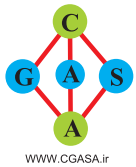
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On a generalization of Conditions (PWP) and $(PWP)_w$

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Abstract. In their foundational work (Commun. Algebra, Vol. 37, No. 6, pp. 1995–2007), Golchin and Rezaei formulated the Conditions (PWP) and $(PWP)_w$. Building on their framework, we present the generalized versions denoted $(GPWP)$ and $(G-(PWP)_w)$. We prove that the former condition entails the latter, although the converse does not necessarily hold. We explore several fundamental properties of these generalized conditions and establish necessary and sufficient criteria under which an amalgamated coproduct satisfies these conditions. Additionally, we examine pomonoids S for which all cyclic or Rees factor S-posets fulfill Condition $(G-(PWP)_w)$ or $(GPWP)$, respectively. We further analyze pomonoids where every Rees factor S-poset satisfying one of the proposed conditions exhibits a particular structural property. Finally, we study the behavior of direct products of S-posets that satisfy these Conditions.

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1 Introduction and preliminaries

The study of flatness conditions for S-posets has a long history, beginning with the pioneering work of Fakhruddin in the 1980s, who established the first basic results on ordered acts and their flatness properties. Subsequently, Golchin and Rezaei introduced weakened versions of classical flatness conditions, namely $(WP)_w$ and $(PWP)_w$, and compared them with the standard notions (WP), (PWP), and po-flatness. Their work clarified the role of weak flatness in the ordered setting. More recently, Liang and Luo continued the study of $(PWP)_w$ and provided deeper structural insights into its behavior. Building on this line of research, our paper introduces two generalized weak conditions, denoted $(GPWP)$ and $(G-(PWP)_w)$, and investigates their fundamental properties.

In this paper, we introduce two new generalized weak conditions, denoted $(GPWP)$ and $(G-(PWP)_w)$. In Section 2, we define these conditions and show that

$$(GPWP) \Rightarrow (G-(PWP)_w) \Rightarrow GPW-po-flatness.$$

We also provide necessary and sufficient conditions for an S-poset of the form $A(K)$ to satisfy these conditions.

Section 3 gives an equivalent description of cyclic (Rees factor) S-posets that satisfy these two conditions. Using this, we identify the pomonoids for which applying these conditions to Rees factor actions results in certain flatness properties, and we also show when those flatness properties imply the conditions.

Section 4 studies when these conditions are preserved under direct products. We determine the pomonoids S for which the product of two S-posets, each satisfying one of the conditions, again satisfies the same condition.

For clarity, we recall several definitions that will be used throughout the paper but not restated later. In what follows, S denotes a pomonoid, while $\mathbb{N}$ denotes the set of natural numbers.

A *pomonoid* is a monoid S equipped with a partial order relation $\leq$ such that for all $s,t,u,v \in S$, if $s \leq t$ and $u \leq v$, then $su \leq tv$. A right S-poset A is defined as a right S-act equipped with a partial order $\leq$ satisfying the following conditions for all $s,t \in S$ and for all $a,b \in A$:

1. If $s \leq t$, then $as \leq at$.

2. If $a \leq b$, then $as \leq bs$.

We say that a subset $K \subseteq S$ of a pomonoid S is convex if, whenever $l, k \in K$ and $i \in S$ satisfy $l \leq i \leq k$, it follows that $i \in K$.

An equivalence relation θ on an S-poset A is considered a *congruence* if it satisfies the properties of an S-act congruence and the factor set A/θ can be endowed with a partial order that aligns with the structure of A, in such a way that the corresponding quotient map from A to A/θ preserves both the order and the S-action.

The *tensor product* $A_S \otimes {}_SB$ of a right S-poset A_S and a left S-poset ${}_SB$ is constructed as the quotient of poset $A \times B$, endowed with the Cartesian order, by a suitable congruence relation. This construction yields a poset structure on $A_S \otimes {}_SB$ such that the canonical map $A_S \times {}_SB \to A_S \otimes {}_SB$, defined by $(a, b) \mapsto a \otimes b$, is balanced and order-preserving in each argument. Moreover, this map is universal among all balanced, order-preserving maps from $A_S \times {}_SB$ into posets.

An S-poset A is called *torsion free* if $\alpha c = \alpha' c$ implies $\alpha = \alpha'$ whenever $\alpha, \alpha' \in A$ and c is a right cancellable element of S.

An S-poset A is called *weakly po-flat* if for all $s, t \in S$ and $\alpha, \alpha' \in A$, $\alpha \otimes s \leq \alpha' \otimes t$ in $A_S \otimes {}_SS$ implies $\alpha \otimes s \leq \alpha' \otimes t$ in $A_S \otimes {}_S(Ss \cup St)$. An S-poset A is called *principally weakly po-flat* if for all $s \in S$ and $\alpha, \alpha' \in A$, $\alpha \otimes s \leq \alpha' \otimes s$ in $A_S \otimes {}_SS$ implies $\alpha \otimes s \leq \alpha' \otimes s$ in $A_S \otimes {}_S(Ss)$. The concepts of *weakly flat* and *principally weakly flat* S-posets are defined in the same manner as their ordered analogues, except that the order relation $\leq$ is replaced by the equality relation $=$ in the relevant definitions.

An S-poset A satisfies

- *Condition* (P) if

$$(\forall\ \alpha, \alpha' \in A)(\forall\ s, t \in S)[(\alpha s \leq \alpha' t) \Rightarrow (\exists\ u, v \in S)(\exists \alpha'' \in A)(\alpha = \alpha'' u,\ \alpha' = \alpha'' v \text{ and } us \leq vt)]$$

- *Condition* (E) if

$$(\forall\ \alpha \in A)(\forall\ s, t \in S)[(\alpha s \leq \alpha t) \Rightarrow (\exists\ u \in S)(\exists \alpha' \in A)(\alpha = \alpha' u \text{ and } us \leq ut)]$$

- *Condition* (E') if

$$(\forall\ \alpha \in A)(\forall\ s,t,z \in S)[(\alpha s \leq \alpha t \text{ and } sz = tz) \Rightarrow (\exists\ u \in S)(\exists \alpha' \in A)(\alpha = \alpha' u \text{ and } us \leq ut)]$$

The S-poset A is called *weakly subpullback flat* whenever it fulfills both Condition (E') and Condition (P).

An S-poset A satisfies

- *Condition* (PWP) if

$$(\forall\ \alpha, \alpha' \in A)(\forall\ s \in S)[(\alpha s \leq \alpha' s) \Rightarrow (\exists\ u, v \in S)(\exists \alpha'' \in A)(\alpha = \alpha'' u,\ \alpha'' v = \alpha' \text{ and } us \leq vs)]$$

- *Condition* $(PWP)_w$ if

$$(\forall\ \alpha, \alpha' \in A)(\forall\ s \in S)[(\alpha s \leq \alpha' s) \Rightarrow (\exists\ u, v \in S)(\exists \alpha'' \in A)(\alpha \leq \alpha'' u,\ \alpha'' v \leq \alpha' \text{ and } us \leq vs)]$$

An S-poset A satisfies *Condition* (WP) if, for every subpullback diagram of the form $P(I, I, f, f, S)$, where I denotes a left ideal of S, the associated map φ is surjective.

