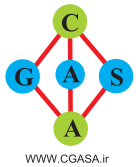
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Doctrinal soft structures and Molodtsov equivalence: A categorical reconstruction of correct soft operations

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Abstract. Molodtsov's soft set theory represents uncertainty by assigning subsets to parameters, but many operations on soft sets may depend on the chosen parameter names rather than on the represented family of subsets. Motivated by Molodtsov's equivalence and correct-operation principles, this paper gives a categorical reconstruction of soft structures over an indexed poset $P : \mathcal{C}^{op} \to \mathbf{Pos}$. A parameterized soft object $S : A \to P(X)$ is treated as a representative, while its invariant content is the image family $\operatorname{Im} S \subseteq P(X)$. We define doctrinal soft structures as such fibrewise families and show that they form the Grothendieck construction of a naturally induced indexed preorder. The canonical projection to $\mathcal{C}$ is proved to be a Grothendieck fibration. We also characterize correct soft operations as reindexing-compatible fibrewise operators, construct pullbacks over base pullbacks under finite-meet assumptions, and extend Molodtsov's internal and external approximation equivalences to finite poset fibres via minimal and maximal antichain representatives.

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1 Introduction and categorical setting

Soft set theory was introduced by Molodtsov as a general mathematical tool for dealing with uncertainty, vagueness and incomplete descriptions of objects [15]. In its original form, a soft set over a universe X is a pair (S, A), where A is a set of parameters and

$$S : A \longrightarrow 2^X$$

assigns to every parameter a subset of X. This is the original Molodtsov definition, and we shall use it below only as the set-theoretic prototype of a more general doctrinal construction. Maji, Biswas and Roy subsequently developed several basic operations and notions such as soft subsets, null soft sets, absolute soft sets, union, intersection and complement [14]. Algebraic, topological and decision-theoretic extensions then appeared rapidly; for instance, Aktas and Cagman introduced soft groups [2], Shabir and Naz introduced soft topological spaces [18], and Cagman and Enginoglu reformulated soft operations for decision-making purposes [9]. Subsequent work also showed that several proposed soft operations require careful reformulation, because operations depending too strongly on the chosen parameter names may lead to incorrect or non-invariant results; see, for instance, the corrections and discussions in [1, 3, 20].

A decisive conceptual point was made by Molodtsov in his discussion of equivalence and correct operations for soft sets [16]. The parameter set in a soft set does not carry, by itself, intrinsic mathematical structure. Its role is primarily to name or indicate members of a family of subsets. Thus, from the viewpoint of the soft-set formalism, the essential datum is not the function $S : A \to 2^X$ together with the specific parameter names in A, but rather the family of subsets represented by this function. In particular, two soft sets over the same universe should be regarded as equivalent whenever they determine the same family of subsets, even if their parameter sets are different or if one representation repeats the same subset several times. Molodtsov therefore introduced an equivalence relation based on equality of represented families and argued that meaningful operations and relations on soft sets should be *correct*, that is, invariant under replacement by equivalent soft sets [16]. This observation changes the mathematical status of a soft set: a parameterized map $S : A \to 2^X$ is not the structure itself but a representative of a family in the power-set fibre over X.

The aim of the present paper is to reconstruct this principle categorically. Existing categorical approaches usually build categories whose objects are ordinary soft sets, soft functions, soft relations or particular algebraic/topological soft structures; see, for example, the category of soft sets studied by Zahiri [21], the categorical properties of soft sets developed by Zhou, Li and Akram [22], and related categorical soft-set constructions in [8]. Our purpose is different: we do not start by forming a category of ordinary soft sets. Instead, we start with an arbitrary category $\mathcal{C}$ and an indexed poset

$$P : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}.$$

The value $P(X)$ is interpreted as a poset of predicates, subobjects, local structures or admissible descriptions over X. For example, $P(X)$ may be the power set of a set, the subobject lattice of an object, the lattice of subgroups of a group, the poset of submodules of a module, the poset of relations on a set, the poset of congruences of an algebra, or another suitable categorical doctrine. In this setting, a parameterized soft object is a map

$$S : A \longrightarrow P(X),$$

but the Molodtsov-invariant soft structure it represents is the image family

$$\mathfrak{F}_S = \operatorname{Im} S \subseteq P(X).$$

Thus a doctrinal soft structure is a pair $(X, \mathfrak{F})$ consisting of an object of $\mathcal{C}$ together with a family of elements of the fibre $P(X)$.

