and* $\mathcal{P}_1 \cap \mathcal{P}_2 \neq 0$ *or* $n = 1$ *and there are distinct non-zero* $a, b \in Z(R)$ *such that* $ab \neq 0$ *and* a, b *don't have a mutual non-zero annihilator.*

Proof. (1). If $\dim(\Gamma(R)) = 0$, then $Z(R) = \{0, a\}$ for some non-zero $a \in Z(R)$. Hence $|Z(R)| = 2$. The converse is clear.

(2). For the forward direction, first notice it is easily seen that $|Z(R)| \geq 3$. Since $\mathrm{diam}(\Gamma(R)) = 1$, hence $\Gamma(R)$ is complete. Thus by Proposition 2.2, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for each $x, y \in Z(R)$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $n = 2$ and $|Z(R)| = 3$. If $xy = 0$ for each $x, y \in Z(R)$, then $Z(R)$ is an ideal (and so $n = 1$) and $(Z(R))^2 = 0$. The backward direction is clear.

(3). For the forward direction, first notice that by Proposition 3.2, we have $n \leq 2$. If $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 \neq 0$, then by Proposition 3.1, $\mathrm{diam}(\Gamma(R)) = 3$, which is a contradiction. Thus $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$. If $|Z(R)| = 3$, then by (1), $\mathrm{diam}(\Gamma(R)) = 1$, which is a contradiction. Hence $|Z(R)| \geq 4$.

Now, let $n = 1$. Since $\mathrm{diam}(\Gamma(R)) = 2$, there are $a, b \in Z(R)$ such that $ab \neq 0$, which implies that $(Z(R))^2 \neq 0$. If $|Z(R)| = 3$, then $\mathrm{diam}(\Gamma(R)) = 1$, which is a contradiction. Therefore $|Z(R)| \geq 4$. Clearly, for each pair of distinct non-zero zero-divisors a, b, if $ab \neq 0$, then a, b have a mutual non-zero annihilator.

For the backward direction, first let $|Z(R)| \geq 4$, $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$. Then by Proposition 3.3, $\mathrm{diam}(\Gamma(R)) = 2$. Now, assume that $|Z(R)| \geq 4$, $n = 1$, $(Z(R))^2 \neq 0$ and for each pair of distinct non-zero zero-divisors a, b, if $ab \neq 0$, then a, b have a mutual non-zero annihilator. If for each pair of distinct non-zero divisors a, b, $ab = 0$, then there exists vertex adjacent to each other vertex and so by Proposition 2.2, $Z(R)$ is an ideal of R with $(Z(R))^2 = 0$, which is a contradiction. Thus there are non-zero zero-divisors $a \neq b$ such that $ab \neq 0$. Hence $d(a, b) = 2$ and so $\mathrm{diam}(\Gamma(R)) = 2$.

(4). Forward direction follows from parts (1), (2) and (3), and the backward direction follows from Propositions 3.1 and 3.2. □

It is often taught in an elementary algebra course that if R is a commutative ring, and $f(x)$ is a zero-divisor in $R[x]$, then there is a non-zero element $r \in R$ with $f(x)r = 0$. This was first proved by McCoy [25, Theorem 2]. Based on this result, Nielsen [26] called a ring R *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some non-zero $c \in R$, where $f(x), g(x)$ are non-zero polynomials in $R[x]$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy then the ring is called a *McCoy ring.*

Let P and Q be nonempty subsets of a monoid M. An element s is called a *u.p.-element* (*unique product element*) of $PQ = \{pq : p \in P \quad , \quad q \in Q\}$ if it is uniquely presented in the form $s = pq$ where $p \in P$ and $q \in Q$. Recall that a monoid M is called a *u.p.-monoid* (*unique product monoid*) if for any two nonempty finite subsets $P, Q \subseteq M$ there exist a u.p.-element in PQ. Following the literature, right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups are all u.p.-monoids (see, e.g., [27], [28]).

Let M be a monoid and R a ring. According to [13], R is called right *M-McCoy* if whenever $0 \neq \alpha = a_1g_1 + \cdots + a_ng_n, 0 \neq \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$, with $g_i, h_j \in M$ and $a_i, b_j \in R$ satisfy $\alpha\beta = 0$, then $\alpha r = 0$ for some non-zero $r \in R$. Left M-McCoy rings are defined similarly. If R is both left and right M-McCoy then R is called *M-McCoy.* Clearly R is right McCoy if and only if R is right M-McCoy, where M is the additive monoid $\mathbb{N} \cup \{0\}$ of non-negative integers.

