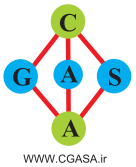
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On the homological classification of monoids by Condition $(G\text{-}PWP_{ssc})$ of right S-acts

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Abstract. Condition (PWP_{ssc}) introduced in (Khamechi, P., Mohammadzadeh Saany, H., Nouri, L., *Classification of monoids by Condition* (PWP_{ssc}) *of right S-acts*, Categ. Gen. Algebr. Struct. Appl. 12(1) (2020), 175-197), is associated with the concept of flatness for acts over monoids. In this paper, we introduce Condition $(G\text{-}PWP_{ssc})$, which serves as a generalization of Condition (PWP_{ssc}). Furthermore, we demonstrate that GP-flat is an extension of this condition, and the converse holds for a G-left PSF monoid. Additionally, we discuss some general properties of Condition $(G\text{-}PWP_{ssc})$ and provide a homological classification of monoids by comparing Condition $(G\text{-}PWP_{ssc})$ with some other properties.

1 Introduction and Preliminaries

For a monoid S, a nonempty set A is called a right S-act (usually denoted A_S) if S acts on A unitarily from the right. This means there exists a

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mapping $A \times S \longrightarrow A$, $(a, s) \longmapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A_S$ and all $s, t \in S$. Henceforth, when we refer to an S-act, we are specifically referring to a right S-act. In this paper, S will consistently denote a monoid and A will represent a right S-act.

For $w \in S$, denote by $\rho(w, 1)$ the right congruence on S generated by the pair $(w, 1)$. For $x, y, s \in S$, let $\rho(xs, ys)$ denote the right congruence on S generated by the pair (xs, ys). For $s \in S$, let $\rho_s : S \to S$ be the right translation defined by $\rho_s(x) = xs$. For $z \in S$, let $\lambda_z : S \to S$ be the left translation defined by $\lambda_z(s) = zs$ for all $s \in S$. We denote by $\ker \lambda_z$ the kernel of this map, i.e., $\ker \lambda_z = \{(x, y) \in S \times S \mid zx = zy\}$.

A right S-act A is called *torsion-free* if for all $a, a' \in A$ and every right cancellable element $c \in S$, the equality $ac = a'c$ implies $a = a'$. Also, a right S-act A is called $\mathfrak{R}$-torsion-free if for all $a, b \in A$, any right cancellable $c \in S$, the conditions $ac = bc$ and $a\mathfrak{R}b$ (where $\mathfrak{R}$ is Green's equivalence, as described in [13]) imply that $a = b$. Note that torsion-freeness implies $\mathfrak{R}$-torsion-freeness; however, the converse does not hold in general (see [13, 25]) For foundational definitions and terminology relating semigroups and acts over monoids, we direct the reader to [11, 13].
As per the definition provided in [13], a right S-act A is called *flat* if the functor $A \otimes_S -$ preserves all monomorphisms of left S-acts. If the functor $A \otimes_S -$ preserves all embeddings of (principal) left ideals into S, then A is called (*principally*) *weakly flat*. A right S-act A satisfies *Condition* (E) if for all $a \in A$ and $s, s' \in S$, $as = as'$ implies the existence of $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = us'$. It satisfies *Condition* (P) if for all $a, a' \in A$ and $s, s' \in S$, $as = a's'$ implies the existence of $a'' \in A$ and $u, u' \in S$ such that $a = a''u, a' = a''u'$, and $us = u's'$. A right S-act A is considered *strongly flat* if it satisfies both Conditions (P) and (E).

Moreover, a right S-act A satisfies Condition (P') if for all $a, a' \in A$ and $s, t, z \in S$, $as = a't$ and $sz = tz$, there exists $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$, and $us = vt$. It is obvious that Condition (P) implies Condition (P'), but the converse need not hold. For more comprehensive information, refer to [9].

Recall from [6, 10, 14, 15, 19] that a right S-act A satisfies Condition (WP) if for all elements $s, t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$ and all $a, a' \in A$, when $af(s) = a'f(t)$, then there exist $a'' \in A$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes_S (Ss \cup St)$,

and $f(us') = f(vt')$. Furthermore, it satisfies *Condition* (P_E) if for all $a, a' \in A$, and $s, s' \in S$, $as = a's'$ implies the existence of $a'' \in A$, $u, u' \in S$ and $e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''u'f, es = s, fs' = s'$ and $us = u's'$. A right S-act A satisfies *Condition* (PWP) if, whenever $as = a's$ with $a, a' \in A$ and $s \in S$, there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. Moreover, a right S-act A satisfies Condition (PWP_E) if, whenever $as = a's$ with $a, a' \in A$ and $s \in S$, there exist $a'' \in A$, $u, v \in S$ and $e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf, es = s = fs$ and $us = vs$, and it satisfies Condition (PWP_{sc}) if, whenever $as = a's$, with $a, a' \in A$ and $s \in S$, there exist $a'' \in A$ and $u, v, r, r' \in S$ such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s = r's$ and $us = vs$.
Recall from [18] that a right S-act A satisfies *Condition* (P_{sc}) if, whenever $as = a't$ for $a, a' \in A$ and $s, t \in S$, there exist $a'' \in A$ and $u, v, r, r' \in S$, such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$ and $us = vt$. We recall from [12] that a right S-act A satisfies Condition (PWP_{ssc}) if, whenever $as = a's$ for $a, a' \in A$ and $s \in S$, there exists $r \in S$ such that $ar = a'r$ and $rs = s$. Recall from [22] that a right S-act A is called GPW-flat if for every $s \in S$, there exists $n = n_{(s,A_S)} \in \mathbb{N}$, such that the functor $A_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$.
The paper is organized as follows. In Section 2 we recall basic notions and preliminary results concerning acts over monoids and introduce Condition (G-PWP_{ssc}). We also establish several basic properties of this condition. In Section 3 we investigate deeper relationships between (G-PWP_{ssc}), GP-flatness and related homological properties, and we obtain a number of classification theorems for monoids.

2 General properties

In this section, we introduce Condition (G-PWP_{ssc}) and demonstrate that a coproduct satisfies the condition if and only if each of its components does. Additionally, we establish that S_S and every retract of an act satisfying Condition (G-PWP_{ssc}) satisfy Condition (G-PWP_{ssc}). Furthermore, we observe that Condition (G-PWP_{ssc}) implies GP-flatness, but the converse does not hold in general. For G-left PSF monoids we prove that the converse holds.

Now we propose a generalization of Condition (PWP_{ssc}) for right acts.

Definition 2.1. A right S-act A satisfies Condition $(G\text{-}PWP_{ssc})$ if, for all $a, a' \in A$ and $s \in S$, whenever $as = a's$, there exist $r \in S$ and $n \in \mathbb{N}$ such that $ar = a'r$ and $rs^n = s^n$.

By comparing Definition 2.1 with the definition of Condition (PWP_{ssc}), it follows that for $n = 1$ the two conditions are equivalent. Additionally, Condition (PWP_{ssc}) for right S-acts implies Condition $(G\text{-}PWP_{ssc})$, but we will demonstrate in Example 2.23 that the converse does not hold in general.

We begin by establishing several fundamental properties of Condition $(G\text{-}PWP_{ssc})$. In particular, we examine the stability of this condition under some standard constructions in the category of right S-acts. These closure properties play an important role in the subsequent results and will be used repeatedly in the classification theorems obtained later in the paper.

Theorem 2.2. *The following statements hold:*

(1) Θ_S *satisfies Condition* $(G\text{-}PWP_{ssc})$.

(2) *If A satisfies Condition $(G\text{-}PWP_{ssc})$, then every subact of A satisfies Condition $(G\text{-}PWP_{ssc})$.*

(3) S_S *satisfies Condition* $(G\text{-}PWP_{ssc})$ *if and only if*

$$(\forall x, y, s \in S)(xs = ys \Rightarrow (\exists r \in S)(\exists n \in \mathbb{N})(xr = yr \wedge rs^n = s^n)).$$

(4) *Suppose that $A = \coprod_{i \in I} A_i$, where each A_i is a right S-act. Then A satisfies Condition $(G\text{-}PWP_{ssc})$ if and only if each A_i satisfies Condition $(G\text{-}PWP_{ssc})$.*

(5) *Let $\{A_i \mid i \in I\}$ be a nonempty chain of right subacts of A. Then, each A_i satisfies Condition $(G\text{-}PWP_{ssc})$ if and only if $\bigcup_{i \in I} A_i$ satisfies Condition $(G\text{-}PWP_{ssc})$.*

(6) *Every retract of S-act A satisfying Condition $(G\text{-}PWP_{ssc})$ also satisfies Condition $(G\text{-}PWP_{ssc})$.*

(7) *Let* $A = \prod_{i\in I} A_i$ *, where* $A_i \in Act - S$ *for every* $i \in I$ *(*I *is a nonempty set). If* $A = \prod_{i\in I} A_i$ *satisfies Condition (G-PWP*$_{ssc}$*), then for every* $i \in I$*,* A_i *satisfies Condition (G-PWP*$_{ssc}$*).*

Proof. The assertions are immediate from Definition 2.1. We provide proofs of parts (5), (6) and (7) for completeness.

(5) Let $B = \bigcup_{i\in I} A_i$. Suppose first that each A_i satisfies Condition (G-PWP$_{ssc}$). For $a, a' \in B$ with $as = a's$, choose $i, j \in I$ such that $a \in A_i$ and $a' \in A_j$. Since $\{A_i \mid i \in I\}$ is a chain, both a and a' belong to one of A_i, A_j. Hence, Condition (G-PWP$_{ssc}$) in that subact yields $r \in S$ and $n \in \mathbb{N}$ such that

$$ar = a'r, \qquad rs^n = s^n.$$

Thus B satisfies Condition (G-PWP$_{ssc}$). The converse follows immediately from $A_i \subseteq B$ for every $i \in I$.

(6) Let B be a retract of A, and let $f : A \to B$ and $g : B \to A$ be S-act morphisms with $fg = 1_B$. Suppose that A satisfies Condition (G-PWP$_{ssc}$). For $b, b' \in B$ with $bs = b's$, we have

$$g(b)s = g(b')s.$$

Hence, there exist $r \in S$ and $n \in \mathbb{N}$ such that

$$g(b)r = g(b')r, \qquad rs^n = s^n.$$

Applying f gives $br = b'r$, while $rs^n = s^n$ remains unchanged. Therefore, B satisfies Condition (G-PWP$_{ssc}$).

(7) Let $A_S = \prod_{i\in I} A_i$ satisfies Condition (G-PWP$_{ssc}$) and for $i \in I$, $a_i, a_i' \in A_i$, $s \in S$, $a_i s = a_i' s$. For every $j \in I \setminus \{i\}$, let a_j be an arbitrary fixed element of A_j. Now, for $k \in I$, define:

$$c_k = \begin{cases} a_i, & k = i, \\ a_k, & k \neq i. \end{cases} \quad , \qquad c_k' = \begin{cases} a_i', & k = i, \\ a_k, & k \neq i. \end{cases}$$

Clearly, $(c_k)_I s = (c_k')_I s$. Since $A_S = \prod_{i\in I} A_i$ satisfies Condition (G-PWP$_{ssc}$), the last equality implies the existence of $r \in S$ and $n \in \mathbb{N}$ such that

$$(c_k)_I r = (c_k')_I r,$$
$$rs^n = s^n.$$

Therefore, $a_i r = a_i' r$ and $rs^n = s^n$, and so A_i satisfies Condition $(G\text{-}PWP_{ssc})$. □

Remark 2.3. Theorem 2.2(7) establishes only the forward implication: if a direct product satisfies Condition $(G\text{-}PWP_{ssc})$, then each of its factors satisfies the same condition. The converse implication has neither been proved nor claimed in this paper. Consequently, unlike the coproduct case in Theorem 2.2(4), no equivalence is asserted for direct products here.