This viewpoint is closely related to the language of categorical logic and fibrations. Indexed posets, doctrines and their associated Grothendieck constructions are standard tools in categorical logic, type theory and the semantics of predicates; see, for example, [7, 11–13]. The fibration terminology used below is also standard in the sense of Grothendieck fibrations [6, 7, 10, 11]. In the present context, the indexed-poset point of view is not used merely as a change of notation. It gives an intrinsic explanation of Molodtsov's equivalence principle and turns correct soft operations into reindexing-compatible fibrewise operations. The resulting category of doctrinal soft structures is not introduced ad hoc; it is obtained as the Grothendieck construction of a naturally induced indexed poset of soft families.

More precisely, for each $X \in \mathcal{C}$ we consider the set of families $\mathfrak{F} \subseteq P(X)$ ordered by the Hoare-type preorder

$$\mathfrak{F} \leq_{\mathrm{H}} \mathfrak{G} \quad \Longleftrightarrow \quad \forall u \in \mathfrak{F}\ \exists v \in \mathfrak{G} \text{ such that } u \leq v.$$

For a morphism $f : X \to Y$ in $\mathcal{C}$, the reindexing map $P(f) : P(Y) \to P(X)$ sends a family $\mathfrak{G} \subseteq P(Y)$ to

$$P(f)[\mathfrak{G}] = \{P(f)(v) : v \in \mathfrak{G}\} \subseteq P(X).$$

These data define an indexed preorder

$$\mathscr{S}_P : \mathcal{C}^{op} \longrightarrow \mathbf{Preord}.$$

The category of doctrinal soft structures is then the Grothendieck construction

$$\int_{\mathcal{C}} \mathscr{S}_P.$$

A morphism

$$f : (X, \mathfrak{F}) \longrightarrow (Y, \mathfrak{G})$$

is therefore a morphism $f : X \to Y$ of $\mathcal{C}$ such that

$$\mathfrak{F} \leq_{\mathrm{H}} P(f)[\mathfrak{G}],$$

or equivalently, for every $u \in \mathfrak{F}$ there is $v \in \mathfrak{G}$ satisfying

$$u \leq P(f)(v).$$

When $\mathcal{C} = \mathbf{Set}$ and $P = P_{\mathrm{pow}}$, where $P_{\mathrm{pow}}(X) = 2^X$, this condition says that for every selected subset U of X there is a selected subset V of Y such that

$$U \subseteq f^{-1}(V),$$

or equivalently $f(U) \subseteq V$. This is precisely the parameter-free form of a soft morphism compatible with Molodtsov equivalence.

The contributions of the paper are as follows.

(i) We distinguish parameterized soft representatives $S : A \to P(X)$ from doctrinal soft structures $\mathfrak{F} \subseteq P(X)$ and show that Molodtsov equivalence is exactly equality of image families.

(ii) We construct the category $\mathrm{Soft}_P(\mathcal{C})$ of doctrinal soft structures as a Grothendieck construction and prove that the projection

$$\pi : \mathrm{Soft}_P(\mathcal{C}) \to \mathcal{C}$$

is a Grothendieck fibration.

(iii) We prove a quotient theorem showing that parameterized representatives descend canonically to doctrinal soft structures, and that operations invariant under Molodtsov equivalence are precisely those factoring through this quotient.

(iv) We characterize correct operations categorically as reindexing-compatible fibrewise operators and show when such operators induce endofunctors over $\mathcal{C}$.