Lemma 3.6. [13, Proposition 1.2] *If M is a u.p.-monoid and R a reversible ring, then R is M-McCoy.*

Lemma 3.7. *If M is a u.p.-monoid, $1 \neq u \in M$ and*

$$s_{11}, \ldots, s_{1m_1}, s_{21}, \ldots, s_{2m_2}, \ldots, s_{\ell 1}, \ldots, s_{\ell m_\ell} \in M$$

for some positive integers $\ell, m_1, \ldots, m_\ell$, then there exist non-negative integer elements $n_1, \ldots, n_\ell$ such that for any $i \neq j$, $s_{ip}u^{n_i} \neq s_{jq}u^{n_j}$ for all p and q.

Proof. Since $1 \neq u$, the set $\{1, u, u^2, \ldots\}$ is infinity. Since M is cancellative, we have for every $q \in \{1, \ldots, m_2\}$ infinity set $\{s_{2q}u, s_{2q}u^2, s_{2q}u^3, \ldots\}$. Thus in particular there exists positive integer d such that $s_{1i} \neq s_{2j}u^d$ for all i and j. Now we set $n_1 = 0$ and $n_2 = d$. Using the above argument finite number of times we get the result. $\square$

According to [18], a commutative ring R has *Property* (A) if every finitely generated ideal of R consisting entirely of zero-divisors has a non-zero annihilator. The class of commutative rings with Property (A) is quite large. For example, the polynomial ring $R[x]$, rings whose classical ring of quotients are von Neumann regular [15], Noetherian rings [19, p. 56] and rings whose prime ideals are maximal [15] are examples of rings with Property (A). Kaplansky [19, p. 56] showed the existence of the non-Noetherian rings without Property (A). Rings with Property (A) was originally studied by Quentel [29]. Quentel used the term Condition (C) for Property (A). Using Property (A), Hinkle and Huckaba [16] extended the concept Kronecker function rings from integral domains to rings with zero divisors.

Hong et. al. [17], extended Property (A) to non-commutative rings as follows: A ring R has *right* (*left*) *Property* (A) if every finitely generated ideal of R consisting entirely of left (right) zero-divisors has a right (left) non-zero annihilator. A ring R is said to have *Property* (A) if R has right and left Property (A). They showed that if R is a reduced ring with finitely many minimal prime ideals, then R has Property (A). Also, if R is a reversible ring and every prime ideal of R is maximal, then R has Property (A).

Theorem 3.8. *Let R be a reversible left* (*right*) *Noetherian ring. Then R has right* (*left*) *Property* (*A*).

Proof. It follows from Remark 2.7, Theorem 2.8 and Corollary 2.9. □

Theorem 3.9. *Let R be a reversible ring and M be a u.p.-monoid. Then the monoid ring $R[M]$ has Property* (*A*).

Proof. Let $1 \neq u \in M$ and $\alpha_i = s_{i1}g_{i1} + \cdots + s_{im_i}g_{im_i} \in R[M]$, for $i = 1, \ldots, \ell$, such that $\alpha_1 R[M] + \cdots + \alpha_\ell R[M] \subseteq Z_\ell(R[M])$. By Lemma 3.7, there exist non-negative integers $n_1, \ldots, n_\ell$ such that for any $i \neq j$, $s_{ip}u^{n_i} \neq s_{jq}u^{n_j}$ for all p and q. Then $\beta = \alpha_1 u^{n_1} + \cdots + \alpha_\ell u^{n_\ell} \in Z(R[M])$, and $\{s_{11}, \ldots, s_{1m_1}, s_{21}, \ldots, s_{2m_2}, \ldots, s_{\ell 1}, \ldots, s_{\ell m_\ell}\}$ is the set of coefficients of β in R. By Lemma 3.6, there exist $0 \neq c \in R$ with $\beta c = 0$, and since R is reversible we have $c \in r_{R[M]}(\alpha_1 R[M] + \cdots + \alpha_\ell R[M])$. Therefore $R[M]$ has right Property (A). By a similar argument one can show that $R[M]$ has left Property (A). □