Also, a right S-poset G_S is called a *generator* if for any S-posets X_S, Y_S and any S-morphisms $f, g : X_S \to Y_S$ with $f \neq g$, there exists an S-morphism $\alpha : G_S \to X_S$ such that $f\alpha \neq g\alpha$.

Let $T \subseteq S$ and take $(T] = \{s \in S |\ s \leq t\ for\ some\ t \in T\}$. A pomonoid S is called *weakly right reversible* if $Ss \cap (St] \neq \emptyset$ for all $s, t \in S$, equivalently, for every $s, t \in S$, $s's \leq t't$ for some $s', t' \in S$.

If B is a sub S-poset of A, the amalgamated coproduct of two copies of A over a proper sub S-poset B, denoted by $A \sqcup^B A$, is defined by $A \sqcup^B A = (\{x, y\} \times (A \setminus B)) \bigcup (\{z\} \times B)$, where $x, y, z \notin A$, with the right S-action given by

$$(y, u)s = \begin{cases} (y, us), & \text{if } us \notin B \\ (z, us), & \text{if } us \in B \end{cases}$$

$$(x, u)s = \begin{cases} (x, us), & \text{if } us \notin B \\ (z, us), & \text{if } us \in B \end{cases}$$

$$(z,u)s = (z,us),$$

with the order

$$\begin{aligned}(\omega_1, s) \leq (\omega_2, t) \Leftrightarrow &(\omega_1 = \omega_2 \text{ and } s \leq t) \text{ or} \\ &(\omega_1 \neq \omega_2 \text{ and } s \leq j \leq t \text{ for some } j \in B).\end{aligned}$$

If K is a proper right ideal of a pomonoid S, we denote the amalgamated coproduct of two copies of S over K by $A(K)$.

We encourage the reader to see [1, 3] for basic results and definitions relating to S-posets over pomonoids.

2 General properties

In this section, we introduce Conditions $(GPWP)$ and $(G-(PWP)_w)$. Moreover, we show that Condition $(GPWP)$ implies Condition $(G-(PWP)_w)$, and also, Condition $(G-(PWP)_w)$ implies GPW-po-flatness. Furthermore, we provide necessary and sufficient criteria for the S-poset $A(K)$ to fulfill these conditions.

Definition 2.1. A right S-poset A is said to satisfy *Condition* $(GPWP)$ if,

$$\begin{aligned}(\forall s \in S)(\exists n \in \mathbb{N})(\forall \alpha, \alpha' \in A)[\alpha s^n \leq \alpha' s^n \Rightarrow \\ (\exists \alpha'' \in A)(\exists u, v \in S)(\alpha = \alpha'' u \wedge \alpha' = \alpha'' v \wedge us^n \leq vs^n)]\end{aligned} \tag{2.1}$$

Definition 2.2. A right S-poset A is said to satisfy *Condition* $(G-(PWP)_w)$ if

$$\begin{aligned}(\forall s \in S)(\exists n \in \mathbb{N})(\forall \alpha, \alpha' \in A)[\alpha s^n \leq \alpha' s^n \Rightarrow \\ (\exists \alpha'' \in A)(\exists u, v \in S)(\alpha \leq \alpha'' u \wedge \alpha'' v \leq \alpha' \wedge us^n \leq vs^n)]\end{aligned} \tag{2.2}$$

Example 2.3. $(G-(PWP)_w)$ does not imply $(GPWP)$.
Let S be an idempotent pomonoid, and let $A = \{\alpha, \alpha'\}$ be a chain consisting of two elements with $\alpha \leq \alpha'$ such that for every $s \in S$, $\alpha s = \alpha$, $\alpha' s = \alpha'$. Then A admits the structure of an S-poset satisfying Condition $(G-(PWP)_w)$, but it fails to satisfy Condition $(GPWP)$.

We first note some basic closure properties of Condition $(GPWP)$.

Proposition 2.4. *Let S be a pomonoid, and let A be a right S-poset. Then*

(1) *S_S satisfies Condition $(GPWP)$.*

(2) *Θ_S satisfies Condition $(GPWP)$.*

(3) *Any retract of an act satisfying Condition $(GPWP)$, satisfies Condition $(GPWP)$.*

(4) *If $A = \coprod A_i$ satisfies Condition $(GPWP)$, then A_i satisfies Condition $(GPWP)$ for every $i \in I$.*

Proof. This follows directly from the definitions. □

Remark 2.5. Proposition 2.4, is also true for Conditions $(G-(PWP)_w)$ and $(G-(PWP)_w)$.

Definition 2.6. [7] A right S-poset A is called

(1) *GPW-flat* if

$$(\forall\ s \in S)(\exists\ n \in \mathbb{N})(\forall\ \alpha, \alpha' \in A)(\alpha s^n = \alpha' s^n \text{ in } A \otimes S\ \Rightarrow \alpha s^n = \alpha' s^n \text{in } A \otimes Ss^n)$$

(2) *GPW-po-flat* if

$$(\forall\ s \in S)(\exists\ n \in \mathbb{N})(\forall\ \alpha, \alpha' \in A)(\alpha s^n \leq \alpha' s^n \text{ in } A \otimes S\ \Rightarrow \alpha s^n \leq \alpha' s^n \text{in } A \otimes Ss^n)$$

Next we show the link between Condition $(G-(PWP)_w)$ and GPW-po-flatness

Proposition 2.7. *Every right S-poset satisfying Condition $(G-(PWP)_w)$ is GPW-po-flat.*

Proof. Suppose that A satisfies Condition $(G-(PWP)_w)$ and let $s \in S$. Thus there exists a natural number $n \in \mathbb{N}$ such that (2.2) holds.
Let $\alpha s^n \leq \alpha' s^n$ for $\alpha, \alpha' \in A$. By assumption $us^n \leq vs^n$, $\alpha \leq \alpha'' u$ and $\alpha'' v \leq \alpha'$ for some $u, v \in S$ and $\alpha'' \in A$. Therefore,

$$\alpha \otimes s^n \leq \alpha'' u \otimes s^n = \alpha'' \otimes us^n \leq \alpha'' \otimes vs^n = \alpha'' v \otimes s^n \leq \alpha' \otimes s^n,$$

in $A \otimes Ss^n$, which implies that A is GPW-po-flat by [7, Theorem 2.3]. □

The flatness properties of S-posets are related as follows:

$$\begin{array}{ccccccc}
(P) & \Longrightarrow & (WP) & \Longrightarrow & (PWP) & \Longrightarrow & (GPWP) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
(P_w) & \Longrightarrow & (WP)_w & \Longrightarrow & (PWP)_w & \Longrightarrow & (G-(PWP)_w) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
po\text{-flat} & \Longrightarrow & w.po\text{-f.} & \Longrightarrow & p.w.po\text{-f.} & \Longrightarrow & GPW\text{-po-flat} \Longrightarrow po\text{-t.f.} \\
\Downarrow & & \Downarrow & & \Downarrow & & (incomparable) \\
flat & \Longrightarrow & w.f. & \Longrightarrow & p.w.f. & \Longrightarrow & GPW\text{-flat} \Longrightarrow t.f.
\end{array}$$

A pomonoid S is called a *left PP pomonoid* if every principal left ideal of S is projective as a left S-poset. For an idempotent $e \in S$, the element a of an S-poset A is called *right e-po-cancellable* if $a = ea$, and for all $s, t \in S$, the inequality $sa \leq ta$ implies $se \leq te$. According to [8, Corollary 2.4] a pomonoid S is an left PP pomonoid if and only if for all $a \in S$ there exists an idempotent $e \in S$ such that a is right e-po-cancellable.