(v) Under finite-meet hypotheses on the fibres, we construct pullbacks in $\mathrm{Soft}_P(\mathcal{C})$ lying over pullbacks in $\mathcal{C}$ and show that the resulting squares are sent by the projection to the chosen pullback squares in $\mathcal{C}$.

(vi) We extend Molodtsov's internal and external approximation relations from families of subsets to arbitrary finite poset fibres and prove that their equivalence classes are represented by minimal and maximal antichains.

The paper is organized as follows. The remainder of this introductory section fixes the categorical setting, separates representatives from invariant image families, and constructs the basic category of doctrinal soft structures. Section 2 develops the Molodtsov quotient from representatives to doctrinal soft structures. Section 3 characterizes correct operations. Section 4 proves the pullback theorem. Section 5 studies internal and external approximation in finite fibres. Section 6 gives examples in sets, subobject doctrines, groups, modules, relations and topological situations. Section 7 summarizes the categorical significance of the main results and indicates further directions.

1.1 Indexed posets and family doctrines We begin with a minimal amount of categorical terminology. Throughout the paper, $\mathcal{C}$ denotes a category, and the word *doctrine* is used in a broad posetal sense: it means an indexed poset of predicates, subobjects or structures, without assuming additional logical operations unless explicitly stated. The use of indexed

posets follows the standard categorical-logic convention in which predicates over an object are organized fibrewise and reindexed contravariantly along morphisms [11, 12]. An indexed poset on $\mathcal{C}$ is a functor

$$P : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}.$$

For a morphism $f : X \to Y$ in $\mathcal{C}$, we write

$$P(f) : P(Y) \longrightarrow P(X)$$

for the corresponding monotone reindexing map. If $u \in P(X)$ and $v \in P(Y)$, the inequality

$$u \leq P(f)(v)$$

is interpreted as saying that the predicate or structure u over X is contained in, or is compatible with, the pullback of v along f.

Example 1.1. If $\mathcal{C} = \mathbf{Set}$ and $P = P_{\mathrm{pow}}$, where $P_{\mathrm{pow}}(X) = 2^X$, then $P_{\mathrm{pow}}(f)(V) = f^{-1}(V)$. Thus $U \leq P_{\mathrm{pow}}(f)(V)$ is exactly $U \subseteq f^{-1}(V)$.

Example 1.2. If $\mathcal{C}$ has pullbacks, then

$$P(X) = \mathrm{Sub}_{\mathcal{C}}(X)$$

with reindexing by pullback defines the usual subobject doctrine of categorical logic [11, 12]. This is the main categorical model behind the present framework.

Definition 1.3. Let L be a poset. Denote by $\mathcal{P}(L)$ the collection of all subsets of L. The *Hoare preorder*, also known from lower-powerdomain semantics of nondeterminism [17, 19], on $\mathcal{P}(L)$ is defined by

$$\mathfrak{F} \leq_{\mathrm{H}} \mathfrak{G} \quad \Longleftrightarrow \quad \forall u \in \mathfrak{F}\ \exists v \in \mathfrak{G} \text{ such that } u \leq v.$$

The use of the Hoare preorder is deliberate. It records the existential refinement principle that every selected predicate in the source family is bounded above by some selected predicate in the target family. In the soft-set situation, this is the parameter-free analogue of an internal approximation direction. The dual Smyth preorder would encode the external direction, while the Egli–Milner comparison combines both directions. The

present category of morphisms is built from the Hoare direction because a soft morphism is intended to send each admissible description of the source into at least one admissible description of the target.

The Hoare preorder is reflexive and transitive. It is generally not antisymmetric. If desired, one may pass to its associated poset quotient. For the categorical results below, the preorder level is sufficient and more natural. We use the Grothendieck construction for preorder-valued indexed categories by regarding each preorder as a thin category.