Corollary 3.10. *Let R be a reversible ring and $M$is a u.p.-monoid. Let $\alpha = a_1g_1 + \cdots + a_mg_m$ and $\beta = b_1h_1 + \cdots + b_nh_m \in R[M]$. Then the following statements are equivalent:*

(1) $(\alpha R[M] + \beta R[M]) \subseteq Z_\ell(R[M])$;

(2) *α and β have a mutual non-zero annihilator in $R[M]$;*

(3) *There is a non-zero* $c \in R$ *such that* $c\alpha = 0 = c\beta$*;*

(4) *If* $v \in M$ *such that* $\{g_1, \ldots, g_m\} \cap \{h_1, \ldots, h_n\}v = \phi$*, then* $\alpha + \beta v$ *is a left zero divisor of* $R[M]$*.*

Proof. It follows from Theorem 3.9. □

Theorem 3.11. *Let* R *be a reversible ring and* M *a u.p.-monoid. Then* $Z(R[M])$ *is an ideal of* $R[M]$ *if and only if* $Z(R)$ *is an ideal of* R *and* R *has right Property* (A)*.*

Proof. For the backward direction, suppose that $Z(R)$ is an ideal of R and R has right Property (A). Let $\alpha = a_1g_1 + \cdots + a_ng_n$ and $\beta = b_1h_1 + \cdots + b_mh_m$ be non-zero zero-divisors of $R[M]$. By Lemma 3.6, there are non-zero elements $r, s \in R$ such that $r\alpha = 0 = s\beta$. Hence $\{a_1, \ldots, a_n, b_1, \ldots, b_m\} \subseteq Z(R)$ and so $\langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle \subseteq Z(R)$. Since R has right Property (A), there exists $0 \neq t \in R$ with $\langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle t = 0$. Since R is reversible, we have $\langle \alpha, \beta\rangle t = 0$ and so $\alpha + \beta \in Z(R[M])$. Now, let $\alpha \in Z(R[M])$ and $\beta \in R[M]$. By Lemma 3.6, there exists $0 \neq r \in R$ with $r\alpha = 0$. Since R is reversible we have $\alpha\beta r = 0 = \beta\alpha r$. Thus $Z(R[M])$ is an ideal of $R[M]$, as wanted.

For the forward direction, first notice that clearly $Z(R) \subseteq Z(R[M])$, and hence $Z(R)[M] \subseteq Z(R[M])$. By Lemma 3.6, $Z(R[M]) \subseteq Z(R)[M]$. Thus $Z(R[M]) = Z(R)[M]$. Now, let $\langle a_1, \ldots, a_n\rangle \subseteq Z(R)$. Let $1 \neq g \in M$, then $a_1g + a_2g^2 + \cdots + a_ng^n \in Z(R)[M] = Z(R[M])$. By Lemma 3.6, there exists $0 \neq r \in R$ such that $(a_1g + a_2g^2 + \cdots + a_ng^n)r = 0$. Then $\langle a_1, \ldots, a_n\rangle r = 0$, since R is reversible. Thus R has right Property (A), and the proof is complete. □

Proposition 3.12. *Let* R *be a reversible ring with diam*$(\Gamma(R)) = 2$ *and* M *a u.p.-monoid. If* $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ *is the union of precisely two maximal primes in* $Z(R)$*, then diam*$(\Gamma(R[M])) = 2$*.*

Proof. By Propositions 3.3 and 3.4, $\mathcal{P}_1$ and $\mathcal{P}_2$ are completely prime ideals and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$, hence R is reduced. Thus $R[M]$ is a reduced ring, by [23, Proposition 1.1]. Let $\alpha = a_1g_1 + \cdots + a_ng_n \in Z(R[M])$. Then $\alpha\beta = 0$, for some $0 \neq \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$. Since R is reduced and M is a u.p.-monoid, $\alpha b_j = 0$, for each j, by [23, Proposition 1.1]. Then $\alpha \in \mathcal{P}_1[M]$ or $\alpha \in \mathcal{P}_2[M]$, which implies that $Z(R[M]) \subseteq \mathcal{P}_1[M] \cup \mathcal{P}_2[M]$.