An element u of S is termed *right semi-cancellable*, if $su = s'u$, for $s, s', u \in S$, implying there exists $r \in S$ such that $u = ru$ and $sr = s'r$. The monoid S is designated as *right semi-cancellative*, if every $s \in S$, is right semi-cancellable. Furthermore, the monoid S is denoted as a *left PSF* if every principal left ideal of S is strongly flat as a left S-act. This condition is synonymous with stating that S is right semi-cancellative (refer to [16]). Similarly, a left semi-cancellative monoid or a right PSF monoid can also be defined.

Definition 2.4. An element $s \in S$ is termed *eventually right semi-cancellable* if there exists $n \in \mathbb{N}$ such that s^n is right semi-cancellable. In other words, $s \in S$ is eventually right semi-cancellable if and only if

$$(\exists n \in \mathbb{N})(\forall x, y \in S)(xs^n = ys^n \Rightarrow (\exists r \in S)(xr = yr \wedge rs^n = s^n)).$$

The monoid S is denoted as eventually right semi-cancellative if all elements of S are eventually right semi-cancellable.

By Definition 2.4, the monoid S is eventually right semi-cancellative when, for every $s \in S$, there exists $n \in \mathbb{N}$ such that principal left ideal Ss^n is strongly flat as a left S-act. Consequently, an eventually right semi-cancellative monoid is also referred to as an *eventually left PSF* monoid.

As mentioned in [3, 17], the monoid S is termed *right(left) generally regular* if for every $s \in S$, there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = sxs^n$ $(s^n = s^n xs)$.
The monoid S is defined as *eventually regular* if for every $s \in S$, there exists $n \in \mathbb{N}$ such that s^n is regular.
Every regular monoid is eventually regular (take $n = 1$), and also, every eventually regular monoid is right (left) generally regular.
While the converse of the above implications is not true in general, it can

be observed that for a commutative monoid S, eventually regular and right generally regular are equivalent.

As per references [13, 21, 22], an element s of monoid S is termed *left almost regular* if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$ and right cancellable elements $c_1, ..., c_m \in S$ such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_mc_m &= s_{m-1}r_m \\ s &= s_mrs. \end{aligned}$$

Furthermore, s is denoted as *generally left almost regular* if there exist $r, r_1, \cdots, , r_m, s_1, \cdots, , s_m \in S$, right cancellable elements $c_1, \cdots, , c_m \in S$ and $n \in \mathbb{N}$ such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_mc_m &= s_{m-1}r_m \\ s^n &= s_mrs^n. \end{aligned}$$

The monoid S is termed as *(generally) left almost regular* if all elements of S are (generally) left almost regular. Every left almost regular monoid is generally left almost regular (taking $n = 1$). The monoid in Example 3.3 of [21] is generally left almost regular but, as verified in that example, it is not left almost regular.

The monoid S is defined *eventually left (right) almost regular* if for every $s \in S$, there exists $n \in \mathbb{N}$ such that s^n is left (right) almost regular.

Every left almost regular monoid is eventually left almost regular (take $n = 1$), and also, every eventually left almost regular monoid is generally left almost regular. However, the converse of the above implications is not always true in general.

It is well known that every regular monoid is left (right) almost regular. Now, let S be a right generally regular monoid and let $s \in S$. Then there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = sxs^n$. Consequently,

$$s \cdot 1 = s \cdot 1 \qquad \text{and} \qquad s^n = sxs^n.$$

Thus, s is generally left almost regular. Similarly, every eventually regular element in S is eventually left (right) almost regular.

Definition 2.5. The monoid S is termed as G-left (right) PSF, if S_S ($_SS$) satisfies Condition (G-PWP_{ssc}).

In the following example we give a G-left PSF monoid which is not left PSF.

Example 2.6. [G-left PSF does not imply left PSF] Let $S = \{1, x, 0\}$ be the monoid with zero where $x^2 = 0$. The monoid S is not left PSF. Indeed, if it were, then the equality $0.x = x.x$ would imply the existence of $r \in S$ such that $0.r = x.r$ and $rx = x$. The last equality implies $r = 1$, and so $x = 0$, which is a contradiction. Thus, S is not left PSF. On the other hand, S is a G-left PSF monoid. Indeed, let $a, b, s \in S$ satisfy $as = bs$. If $s = 1$, then $a = b$, and we may take $r = 1$ and $n = 1$. If $s = 0$, take $r = 0$ and $n = 1$. Finally, if $s = x$, then $x^2 = 0$; hence, taking $r = 0$ and $n = 2$, we obtain

$$ar = br = 0 \qquad \text{and} \qquad rs^n = 0x^2 = 0 = x^2 = s^n.$$

Thus, in every case, there exist $r \in S$ and $n \in \mathbb{N}$ such that

$$ar = br \qquad \text{and} \qquad rs^n = s^n.$$

Consequently, S_S satisfies Condition (G-PWP_{ssc}), and hence S is a G-left PSF monoid.

Theorem 2.7. *For the monoid S:*

(1) *Left PSF $\Longrightarrow$ eventually left PSF $\Longrightarrow$ G-left PSF.*

(2) *Generally left almost regular $\Longrightarrow$ G-left PSF.*

Proof. (1) Every left PSF monoid is eventually left PSF (take $n = 1$). Assume that S is eventually left PSF and suppose $xs = ys$, for $x, y, s \in S$. By definition of eventually left PSF, there exists $n \in \mathbb{N}$ such that the principal left ideal Ss^n is strongly flat. Now, from $xs = ys$, multiplying on the right by s^{n-1} (if $n \geq 1$) gives $xs^n = ys^n$. Since Ss^n is strongly flat, it satisfies Condition (E), and hence there exists $r \in S$ such that $xr = yr$ and

$rs^n = s^n$. Thus, for the given s, we have found $r \in S$ and $n \in \mathbb{N}$ satisfying the required conditions. Therefore S is a G-left PSF monoid.

(2) Assume that S is generally left almost regular and $xs = ys$, for $x, y, s \in S$. Then, there exist $r, r_1, \cdots, , r_m, s_1, \cdots, , s_m \in S$, right cancellable elements $c_1, \cdots, , c_m \in S$ and $n \in \mathbb{N}$ such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\cdots \\ s_mc_m &= s_{m-1}r_m \\ s^n &= s_mrs^n. \end{aligned}$$

Now, the equality $xs = ys$ implies $xsr_1 = ysr_1$, and so $xs_1c_1 = ys_1c_1$. Since c_1 is right cancellable, $xs_1 = ys_1$. Therefore, $xs_1r_2 = ys_1r_2$ implies $xs_2c_2 = ys_2c_2$. Since c_2 is right cancellable, $xs_2 = ys_2$. Continuing this procedure, for every i, $(1 \leq i \leq m)$, $xs_i = ys_i$, and so $xs_m = ys_m$ which implies $xs_mr = ys_mr$. Let $s_mr = r^\star$. Then $xr^\star = yr^\star$ and $r^\star s^n = s_mrs^n = s^n$, indicating that S is G-left PSF. □

By employing a similar argument to the proof of Theorem 2.7, we can demonstrate that for every monoid S,

(1) Right $PSF \Longrightarrow$ eventually right $PSF \Longrightarrow$ G-right PSF.

(2) Generally right almost regular $\Longrightarrow$ G-right PSF.

The following theorem on cyclic right S-acts follows immediately from the definition of Condition (*G-PWP$_{ssc}$*).

Theorem 2.8. *Let ρ be a right congruence on S. Then the right S-act S/ρ satisfies Condition (G-PWP$_{ssc}$) if and only if*

$$(\forall x, y, s \in S)((xs)\rho(ys) \Longrightarrow (\exists r \in S)(\exists n \in \mathbb{N})((xr)\rho(yr) \wedge rs^n = s^n)).$$

Corollary 2.9. *For $z \in S$, the principal right ideal zS satisfies Condition (G-PWP$_{ssc}$) if and only if*

$$(\forall x, y, s \in S)(zxs = zys \Longrightarrow (\exists r \in S)(\exists n \in \mathbb{N})(zxr = zyr \wedge rs^n = s^n))).$$

Proof. Using the property $zS \cong S/\ker\lambda_z$, for every $z \in S$, it is sufficient to substitute $\rho = \ker\lambda_z$ into Theorem 2.8. □

Theorem 2.10. *For $w \in S$, the right S-act $S/\rho(w,1)$ satisfies Condition (G-PWP_{ssc}) if and only if*

$$(\forall x,y,s \in S)(\forall m,n \in \mathbb{N}_0)(w^m xs = w^n ys \Longrightarrow (\exists p,q \in \mathbb{N}_0)(\exists l \in \mathbb{N})(\exists r \in S)(w^p xr = w^q yr \wedge rs^l = s^l)).$$

Proof. Let $\rho = \rho(w,1)$.
Necessity: If $w^m xs = w^n ys$, for $x,y,s \in S$ and $m,n \in \mathbb{N}_0$, then by [13, Corollary 3.8.7], $(xs)\rho(ys)$. According to Theorem 2.8, there exist $r \in S$ and $l \in \mathbb{N}$ such that $(xr)\rho(yr)$ and $rs^l = s^l$. Hence, by [13, Corollary 3.8.7], there exist $p,q \in \mathbb{N}_0$ such that $w^p xr = w^q yr$.
Sufficiency: If $(xs)\rho(ys)$, for $x,y,s \in S$, then by [13, Corollary 3.8.7], there exist $m,n \in \mathbb{N}_0$ such that $w^m xs = w^n ys$. By assumption, there exist $r \in S$, $p,q \in \mathbb{N}_0$ and $l \in \mathbb{N}$ such that $rs^l = s^l$ and $w^p xr = w^q yr$. This implies $(xr)\rho(yr)$, by [13, Corollary 3.8.7], and so S/ρ satisfies Condition (G-PWP_{ssc}), according to Theorem 2.8. □

Corollary 2.11. *Let $e^2 = e \in E(S)$. The following statements are equivalent:*

(1) *The monocyclic right S-act $S/\rho(e,1)$ satisfies Condition (G-PWP_{ssc}).*

(2) *eS satisfies Condition (G-PWP_{ssc}).*

(3) $(\forall x,y,s \in S)(exs = eys \Longrightarrow (\exists r \in S)(\exists n \in \mathbb{N})(exr = eyr \wedge rs^n = s^n))$.

Proof. (1) $\Leftrightarrow$ (3). It follows by Theorem 2.10.
(2) $\Leftrightarrow$ (3). It follows by Corollary 2.9. □

Specifically, setting $w = 1$ in Theorem 2.10 yields $S/\rho = S/\rho(w,1) = S/\rho(1,1) = S/\Delta_S \cong S_S$. Hence, part (3) of Theorem 2.2 is obtained.

Corollary 2.12. *Let $e^2 = e \in E(S)$ be a left zero element of S. Then the cyclic right S-act eS and the monocyclic right S-act $S/\rho(e,1)$ satisfy Condition (G-PWP_{ssc}).*

Proof. This follows immediately from Corollary 2.11. □

As per [21], a right S-act A is considered GP-flat, if $a \otimes s = a' \otimes s$ in $A \otimes_S S$, for $s \in S$ and $a, a' \in A$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A \otimes_S Ss^n$. It follows from the definitions that every principally weakly flat act is GP-flat, and it is proved that every GP-flat act is torsion-free, but the converse of both implications is not generally true.
Now, in the following theorem, we show that Condition (*G-PWP$_{ssc}$*) implies GP-flat, but the converse is not true. Additionally, for a G-left PSF monoid S, we establish that the converse is also true.