Lemma 1.4. *For every poset L, the relation $\leq_{\mathrm{H}}$ is a preorder on $\mathcal{P}(L)$.*

Proof. Reflexivity follows by taking $v = u$ for every $u \in \mathfrak{F}$. For transitivity, suppose $\mathfrak{F} \leq_{\mathrm{H}} \mathfrak{G}$ and $\mathfrak{G} \leq_{\mathrm{H}} \mathfrak{H}$. If $u \in \mathfrak{F}$, then there exists $v \in \mathfrak{G}$ with $u \leq v$. Since $\mathfrak{G} \leq_{\mathrm{H}} \mathfrak{H}$, there exists $w \in \mathfrak{H}$ with $v \leq w$. Hence $u \leq w$, so $\mathfrak{F} \leq_{\mathrm{H}} \mathfrak{H}$. □

Construction 1.5. Let

$$P : \mathcal{C}^{op} \to \mathbf{Pos}$$

be an indexed poset. Define an indexed preorder

$$\mathscr{S}_P : \mathcal{C}^{op} \to \mathbf{Preord}$$

as follows. On objects,

$$\mathscr{S}_P(X) = (\mathcal{P}(P(X)), \leq_{\mathrm{H}}).$$

For a morphism $f : X \to Y$, define

$$\mathscr{S}_P(f) : \mathscr{S}_P(Y) \to \mathscr{S}_P(X)$$

by

$$\mathscr{S}_P(f)(\mathfrak{G}) = P(f)[\mathfrak{G}] = \{P(f)(v) : v \in \mathfrak{G}\}.$$

Proposition 1.6. *The assignment $\mathscr{S}_P$ is a functor*

$$\mathcal{C}^{op} \longrightarrow \mathbf{Preord}.$$

Proof. First, $\mathscr{S}_P(f)$ is monotone. Suppose $\mathfrak{G} \leq_{\mathrm{H}} \mathfrak{H}$ in $\mathcal{P}(P(Y))$. Let $P(f)(v) \in P(f)[\mathfrak{G}]$, where $v \in \mathfrak{G}$. There exists $w \in \mathfrak{H}$ such that $v \leq w$. Since $P(f)$ is monotone, $P(f)(v) \leq P(f)(w)$, and $P(f)(w) \in P(f)[\mathfrak{H}]$. Hence $P(f)[\mathfrak{G}] \leq_{\mathrm{H}} P(f)[\mathfrak{H}]$.

For identities,

$$\mathscr{S}_P(1_X)(\mathfrak{F}) = P(1_X)[\mathfrak{F}] = \mathfrak{F}.$$

For composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\mathscr{S}_P(g \circ f)(\mathfrak{H}) = P(g \circ f)[\mathfrak{H}] = P(f)[P(g)[\mathfrak{H}]] = \mathscr{S}_P(f)(\mathscr{S}_P(g)(\mathfrak{H})).$$

Thus $\mathscr{S}_P$ is functorial. □

1.2 Parameterized representatives and Molodtsov equivalence We now separate the two levels that are often conflated in soft set theory: parameterized representatives and invariant soft structures. The terminology "representative" is chosen deliberately: in Molodtsov's original theory a soft set is a parameterized family of subsets [15], whereas Molodtsov's later equivalence principle identifies representatives that determine the same family of subsets [16].

Definition 1.7. Let $P : \mathcal{C}^{op} \to \mathbf{Pos}$ be an indexed poset. A *P-soft representative* is a triple

$$(X, A, S),$$

where $X \in \mathrm{Ob}(\mathcal{C})$, A is a set of parameters, and

$$S : A \longrightarrow P(X)$$

is a map.

Definition 1.8. Let (X, A, S) and (X, B, T) be two P-soft representatives over the same object X. Following Molodtsov's equivalence idea for ordinary soft sets [16], they are called *Molodtsov-equivalent*, written

$$(X, A, S) \sim_M (X, B, T),$$

if

$$\mathrm{Im}\, S = \mathrm{Im}\, T$$

as subsets of $P(X)$.