The following result shows that three notions of flatness coincide when S is po-cancellative.

Proposition 2.8. *Let S be a right po-cancellative pomonoid and A a right S-poset. The following are equivalent:*

(1) *A is principally weakly po-flat.*

(2) *A is GPW-po-flat.*

(3) *A satisfies Condition $(G-(PWP)_w)$.*

Proof. The implication (3) $\Rightarrow$ (2) follows immediately from Proposition 2.7.

(2) $\Leftrightarrow$ (1) By definition, GPW-po-flatness coincides with principal weak po-flatness when S is po-cancellative.

(2) $\Rightarrow$ (3) Suppose A is GPW-po-flat and fix $s \in S$. Then there exists $n \in \mathbb{N}$ such that $A \otimes -$ preserves the order-embedding $s(Ss) \hookrightarrow sS$ at level s^n. Now assume $as^n \leq a's^n$ for some $a, a' \in A$. By [5, Proposition 2.9], e-cancellativity of a implies the existence of an idempotent $e \in E(S)$ and $a'' \in A$ with

$$a \leq a''e, \qquad a''e \leq a', \qquad es^n = s^n.$$

From po-cancellativity, $e = 1$. Therefore, $a \leq a'' \leq a'$, showing that A satisfies Condition $(G-(PWP)_w)$. □

The following two results discuss when the amalgamated coproduct $A(K)$ can satisfy Conditions $(GPWP)$ and $(G-(PWP)_w)$.

Proposition 2.9. *For any pomonoid S and any proper right ideal K of S, $A(K)$ does not satisfy Condition $(GPWP)$.*

Proof. Suppose Condition $(GPWP)$ holds for $A(K)$. Since for every $k \in K$, $(x,1)k \leq (y,1)k$, it follows from the assumption that there exist $(w,t) \in A(K)$ and $u, v \in S$ such that $(x,1) = (w,t)u$, $(y,1) = (w,t)v$ and $uk \leq vk$. By the first two equalities, we deduce that $x = y$, a contradiction. □

Proposition 2.10. *Let S be a pomonoid and K be a proper right ideal of a pomonoid S. Then $A(K)$ satisfies Condition $(G-(PWP)_w)$ if and only if*

$$(\forall\ s \in S)(\exists\ n \in \mathbb{N})(\forall\ u, v \in S)(\forall\ k \in K)[us^n \leq k \leq vs^n \Rightarrow (\exists\ l \in K)(us^n \leq ls^n \wedge l \leq v) \vee (ls^n \leq vs^n \wedge u \leq l)] \tag{2.3}$$

Proof. Necessity: Let $A(K)$ satisfy Condition $(G-(PWP)_w)$ and let $s \in S$. Thus (2.2) holds for some $n \in \mathbb{N}$, for $A(K)$. If for any $k \in K$ and $u, v \in S$, $us^n \leq k \leq vs^n$, then $(x,1)us^n \leq (y,1)vs^n$. We consider the following four cases:

Case 1. If $u, v \notin K$, then $(x,u)s^n \leq (y,v)s^n$. Assuming $A(K)$ satisfies Condition $(G-(PWP)_w)$, there exist elements such that

$$(x,u) \leq (\omega,\omega')u', \quad (\omega,\omega')v' \leq (y,v) \quad \text{and} \quad u's^n \leq v's^n,$$

for some $(\omega,\omega') \in A(K)$ and $u', v' \in S$. This case splits into three subcases:

Subcase 1. If $\omega = x$ and $\omega'v' \notin K$, then

$$(x,u) \leq (x,\omega')u', (x,\omega'v') \leq (y,v), \text{ and } u's^n \leq v's^n.$$

It follows from how $A(K)$ is defined that there exists $l \in K$ such that $u \leq \omega'u'$, $\omega'v' \leq l \leq v$. Now from $u's^n \leq v's^n$, we have $(\omega'u')s^n \leq (\omega'v')s^n$ and so $us^n \leq (\omega'u')s^n \leq (\omega'v')s^n \leq ls^n$.
If $\omega'v' \in K$, we may choose $l = \omega'v'$.

Subcase 2. If $\omega = y$ and $\omega' u' \notin K$, then

$$(x,u) \leq (y,\omega' u'), (y,\omega')v' \leq (y,v), \text{ and } u's^n \leq v's^n.$$

By the definition of $A(K)$, there exist $l \in K$ such that $u \leq l \leq \omega' v'$ and $\omega' v' \leq v$. Now from $u's^n \leq v's^n$, we have $(\omega' u')s^n \leq (\omega' v')s^n$ and so $ls^n \leq (\omega' u')s^n \leq (\omega' v')s^n \leq vs^n$. If $\omega' u' \in K$, then we can take $l = \omega' u'$.

Subcase 3. If $\omega = z$, then we can choose $l = \omega' u'$ or $l = \omega' v'$.

Case 2. If $u \notin K$, $v \in K$, this case is similar to case 1.

Case 3. If $u \in K$, $v \notin K$, this case is similar to case 1.

Case 4. If $u, v \in K$, then $(z,u)s^n \leq (z,v)s^n$. Because $A(K)$ satisfies Condition $(G - (PWP)_w)$, there exist $(\omega, \omega') \in A(K)$ and $u', v' \in S$ such that $(z,u) \leq (\omega,\omega')u'$, $(\omega,\omega')v' \leq (z,v)$, and $u's^n \leq v's^n$. From this, we deduce $(z,us^n) = (z,u)s^n \leq (\omega,\omega')u's^n \leq (\omega,\omega')v's^n \leq (z,v)s^n = (z,vs^n)$, so $us^n \leq vs^n$. Thus we may choose $l = u$ or $l = v$.

Sufficiency: Let $s \in S$. Thus by assumption there exists $n \in \mathbb{N}$ such that (2.3) holds. Let $(\omega,u), (\omega',v) \in A(K)$ be such that $(\omega,u)s^n \leq (\omega',v)s^n$. We consider the following six cases:

Case 1. If $\omega = \omega' = x$, then $(x,u)s^n \leq (x,v)s^n$. Therefore $us^n \leq vs^n$, $(x,u) \leq (x,1)u$ and $(x,1)v \leq (x,v)$.