Since $\mathcal{P}_1\mathcal{P}_2 = 0 = \mathcal{P}_2\mathcal{P}_1$, we have $\mathcal{P}_1[x] \cup \mathcal{P}_2[M] \subseteq Z(R[M])$. Therefore $Z(R[M]) = \mathcal{P}_1[M] \cup \mathcal{P}_2[M]$, which implies that $\mathrm{diam}(\Gamma(R[M])) = 2$. $\square$

Note that polynomial rings over reversible rings need not be reversible in general, then so is u.p.-monoid rings.

Theorem 3.13. *Let R be a reversible ring with non-zero zero-divisors, M a u.p.-monoid and $Z(R) = \cup_{i=1}^{n}\mathcal{P}_i$. Then*

(1) *diam*$(\Gamma(R[M])) \geq 1$.

(2) *diam*$(\Gamma(R[M])) = 1$ *if and only if* $n = 1$ *and* $(Z(R))^2 = 0$.

In addition if R has right Property (A), then

(3) *diam*$(\Gamma(R[M])) = 2$ *if and only if (i)* $n = 2$ *and* $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$ *or (ii)* $|Z(R)| \geq 4$, $n = 1$, $(Z(R))^2 \neq 0$ *and for each pair of distinct non-zero zero-divisors a, b, if $ab \neq 0$, then a, b have a non-zero mutual annihilator.*

(4) *diam*$(\Gamma(R[M])) = 3$ *if and only if* $n \geq 3$ *or* $n = 2$ *and* $\mathcal{P}_1 \cap \mathcal{P}_2 \neq 0$.

Proof. (1). Let $0 \neq a \in Z(R)$ and $1 \neq g \in M$. Then a_1g, a_1g^2 are distinct non-zero zero-divisors of $R[M]$. Since $d(a_1g, a_1g^2) \geq 1$, hence $\mathrm{diam}(\Gamma(R[M])) \geq 1$.

(2). For the forward direction, since $\mathrm{diam}(\Gamma(R[M])) = 1$, hence $\mathrm{diam}(\Gamma(R)) = 0$ or $\mathrm{diam}(\Gamma(R)) = 1$. If $\mathrm{diam}(\Gamma(R)) = 0$, then $Z(R) = \{0, a\}$, for some non-zero $a \in Z(R)$. Hence $n = 1$ and $(Z(R))^2 = 0$.

If $\mathrm{diam}(\Gamma(R)) = 1$, then $n = 2$ and $|Z(R)| = 3$ or $n = 1$, $(Z(R))^2 = 0$ and $|Z(R)| \geq 3$, by Theorem 3.5. Let $n = 2$ and $|Z(R)| = 3$. We can assume that $\mathcal{P}_1 = \{0, a\}$ and $\mathcal{P}_2 = \{0, b\}$. Let $1 \neq g \in M$, then ag, ag^2 are distinct non-zero zero divisors of $R[M]$ and $agag^2 \neq 0$. Hence $d(ag, ag^2) \geq 2$, and so $\mathrm{diam}(\Gamma(R[M])) \geq 2$, which is a contradiction. Therefore $n = 1$, $(Z(R))^2 = 0$ and $|Z(R)| \geq 3$.

For the backward direction, let a be a non-zero zero-divisor of R and $1 \neq g \in M$. Then ag, ag^2 are distinct zero-divisors of $Z(R[M])$. Hence $\mathrm{diam}(\Gamma(R[M])) \geq 1$. If $\alpha = a_1g_1 + \cdots + a_mg_m, \beta = b_1h_1 + \cdots + b_nh_n \in Z(R[M])$, then by Lemma 3.6, $a_1, \cdots, a_m, b_1, \cdots, b_n \in Z(R)$, and since $(Z(R))^2 = 0$, we have $\alpha\beta = 0$, which implies that $\mathrm{diam}(\Gamma(R[M])) = 1$.

(3). For forward direction, since $\operatorname{diam}(\Gamma(R[M])) = 2$, we have $\operatorname{diam}(\Gamma(R)) = 0$ or $\operatorname{diam}(\Gamma(R)) = 1$ or $\operatorname{diam}(\Gamma(R)) = 2$. Let $\operatorname{diam}(\Gamma(R)) = 0$, then by (1), $\operatorname{diam}(\Gamma(R[M])) = 1$, which is a contradiction. Let $\operatorname{diam}(\Gamma(R)) = 1$, then $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$ or $n = 1$ and $(Z(R))^2 = 0$ and $|Z(R)| \geq 3$, by Theorem 3.5. If $n = 1$ and $(Z(R))^2 = 0$, then by using Lemma 3.6, we have $\operatorname{diam}(\Gamma(R[M])) = 1$, which is a contradiction. Thus $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$.