Proposition 1.9. *Molodtsov equivalence is an equivalence relation on the class of P-soft representatives over a fixed object X.*

Proof. It is equality of image subsets of $P(X)$. Reflexivity, symmetry and transitivity therefore follow from equality of subsets. □

Definition 1.10. A *doctrinal P-soft structure* on an object X is a subset

$$\mathfrak{F} \subseteq P(X).$$

A *doctrinal P-soft object* is a pair

$$(X, \mathfrak{F})$$

where $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathfrak{F} \subseteq P(X)$.

Remark 1.11. The terminology reflects the following distinction. A map $S : A \to P(X)$ is a parameterized presentation. The represented soft structure is the family $\operatorname{Im} S \subseteq P(X)$. Repetition of the same fibre element under different parameter names has no effect on the doctrinal soft structure.

Example 1.12. If $\mathcal{C} = \mathbf{Set}$ and $P = P_{\mathrm{pow}}$, where $P_{\mathrm{pow}}(X) = 2^X$, then a doctrinal soft object is a pair

$$(X, \mathfrak{F}), \qquad \mathfrak{F} \subseteq 2^X.$$

An ordinary soft set $S : A \to 2^X$ is a representative of the family $\mathfrak{F} = \operatorname{Im} S$. This is precisely Molodtsov's interpretation of equivalent soft sets as different parameterizations of the same family of subsets.

1.3 The category of doctrinal soft structures We now define morphisms between doctrinal soft structures. Soft mappings and morphisms have been studied in several set-theoretic and algebraic forms, including soft functions and categorical soft-set morphisms [8, 21, 22]. The guiding principle here is different but compatible with Molodtsov equivalence: a morphism must be invariant under parameter renaming and must use only the reindexing structure of the underlying doctrine.

Definition 1.13. Let $(X, \mathfrak{F})$ and $(Y, \mathfrak{G})$ be doctrinal P-soft objects. A *P-soft morphism*

$$f : (X, \mathfrak{F}) \longrightarrow (Y, \mathfrak{G})$$

is a morphism $f : X \to Y$ in $\mathcal{C}$ such that

$$\mathfrak{F} \leq_{\mathrm{H}} P(f)[\mathfrak{G}].$$

Equivalently, for every $u \in \mathfrak{F}$, there exists $v \in \mathfrak{G}$ with

$$u \leq P(f)(v).$$

Remark 1.14. The preceding objects and morphisms form a category, denoted $\mathrm{Soft}_P(\mathcal{C})$. Indeed, $1_X : (X, \mathfrak{F}) \to (X, \mathfrak{F})$ is soft because $P(1_X)[\mathfrak{F}] = \mathfrak{F}$ and $\mathfrak{F} \leq_{\mathrm{H}} \mathfrak{F}$. If $f : (X, \mathfrak{F}) \to (Y, \mathfrak{G})$ and $g : (Y, \mathfrak{G}) \to (Z, \mathfrak{H})$ are soft morphisms, then $\mathfrak{F} \leq_{\mathrm{H}} P(f)[\mathfrak{G}]$ and $\mathfrak{G} \leq_{\mathrm{H}} P(g)[\mathfrak{H}]$ imply, by monotonicity of $\mathscr{S}_P(f)$, that $\mathfrak{F} \leq_{\mathrm{H}} P(g \circ f)[\mathfrak{H}]$. Hence $g \circ f$ is soft, and the identity and associativity laws are inherited from $\mathcal{C}$.

Theorem 1.15 (Grothendieck representation)**.** *Let $P : \mathcal{C}^{op} \to \mathbf{Pos}$ be an indexed poset. Then the category $\mathrm{Soft}_P(\mathcal{C})$ is isomorphic to the Grothendieck construction*

$$\int_{\mathcal{C}} \mathscr{S}_P$$

of the indexed preorder $\mathscr{S}_P : \mathcal{C}^{op} \to \mathbf{Preord}$.