Case 2. If $\omega = \omega' = y$ or $\omega = \omega' = z$, this is similar to Case 1.

Case 3. If $\omega = x$, $\omega' = y$, then we have $(x,u)s^n \leq (y,v)s^n$. By the definition of $A(K)$, there exists $k \in K$ such that $us^n \leq k \leq vs^n$. By assumption, there exists $l \in K$ such that $us^n \leq ls^n$ and $l \leq v$, or $ls^n \leq vs^n$ and $u \leq l$. There are two subcases:

Subcase 1. If $us^n \leq ls^n$, $l \leq v$, then

$$(x,u) \leq (x,1)u \text{ and } (x,1)l = (z,l) = (y,1)l \leq (y,1)v = (y,v).$$

Subcase 2. If $ls^n \leq vs^n$, $u \leq l$, then

$$(y,1)v \leq (y,v) \text{ and } (x,u) = (x,1)u \leq (x,1)l = (z,l) = (y,1)l.$$

Therefore, $A(K)$ satisfies Condition $(G - (PWP)_w)$.

Case 4. If $\omega = y$, $\omega' = x$, this case is similar to Case 3.

Case 5. If $\omega = x$, $\omega' = z$, then $(x,u)s^n \leq (z,v)s^n$. Hence, $(x,u) \leq (x,1)u$, $(x,1)v \leq (z,v)$, and $us^n \leq vs^n$.

Case 6. If $\omega = y$, $\omega' = z$, this case is similar to Case 5. □

The next two propositions show that Conditions $(GPWP)$ and $(G-(PWP)_w)$ are inherited from products of S-posets to each component.

Proposition 2.11. *For any family $\{A_i\}_{i\in I}$ of right S-posets, if $\prod_{i\in I} A_i$ satisfies Condition $(GPWP)$, then A_i satisfies Condition $(GPWP)$, for every $i \in I$.*

Proof. Let $\prod_{i\in I} A_i$ satisfy Condition $(GPWP)$ and let $s \in S$ and $i \in I$. By assumption there exists a natural number $n \in \mathbb{N}$ such that for any $(a_k)_I, (a'_k)_I \in \prod_{i\in I} A_i$, $(a_k)_I s^n \leq (a'_k)_I s^n$ implies that there exist $(a''_k)_I \in \prod_{i\in I} A_i$ and $u, v \in S$ such that $(a_k)_I = (a''_k)_I u, (a'_k)_I = (a''_k)_I v$ and $us^n \leq vs^n$. Let $bs^n \leq cs^n$ for any $b, c \in A_i$ and fix $a_j \in A_j$ for each $j \neq i$. If

$$a_k = \begin{cases} b & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

$$a'_k = \begin{cases} c & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

Then $(a_k)_I s^n \leq (a'_k)_I s^n$. By assumption, $(a_k)_I = (a''_k)_I u, (a'_k)_I = (a''_k)_I v$ and $us^n \leq vs^n$ for some $(a''_k)_I \in \prod_{i\in I} A_i$ and $u, v \in S$. Now we have $b = a''_i u$, $c = a''_i v$ which means that A_i satisfies Condition $(GPWP)$, as required. □

Using an argument similar to the proof of Proposition 2.11, we have the following proposition.

Proposition 2.12. *For any family $\{A_i\}_{i\in I}$ of right S-posets, if $\prod_{i\in I} A_i$ satisfies Condition $(G-(PWP)_w)$, then A_i satisfies Condition $(G-(PWP)_w)$, for every $i \in I$.*

As demonstrated in [2], when the pomonoid S is written as a disjoint union $S = G\dot{\cup}I$, with G being a pogroup and I a poideal, and if a right S-poset A possesses certain flatness properties in its structure as a right I^1-poset, then those properties are preserved when A is considered as a right S-poset. The results that follow extend this observation to the cases of Conditions $(G-(PWP)_w)$ and $(GPWP)$.

Theorem 2.13. *Suppose that the pomonoid S can be expressed as the disjoint union $S = G\dot{\cup}I$, and let A be a right S-poset. If A satisfies Condition $(GPWP)$ when considered as a right I^1-poset, then it also satisfies this condition in the context of right S-posets.*

Proof. Assume that A satisfies Condition $(GPWP)$ as a right I^1-poset. To prove that A also satisfies this condition as a right S-poset, let $s \in S$. We consider two cases:

Case 1: Suppose $s \in G$. In this case, since $Ss = S$, it follows that for every $n \in \mathbb{N}$, $Ss^n = Ss = S$. For any $\alpha, \alpha' \in A$, if $\alpha s \leq \alpha' s$, then we have $(\alpha s) \cdot 1 \leq (\alpha' s) \cdot 1$. Because A satisfies Condition $(GPWP)$ as a right I^1-poset, thus $\alpha s = \alpha'' u$, $\alpha' s = \alpha'' v$ and $u \cdot 1 \leq v \cdot 1$ for some $\alpha'' \in A$ and $u, v \in I^1$. Consequently, $\alpha = \alpha''(us^{-1})$, $\alpha' = \alpha''(vs^{-1})$, which implies that A satisfies Condition $(GPWP)$ as a right S-poset.

Case 2: Suppose $s \in I \subseteq I^1$. Since A satisfies Condition $(GPWP)$ as a right I^1-poset, there exists $n \in \mathbb{N}$ such that the relation (2.1) holds. Let $\alpha s^n \leq \alpha' s^n$ for $\alpha, \alpha' \in A$. Under the given assumption, there exist $\alpha'' \in A$ and $u, v \in I^1$ such that $\alpha = \alpha'' u$, $\alpha' = \alpha'' v$ and $us^n \leq vs^n$. Since $u, v \in S$, it follows that A satisfies Condition $(GPWP)$ as a right S-poset. □

Similar to the argument used in the proof of Theorem 2.13, we have the following theorem.

Theorem 2.14. *Suppose that the pomonoid S is given by the disjoint union $S = G \dot{\cup} I$, and let A be a right S-poset. If A satisfies Condition $(G - (PWP)_w)$ when regarded as a right I^1-poset, then it also satisfies this condition as a right S-poset.*

3 Cyclic (Rees factor) S-posets satisfying Condition $(G - (PWP)_w)$ $((GPWP))$

In this section, we present an equivalent description of cyclic (Rees factor) S-posets that satisfy Conditions $(GPWP)$ and $(G - (PWP)_w)$. Furthermore, we offer a classification of pomonoids for which these conditions, when applied to Rees factor actions, lead to certain flatness properties and, conversely, when such flatness conditions imply the given poset properties.

Given a right order congruence ρ on a pomonoid S, we define the relation $\hat{\rho}$ as follows:

$$\hat{\rho} = \{(u, v) \mid [u]_\rho \leq [v]_\rho \text{ in } S/\rho\},$$

where S/ρ denotes the cyclic right S-poset arising from the quotient.

We first study cyclic (Rees factor) S-posets under Condition $(GPWP)$.