Now, let $\operatorname{diam}(\Gamma(R)) = 2$. Then $|Z(R)| \geq 4$ and (i) $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$ or (ii) $n = 1$, $(Z(R))^2 \neq 0$ and for each pair of distinct non-zero zero-divisors a, b, if $ab \neq 0$, then a, b have a non-zero mutual annihilator, by Theorem 3.5.

For the backward direction, first let $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$. Then $\operatorname{diam}(R[M]) = 2$, by Proposition 3.12. Now, let $|Z(R)| \geq 4$, $n = 1$, $(Z(R))^2 \neq 0$ and for each pair of distinct non-zero zero-divisors a, b, if $ab \neq 0$, then a, b have a non-zero mutual annihilator. By Theorem 3.5, $\operatorname{diam}(\Gamma(R)) = 2$, hence $\operatorname{diam}(\Gamma(R[M])) \geq 2$. Since R has right Property (A), hence $Z(R[M])$ is an ideal of $R[M]$ and $Z(R[M]) = Z(R)[M]$, by Theorem 3.11. Let $\alpha = a_1g_1 + \cdots + a_mg_m$ and $\beta = b_1h_1 + \cdots + b_nh_n \in Z(R[M])$. Let $v \in M$ such that $\{g_1, \ldots, g_m\} \cap \{h_1, \ldots, h_n\}v = \emptyset$. Then $\alpha + \beta v \in Z(R[M])$, since $Z(R[M])$ is an ideal of $R[M]$. There is a non-zero $\delta \in R[M]$ such that $(\alpha + \beta v)\delta = 0$, hence δ is a mutual annihilator for α and β, by Corollary 3.10. Thus $d(\alpha, \beta) \leq 2$, which implies that $\operatorname{diam}(\Gamma(R[M])) = 2$.

(4). For proving the forward direction, first we claim that $\operatorname{diam}(\Gamma(R)) = 3$. Assume that $\operatorname{diam}(\Gamma(R)) = 2$. Since R has right Property (A), hence $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$, by Theorems 3.5 and 3.11. Then $\operatorname{diam}(\Gamma(R[M])) = 2$, by Proposition 3.12, which is a contradiction. Thus $\operatorname{diam}(\Gamma(R)) = 3$, and so (i) $n \geq 3$ or (ii) $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 \neq 0$, by Theorem 3.5.

For the backward direction, assume that $n \geq 3$ or $n = 2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 \neq 0$. There exist $a \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $b \in \mathcal{P}_2 \setminus \mathcal{P}_1$ such that $ab \neq 0$ and a, b don't have mutual non-zero annihilator. If $\alpha = a_1g_1 + \cdots + a_mg_m \in R[M]$ is a mutual annihilator of a, b, then $a \neq a_i \neq b$, for each i, since $ab \neq 0$. Thus $\alpha = 0$, since a, b don't have mutual non-zero annihilator. Therefore $d(a, b) = 3$ in $\Gamma(R[M])$, which implies that $\operatorname{diam}(\Gamma(R[M])) = 3$. □

Corollary 3.14. *Let M be a u.p.-monoid and R a reversible and right or left Noetherian ring. Then $0 \leq diam(\Gamma(R)) \leq diam(\Gamma(R[M])) \leq 3$.*

Proof. It follows from Remark 2.7, Theorem 3.5 and Theorem 3.13. □

Corollary 3.15. *Let M be a u.p.-monoid and R a reversible and right or left Noetherian ring with diam$(\Gamma(R)) = 2$. Then diam$(\Gamma(R[M])) = 2$.*

Proof. Since R is reversible and right or left Noetherian, R has right Property (A), by Theorem 3.8. Now the result follows from Theorems 3.5 and 3.13. □

Proposition 3.16. *Let R be a reversible ring and M a u.p.-monoid. If, for some $n \in \mathbb{Z}$ with $n > 2$, $(Z(R))^n = 0$, then diam$(\Gamma(R[M]))$ = diam$(\Gamma(R)) = 2$.*