Theorem 2.13. *The following statements hold:*

(1) *Every right S-act satisfying Condition (G-PWP$_{ssc}$) is GP-flat.*

(2) *For a G-left PSF monoid S, the right S-act A is GP-flat if and only if A satisfies Condition (G-PWP$_{ssc}$).*

Proof. (1) Let A be a right S-act satisfying Condition (*G-PWP$_{ssc}$*) and $a \otimes s = a' \otimes s$ in $A \otimes_S S$, for $s \in S$ and $a, a' \in A$. According to [13, Proposition 2.5.13], we have $as = a's$. By assumption, there exist $r \in S$ and $n \in \mathbb{N}$, such that $ar = a'r$ and $rs^n = s^n$. Now, we can observe:

$$a \otimes s^n = a \otimes rs^n = ar \otimes s^n = a'r \otimes s^n = a' \otimes rs^n = a' \otimes s^n,$$

in $A \otimes_S Ss^n$. Thus A is GP-flat.

(2) Necessity: Let $as = a's$, for $s \in S$ and $a, a' \in A$. Then $a \otimes s = a' \otimes s$ in $A \otimes_S S$, by [13, Proposition 2.5.13]. Moreover, based on [21, Lemma 2.2], there exist $m, n \in \mathbb{N}$ and elements $a_1, \cdots a_m \in A$, $s_1, t_1, \cdots, , s_m, t_m \in S$ such that

$$\begin{array}{rr} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 s^n = t_1 s^n \\ a_2 t_2 = a_3 s_3 & s_2 s^n = t_2 s^n \\ \vdots & \vdots \\ a_m t_m = a^{'} & s_m s^n = t_m s^n. \end{array}$$

Since S is G-left PSF, the equality $s_1 s^n = t_1 s^n$ implies the existence of $r_1 \in S$ and $n_1 \in \mathbb{N}$, such that $s_1 r_1 = t_1 r_1$ and $r_1 (s^n)^{n_1} = (s^n)^{n_1}$. Subsequently, $s_2 r_1 s^{nn_1} = t_2 r_1 s^{nn_1}$ implies the existence of $r_2 \in S$ and $n_2 \in \mathbb{N}$ such that $s_2 r_1 r_2 = t_2 r_1 r_2$ and $r_2 (s^{nn_1})^{n_2} = (s^{nn_1})^{n_2}$. If $r = r_1 r_2$, then $rs^{nn_1 n_2} =$

$r_1r_2s^{nn_1n_2} = r_1s^{nn_1n_2} = s^{nn_1n_2}$, $s_1r = s_1r_1r_2 = t_1r_1r_2 = t_1r$ and $s_2r = t_2r$. By continuing this procedure, we can find $l \in S$ and $k \in \mathbb{N}$ such that $ls^k = s^k$ and $s_il = t_il$, for every $1 \leq i \leq m$. Thus, $al = a_1s_1l = a_1t_1l = a_2s_2l = \cdots = a_mt_ml = a'l$, as required.

Sufficiency: It is obvious by (1). □

According to Theorem 2.13, Condition (G-PWP_{ssc}) implies GP-flatness. On the other hand, GP-flatness implies torsion-freeness. Hence, Condition (G-PWP_{ssc}) implies torsion-freeness.
From Theorem 2.13, we have the following corollary.

Corollary 2.14. *For a G-left PSF monoid S, the right S-act A is GP-flat if and only if for $s \in S$, and $a, a' \in A$ with $as = a's$, there exist $u \in S$ and $n \in \mathbb{N}$ such that $us^n = s^n$ and $au = a'u$.*

Obviously, by Theorem 2.7, the above corollary is true for eventually left PSF monoids and also generally left almost regular.

Proposition 2.15. *For a left PSF monoid S, the right S-act A is GP-flat if and only if for $s \in S$, and $a, a^{'} \in A$ with $as = a^{'}s$, there exist $u \in S$ and $n \in \mathbb{N}$ such that $us^n = s^n$ and $au = a^{'}u$.*

This recovers Proposition 2.5 of [20] as a direct consequence of the above.
From Theorem 2.13, we have the following corollary.

Corollary 2.16. *Let $n \in \mathbb{N}$ and let $A_1, \ldots, A_n$ be right S-acts. For a G-left PSF monoid S, the following statements are equivalent:*

(1) $\prod_{i=1}^{n} A_i$ *is GP-flat;*

(2) *for every $s \in S$ and every $a_i, a_i' \in A_i$ (where $i = 1, \ldots, n$),*

$$(a_1, \ldots, a_n)s = (a_1', \ldots, a_n')s \implies \exists u \in S,\ \exists m \in \mathbb{N} : \\ (a_1, \ldots, a_n)u = (a_1', \ldots, a_n')u \quad \text{and} \quad us^m = s^m.$$

Proof. This is an immediate consequence of Corollary 2.14 applied to the product act $\prod_{i=1}^{n} A_i$. □

Theorem 2.17. *For a G-left PSF monoid S, $\prod_{i=1}^{n} A_i$ is GP-flat if and only if A_i is GP-flat, for every i, $1 \leq i \leq n$.*

Proof. Necessity: Suppose that $\prod_{i=1}^{n} A_i$ is GP-flat. Since [20, Lemma 2.8] applies to any family of right S-acts (in particular, to the finite family $\{A_1, \ldots, A_n\}$), it follows immediately that each A_i is GP-flat.

Sufficiency: Let $(a_1, \cdots, , a_n)s = (a'_1, \cdots, , a'_n)s$, for $a_i, a'_i \in A_i, 1 \leq i \leq n$, and $s \in S$. Thus $a_i s = a'_i s$, for every i, $(1 \leq i \leq n)$. By the assumption and Corollary 2.14, there exist $r_1 \in S$ and $k_1 \in \mathbb{N}$ such that $a_1 r_1 = a'_1 r_1$ and $r_1 s^{k_1} = s^{k_1}$. Consequently, $a_2 s^{k_1} = a'_2 s^{k_1}$ and $a_2 r_1 s^{k_1} = a'_2 r_1 s^{k_1}$, implying the existence of $r_2 \in S$ and $k_2 \in \mathbb{N}$ such that $a_2 r_1 r_2 = a'_2 r_1 r_2$ and $r_2 (s^{k_1})^{k_2} = (s^{k_1})^{k_2}$. Therefore $a_1 r_1 r_2 = a'_1 r_1 r_2$, $a_2 r_1 r_2 = a'_2 r_1 r_2$ and $r_1 r_2 s^{k_1 k_2} = s^{k_1 k_2}$. This progression continues until we have $r_1, \cdots r_n \in S$, $k_1, \cdots, , k_n \in \mathbb{N}$ such that for every i, $(1 \leq i \leq n)$, $a_i r_1 \cdots r_n = a'_i r_1 \cdots r_n$, $r_1 \cdots r_n s^{k_1 \cdots k_n} = s^{k_1 \cdots k_n}$. Let $r = r_1 \cdots r_n$ and $k = k_1 \cdots k_n$. Then, for every i, $(1 \leq i \leq n)$, $a_i r = a'_i r$ and $r s^k = s^k$. Thus $(a_1, \cdots, , a_n)r = (a'_1, \cdots, , a'_n)r$, as desired. □

Since S_S is GP-flat, the following corollary can be derived from the above theorem.

Corollary 2.18. *For a G-left PSF monoid S, the right S-act $(S_S)^n$ is GP-flat, for every $n \in \mathbb{N}$.*

In the following example, we show that Condition (*G-PWP$_{ssc}$*) does not imply PWF .

Example 2.19. [Condition (*G-PWP$_{ssc}$*) does not imply PWF] For this purpose, consider the monoid $S = \{1, x, 0\}$ with $x^2 = 0$. By Example 2.6, S is G-left PSF monoid, but not left PSF. Hence, S_S satisfies (*G-PWP$_{ssc}$*). Therefore, by Theorem 2.13(2), the class of GP-flat S-acts coincides with the class of right S-acts satisfying Condition (*G-PWP$_{ssc}$*). By Theorem 2.17, the right S-act $(S_S)^2 = (S \times S)_S$ is GP-flat and thus also satisfies Condition (*G-PWP$_{ssc}$*). However, according to [1, Example 4], the right S-act $(S_S)^2 = (S \times S)_S$ is not PWF.

In the following example, we give a monoid that it is not a G-left PSF monoid.

Example 2.20. Let G be a nontrivial group with identity element e, and let $N' = \mathbb{N} \setminus \{1\}$. Consider the set $S = G \cup N'$. Define a binary operation on S as follows:

- For $g, h \in G$, the product gh is the usual multiplication in the group G.
- For $m, n \in N'$, the product mn is the usual multiplication of natural numbers (so $mn \in N'$ since $m, n \geq 2$).
- For $g \in G$ and $n \in N'$, define $ng = gn = n$.

The identity element of S is the identity element e of G. Indeed, for any $g \in G$, $eg = ge = g$, and for any $n \in N'$, $en = ne = n$ by the mixed product rule. The operation is associative: multiplication in G and in N' are associative, and any mixed product involving an element of G and an element of N' reduces to the element of N', so associativity holds in all cases. Thus, S is a monoid.
We show that S is not a G-left PSF monoid. Let $x, y \in G$ and $s \in N'$ with $x \neq y$, and suppose that $xs = ys$. We claim that for every $r \in S$ and every $n \in \mathbb{N}$,

$$(xr = yr) \quad \text{and} \quad (rs^n = s^n)$$

do not hold simultaneously. Let $r \in S$. If $r \in G$, then $xr \neq yr$, and hence the claim follows. If $r \in N'$, then for every $n \in \mathbb{N}$, we have $rs^n \neq s^n$, and therefore the claim follows again. Consequently, the monoid S is not a G-left PSF monoid.

Example 2.21. [Freeness does not imply Condition (G-PWP_{ssc})] By the previous example, there exist monoids S that are not G-left PSF while S_S is free. Hence, we conclude that freeness does not imply Condition (G-PWP_{ssc}).

By the above example, we conclude that all flatness properties that follow from freeness do not hold under Condition (G-PWP_{ssc}).

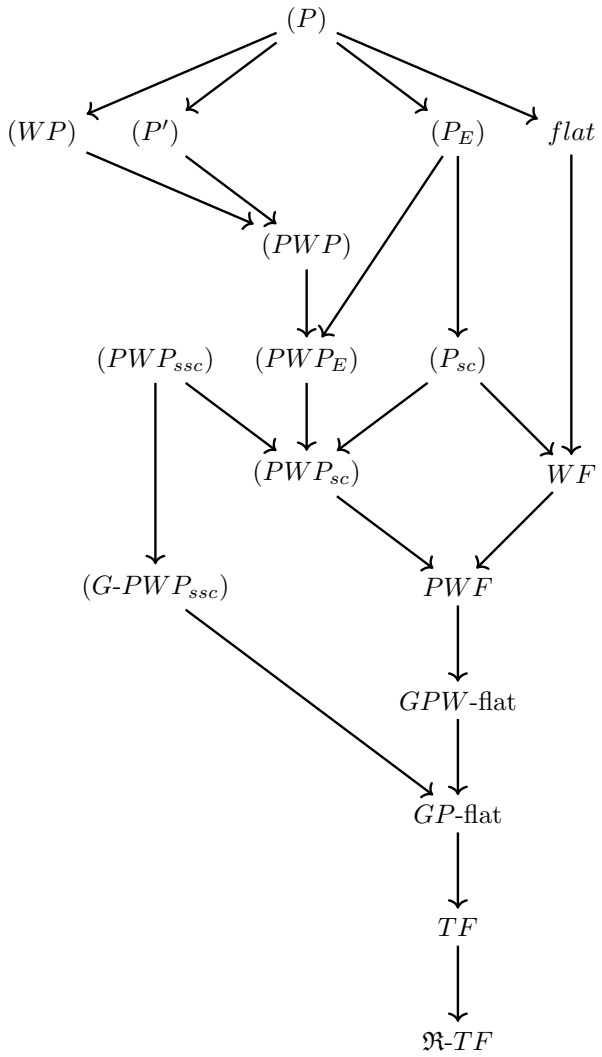
As a reminder from [13], S is called *right reversible* if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'$.