Proposition 3.1. *Let ρ be a right S-poset congruence on a pomonoid S. Then S/ρ satisfies Condition $(GPWP)$ if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for all $x, y \in S$, the inequality $[x]_\rho s^n \leq [y]_\rho s^n$ implies that there exist $u, v \in S$ with $x\rho u$, $y\rho v$, and $us^n \leq vs^n$.*

Proof. Necessity. Fix an arbitrary $s \in S$. Since S/ρ fulfills Condition $(GPWP)$, there exists a natural number n such that inequality (2.1) is satisfied. Suppose $[x]_\rho s^n \leq [y]_\rho s^n$ for some $x, y \in S$. Then, by the assumption, we can find elements $u_1, v_1, z \in S$ such that $[x]_\rho = [z]_\rho u_1$, $[y]_\rho = [z]_\rho v_1$, and $u_1 s^n \leq v_1 s^n$. Setting $u = zu_1$ and $v = zv_1$, it follows that $x\rho u$, $y\rho v$, and $us^n \leq vs^n$ as required.

Sufficiency. Again, let $s \in S$. By assumption, there exists $n \in \mathbb{N}$ such that for all $x, y \in S$, whenever $[x]_\rho s^n \leq [y]_\rho s^n$, there are $u, v \in S$ with $x\rho u$, $y\rho v$, and $us^n \leq vs^n$. Thus, $[x]_\rho = [1]_\rho u$ and $[y]_\rho = [1]_\rho v$, which implies that S/ρ satisfies Condition $(GPWP)$, completing the proof. □

Following the approach used in the proof of Proposition 3.1, we can establish the following result.

Proposition 3.2. *Let ρ be a right order congruence on a pomonoid S. Then the quotient S/ρ satisfies Condition $(G\text{-}(PWP)_w)$ if and only if, for each $s \in S$, there exists some $n \in \mathbb{N}$ such that for all $x, y \in S$, the inequality $[x]_\rho s^n \leq [y]_\rho s^n$ guarantees the existence of elements $u, v \in S$ satisfying $x \,\hat{\rho}\, u$, $y \,\hat{\rho}\, v$, and $us^n \leq vs^n$.*

Remark 3.3. In this section, an *ideal* always means a convex and proper right ideal of the pomonoid S, unless stated otherwise.

Definition 3.4. [5] An ideal K of a pomonoid S is said to be:

(1) *strongly left stabilizing* (SLS) if for all $s \in S$ and $l \in K$, the following conditions hold:

$$l \leq s \;\Rightarrow\; \exists\, l' \in K \text{ such that } l's \leq s,$$
$$s \leq l \;\Rightarrow\; \exists\, l' \in K \text{ such that } s \leq l's.$$

(2) *strongly left annihilating* (SLA) if for every $s \in S$ and all $x, x' \in S \setminus K$, the implication

$$[x]_{\rho_K} s \leq [x']_{\rho_K} s \;\Rightarrow\; xs \leq x's$$

holds.

(3) *D-strongly left annihilating* (*D-SLA*) if for all $s, s' \in S \setminus K$ and for every homomorphism $f : {}_S(Ss \cup Ss') \to {}_SS$, we have:

$$[f(s)]_{\rho_K} \leq [f(s')]_{\rho_K} \;\Rightarrow\; f(s) \leq f(s').$$

Definition 3.5. An ideal K of a pomonoid S is called *eventually strongly left stabilizing* (E-SLS) if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for all $l \in K$, the following conditions are satisfied:

$$\begin{aligned} l \leq s^n &\;\Rightarrow\; \exists\, l' \in K \text{ such that } l's^n \leq s^n, \\ s^n \leq l &\;\Rightarrow\; \exists\, l' \in K \text{ such that } s^n \leq l's^n. \end{aligned} \tag{3.1}$$

Lemma 3.6. *[7, Proposition 3.3] For an ideal K, S/K is GPW-po-flat if and only if K is E-SLS.*

Definition 3.7. An ideal K of a pomonoid S is said to be *eventually strongly left annihilating* (E-SLA) if for every $s \in S$, there exists a natural number n such that for all $x, x' \in S \setminus K$, the following implication holds:

$$[x]_{\rho_K} s^n \leq [x']_{\rho_K} s^n \;\Rightarrow\; xs^n \leq x's^n. \tag{3.2}$$

Definition 3.8. An ideal K of a pomonoid S is said to be *eventually w-strongly left annihilating* (E-w-SLA) if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for all $x, x' \in S \setminus K$, the inequality

$$[x]_{\rho_K} s^n \leq [x']_{\rho_K} s^n$$

implies the existence of elements $u, v \in S$ and $l, l', j, j' \in K$ such that exactly one of the following conditions holds:

(i) $us^n \leq vs^n$ with $x \leq u$ and $v \leq x'$;

(ii) $us^n \leq vs^n$ with $x \leq u$, $v \leq l$, and $l' \leq x'$;

(iii) $us^n \leq vs^n$ with $x \leq j$, $j' \leq u$, and $v \leq x'$;

(iv) $us^n \leq vs^n$ with $x \leq j$, $j' \leq u$, $v \leq l$, and $l' \leq x'$.

The following theorem gives an equivalent condition for S/K to satisfy Condition $(GPWP)$.

Theorem 3.9. *For an ideal K, S/K satisfies Condition $(GPWP)$ if and only if K is both E-SLS and E-SLA.*

Proof. Necessity. Assume that S/K meets the Condition $(GPWP)$. By Proposition 2.7, this implies that S/K is GPW-po-flat. Consequently, applying Lemma 3.6, we conclude that K must be an E-SLS ideal. Now, take an arbitrary element $s \in S$. Then there exists a natural number n such that (2.1) is satisfied. Suppose that for elements $x, x' \in S \setminus K$, we have:

$$[x]_{\rho_K} s^n \leq [x']_{\rho_K} s^n.$$

Since S/K satisfies $(GPWP)$, by Proposition 3.1, there exist elements $u, v \in S$ such that $x\rho_K u$, $x'\rho_K v$, and $us^n \leq vs^n$. Given that x, x' do not belong to K, we conclude that $x = u$ and $x' = v$. Therefore, $xs^n \leq x's^n$.
Sufficiency. Now suppose that K satisfies both E-SLS and E-SLA properties. Fix $s \in S$. Then there exists $n \in \mathbb{N}$ such that both (3.1) and (3.2) are satisfied. Let $x, x' \in S \setminus K$ such that:

$$[x]_{\rho_K} s^n \leq [x']_{\rho_K} s^n.$$

By [1, Lemma 3], one of the following two situations occurs: either $xs^n \leq x's^n$, or there exist $l_1, l_2 \in K$ such that $xs^n \leq l_1$ and $l_2 \leq x's^n$.

If $xs^n \leq x's^n$, then Proposition 3.1 applies with the choice $u = x$, $v = x'$.

Otherwise, we consider the following cases:

Case 1. Both x and x' belong to K. Then choosing $u = v = x$ suffices to apply Proposition 3.1.

Case 2. If neither x nor x' belongs to K, then since K is E-SLA, we deduce that $xs^n \leq x's^n$, and again, setting $u = x$, $v = x'$ suffices.