Proof. The proof is essentially same as that of [7, Proposition 3.12]. We assume n is the minimal number such that $(Z(R))^n = 0$. By hypothesis and Proposition 2.2, $\Gamma(R)$ is not complete. Hence, there exist distinct $a, b \in Z(R)$ with $ab \neq 0$. Since $(Z(R))^{n-1} \neq 0$, there exist $c_1, \ldots, c_{n-1} \in Z(R)$ with $c = \prod_{i=1}^{n-1} c_i \neq 0$. However, $ac = bc = 0$, so *diam*$(\Gamma(R)) = 2$. Since $(Z(R))^n = 0$ and the collection of zero-divisors of R is the set-theoretic union of prime ideals by Theorem 2.7, hence $Z(R) = \mathcal{P}$ for some prime ideal $\mathcal{P}$. By Lemma 3.6, R is M-McCoy, hence $Z(R[M])) \subseteq \mathcal{P}[M]$. Now, let $\alpha, \beta \in Z^*(R[M])$. Then all coefficients of α and β belong to $Z(R)$. Thus either $\alpha - \beta$ or $\beta - \alpha$ or $\alpha - c - \beta$. Therefore we get diam$(\Gamma(R[M])) = 2$, as desired. □

Corollary 3.17. *Let R be a reversible ring. If for some $n \in \mathbb{Z}$ with $n > 2$, $(Z(R))^n = 0$, then diam$(\Gamma(R[x, x^{-1}]))$ = diam$(\Gamma(R[x]))$ = diam$(\Gamma(R)) = 2$.*

Theorem 3.18. *If $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a reversible ring and M is a u.p.-monoid, then the following statements are equivalent:*

(1) $\Gamma(R[M])$ *is complete.*

(2) $\Gamma(R)$ *is complete.*

Proof. (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (1), since $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$ we have $xy = 0$ for every $x, y \in Z^*(R)$, by Proposition 2.2. Therefore $\Gamma(R)$ complete implies $(Z(R))^2 = 0$. Let $\alpha, \beta \in Z^*(R[M])$. By Lemma 3.5, R is M-McCoy, hence all coefficients of α and β are zero-divisors in R. Since $\Gamma(R)$ is complete, $\alpha\beta = 0$, and hence $\Gamma(R[M])$ is complete, as wanted. □

Corollary 3.19. *If $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a reversible ring, then the following conditions are equivalent:*

(1) $\Gamma(R[x])$ *is complete;*

(2) $\Gamma(R[x, x^{-1}])$ *is complete;*

(3) $\Gamma(R)$ *is complete.*

Proof. It follows from Theorem 3.18. □

Lemma 3.20. [23, Lemma 1.13] *Let M and N be u.p.-monoids. Then so is the $M \times N$.*

Now, we bring a remark which will be used frequently in the sequel; we include it and its proof for the sake of completeness.

Remark 3.21. Let M and N be u.p.-monoids and R a ring. Then

$$(R[M])[N] \cong R[M \times N].$$

Proof. Suppose that $\sum_{i=1}^{s} a_i(m_i, n_i)$ is in $R[M \times N]$. Without loss of generality, we assume that $\{n_1, n_2, \cdots, n_s\} = \{n_1, n_2, \cdots, n_t\}$ with $n_i \neq n_j$, when $1 \leq i \neq j \leq t$. For any $1 \leq p \leq t$, denote $A_p = \{i | 1 \leq i \leq s, n_i \neq n_p\}$. Then $\sum_{p=1}^{t}(\sum_{i \in A_p} a_i m_i)n_p \in R[M][N]$. Note that $m_i \neq m_{i'}$ for any $i, i' \in A_p$ with $i \neq i'$. Now it is easy to see that there exists an isomorphism of rings $R[M \times N] \longrightarrow R[M][N]$ defined by

$$\sum_{i=1}^{s} a_i(m_i, n_i) \longrightarrow \sum_{p=1}^{t}(\sum_{i \in A_p} a_i m_i)n_p.$$

□

Theorem 3.22. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a reversible and right or left Noetherian ring with non-trivial zero-divisors. Let M, N be u.p.-monoids. The following conditions are equivalent:*

(1) *diam*$(\Gamma(R)) = 2$;

(2) *diam*$(\Gamma(R[M])) = 2$;

(3) *diam*$(\Gamma((R[M])[N])) = 2$;

(4) *$Z(R)$ is either the union of two primes with intersection $\{0\}$ and $|Z(R)| \geq 4$, or $Z(R)$ is prime and $(Z(R))^2 \neq 0$.*

Proof. (1)⇒(2). It was proven in Corollary 3.15.