Example 2.22. [Condition (G-PWP_{ssc}) does not imply flatness] Since there exist monoids S which are not right reversible, Θ_S is not flat. But Θ_S satisfies Condition (G-PWP_{ssc}). For example, let T be a right zero semigroup such that $|T| > 1$ and let $S = T^1$. Then S is not right reversible.

Now, in the following example, we show that Condition (G-PWP_{ssc}) does not imply Condition (PWP_{ssc}).

Example 2.23. [Condition (*G-PWP*$_{ssc}$) does not imply *PWP*$_{ssc}$)] By Theorem 2.8 of [12], Condition (*PWP*$_{ssc}$) implies (*PWF*). Therefore, Example 2.19 shows that Condition (*G-PWP*$_{ssc}$) does not imply Condition (*PWP*$_{ssc}$) in general.

The following diagram illustrates how the conditions are related to the properties already studied.



Here, the abbreviations stand for the corresponding properties of right S-acts defined in the Introduction. WF = weak flatness, PWF = principal weak flatness, TF = torsion-freeness, $\mathfrak{R}$-TF=being $\mathfrak{R}$-torsion-free. Moreover, each arrow in the diagram denotes an implication only; the converse implications are not claimed unless explicitly proved elsewhere in the paper.

3 Homological classifications of monoids by Condition (G-PWP_{ssc})

In this section, we present some results regarding homological classifications. We begin by examining scenarios where a certain property implies Condition (G-PWP_{ssc}) for (finitely generated, cyclic, monocyclic) right S-acts. Additionally, we offer a classification of monoids in which acts with this characteristic demonstrate other flatness properties.

In reference to [13], an element $a \in A$ is termed *divisible by* $s \in S$ if there exists $b \in A$ such that $bs = a$. A right S-act A is called *divisible* if every element of A is divisible by any left cancellable element of S.
Furthermore, a right S-act Q is called *injective* if for any monomorphism $\iota : A \to B$ and any homomorphism $f : A \to Q$ there exists a homomorphism $\bar{f} : B \to Q$ such that $f = \bar{f}\iota$. It is termed (fg-)*weakly injective*, if it is injective with respect to all embeddings of (finitely generated) right ideals into S. A right S-act Q is called *principally weakly injective* if it is injective with respect to all embeddings of principal right ideals into S.

Theorem 3.1. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition (G-PWP_{ssc});*

(2) *all finitely generated right S-acts satisfy Condition (G-PWP_{ssc});*

(3) *all right S-acts generated by at most two elements satisfy Condition (G-PWP_{ssc});*

(4) *all right S-acts generated by exactly two elements satisfy Condition (G-PWP_{ssc});*

(5) *all cyclic right S-acts satisfy Condition (G-PWP_{ssc});*

(6) *all monocyclic right S-acts satisfy Condition (G-PWP_{ssc});*

(7) *all monocyclic right S-acts of the form $S/\rho(s, s^2)$, $s \in S$, satisfy Condition (G-PWP_{ssc});*

(8) $(\forall x, y, s \in S)(\exists r \in S)(\exists n \in \mathbb{N})(rs^n = s^n \wedge (xr, yr) \in \rho(xs, ys))$;

(9) *all right Rees factor acts of S satisfy Condition (G-PWP$_{ssc}$);*

(10) *all right Rees factor acts of S of the form S/sS, $s \in S$, satisfy Condition (G-PWP$_{ssc}$);*

(11) *all cofree right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(12) *S is right generally regular.*

Proof. Implications (1) $\Rightarrow$ (11), (1) $\Rightarrow$ (5), (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6) $\Rightarrow$ (7) and (5) $\Rightarrow$ (9) $\Rightarrow$ (10) are obvious.

(10) $\Rightarrow$ (12) Let $sS = K$. Here ρ_K denotes the Rees congruence on S with respect to the right ideal $K = sS$. Given that $s\rho_K s^2$, we can find $r \in S$ and $n \in \mathbb{N}$ such that $(1.r)\rho_K(s.r)$ and $rs^n = s^n$, as per Theorem 2.8. Since $r\rho_K(s.r)$, we have $r = sr$ or $r, sr \in K$. If $r = sr$, then $s^n = rs^n = srs^n$. If $r \in K = sS$, then there exists $x \in S$ such that $r = sx$, thus $s^n = rs^n = sxs^n$, which satisfies the condition.

(7) $\Rightarrow$ (12) Let $\rho = \rho(s, s^2)$. Since $s\rho s^2$, we can find $r \in S$ and $n \in \mathbb{N}$ such that $(1.r)\rho(s.r)$ and $rs^n = s^n$, based on Theorem 2.8. Hence, according to [13, Corollary 3.8.6], either $r = sr$ or $r = sp, sr = sq$ and $s^{m_1}r = s^{m_2}sr$, for some $p, q \in S$ and $m_1, m_2 \in \mathbb{N}$. If $r = sr$, then $s^n = rs^n = srs^n$; otherwise, $s^n = rs^n = sps^n$, meeting the required condition.

(6) $\Rightarrow$ (8) Let $x, y, s \in S$. By assumption, the monocyclic right S-act of the form $S/\rho(xs, ys)$ satisfies Condition (G-PWP$_{ssc}$). Since $xs\ \rho(xs, ys)\ ys$, by Theorem 2.8, the result follows.

(8) $\Rightarrow$ (5) Let ρ be a right congruence on S, and $(xs)\ \rho\ (ys)$ for $x, y, s \in S$. Then $\rho(xs, ys) \subseteq \rho$. By assumption, there exist $r \in S$ and $n \in \mathbb{N}$ such that $rs^n = s^n$ and $xr\ \rho(xs, ys)\ yr$, it follows that $xr\ \rho\ yr$. Consequently, by Theorem 2.8, S/ρ satisfies Condition (G-PWP$_{ssc}$), and so, all cyclic right S-acts satisfy Condition (G-PWP$_{ssc}$).

(11) $\Rightarrow$ (12) As every right S-act can be embedded in a cofree right S-act, by assumption, every right S-act is a subact of a right S-act satisfying Condition (G-PWP$_{ssc}$). Consequently, according to part (2) of Theorem 2.2, all right S-acts satisfy Condition (G-PWP$_{ssc}$), and so, by part (1) of Theorem 2.13, all right S-acts are GP-flat. Hence, based on [21, Theorem 3.4], S is right generally regular.

(12) $\Rightarrow$ (1) Consider A as a right S-act where $as = a's$, for $a, a' \in A$ and $s \in S$. By assumption, there exist $x \in S$ and $n \in \mathbb{N}$ such that $s^n = sxs^n$.

Let $sx = r$. It follows that $ar = a'r$ and $rs^n = sxs^n = s^n$. Consequently, all right S-acts satisfy Condition (G-PWP_{ssc}).

(4) $\Rightarrow$ (12) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sxs = s$. This implies that s is regular, and so, s is right generally regular. Now, consider the case where $sS \neq S$. Let

$$A = S \coprod^{sS} S = \{(l, x) \mid l \in S \setminus sS\} \dot{\cup} sS \dot{\cup} \{(t, y) \mid t \in S \setminus sS\}.$$

By definition, we have the isomorphisms

$$B = \{(l, x) \mid l \in S \setminus sS\} \dot{\cup} sS \cong S_S \cong \{(t, y) \mid t \in S \setminus sS\} \dot{\cup} sS = C.$$

Since $A = B \cup C$, A is generated by two distinct elements $(1, x)$ and $(1, y)$, thus A satisfies Condition (G-PWP_{ssc}). Hence, the equality $(1, x)s = (1, y)s$ implies the existence of $r \in S$ and $n \in \mathbb{N}$ such that $rs^n = s^n$ and $(1, x)r = (1, y)r$. The last equation implies $r \in sS$, leading to the existence of $u \in S$ such that $r = su$. Therefore $s^n = rs^n = sus^n$. Thus, S is right generally regular, as required. □

As a consequence of Theorem 3.1, we obtain the following classification for several well-known classes of right S-acts, including injective, weakly injective, fg-weakly injective, principally weakly injective and divisible acts. This extends the characterization of right generally regular monoids beyond the class of all acts considered in Theorem 3.1.

Corollary 3.2. *Let ($\mathcal{C}$) be any one of the following classes of right S-acts: injective, weakly injective, fg-weakly injective, principally weakly injective, or divisible. Then the following statements are equivalent:*

(i) *Every right S-act satisfies Condition (G-PWP_{ssc});*

(ii) *Every right S-act in ($\mathcal{C}$) satisfies Condition (G-PWP_{ssc});*

(iii) *Every cofree right S-act satisfies Condition (G-PWP_{ssc});*

(iv) *S is right generally regular.*

Proof. The implication (i)$\Rightarrow$(ii) is immediate. Since every cofree right S-act belongs to each of the classes listed above, (ii)$\Rightarrow$(iii). By Theorem 3.1, (iii)$\Rightarrow$(iv)$\Rightarrow$(i). Hence all four conditions are equivalent. □

Corollary 3.3 can be viewed as the $n = 1$ specialization of the characterization in Theorem 3.1. The fact that this specialization characterizes regular monoids is verified separately below.

Corollary 3.3. *The monoid S is regular if and only if*

$$(\forall x, y, s \in S)(\exists r \in S)(rs = s \wedge xr \ \rho(xs, ys) \ yr),$$

where $\rho(xs, ys)$ denotes the smallest right congruence on S containing (xs, ys).

Proof. If S is regular, then for each $s \in S$ there exists $t \in S$ such that $s = sts$. Given $x, y, s \in S$, set $r = st$. Then $rs = s$, and, since $\rho(xs, ys)$ is a right congruence containing (xs, ys),

$$xr = xst \ \rho(xs, ys) \ yst = yr.$$

Hence the stated condition holds.

Conversely, suppose that the stated condition holds. Fix $s \in S$ and take $x = s$ and $y = 1$. Then, for some $r \in S$,

$$rs = s \qquad \text{and} \qquad sr \ \rho(s^2, s) \ r.$$

Since $\rho(s^2, s)$ is the smallest right congruence containing (s^2, s), both components of every related pair belong to sS. Thus $r \in sS$, say $r = st$ for some $t \in S$. Consequently,

$$s = rs = sts,$$

and hence s is regular. Therefore, S is regular. □

Theorem 3.4. *Let $(\star)$ be a property of right S-acts such that every right S-act satisfying Property $(\star)$ is GP-flat and S_S satisfies Property $(\star)$. Then the following statements are equivalent:*

(1) *All right S-acts satisfying property $(\star)$, satisfy Condition (G-PWP$_{ssc}$);*

(2) *all finitely generated right S-acts satisfying property $(\star)$, satisfy Condition (G-PWP$_{ssc}$);*

(3) *all cyclic right S-acts satisfying property $(\star)$, satisfy Condition (G-PWP$_{ssc}$);*

(4) *all monocyclic right S-acts satisfying property $(\star)$, satisfy Condition $(G\text{-}PWP_{ssc})$;*

(5) *S is G-left PSF.*

Proof. Implications (1) $\Rightarrow$ (2)) $\Rightarrow$ (3) $\Rightarrow$ (4) are straightforward.