Proof. The Grothendieck construction of $\mathscr{S}_P$ has objects pairs

$$(X, \mathfrak{F}),$$

where $X \in \mathcal{C}$ and $\mathfrak{F} \in \mathscr{S}_P(X)$. A morphism

$$(X, \mathfrak{F}) \to (Y, \mathfrak{G})$$

is a morphism $f : X \to Y$ in $\mathcal{C}$ such that

$$\mathfrak{F} \leq_{\mathrm{H}} \mathscr{S}_P(f)(\mathfrak{G}) = P(f)[\mathfrak{G}].$$

This is exactly the definition of a P-soft morphism. Identities and compositions coincide with those already described. Hence the two categories are identical up to the standard notational identification. □

A Grothendieck fibration may be understood here as a formal way of saying that soft structures can be pulled back coherently along morphisms of the base category. Given a base morphism $f : X \to Y$ and a soft structure over Y, the fibration property supplies a universal soft structure over X obtained by reindexing along f. Thus the theorem below is not merely a formal categorical statement: it asserts that doctrinal soft structures vary functorially over the underlying objects and that their change of base is controlled by the doctrine P.

Theorem 1.16 (Fibration theorem)**.** *The canonical projection*

$$\pi : \mathrm{Soft}_P(\mathcal{C}) \longrightarrow \mathcal{C}, \qquad \pi(X, \mathfrak{F}) = X,$$

is a Grothendieck fibration. For a morphism $f : X \to Y$ in $\mathcal{C}$ and a doctrinal soft object $(Y, \mathfrak{G})$, a cartesian lift of f at $(Y, \mathfrak{G})$ is

$$f : (X, P(f)[\mathfrak{G}]) \longrightarrow (Y, \mathfrak{G}).$$

Proof. First, the displayed arrow is a soft morphism because

$$P(f)[\mathfrak{G}] \leq_{\mathrm{H}} P(f)[\mathfrak{G}].$$

We prove that it is cartesian. Let

$$h : (Z, \mathfrak{H}) \to (Y, \mathfrak{G})$$

be a soft morphism in $\mathrm{Soft}_P(\mathcal{C})$ and suppose that in $\mathcal{C}$ the morphism $h : Z \to Y$ factors as

$$h = f \circ k$$

for a morphism $k : Z \to X$. Since h is a soft morphism,

$$\mathfrak{H} \leq_{\mathrm{H}} P(h)[\mathfrak{G}].$$

Using functoriality of P,

$$P(h)[\mathfrak{G}] = P(k)[P(f)[\mathfrak{G}]].$$

Hence

$$\mathfrak{H} \leq_{\mathrm{H}} P(k)[P(f)[\mathfrak{G}]],$$

so k is a soft morphism

$$k : (Z, \mathfrak{H}) \to (X, P(f)[\mathfrak{G}]).$$

It is the unique morphism over k whose composite with the displayed cartesian lift is h, because the underlying morphism in $\mathcal{C}$ is unique. Thus the displayed arrow is cartesian. Therefore π is a Grothendieck fibration. □

2 The Molodtsov quotient from representatives

We now formalize the passage from parameterized representatives to doctrinal soft structures. This is the categorical version of Molodtsov's assertion that equivalent soft sets differ only by the choice of names, or by repeated names, for the represented subsets [16]. There are two possible levels of representation. A strict representative morphism may carry an explicit parameter map, whereas the Molodtsov viewpoint treats parameter names as inessential. In this paper we therefore use an *extensional representative category*: morphisms are determined by the underlying categorical map and by the induced image-family condition.