Case 3. If $x \in K$ but $x' \notin K$, then from $l_2 \leq x's^n$ and the E-SLS property of K, there exists $j \in K$ such that $jx's^n \leq x's^n$. Thus, taking $u = jx'$, $v = x'$ works.

Case 4. If $x \notin K$ and $x' \in K$, then since $xs^n \leq l_1$, there exists $j' \in K$ with $xs^n \leq j'xs^n$. Therefore, choosing $u = x$ and $v = j'x$ completes the requirement.

In all cases, the condition in Proposition 3.1 is satisfied, and hence S/K fulfills $(GPWP)$. □

Similarly, we consider when S/K satisfies Condition $(G-(PWP)_w)$.

Theorem 3.10. *For an ideal K, S/K satisfies Condition $(G-(PWP)_w)$ if and only if K is both E-SLS and E-w-SLA.*

Proof. Necessity. Assume that S/K fulfills Condition $(G-(PWP)_w)$. By Proposition 2.7, it follows that S/K is GPW-po-flat, and hence, by Lemma 3.6, the ideal K must be E-SLS. Now fix an element $s \in S$. Then there exists a natural number n such that (2.2) is satisfied. Suppose $[x]_{\rho_K} s^n \leq [x']_{\rho_K} s^n$ for some $x, x' \in S \setminus K$. Because S/K satisfies Condition $(G-(PWP)_w)$, Proposition 3.2 guarantees the existence of $u, v \in S$ such that $x \hat{\rho} u$, $v \hat{\rho} x'$, and $us^n \leq vs^n$. This implies that $[x]_{\rho_K} \leq [u]_{\rho_K}$ and $[v]_{\rho_K} \leq [x']_{\rho_K}$. Using [1, Lemma 3], the first inequality shows that either $x \leq u$, or there exist $l, l' \in K$ with $x \leq l$ and $l' \leq u$. Similarly, the second inequality yields either $v \leq x'$, or $v \leq j$ and $j' \leq x'$ for some $j, j' \in K$. Thus, all four possible situations from Definition 3.8 are realized, which establishes that K is E-w-SLA.
Sufficiency. Conversely, suppose that K is both E-SLS and E-w-SLA. Let $s \in S$. Then there exists $n \in \mathbb{N}$ such that the four cases in Definition 3.8 are satisfied. Consider $x, x' \in S$ with

$$[x]_{\rho_K} s^n \leq [x']_{\rho_K} s^n,$$

which means that $[xs^n]_{\rho_K} \leq [x's^n]_{\rho_K}$. According to [1, Lemma 3], either $xs^n \leq x's^n$, or there exist $l_1, l_2 \in K$ such that $xs^n \leq l_1$ and $l_2 \leq x's^n$.

If $xs^n \leq x's^n$, then Proposition 3.2 applies directly with $u = x$ and $v = x'$. Otherwise, one of the following four cases occurs:

Case 1. If $x, x' \in K$, then setting $u = v = x$ suffices.

Case 2. If both $x, x' \notin K$, since S is E-w-SLA, there exist $u, v \in S$ and $l, l', j, j' \in K$ such that one of the conditions of Definition 3.8 is satisfied. In every instance, we still obtain $x \rho_K u$, $v \rho_K x'$, and $xs^n \leq x's^n$.

Case 3. If $x \in K$ but $x' \notin K$, then from $l_2 \leq x's^n$ and since K is E-SLS, we find $k \in K$ such that $kx's^n \leq x's^n$. Hence, Proposition 3.2 applies with $u = kx'$ and $v = x'$.

Case 4. If $x \notin K$ while $x' \in K$, then from $xs^n \leq l_1$ and the fact that K is E-SLS, there exists $k \in K$ such that $xs^n \leq kxs^n$. Thus, in Proposition 3.2 we may take $u = x$ and $v = kx$.

In all cases, the condition required by Proposition 3.2 is fulfilled, so S/K indeed satisfies $(G-(PWP)_w)$. □

We next connect these conditions with weak po-flatness.

Theorem 3.11. *For any pomonoid S, the following assertions are equivalent:*

(1) *S/K satisfying Condition $(G-(PWP)_w)$ $((GPWP))$ is weakly po-flat.*

(2) Every ideal K that is both *E-SLS* and *E-w-SLA* (or *E-SLA*) is in fact SLA, and moreover, S is weakly right reversible.

Proof. (1) $\Rightarrow$ (2) Assume K is an *E-SLS* and *E-w-SLA* ideal. Then, by Theorem 3.10, S/K satisfies Condition $(G-(PWP)_w)$. By hypothesis, this implies that S/K is weakly po-flat. Consequently, by [5, Lemma 3.7], the ideal K must be SLA. Furthermore, since Θ_S also satisfies Condition $(G-(PWP)_w)$, it is weakly po-flat by assumption. Hence, using [5, Lemma 3.12], we deduce that S is weakly right reversible.

(2) $\Rightarrow$ (1) Conversely, let S be weakly right reversible, and consider an ideal K of S (not necessarily proper) such that S/K satisfies Condition $(G-(PWP)_w)$. We distinguish two cases:

Case 1. If $K=S$, then $S/K \cong \Theta_S$. Since S is weakly right reversible, [5, Lemma 3.12] implies that S/K is weakly po-flat.

Case 2. If $K \neq S$, then by Theorem 3.10, K is *E-SLS* and *E-w-SLA*. By assumption, such an ideal K is also SLA. Given that S is weakly right reversible, it follows from [5, Lemma 3.7] that S/K must be weakly po-flat. □

Theorem 3.12. *The following assertions are equivalent for any pomonoid S:*

(1) *Whenever S/K fulfills Condition $(G-(PWP)_w)$ $((GPWP))$, it also satisfies Condition (WP).*

(2) Every *E-SLS* and *E-w-SLA* (*E-SLA*) ideal K is *D-SLA*, and moreover, S is weakly right reversible.

Proof. (1) $\Rightarrow$ (2) Let K be an *E-SLS* and *E-w-SLA* ideal. By Theorem 3.10, S/K satisfies Condition $(G-(PWP)_w)$. By assumption, it follows that S/K also satisfies Condition (WP). Hence, by [5, Lemma 3.8], K must be a *D-SLA* ideal. Furthermore, since Θ_S satisfies Condition

$(G-(PWP)_w)$, it also satisfies Condition (WP) by assumption. Therefore, by [5, Lemma 3.12], S is weakly right reversible.

(2) $\Rightarrow$ (1) Consider an ideal K (possibly improper) of a weakly right reversible pomonoid S, and assume that S/K satisfies Condition $(G-(PWP)_w)$. We distinguish two cases:

Case 1. If $K = S$, then $S/K \cong \Theta_S$. Since S is weakly right reversible, it follows from [5, Lemma 3.12] that S/K satisfies Condition (WP).

Case 2. If $K \neq S$, then by Theorem 3.10, K is both E-SLS and E-w-SLA. By assumption, this implies that K is D-SLA. Since S is weakly right reversible, it follows from [5, Lemma 3.8] that S/K satisfies Condition (WP). □

Using Theorem 3.10 together with [5, Lemma 3.12], we obtain the following result.