(4) $\Rightarrow$ (5) Since S_S is a monocyclic right S-act and S_S satisfies Property $(\star)$, then S_S satisfies Condition $(G\text{-}PWP_{ssc})$, by assumption. Hence S is G-left PSF.

(5) $\Rightarrow$ (1) Given that every right S-act satisfying Property $(\star)$ is GP-flat. Since S is G-left PSF, by part (2) of Theorem 2.13, GP-flatness and Condition $(G\text{-}PWP_{ssc})$ are equivalent. Consequently, all right S-acts satisfying Property $(\star)$ also satisfy Condition $(G\text{-}PWP_{ssc})$. □

For left S-acts X, Y, Z and S-homomorphisms $f : X \to Z$ and $g : Y \to Z$, let $P = P(X, Y, f, g, Z)$ denote their pullback, with the canonical projections $p_X : P \to X$ and $p_Y : P \to Y$. Thus,

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

For a right S-act A, tensoring the above pullback diagram with A induces the canonical map

$$\varphi_A : A \otimes_S P \longrightarrow P(A \otimes_S X, A \otimes_S Y, 1_A \otimes f, 1_A \otimes g, A \otimes_S Z),$$

defined by

$$\varphi_A(a \otimes (x, y)) = (a \otimes x, a \otimes y).$$

The map φ_A is the corresponding morphism used in the definitions of WPF, WKF, $PWKF$, and TKF (see [13, 15]).
A right S-act A is called:

- *weakly pullback flat (WPF)* if the corresponding map φ is bijective for every pullback diagram $P(S, S, f, g, S)$;
- *weakly kernel flat (WKF)* if the corresponding map φ is bijective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S;
- *principally weakly kernel flat (PWKF)* if the corresponding map φ is bijective for every pullback diagram $P(Ss, Ss, f, f, S)$, where $s \in S$;

- *translation kernel flat (TKF)* if the corresponding map φ is bijective for every pullback diagram $P(S, S, f, f, S)$.

We next indicate several properties that can be used as Property ($\star$) in Theorem 3.4. For each of these properties, the two hypotheses of Theorem 3.4 must be satisfied:

(a) every right S-act with that property is GP-flat;

(b) the right S-act S_S itself has that property. The following implications are well known for right S-acts:

$$\text{WPF} \Rightarrow \text{WKF} \Rightarrow \text{PWKF} \Rightarrow \text{TKF} \Rightarrow \text{(PWP)}.$$

For the properties appearing in the implication diagram on page 14, condition (a) follows from the diagram together with the fact that Condition (G-PWP_{ssc}) implies GP-flatness (Theorem 2.13). Condition (b) holds for all of them since S_S is free, and hence satisfies every property in the diagram. For the remaining properties not shown in the diagram, we have:

- **Free, projective, strongly flat, flat, weakly flat, and principally weakly flat:** Each implies GP-flatness by the standard hierarchy of flatness properties (see [13]); S_S satisfies all of them.
- **Equalizer flat:** Equalizer flatness implies flatness by [13, Lemma 3.9.2] and flatness implies GP-flatness, thus, Equalizer flatness implies GP-flatness; S_S is equalizer flat since it is free.
- **GPW-flat:** GPW-flatness implies GP-flatness by [23]; S_S is GPW-flat by definition.
- Condition (P_E) implies Condition (P_{sc}) and Condition (P_{sc}) implies weakly flatness, by [18, Theorem 2.5]. Also, Condition (PWP_{sc}) implies PWF, by [19, Theorem 2.4]. Therefore, these imply GP-flatness, respectively; S_S satisfies each of them since it is free.
- **GP-flat:** Trivial; S_S is GP-flat by [17, Remark 2.5].

Thus all the listed properties satisfy both hypotheses of Theorem 3.4 and are legitimate choices for ($\star$).

To compare our main result with known characterizations in the literature, we recall the following theorem from [21]. It shows that for torsion-free acts, the property of being GP-flat is equivalent to the monoid being generally left almost regular.

Theorem 3.5. [21, Theorem 3.9] *The following conditions on a monoid S are equivalent:*

(1) *All torsion-free right S-acts are GP-flat.*

(2) *All cyclic torsion-free right S-acts are GP-flat.*

(3) *All torsion-free right Rees factor acts of S are GP-flat.*

(4) *S is a generally left almost regular monoid.*

The next theorem shows that the same characterization remains valid if GP-flatness is replaced by the stronger Condition (G-PWP_{ssc}).

Theorem 3.6. *The following statements are equivalent:*

(1) *All torsion-free right S-acts satisfy Condition (G-PWP_{ssc});*

(2) *all torsion-free cyclic right S-acts satisfy Condition (G-PWP_{ssc});*

(3) *all torsion-free right Rees factor acts of S satisfy Condition (G-PWP_{ssc});*

(4) *S is generally left almost regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By assumption and part (1) of Theorem 2.13, it follows that all torsion-free right Rees factor acts of S are GP-flat, and so, S is generally left almost regular, by Theorem 3.5.

(4) $\Rightarrow$ (1). Since S is generally left almost regular, it is G-left PSF by Theorem 2.7. Thus, according to part (2) of Theorem 2.13, GP-flatness and Condition (G-PWP_{ssc}) are equivalent. Consequently, the result can be derived from Theorem 3.5. □

The theorem also holds for finitely generated right S-acts.

Theorem 3.7. *The following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion-free right S-acts satisfy Condition (G-PWP_{ssc});*

(2) *all $\mathfrak{R}$-torsion-free finitely generated right S-acts satisfy Condition (G-PWP_{ssc});*

(3) *all $\mathfrak{R}$-torsion-free right S-acts generated by exactly two elements satisfy Condition (G-PWP_{ssc});*

(4) *S is right generally regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are evident.

(3) $\Rightarrow$ (4) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sxs = s$. This implies that s is regular, and so, s is right generally regular. Now, consider the case where $sS \neq S$. Let

$$A = S \overset{sS}{\coprod} S = \{(l, x) \mid l \in S \setminus sS\} \dot{\cup} sS \dot{\cup} \{(t, y) \mid t \in S \setminus sS\}.$$

By definition, we have the isomorphisms

$$B = \{(l, x) \mid l \in S \setminus sS\} \dot{\cup} sS \cong S_S \cong \{(t, y) \mid t \in S \setminus sS\} \dot{\cup} sS = C.$$

Since $A = B \cup C$, A is generated by two distinct elements $(1, x)$ and $(1, y)$. Through the given isomorphisms, B and C satisfy Condition (E), thus A satisfies Condition (E). Therefore, based on [25, Proposition 1.2], A is $\mathfrak{R}$-torsion-free. Consequently, by assumption, satisfies Condition (G-PWP_{ssc}). Hence, the equality $(1, x)s = (1, y)s$ implies the existence of $r \in S$ and $n \in \mathbb{N}$ such that $rs^n = s^n$ and $(1, x)r = (1, y)r$. The last equation implies $r \in sS$, leading to the existence of $u \in S$ such that $r = su$. Therefore $s^n = rs^n = sus^n$. Thus, S is right generally regular, as required.

(4) $\Rightarrow$ (1) The result can be deduced from Theorem 3.1. □

The theorem also holds for right S-acts generated by at most two elements.

Recall from [5, 8, 15] that a right S-act A satisfies Condition (E') if, for all $a \in A$, $s, s', z \in S$,

$$(as = as' \wedge sz = s'z) \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \wedge us = us').$$

A right S-act A satisfies Condition (EP) if, for all $a \in A$, $s, t \in S$,

$$as = at \Rightarrow (\exists a' \in A)(\exists u, u' \in S)(a = a'u = a'u' \wedge us = u't).$$

A right S-act A satisfies Condition $(E'P)$ if, for all $a \in A$, $s, t, z \in S$,

$$(as = at \wedge sz = tz) \Rightarrow (\exists a' \in A)(\exists u, u' \in S)(a = a'u = a'u' \wedge us = u't).$$

Theorem 3.8. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition (G-PWP_{ssc});*

(2) *all right S-acts satisfying Condition $(E^{'}P)$, satisfy Condition $(G\text{-}PWP_{ssc})$;*

(3) *all right S-acts satisfying Condition (EP), satisfy Condition $(G\text{-}PWP_{ssc})$;*

(4) *all right S-acts satisfying Condition $(E^{'})$, satisfy Condition $(G\text{-}PWP_{ssc})$;*

(5) *all right S-acts satisfying Condition (E), satisfy Condition $(G\text{-}PWP_{ssc})$;*

(6) *S is right generally regular.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious, because $(E) \Rightarrow (EP) \Rightarrow (E^{'}P)$ and $(E) \Rightarrow (E^{'}) \Rightarrow (E^{'}P)$.

(5) $\Rightarrow$ (6) If $s \in S$ and $sS = S$, then there exists $x \in S$ such that $sxs = s$. This means s is regular, and thus, s is right generally regular. If $sS \neq S$, consider $A = S \coprod^{sS} S$. By (3) $\Rightarrow$ (4) of Theorem 3.7, A satisfies Condition (E). Therefore, by assumption, A satisfies Condition $(G\text{-}PWP_{ssc})$. Consequently, by (3) $\Rightarrow$ (4) of Theorem 3.7, S is right generally regular.

(6) $\Rightarrow$ (1) From Theorem 3.1, the desired conclusion follows. □

Based on the proof of Theorem 3.8, we can conclude that the above theorem is true for finitely generated right S-acts, right S-acts generated by at most two elements and right S-acts generated by exactly two elements.
Recall from [13] that A_S is a *(strongly) faithful* right S-act if for $s, t \in S$ the equality $as = at$ for (some) all $a \in A$ implies $s = t$. It is straightforward that every strongly faithful right S-act is faithful, but the converse does not hold in general.

Theorem 3.9. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition $(G\text{-}PWP_{ssc})$;*

(2) *all faithful right S-acts satisfy Condition $(G\text{-}PWP_{ssc})$;*

(3) *all finitely generated faithful right S-acts satisfy Condition $(G\text{-}PWP_{ssc})$;*

(4) *all faithful right S-acts generated by at most two elements satisfy Condition $(G\text{-}PWP_{ssc})$;*

(5) *all faithful right S-acts generated by exactly two elements satisfy Condition (G-PWP$_{ssc}$);*

(6) *S is right generally regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6) Consider $s \in S$. If $sS = S$, then s is right generally regular. In the case where $sS \neq S$, let's consider $A = S \coprod^{sS} S$. By (3) $\Rightarrow$ (4) of Theorem 3.7, $A = B \cup C$ where $B \cong S_S \cong C$. Since S_S is faithful, thus B and C are faithful. Thus, A is faithful and so A is a right S-act generated by two elements $(1, x)$ and $(1, y)$. Therefore, A satisfies Condition (G-PWP$_{ssc}$), by assumption. Thus, by (3) $\Rightarrow$ (4) of Theorem 3.7, S is a right generally regular monoid.