Definition 2.1. Define the *extensional representative category* $\mathrm{RepSoft}_P(\mathcal{C})$ as follows. Objects are P-soft representatives (X, A, S). For such an object write $\mathfrak{F}_S := \operatorname{Im} S \subseteq P(X)$ for its represented image family. A morphism $f : (X, A, S) \longrightarrow (Y, B, T)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that, with $\mathfrak{F}_S = \operatorname{Im} S$ and $\mathfrak{G}_T = \operatorname{Im} T$, one has

$$\mathfrak{F}_S \leq_{\mathrm{H}} P(f)[\mathfrak{G}_T].$$

Remark 2.2. One could instead define a stricter representative category whose morphisms contain both a morphism $f : X \to Y$ in $\mathcal{C}$ and a parameter map $\alpha : A \to B$ satisfying

$$S(a) \leq P(f)(T(\alpha(a))) \qquad (a \in A).$$

That stricter version records a particular matching of parameter names. The extensional category used here forgets this extra presentation data and retains only the condition invariant under Molodtsov equivalence. This choice is essential for the quotient theorem below to reflect Molodtsov's principle rather than a parameter-sensitive variant of it.

Remark 2.3. The extensional representatives form a category. The identity on (X, A, S) is 1_X, and if $f : (X, A, S) \to (Y, B, T)$ and $g : (Y, B, T) \to (Z, C, R)$ are morphisms, then $\operatorname{Im} S \leq_{\mathrm{H}} P(f)[\operatorname{Im} T]$ and $\operatorname{Im} T \leq_{\mathrm{H}} P(g)[\operatorname{Im} R]$ imply $\operatorname{Im} S \leq_{\mathrm{H}} P(g \circ f)[\operatorname{Im} R]$. Thus composition is inherited from $\mathcal{C}$.

Definition 2.4. The *Molodtsov quotient functor*

$$Q : \mathrm{RepSoft}_P(\mathcal{C}) \longrightarrow \mathrm{Soft}_P(\mathcal{C})$$

is defined on objects by

$$Q(X, A, S) = (X, \operatorname{Im} S)$$

and on morphisms by

$$Q(f) = f.$$

Theorem 2.5 (Molodtsov quotient theorem)**.** *The functor*

$$Q : \mathrm{RepSoft}_P(\mathcal{C}) \to \mathrm{Soft}_P(\mathcal{C})$$

is well-defined, full, and essentially surjective. Moreover, two representatives over the same object are identified by Q exactly when they are Molodtsov-equivalent.

Proof. If $f : (X, A, S) \to (Y, B, T)$ is a morphism in $\mathrm{RepSoft}_P(\mathcal{C})$, then $\operatorname{Im} S \leq_{\mathrm{H}} P(f)[\operatorname{Im} T]$ by definition. Hence the same underlying morphism $f : X \to Y$ is a morphism $(X, \operatorname{Im} S) \to (Y, \operatorname{Im} T)$ in $\mathrm{Soft}_P(\mathcal{C})$, so Q is well-defined on morphisms and plainly preserves identities and compositions.

To prove fullness, fix two representatives $R = (X, A, S)$ and $R' = (Y, B, T)$, and write $\mathfrak{F}_S = \operatorname{Im} S$ and $\mathfrak{G}_T = \operatorname{Im} T$. Let $f : Q(R) = (X, \mathfrak{F}_S) \to Q(R') = (Y, \mathfrak{G}_T)$ be any morphism in $\mathrm{Soft}_P(\mathcal{C})$. By the definition of a soft morphism, $\mathfrak{F}_S \leq_{\mathrm{H}} P(f)[\mathfrak{G}_T]$. This is exactly the defining condition for the same underlying morphism $f : X \to Y$ to be a morphism $R \to R'$ in the extensional representative category. Therefore every morphism between $Q(R)$ and $Q(R')$ is in the image of the map on hom-sets, and Q is full. Notice that this statement is made for the extensional representative category of Definition 2.1; the stricter parameter-sensitive category of Remark 2.2 would impose additional matching data and would not give this fullness statement in general.