Theorem 3.13. *For any pomonoid S, the following assertions are equivalent:*

(1) *Whenever S/K fulfills Condition $(G-(PWP)_w)$ $((GPWP))$, it also satisfies Condition (PWP).*

(2) Every ideal of type E-SLS and E-w-SLA (E-SLA) ideal is in fact both SLS and SLA.

Likewise, applying Theorem 3.10 together with [5, Theorem 3.10], we arrive at the next theorem.

Theorem 3.14. *For any pomonoid S, the following assertions are equivalent:*

(1) *Whenever S/K fulfills Condition $(G-(PWP)_w)$ $((GPWP))$, it also satisfies Condition $(PWP)_w$.*

(2) Every E-SLS and E-w-SLA (E-SLA) ideal turns out to be both SLS and w-SLA.

Theorem 3.15. *For any pomonoid S, the following assertions are equivalent:*

(1) *Whenever S/K fulfills Condition $(G-(PWP)_w)$ $((GPWP))$, then it also fulfills Condition (P).*

(2) The pomonoid S admits no E-SLS or E-w-SLA (E-SLA) ideal K with $|K| > 1$, and moreover S is weakly right reversible.

Proof. (1) $\Rightarrow$ (2) Assume that K is an E-SLS and also an E-w-SLA ideal. By Theorem 3.10, the quotient S/K satisfies $(G-(PWP)_w)$. According to our assumption, this implies that S/K also satisfies Condition (P). Hence, [5, Lemma 3.13] implies $|K| = 1$.

(2) $\Rightarrow$ (1) Let us suppose that S/K fulfills Condition $(G-(PWP)_w)$ for some ideal K (which may or may not be proper). We distinguish two situations:

Case 1. If $K = S$, then since S is weakly right reversible and $S/K \cong \Theta_S$, it follows from [5, Lemma 3.8] that S/K satisfies Condition (P).

Case 2. If $K \neq S$, then Theorem 3.10 ensures that K is both E-SLS and E-w-SLA. By assumption, this forces $|K| = 1$. Therefore, S/K satisfies Condition (P) by [5, Lemma 3.13]. □

It is evident from [5, Example 3.18] that satisfying Condition $(G-(PWP)_w)$ does not necessarily guarantee that Condition (P) holds.

Theorem 3.16. *For any pomonoid S, the following assertions are equivalent:*

(1) *Whenever S/K satisfies Condition $(G-(PWP)_w)$ ($(GPWP)$), it is weakly subpullback flat.*

(2) S admits neither E-SLS nor E-w-SLA (E-SLA) ideal K with cardinality $|K| > 1$, and, in addition, S is both weakly right reversible and weakly left collapsible.

Proof. (1) $\Rightarrow$ (2) Suppose that K is an ideal which is at the same time E-SLS and E-w-SLA. By Theorem 3.10, S/K satisfies Condition $(G-(PWP)_w)$. Our hypothesis then ensures that S/K is weakly subpullback flat, and by [5, Theorem 3.19] it follows that $|K| = 1$. Moreover, as Θ_S itself satisfies $(G-(PWP)_w)$, it is weakly subpullback flat under the assumption, hence [5, Theorem 3.19] yields that S must be weakly left collapsible and weakly right reversible.

(2) $\Rightarrow$ (1) Assume that for some (possibly improper) ideal K, S/K satisfies $(G-(PWP)_w)$. Two cases may occur:

Case 1. If $K = S$, then by [5, Lemma 3.12], $S/K \cong \Theta_S$ satisfies both Conditions (P) and (E'). Thus, Θ_S is weakly subpullback flat.

Case 2. If $K \neq S$, then by Theorem 3.10, K is *E-SLS* and *E-w-SLA*. The assumption thus forces $|K| = 1$, and consequently, by [5, Theorem 3.19], S/K is weakly subpullback flat. □

Theorem 3.17. *For a pomonoid S, the following assertions are equivalent:*

(1) *Whenever S/K satisfies Condition $(G - (PWP)_w)$ $((GPWP))$, it is strongly flat.*

(2) S admits neither *E-SLS* nor *E-w-SLA* (*E-SLA*) ideal K with $|K| > 1$, and in addition, S is left collapsible.

Proof. (1) ⇒ (2) Suppose K is an ideal which is simultaneously *E-SLS* and *E-w-SLA*. Then, by Theorem 3.10, S/K satisfies Condition $(G-(PWP)_w)$. By our assumption, this implies that S/K is strongly flat. Consequently, [5, Theorem 3.19] yields $|K| = 1$. Moreover, since Θ_S itself satisfies Condition $(G - (PWP)_w)$, it is strongly flat under the assumption, and therefore satisfies Condition (E). Hence [5, Lemma 3.12] yields that S must be left collapsible.

(2) ⇒ (1) Assume that for some ideal K (possibly improper), S/K fulfills Condition $(G - (PWP)_w)$. We consider two cases:

Case 1. If $K = S$, then since S is left collapsible, it follows from [5, Lemma 3.12] that $S/K \cong \Theta_S$ satisfies Condition (E). Hence, Θ_S is strongly flat.

Case 2. If $K \neq S$, then by Theorem 3.10, the ideal K must be both *E-SLS* and *E-w-SLA*. By hypothesis this forces $|K| = 1$, and thus, by [5, Theorem 3.19], S/K is strongly flat. □

Using an argument analogous to the proof of Theorem 3.17, the subsequent theorems hold.

Theorem 3.18. *For any pomonoid S, the following statements are equivalent:*

(1) *Whenever S/K satisfies Condition $(G - (PWP)_w)$ $((GPWP))$, it is projective.*

(2) S admits neither *E-SLS* nor *E-w-SLA* (*E-SLA*) ideal K with cardinality $|K| > 1$.

Theorem 3.19. *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(G-(PWP)_w)$ $((GPWP))$ is free.*

(2) *S admits neither E-SLS nor E-w-SLA (E-SLA)* ideal K with $|K| > 1$, and S has a left zero element.

Proof. (1) $\Rightarrow$ (2) Observe that Θ_S satisfies Condition $(G-(PWP)_w)$, hence, by the hypothesis, it must be free. It then follows from [1, Theorem 1] that the cardinality of K is 1.

(2) $\Rightarrow$ (1) This result is established by [6, Theorem 2.13]. □

4 Characterization by condition $(GPWP)$ on diagonal acts

In [4], the author examines direct products of S-posets under various flatness properties.

In this section, we focus on pomonoids S for which the direct product of two S-posets, each satisfying Conditions $(G-(PWP)_w)$ and $(GPWP)$, also satisfies these conditions.

A right S-poset A is said to be *weakly locally cyclic* if every finitely generated S-subposet of A is contained within a cyclic S-subposet. Moreover, a left ideal of S that is weakly locally cyclic is referred to as a *weakly locally principal left ideal*.