(6) $\Rightarrow$ (1). It is evident based on Theorem 3.1. □

A right S-act A is termed *decomposable* if there exist two subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \emptyset$. In this case $A = B \cup C$ represents a decomposition of A. If this condition is not met, A is referred to as indecomposable (see [13]).

Recall from [26, 27] that a right S-act B is called *locally cyclic* if every finite elements (equivalently, every two-elements) of B is contained in a cyclic subact of B. The right S-act A satisfies *Condition* (L_P) if $aS \cap a'S \neq \emptyset$ for $a, a' \in A$, then there exists $a'' \in A$ such that $a, a' \in a''S$.

Also, the right S-act A satisfies *Condition* (L_{PWP}) if for $a, a' \in A$, $s \in S$, the equality $as = a's$ implies that there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$ and $a' = a''v$.

Theorem 3.10. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(2) *all indecomposable right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(3) *all (L_{PWP}) right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(4) *all (L_P) right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(5) *all locally cyclic right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(6) *S is right generally regular.*

Proof. The implication (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious, because locally cyclic $\Rightarrow (L_P) \Rightarrow (L_{PWP})$.

(2) $\Rightarrow$ (5) By [24, Lemma 3.4], every locally cyclic right S-act is indecomposable, and so, the result follows.

(5) $\Rightarrow$ (6) By hypothesis, all cyclic right S-acts satisfy Condition (G-PWP_{ssc}); thus, according to Theorem 3.1, S is right generally regular.

(6) $\Rightarrow$ (1) It is obvious, as per Theorem 3.1. □

The theorem also holds for finitely generated indecomposable right S-acts, indecomposable right S-acts generated by at most two elements, indecomposable right S-acts generated by exactly two elements, finitely generated locally cyclic right S-acts, finitely generated (L_P) right S-acts and finitely generated (L_{PWP}) right S-acts.

Notation: Let

$$C_l = \{s \in S \mid s \text{ is left cancellable}\},$$

and

$$C_r = \{s \in S \mid s \text{ is right cancellable}\}.$$

Observe that both C_l and C_r are nonempty, since $1 \in C_l \cap C_r$.

Lemma 3.11. *The following statements are equivalent:*

(1) *There exists at least one strongly faithful right S-act;*

(2) *there exists at least one cyclic strongly faithful right S-act;*

(3) *there exists at least one monocyclic strongly faithful right S-act;*

(4) *there exists at least one finitely generated strongly faithful right S-act;*

(5) *for every $s \in S$, sS is a strongly faithful right S-act;*

(6) *there exists $s \in S$ such that sS is a strongly faithful right S-act;*

(7) *S_S is a strongly faithful right S-act;*

(8) *for every $s \in S$, every element of sS is left cancellable;*

(9) *there exists $s \in S$ such that all elements of sS are left cancellable;*

(10) *S is left cancellative.*

Proof. The equivalences (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (10) are precisely [12, Lemma 3.7]. The remaining equivalences follow immediately from the definitions:

((3) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1) every monocyclic act is cyclic, every cyclic act is finitely generated, and every finitely generated act is an act.

(5) $\Rightarrow$ (6) $\Rightarrow$ (2) If sS is strongly faithful for every s, then certainly for some s; moreover, each sS is cyclic.

(8) $\Rightarrow$ (9) It is clear.

(7) $\Rightarrow$ (2) S_S is itself a cyclic right S-act (generated by 1).

(7) $\Rightarrow$ (3) Since $S/\rho(s,s) = S/\Delta_S \cong S_S$, the right S-act S_S is monocyclic.

(9) $\Rightarrow$ (10) Suppose there exists $s \in S$ such that $sS \subseteq C_l$. If $rt = rl$ for $r, t, l \in S$, then $(sr)t = (sr)l$. Since $sr \in sS$ is left cancellable, it follows that $t = l$. Hence S is left cancellative. □

By Lemma 3.11, there is no strongly faithful right S-act for S if and only if S is not left cancellative.

Theorem 3.12. *The following statements are equivalent:*

(1) *All strongly faithful right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(2) *all finitely generated strongly faithful right S-acts satisfy Condition (G-PWP$_{ssc}$);*

(3) *all strongly faithful right S-acts generated by at most two elements satisfy Condition (G-PWP$_{ssc}$);*

(4) *all strongly faithful right S-acts generated by exactly two elements satisfy Condition (G-PWP$_{ssc}$);*

(5) *S is not left cancellative or S is right generally regular.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are evident.

(4) $\Rightarrow$ (5) Assume that S is left cancellative and $s \in S$. If $sS = S$, then s is regular, and so s is right generally regular. In case where $sS \neq S$, consider $A = S \coprod_{sS} S$. By proof (3) $\Rightarrow$ (4) of Theorem 3.7, A is a right S-act generated by two distinct elements $(1,x)$ and $(1,y)$, with $(1,x)S = B \cong S_S \cong C = (1,y)S$ and $A = B \cup C$, where B and C are subacts of A. Since S is left cancellative, as per Lemma 3.11, S_S is strongly faithful. Consequently, through the aforementioned isomorphisms, subacts B and C of A

are strongly faithful, which implies A is strongly faithful, too. Therefore, based on the assumption, A satisfies Condition (G-PWP_{ssc}), leading to the conclusion that S is a right generally regular monoid, as proven in the proof (3) $\Rightarrow$ (4) of Theorem 3.7.

(5) $\Rightarrow$ (1) If S is not left cancellative, then, by Lemma 3.11, there are no strongly faithful right S-acts, and the assertion is vacuously true. Otherwise, if S is right generally regular, then by Theorem 3.1, all right S-acts satisfy Condition (G-PWP_{ssc}). □

Theorem 3.13. *The following statements are equivalent:*

(1) *All cyclic strongly faithful right S-acts satisfy Condition (G-PWP_{ssc});*

(2) *all monocyclic strongly faithful right S-acts satisfy Condition (G-PWP_{ssc});*

(3) *S is not left cancellative or S is G-left PSF.*

Proof. Implication (1) $\Rightarrow$ (2) is clearly evident.

(2) $\Rightarrow$ (3) Suppose S is left cancellative. Then, by Lemma 3.11, S_S is strongly faithful. Therefore, based on the assumption, $S/\rho(1,1) = S/\Delta_S \cong S_S$ satisfies Condition (G-PWP_{ssc}), indicating that S is G-left PSF.

(3) $\Rightarrow$ (1) If S is not left cancellative, then according to Lemma 3.11, there is no strongly faithful right S-act, which leads to the desired conclusion. On the other hand, if S is left cancellative then S is G-left PSF, and $A = aS$ is a cyclic strongly faithful right S-act, Defining $f : aS \longrightarrow S_S$ as $f(as) = s$ establishes an isomorphism of right S-acts. Since S is G-left PSF, S_S satisfies Condition (G-PWP_{ssc}), thereby implying that aS also satisfies Condition (G-PWP_{ssc}). □

Lemma 3.14. [12, Lemma 3.9] *Let ρ be a right congruence on S. Then the following statements are equivalent:*

(1) *S/ρ is strongly faithful;*

(2) *$\rho = \Delta_S$ and S is left cancellative.*

Theorem 3.15. *The following statements are equivalent:*

(1) *There exists at least one strongly faithful monocyclic right S-act satisfying Condition (G-PWP_{ssc});*

(2) *there exists at least one strongly faithful cyclic right S-act satisfying Condition (G-PWP_{ssc});*

(3) *S is left cancellative and every strongly faithful cyclic right S-act satisfies Condition (G-PWP_{ssc});*

(4) *S is left cancellative and every strongly faithful monocyclic right S-act satisfies Condition (G-PWP_{ssc});*

(5) *S is left cancellative and S is G-left PSF.*

Proof. Implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are evident.

(2) $\Rightarrow$ (3) By assumption and Lemma 3.11, S is left cancellative. Let S/ρ be a strongly faithful cyclic right S-act satisfying Condition (G-PWP_{ssc}). Thus, by Lemma 3.14, $\rho = \Delta_S$, and so $S/\rho = S/\Delta_S \cong S_S$ satisfies Condition (G-PWP_{ssc}) and so S is G-left PSF. Since S is left cancellative, by Lemma 3.11, there exists at least one strongly faithful cyclic right S-act. This implies, by Theorem 3.13, that every strongly faithful cyclic right S-act satisfies Condition (G-PWP_{ssc}).

(4) $\Rightarrow$ (5) By Theorem 3.13, it is obvious.

(5) $\Rightarrow$ (1) Since S is left cancellative, by Lemma 3.11, there exists at least one strongly faithful monocyclic right S-act. If S/ρ is strongly faithful monocyclic right S-act, then by Lemma 3.14, $\rho = \Delta_S$, and so, $S/\rho = S/\Delta_S \cong S_S$. Since S is G-left PSF, $S/\rho \cong S_S$ satisfies Condition (G-PWP_{ssc}). □

Having characterized strongly faithful cyclic acts in Theorems 3.12–3.15, we now turn our attention to the more general class of faithful cyclic acts. This transition is natural, since every strongly faithful act is faithful, but the converse need not hold. Investigating the faithful case allows us to compare the resulting conditions on the monoid S with those obtained for strongly faithful acts, and in particular, to see whether the weaker faithfulness assumption yields a different classification.

If ρ is a right congruence on S and $s \in S$, then by ρs we denote the right congruence on S defined by

$$x(\rho s)y \iff (sx)\rho(sy),$$

for $x, y \in S$. Also if λ is a left congruence on S and $s \in S$, then by $s\lambda$ we denote the left congruence on S defined by

$$x(s\lambda)y \Longleftrightarrow (xs)\lambda(ys),$$

for $x, y \in S$ (see [13]).

Lemma 3.16. *For the right congruence ρ on S, the following statements are equivalent:*

(1) *The cyclic right S-act S/ρ is faithful;*

(2) *ρ does not contain any left congruence τ on S such that $\tau \neq \Delta_S$;*

(3) $\bigcap_{u\in S} \rho u = \Delta_S$.

Proof. (1) $\Leftrightarrow$ (2) It is obvious, by [13, Proposition 1.5.24].

(2) $\Rightarrow$ (3) Let $\sigma = \bigcap_{u\in S} \rho u$. Obviously, σ is a right congruence on S. Now we show that σ is a left congruence on S. For $x, y \in S$, we have:

$$(x, y) \in \sigma \Leftrightarrow (\forall u \in S)(x, y) \in \rho u \Leftrightarrow (\forall u \in S)(ux, uy) \in \rho.$$

Now if $l \in S$, then we have:

$$(x, y) \in \sigma \Leftrightarrow (\forall u \in S)(ux, uy) \in \rho \Rightarrow (\forall u \in S)(ulx, uly) \in \rho \Rightarrow (\forall u \in S)(lx, ly) \in \rho u \Rightarrow (lx, ly) \in \bigcap_{u\in S} \rho u = \sigma.$$

Therefore, σ is a left congruence on S. Since $\sigma = \bigcap_{u\in S} \rho u \subseteq \rho$, by assumption, $\sigma = \Delta_S$.