(2)⇒(1). Assume that $\mathrm{diam}(\Gamma(R)) \neq 2$. If $\mathrm{diam}(\Gamma(R)) = 1$, then by Theorems 3.5 and 3.13, $\mathrm{diam}(\Gamma(R[M])) = 1$, which is a contradiction.

Since M and N are u.p.-monoids, hence by Lemma 3.20, $M \times N$ is u.p.-monoid and by Remark 3.21, $(R[M])[N] \cong R[M \times N]$. Now by a similar way as used in the proof of (1)⇔(2) one can prove (1)⇔(3).

(1)⇒(4). It follows from Theorem 3.5.

(4)⇒(1). First suppose that $Z(R)$ is the union of two primes $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$ and $|Z(R)| \geq 4$. Then $\mathrm{diam}(\Gamma(R)) = 2$, by Theorem 3.5. If $Z(R)$ is prime, there exists $z \in Z(R)$ such that $Z(R) = ann(z)$. Let a, b be distinct non-zero zero divisors and $ab \neq 0$. Then $az = 0 = bz = 0$; so z is a mutual non-zero annihilator for a, b. Thus $\mathrm{diam}(\Gamma(R)) = 2$, by Theorem 3.5. □

Corollary 3.23. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a reversible and right or left Noetherian ring with non-trivial zero-divisors. Then the following statements are equivalent:*

(1) *$diam(\Gamma(R)) = 2$;*

(2) *$diam(\Gamma(R[x])) = 2$;*

(3) *$diam(\Gamma(R[x_1, \ldots, x_n])) = 2$, for all $n > 0$;*

(4) *$diam(\Gamma(R[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}])) = 2$, for all $n > 0$;*

(5) *$Z(R)$ is either the union of two primes with intersection $\{0\}$ and $|Z(R)| \geq 4$, or $Z(R)$ is prime and $(Z(R))^2 \neq 0$.*

We conclude the paper by giving some results about the girth of a zero divisor graph of u.p.-monoid rings over reversible rings.

4 Relations between the girths of $\Gamma(R)$ and $\Gamma(R[M])$

Proposition 4.1. *Let M be a u.p.-monoid and R a reversible ring not necessarily with identity. If $\alpha, \beta \in Z^*(R[M])$ are distinct non-constant* *elements with $\alpha\beta = 0$, then there exist $a, b \in Z^*(R)$ such that $a-\alpha-\beta-b-a$ is a cycle in $\Gamma(R[M])$, or $b-\alpha-\beta-b$ is a cycle in $\Gamma(R[M])$.*

Proof. If $\alpha, \beta \in Z^*(R[M])$, then there exist $a, b \in Z^*(R)$ such that $a\alpha = 0 = b\beta$, by Lemma 3.6. Now, by a similar argument as used in the proof of [7, Proposition 4.1], one can prove it. $\square$

Corollary 4.2. *Let M be a u.p.-monoid and R a reversible ring and $\alpha \in Z^*(R[M])$ a non-constant element. Then there exists a cycle of length 3 or 4 in $\Gamma(R[M])$ with α as one vertex and some $a \in Z^*(R)$ as another.*

Theorem 4.3. *Let M be a u.p.-monoid and R is a reversible ring not necessarily with identity. Then $\mathrm{gr}(\Gamma(R)) \geq \mathrm{gr}(\Gamma(R[M]))$. In addition, if R is a reduced ring and $\Gamma(R)$ contains a cycle, then $\mathrm{gr}(\Gamma(R)) = \mathrm{gr}(\Gamma(R[M]))$.*

Proof. By using Corollary 4.2 and a similar argument as used in the proof of [7, Theorem 4.3] one can prove it. $\square$

Corollary 4.4. *Let M be a u.p.-monoid, R be a reduced ring, and $\mathrm{gr}(\Gamma(R[M])) = 3$. Then $\mathrm{gr}(\Gamma(R)) = 3$.*

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