Following [4], for each element $s \in S$, define

$$L(s,s) = \{(x,y) \in D(S) \mid xs \leq ys\},$$

which constitutes a left S-subposet of $D(S)$, where $D(S)$ denotes the S-poset $S \times S$. Furthermore, for any $(p,q) \in D(S)$, let

$$\widehat{S(p,q)} = \{(x,y) \in D(S) \mid x \leq wp \text{ and } wq \leq y \text{ for some } w \in S\},$$

which also forms a left S-poset.

Proposition 4.1. *The following assertions are equivalent for any pomonoid S:*

(1) *Any finite product of S-posets satisfying Condition $(GPWP)$ satisfies Condition $(GPWP)$.*

(2) *$D(S)$ satisfies Condition $(GPWP)$.*

(3) *For every $s \in S$ there exists $n \in \mathbb{N}$ such that $L(s^n, s^n)$ is a weakly locally cyclic left S-poset.*

Proof. Implication (1) $\Rightarrow$ (2) holds trivially.

(2) $\Rightarrow$ (3) Suppose that $D(S)$ satisfies Condition $(GPWP)$ and let $s \in S$. Thus (2.1) holds for some $n \in \mathbb{N}$. Let $(a, b), (a', b') \in L(s^n, s^n)$. Then we have $(a, a')s^n \leq (b, b')s^n$. Because $D(S)$ satisfies Condition $(GPWP)$, it follows that $(a, a') = (x, y)u$, $(b, b') = (x, y)v$ and $us^n \leq vs^n$, for some $(x, y) \in D(S)$ and $u, v \in S$. Thus $(a, b), (a', b') \in S(u, v) \subseteq L(s^n, s^n)$, and therefore $L(s^n, s^n)$ is weakly locally cyclic.

(3) $\Rightarrow$ (1) Suppose that $A_1, \ldots, A_m$ are S-posets each satisfying Condition $(GPWP)$. Let $s \in S$. Then there exists a natural number $n \in \mathbb{N}$ such that $L(s^n, s^n)$ is a weakly locally cyclic left S-poset. Now let $(a_1, \ldots, a_m)s^n \leq (a'_1, \ldots, a'_m)s^n$ for $(a_1, \ldots, a_m), (a'_1, \ldots, a'_m) \in \prod_{i=1}^m A_i$. For each $1 \leq i \leq m$, $a_i s^n \leq a'_i s^n$ implies the existence of $a''_i \in A_i$, $p_i, q_i \in S$ such that $a''_i p_i = a_i$, $a''_i q_i = a'_i$, and $p_i s^n \leq q_i s^n$. Then $(p_i, q_i) \in L(s^n, s^n)$ for each $1 \leq i \leq m$. According to our assumption, $(p_i, q_i) \in S(p, q)$ for some $(p, q) \in L(s^n, s^n)$. Let us assume that $(p_i, q_i) = w_i(p, q)$ for $w_i \in S$, $1 \leq i \leq m$. Then $ps^n \leq qs^n$, $(a_1, \ldots, a_n) = (a''_1 w_1, \ldots, a''_m w_m)p$ and $(a'_1, \ldots, a'_m) = (a''_1 w_1, \ldots, a''_m w_m)q$, which shows that $\prod_{i=1}^m A_i$ satisfies Condition $(GPWP)$. □

The following proposition follows by an argument analogous to the proof of Proposition 4.1.

Proposition 4.2. *The following assertions are equivalent for any pomonoid S:*

(1) *Any finite product of S-posets satisfying Condition $(G-(PWP)_w)$, satisfies Condition $(G-(PWP)_w)$.*

(2) *$D(S)$ satisfies Condition $(G-(PWP)_w)$.*

(3) *For every $s \in S$, there exists $n \in \mathbb{N}$ such that either for each 2 elements $(x, y), (x', y') \in L(s^n, s^n)$, $(x, y), (x', y') \in \widehat{S(p, q)}$ for some $(p, q) \in L(s^n, s^n)$ or $L(s^n, s^n)$ is empty.*

Proposition 4.3. *The following assertions are equivalent for any pomonoid S:*

(1) *The direct product of every nonempty family of S-posets satisfying Condition $(G-(PWP)_w)$, satisfies Condition $(G-(PWP)_w)$.*

(2) *$(S^I)_S$ satisfies Condition $(G-(PWP)_w)$ for every nonempty set I.*

(3) *For every $s \in S$, there exists $n \in \mathbb{N}$ such that either $L(s^n, s^n) = \widehat{S(p,q)}$ for some $(p,q) \in L(s^n, s^n)$ or $L(s^n, s^n)$ is empty.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Let $s \in S$. Since for every nonempty set I, $(S^I)_S$ satisfies Condition $(G-(PWP)_w)$, then there exists $n \in \mathbb{N}$ such that (2.1) holds. Let $L(s^n, s^n) \neq \emptyset$. Write $L(s^n, s^n) = \{(x_i, y_i) | i \in I\}$. Let x_i and y_i be the i'th components of the elements u and v of S^I, respectively. So we have $us^n \leq vs^n$ in S^I. Because S^I satisfies Condition $(G-(PWP)_w)$, it follows that there exist $p, q \in S$ and $w \in S^I$ such that $u \leq wp$, $wq \leq v$ and $ps^n \leq qs^n$. Hence $(p,q) \in L(s^n, s^n)$, and for each $i \in I$ we obtain $x_i \leq w_i p$, $w_i q \leq y_i$ where w_i is the i'th component of w. Consequently, $L(s^n, s^n) = \widehat{S(p,q)}$, completing the proof.

(3) $\Rightarrow$ (1) Let $\{A_j | j \in I\}$ be a family of S-posets satisfying Condition $(G-(PWP)_w)$ and $A = \prod_{i \in I} A_i$. Let $s \in S$. As assumed there exists $n \in \mathbb{N}$ such that either $L(s^n, s^n) = \widehat{S(p,q)}$ for some $(p,q) \in L(s^n, s^n)$ or $L(s^n, s^n)$ is empty. Suppose $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$ be the elements of A such that $as^n \leq bs^n$. Then we have $a_i s^n \leq b_i s^n$ for each $i \in I$. Under the assumption that A_i satisfies Condition $(G-(PWP)_w)$, there exist $u_i, v_i \in S$ and $c_i \in A_i$ with $a_i \leq c_i u_i$, $c_i v_i \leq b_i$, and $u_i s^n \leq v_i s^n$. Therefore, $(u_i, v_i) \in L(s^n, s^n) \neq \emptyset$ and as assumed $L(s^n, s^n) = \widehat{S(p,q)}$ for some $(p,q) \in L(s^n, s^n)$. Then for every $(u_i, v_i) \in L(s^n, s^n)$, $u_i \leq w_i p$ and $w_i q \leq v_i$ for some $w_i \in S$. Thus, $ps^n \leq qs^n$, and for each $i \in I$ we have $a_i \leq c_i w_i p$ and $c_i w_i q \leq b_i$. Put $a' = (c_i w_i)_{i \in I} \in A$. Thus $a \leq a'p$ and $a'q \leq b$. □

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