(3) $\Rightarrow$ (1) Assume that S/ρ is not faithful. Then there exist $x, y \in S$, such that $x \neq y$ and $[u]_\rho x = [u]_\rho y$, for every $u \in S$. Thus $(x, y) \in \rho u$, for every $u \in S$, and so $(x, y) \in \bigcap_{u\in S} \rho u$, which is a contradiction. Therefore S/ρ is faithful, as required. □

Theorem 3.17. *The following statements are equivalent:*

(1) *All faithful cyclic right S-acts satisfies Condition (G-PWP_{ssc});*

(2) *For every right congruence ρ, ρ contains a nontrivial left congruence τ on S or the right S-act S/ρ satisfies Condition (G-PWP_{ssc});*

(3) $\bigcap_{u\in S} \rho u \neq \Delta_S$ *or the right S-act S/ρ satisfies Condition (G-PWP_{ssc}), for every right congruence ρ.*

Proof. (1) $\Rightarrow$ (2) Let ρ be a right congruence on S that does not contain any nontrivial left congruence on S. By Lemma 3.16, S/ρ is faithful. Therefore, based on the assumption, S/ρ satisfies Condition (G-PWP_{ssc}).

(2) $\Rightarrow$ (3) Let ρ be a right congruence on S such that $\bigcap\limits_{u\in S} \rho u = \Delta_S$. By Lemma 3.16, ρ does not contain any nontrivial left congruence on S. Therefore, based on the assumption, S/ρ satisfies Condition (G-PWP_{ssc}).

(3) $\Rightarrow$ (1) Let ρ be a right congruence on S such that the right S-act S/ρ is faithful. According to Lemma 3.16, $\bigcap\limits_{u\in S} \rho u = \Delta_S$. Therefore, by assumption S/ρ satisfies Condition (G-PWP_{ssc}). □

Theorem 3.18. *If all faithful monocyclic right S-acts satisfy Condition (G-PWP_{ssc}), then S is G-left PSF.*

Proof. Since $S/\rho(1,1) = S/\Delta_S \cong S_S$, thus S_S satisfies Condition (G-PWP_{ssc}), by assumption. Hence, S is G-left PSF. □

Theorem 3.19. *The following statements are equivalent:*

(1) *All right Rees factor acts of S satisfying Condition (P) satisfy Condition (G-PWP_{ssc});*

(2) *all right Rees factor acts of S that are WPF satisfy Condition (G-PWP_{ssc});*

(3) *all strongly flat right Rees factor acts of S satisfy Condition (G-PWP_{ssc});*

(4) *all projective right Rees factor acts of S satisfy Condition (G-PWP_{ssc});*

(5) *all projective generator right Rees factor acts of S satisfy Condition (G-PWP_{ssc});*

(6) *all free right Rees factor acts of S satisfy Condition (G-PWP_{ssc});*

(7) *S does not contain left zero or S is G-left PSF.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious, because free $\Rightarrow$ projective generator $\Rightarrow$ projective $\Rightarrow$ strongly flat $\Rightarrow$ WPF $\Rightarrow$ Condition (P).

(6) $\Rightarrow$ (7) If $z \in S$ is a left zero element, then $K_S = zS = \{z\}$ is a right ideal. This implies $S/K_S \cong S_S$ is free and by assumption, $S/K_S \cong S_S$ satisfies Condition (G-PWP_{ssc}). Thus S is G-left PSF.

(7) $\Rightarrow$ (1) If K_S is a right ideal of S such that S/K_S satisfies Condition (P), two cases arise:

Case 1. If $K_S = S$, then $S/K_S = S/S \cong \Theta_S$, and by Theorem 2.2, $S/K_S \cong \Theta_S$ satisfies Condition (G-PWP_{ssc}).

Case 2. If $K_S \neq S$, then by [13, Proposition 3.13.9] (which states that a proper right ideal K of S with S/K satisfying Condition (P) must have exactly one element), we have $|K_S| = 1$. If $z \in K_S$, then $K_S = \{z\} = zS$, indicating z is a left zero of S. Therefore, by assumption, S is G-left PSF, and consequently, $S/K_S \cong S_S$ satisfies Condition (G-PWP_{ssc}). □

In the category $\mathbf{C}$, an object $G \in \mathbf{C}$ is termed a *generator* if the functor $Mor_{\mathbf{C}}(G, -)$ is faithful. This means that for any objects $X, Y \in \mathbf{C}$ and any $f, g \in Mor_{\mathbf{C}}(X, Y)$ where $f \neq g$, there exists $\alpha \in Mor_{\mathbf{C}}(G, X)$ such that $f\alpha \neq g\alpha$.
As a reminder from [13, Theorem 2.3.16], G_S is a generator if and only if there exists an epimorphism $\pi : G_S \longrightarrow S_S$. Therefore, S_S is a generator in **Act**-S.

Theorem 3.20. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition (G-PWP_{ssc});*

(2) *all generator right S-acts satisfy Condition (G-PWP_{ssc});*

(3) *$S \times A$ satisfies Condition (G-PWP_{ssc}), for every generator right S-act A;*

(4) *$S \times A$ satisfies Condition (G-PWP_{ssc}), for every right S-act A;*

(5) *the right S-act A satisfies Condition (G-PWP_{ssc}), if $Hom(A, S) \neq \emptyset$;*

(6) *S is a right generally regular monoid.*

Proof. Implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (5) are clearly evident.

(3) $\Rightarrow$ (2) Let A be a generator right S-act where $as = a's$, for $a, a' \in A$ and $s \in S$. Then, there exists an epimorphism $\pi : A \longrightarrow S_S$. Define $\pi(a'') = 1$ and set $B = aS \cup a'S \cup a''S$. Then B is a subact of A generated by at most three elements. Therefore, the restriction of π to B, defined as $\pi^\star = \pi \mid_B : B \longrightarrow S_S$ is an epimorphism. Consequently, B is a generator right S-act and so, by assumption, $S \times B$ satisfies Condition (G-PWP_{ssc}). Then the equality $(\pi(a), a)s = (\pi(a'), a')s$ implies the existence of $r \in S$

and $n \in \mathbb{N}$ such that $(\pi(a), a)r = (\pi(a'), a')r$ and $rs^n = s^n$. Consequently, $ar = a'r$ and $rs^n = s^n$, which imply A satisfies Condition (*G-PWP$_{ssc}$*).

(5) $\Rightarrow$ (2) Let A be a generator in Act-S, where $as = a's$, for $a, a' \in A$ and $s \in S$. In this case, there exists an epimorphism $\pi : A \longrightarrow S_S$. By defining $B = aS \cup a'S$, it can be observed that B, generated by at most two elements, is a subact of A. The restriction of π to B, denoted as $\pi^\star = \pi \mid_B : B \longrightarrow S_S$, is a homomorphism. This leads to $Hom(B, S) \neq \emptyset$, and based on the assumption, B satisfies Condition (*G-PWP$_{ssc}$*). Therefore, there exist $r \in S$ and $n \in \mathbb{N}$ such that $ar = a'r$ and $rs^n = s^n$, demonstrating that A satisfies Condition (*G-PWP$_{ssc}$*), as required.

(4) $\Rightarrow$ (6) If $s \in S$, and $sS = S$, then s is a regular element, making s right generally regular. Now let $sS \neq S$ and $A = S \coprod_{sS} S$. Then, the right S-act A is generated by two distinct elements $(1, x)$ and $(1, y)$. Based on the assumption, $S \times A$ satisfies Condition (*G-PWP$_{ssc}$*). Then the equality $(1, (1, x))s = (1, (1, y))s$ implies the existence of $r \in S$ and $n \in \mathbb{N}$ such that $(1, (1, x))r = (1, (1, y))r$ and $rs^n = s^n$. Consequently, $(1, x)r = (1, y)r$, indicating that $r \in sS$. Therefore, there exists $t \in S$ such that $r = st$, and so, $s^n = rs^n = sts^n$. Thus s is right generally regular.

(2) $\Rightarrow$ (4) Consider A as a right S-act. Define the mapping

$$\begin{cases} \pi : S \times A \longrightarrow S_S \\ \quad (s, a) \mapsto s \end{cases}.$$

Since π is surjective by construction, it is an epimorphism. Thus $S \times A$ is a generator. Therefore, based on the assumption, $S \times A$ satisfies Condition (*G-PWP$_{ssc}$*).

(6) $\Rightarrow$ (1) From Theorem 3.1, the desired conclusion follows. □

By the proof of Theorem 3.20, we conclude that the above theorem is true for finitely generated (generator) right S-acts as well as we can conclude that parts (2) and (3) are true for generator right S-acts generated by at most three elements and parts (4) and (5) are true for right S-acts generated by at most two elements.

Theorem 3.21. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (G-PWP$_{ssc}$) are free;*

(2) *all cyclic right S-acts satisfying Condition (G-PWP$_{ssc}$) are free;*

(3) *all right Rees factor acts of S satisfying Condition (G-PWP_{ssc}) are free;*

(4) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clearly evident.

(3) $\Rightarrow$ (4) According to Theorem 2.2, the right S-act Θ_S satisfies Condition (G-PWP_{ssc}), and thus, by assumption, $\Theta_S \cong S/S$ is free. Hence, $|S| = 1$.

(4) $\Rightarrow$ (1) Since all right S-acts are free, the result follows. □

The theorem also holds for finitely generated right S-acts. Furthermore, "free" can be replaced by "projective generator" in the theorem, since these two properties coincide for all right Rees factor acts of S.

Theorem 3.22. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (G-PWP_{ssc}) are generator;*

(2) *all cyclic right S-acts satisfying Condition (G-PWP_{ssc}) are generator;*

(3) *all right Rees factor acts of S satisfying Condition (G-PWP_{ssc}) are generator;*

(4) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) The right S-act Θ_S satisfies Condition (G-PWP_{ssc}), by Theorem 2.2. Therefore, based on the assumption, $\Theta_S \cong S/S$ is a generator. Thus, there exists an epimorphism $\pi : \Theta_S \longrightarrow S_S$, leading to $S = \{1\}$.

(4) $\Rightarrow$ (1) If $S = \{1\}$, then every right S-act is a generator. □

The theorem also holds for finitely generated right S-acts.

Theorem 3.23. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (G-PWP_{ssc}) are (Strongly) faithful;*

(2) *all cyclic right S-acts satisfying Condition (G-PWP_{ssc}) are (Strongly) faithful;*

(3) *all right Rees factor acts of S satisfying Condition (G-PWP_{ssc}) are (Strongly) faithful;*

(4) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) According to Theorem 2.2, Θ_S satisfies Condition (*G-PWP$_{ssc}$*). By assumption, since $\Theta_S \cong S/S$ is faithful, it implies that $S = \{1\}$.

(4) $\Rightarrow$ (1) If $S = \{1\}$, then all right S-acts are strongly faithful, and consequently, the conclusion follows. □

The theorem also holds for finitely generated right S-acts.
We recall from [13] that a right ideal K of S is said to satisfy Condition (LU) (or is left stabilizing) if for every $k \in K$, there exists $l \in K$ such that $lk = k$. Similarly, we provide the following definition.

Definition 3.24. The right ideal K of S is called *G-left stabilizing* or satisfies *Condition* (*G-LU*) if for every $s \in S$ and $r \in S \setminus K$, $rs \in K$ implies there exist $k \in K$ and $n \in \mathbb{N}$ such that $rs^n = ks^n$.

The following lemma, due to Qiao and Wei [21], provides a useful criterion for GP-flatness of the amalgamated coproduct $A(K) = S \coprod\limits^{K} S$. We include it here for the convenience of the reader.

Lemma 3.25. *[21, Lemma 2.4] Let S be a monoid and K be a proper right ideal of S. Then the following conditions are equivalent:*

(1) $A(K) = S \coprod\limits^{K} S$ *is GP-flat.*

(2) *For every $s \in S$, if there exists $r \in S \setminus K$ such that $rs \in K$, then there exist $n \in \mathbb{N}$ and $j \in K$ such that $rs^n = js^n$.*

Now, using the above lemma, we characterize when the amalgamated coproduct $A(K) = S \coprod\limits^{K} S$ satisfies Condition (*G-PWP$_{ssc}$*).

Theorem 3.26. *For a proper right ideal K of S, the following statements are equivalent:*

(1) $A(K) = S \coprod\limits^{K} S$ *satisfies Condition (G-PWP$_{ssc}$);*

(2) *K is G-left stabilizing and S is G-left PSF.*

Proof. (1) $\Rightarrow$ (2) By part (1) of Theorem 2.13, A is GP-flat. Therefore, according to Lemma 3.25, K is G-left stabilizing. On the other hand, $A = B \cup C$, where $B = \{(l, x) \mid l \in S \setminus K\} \cup K$, $C = \{(t, y) \mid t \in S \setminus K\} \cup K$, $B, C \leq A$ and $B \cong S_S \cong C$. By part (2) of Theorem 2.2, $B \cong S_S$ satisfies Condition (G-PWP_{ssc}), hence S is G-left PSF.

(2) $\Rightarrow$ (1) Since K is G-left stabilizing, according to Lemma 3.25, A is GP-flat. As S is G-left PSF, based on part (2) of Theorem 2.13, GP-flatness and Condition (G-PWP_{ssc}) are equivalent. Therefore, A satisfies Condition (G-PWP_{ssc}). □

We shall need the following auxiliary fact. If $S \neq C_r$, then, by [12, Lemma 3.12], $I = S \setminus C_r$ is a proper right ideal of S and the Rees factor S/I is torsion-free. In the case where S is a G-left PSF monoid, this observation yields the following lemma.

Lemma 3.27. *If S is a G-left PSF monoid such that $S \neq C_r$, then $I = S \setminus C_r$ is a G-left stabilizing right ideal. Consequently, $A = S \coprod_I S$ satisfies Condition (G-PWP_{ssc}).*

Proof. Let $s \in S$ and $r \in S \setminus I = C_r$ such that $rs \in I$. Since $I = S \setminus C_r$, rs is not right cancellable. Therefore, there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 rs = l_2 rs$. By assumption, there exist $r' \in S$ and $n \in \mathbb{N}$ such that $l_1 rr' = l_2 rr'$ and $r's^n = s^n$. Due to $l_1 \neq l_2$, the equality $l_1 rr' = l_2 rr'$ implies that $rr' \in S \setminus C_r = I$. If $i = rr' \in I$, then $rs^n = is^n$, and so, $I = S \setminus C_r$ is a G-left stabilizing right ideal. Consequently, by Theorem 3.26, A satisfies Condition (G-PWP_{ssc}). □

By following a similar argument as in the proof of [12, Lemma 3.13], we can incorporate Condition (G-PWP_{ssc})into this lemma.

Theorem 3.28. *Let $(\star)$ be a property of right S-acts such that GP-flat $\Rightarrow$ Property $(\star)$ $\Rightarrow$ torsion-free. Then the following statements are equivalent:*

(1) *S is G-left PSF and Property $(\star)$ implies $PWKF$;*

(2) *S is G-left PSF and Property $(\star)$ implies TKF;*

(3) *S is G-left PSF and property $(\star)$ implies Condition (PWP);*

(4) *S is G-left PSF and Property $(\star)$ implies Condition (P');*

(5) *S is right Cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are obvious, because $PWKF \Rightarrow TKF \Rightarrow$ Condition (PWP) and Condition $(P') \Rightarrow$ Condition (PWP).

(3) $\Rightarrow$ (5) Let $S \neq C_r$ and $I = S \setminus C_r$, according to Lemma 3.27, I is a G-left stabilizing right ideal. The set

$$A = S \overset{I}{\coprod} S = \{(l, x) \mid l \in S \setminus I\} \dot\cup I \dot\cup \{(t, y) \mid t \in S \setminus I\} = (1, x)S \cup (1, y)S,$$

is GP-flat. As per the assumption, A satisfies Condition (PWP). If $i \in I$, then the equality $(1, x)i = (1, y)i$ implies the existence of $a \in A$ and $u, v \in S$ such that $(1, x) = au$, $(1, y) = av$ and $ui = vi$. Consequently, there exist $t, l \in S \setminus I$ such that $(l, x) = a = (t, y)$, leading to a contradiction and establishing that S is right cancellative.

(5) $\Rightarrow$ (1) Since S is right cancellative, it is left PSF and so is G-left PSF. Moreover, according to [12, Lemma 3.13], properties of being torsion-free and $PWKF$ are equivalent. Therefore, based on the assumption, any right S-act satisfies Property $(\star)$ is $PWKF$.

(5) $\Rightarrow$ (4) Since S is right cancellative, it implies that S is G-left PSF. Additionally, as per [12, Lemma 3.13], torsion-free and Condition (P') are equivalent. Hence, based on the assumption, any right S-act satisfying Property $(\star)$, satisfies Condition (P'). □

As the proof shows, the theorem also holds for finitely generated right S-acts and right S-acts generated by at most (exactly) two elements.

Theorem 3.29. [21, Theorem 3.11] *The following statements are equivalent:*

(1) *S is right cancellative;*

(2) *S is left PSF and all GP-flat right S-acts satisfy Condition (PWP);*

(3) *S is left PSF and all GP-flat right S-acts are TKF;*

(4) *S is left PSF and all GP-flat right S-acts are $PWKF$.*

As a corollary of Theorem 3.28, we show that in items (2), (3) and (4) of Theorem 3.29, we can replace the left PSF by a weaker condition. Furthermore, we add a new item to this Theorem.

Corollary 3.30. *The following statements are equivalent:*

(1) *S is right cancellative;*

(2) *S is G-left PSF and all GP-flat right S-acts satisfy Condition (PWP);*

(3) *S is G-left PSF and all GP-flat right S-acts are TKF;*

(4) *S is G-left PSF and all GP-flat right S-acts are $PWKF$;*

(5) *S is G-left PSF and all GP-flat right S-acts satisfy Condition (P').*

It is important to note that Corollary 3.30 holds true for both finitely generated right S-acts and right S-acts generated by at most (exactly) two elements.
An element s in S is called right e-cancellable (for an idempotent $e \in S$), if $s = es$ and $\ker \rho_s \leq \ker \rho_e$. The monoid S is called left PP if every principal left ideal of S is projective, as a left S-act. This is equivalent to saying that every element $s \in S$ is right e-cancellable, for some idempotent $e \in S$ (see [4], [13, Proposition 3.17.16]).

Definition 3.31. The monoid S is called *eventually left PP* if for every $s \in S$, there exist $n \in \mathbb{N}$ and an idempotent $e \in S$ such that s^n is right e-cancellable.

In Theorem 3.28 and Corollary 3.30, the condition G-left PSF can be replaced by left almost regular, eventually left almost regular, generally left almost regular, left PP, eventually left PP, left PSF, or eventually left PSF, because

$$\begin{array}{lccccc} \text{Right cancellative} \Rightarrow & \text{L.A. regular} & \Rightarrow & \text{E.L.A. regular} & \Rightarrow & \text{G.L.A. regular} \\ & \Downarrow & & \Downarrow & & \\ & \text{left } PP & \Rightarrow & \text{eventually left } PP & & \Downarrow \\ & \Downarrow & & \Downarrow & & \\ & \text{left } PSF & \Rightarrow & \text{eventually left } PSF & \Rightarrow & G\text{-left } PSF \end{array}$$

Here, the abbreviations stand for the following properties of right S-acts. L. A. regular=left almost regular, E. L. A. regular =eventually left almost regular, G. L. A. regular= generally left almost regular.

Theorem 3.32. *Let $(\star)$ be a property of right S-acts such that GP-flat $\Rightarrow$ Property $(\star) \Rightarrow$ torsion-free. Then the following statements are equivalent:*

(1) *All right S-acts satisfying Property ($\star$) are $PWKF$ and satisfy Condition (G-PWP_{ssc});*

(2) *all right S-acts satisfying Property ($\star$) are TKF and satisfy Condition (G-PWP_{ssc});*

(3) *all right S-acts satisfying Property ($\star$) satisfy Conditions (PWP) and (G-PWP_{ssc});*

(4) *all right S-acts satisfying Property ($\star$) satisfy Conditions (P') and (G-PWP_{ssc});*

(5) *S is right Cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are obvious, because $(PWKF) \Rightarrow (TKF) \Rightarrow$ Condition (PWP) and Condition $(P') \Rightarrow$ Condition (PWP).

(3) $\Rightarrow$ (5) By hypothesis, S_S satisfies Condition (G-PWP_{ssc}), hence S is G-left PSF. Moreover, since every right S-act satisfying Property ($\star$) also satisfies Condition (PWP), according to Theorem 3.28, S is right cancellative.

(5) $\Rightarrow$ (1) According to Theorem 3.28, every right S-act satisfying Property ($\star$) is $(PWKF)$. Additionally, since S is generally left almost regular, by Theorem 3.6, every torsion-free right S-act satisfies Condition (G-PWP_{ssc}). Consequently, every right S-act satisfying Property ($\star$) satisfies Condition (G-PWP_{ssc}).

(5) $\Rightarrow$ (4) Since S is right cancellative, thus by [12, Lemma 3.13], Condition (P') and $(PWKF)$ are equivalent. Hence by the proof of (5) $\Rightarrow$ (1), the result follows. □

Recall from the discussion after Theorem 2.13 that every GP-flat right S-act is torsion-free. Therefore, by substituting GP-flat for the property ($\star$) in Theorem 3.32, we obtain the following.

Corollary 3.33. *The following statements are equivalent:*

(1) *All GP-flat right S-acts are $PWKF$ and satisfy Condition (G-PWP_{ssc});*

(2) *all GP-flat right S-acts are TKF and satisfy Condition (G-PWP_{ssc});*

(3) *all GP-flat right S-acts satisfy Conditions (PWP) and (G-PWP_{ssc});*

(4) *all GP-flat right S-acts satisfy Conditions* (P') *and* $(G\text{-}PWP_{ssc})$*;*

(5) S *is right cancellative.*

The right S-act $S \times S$ equipped with the right S-action $(s,t)u = (su, tu)$, for $s, t, u \in S$ is called the *diagonal act* of S and denoted by $D(S)$.

Theorem 3.34. *The following statements are equivalent:*

(1) S_S^n *satisfies Condition* $(G\text{-}PWP_{ssc})$*, for every* $n \in \mathbb{N}$*;*

(2) *there exists* $m \in \mathbb{N}$ *such that* S_S^m *satisfies Condition* $(G\text{-}PWP_{ssc})$*;*

(3) $D(S)$ *satisfies Condition* $(G\text{-}PWP_{ssc})$*;*

(4) S *is G-left* PSF*.*

Proof. Implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (4) The mapping $\psi : S_S \longrightarrow S_S^m$ defined by $\psi(s) = (s, \cdots, , s)$ is an obvious S-monomorphism. Therefore, $S_S \cong Im\psi \leq S_S^m$. By part (2) of Theorem 2.2, $Im\psi$ satisfies Condition $(G\text{-}PWP_{ssc})$, implying that S is a G-left PSF.

(3) $\Rightarrow$ (4) This is similar to the proof of (2) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1) According to part (2) of Theorem 2.13, GP-flatness and Condition $(G\text{-}PWP_{ssc})$ are equivalent. As stated in Corollary 2.18, S_S^n is GP-flat for every $n \in \mathbb{N}$. Hence, S_S^n, satisfies Condition $(G\text{-}PWP_{ssc})$ for every $n \in \mathbb{N}$. □

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