For essential surjectivity, let $(X, \mathfrak{F})$ be a doctrinal soft object. Choose the canonical representative $(X, \mathfrak{F}, \iota_{\mathfrak{F}})$, where $\iota_{\mathfrak{F}} : \mathfrak{F} \hookrightarrow P(X)$ is the inclusion. Since $\operatorname{Im} \iota_{\mathfrak{F}} = \mathfrak{F}$, we have $Q(X, \mathfrak{F}, \iota_{\mathfrak{F}}) = (X, \mathfrak{F})$.

Finally, for two representatives over the same object X, $Q(X, A, S) = Q(X, B, T)$ holds precisely when $\operatorname{Im} S = \operatorname{Im} T$ as subsets of $P(X)$, which is exactly Molodtsov equivalence. □

Corollary 2.6. *The category* $\mathrm{Soft}_P(\mathcal{C})$ *is a canonical parameter-free realization of the theory of* P*-soft representatives modulo Molodtsov equivalence.*

3 Correct operations as reindexing-compatible operators

Molodtsov's notion of correct operation requires invariance under equivalent soft-set representatives [16]. This is also consistent with the later literature correcting or refining operations on soft sets, where the dependence of operations on parameter choices became a central issue [1, 3, 20]. In the doctrinal framework, this becomes the statement that operations are defined on image families and are compatible with the indexed structure.

Definition 3.1. A *fibrewise soft operator* on $\mathscr{S}_P$ is a family of monotone maps

$$\Phi_X : \mathscr{S}_P(X) \to \mathscr{S}_P(X), \qquad X \in \mathrm{Ob}(\mathcal{C}).$$

It is called *lax reindexing-compatible* if for every morphism $f : X \to Y$ in $\mathcal{C}$ and every $\mathfrak{G} \subseteq P(Y)$,

$$\Phi_X(P(f)[\mathfrak{G}]) \leq_{\mathrm{H}} P(f)[\Phi_Y(\mathfrak{G})].$$

It is called *strictly reindexing-compatible* if equality holds.

Theorem 3.2 (Correct-operation theorem)**.** *Let* $\Phi = \{\Phi_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ *be a fibrewise soft operator. If* Φ *is lax reindexing-compatible, then it induces an endofunctor*

$$\widehat{\Phi} : \mathrm{Soft}_P(\mathcal{C}) \to \mathrm{Soft}_P(\mathcal{C})$$

over $\mathcal{C}$ *by*

$$\widehat{\Phi}(X, \mathfrak{F}) = (X, \Phi_X(\mathfrak{F})), \qquad \widehat{\Phi}(f) = f.$$

Conversely, if this assignment defines an endofunctor over $\mathcal{C}$*, then* Φ *is lax reindexing-compatible.*

Proof. Assume first that Φ is lax reindexing-compatible. Let

$$f : (X, \mathfrak{F}) \to (Y, \mathfrak{G})$$

be a soft morphism. Then

$$\mathfrak{F} \leq_{\mathrm{H}} P(f)[\mathfrak{G}].$$

Since Φ_X is monotone,

$$\Phi_X(\mathfrak{F}) \leq_{\mathrm{H}} \Phi_X(P(f)[\mathfrak{G}]).$$

By lax reindexing-compatibility,

$$\Phi_X(P(f)[\mathfrak{G}]) \leq_{\mathrm{H}} P(f)[\Phi_Y(\mathfrak{G})].$$

Therefore

$$\Phi_X(\mathfrak{F}) \leq_{\mathrm{H}} P(f)[\Phi_Y(\mathfrak{G})],$$

so the same underlying morphism $f : X \to Y$ is a soft morphism

$$(X, \Phi_X(\mathfrak{F})) \to (Y, \Phi_Y(\mathfrak{G})).$$

Thus $\widehat{\Phi}$ is well-defined on morphisms. Since it does not change the underlying morphisms of $\mathcal{C}$, it preserves identities and compositions.

Conversely, suppose that the displayed assignment defines an endofunctor over $\mathcal{C}$. For a morphism $f : X \to Y$ and a family $\mathfrak{G} \subseteq P(Y)$, the canonical cartesian morphism

$$f : (X, P(f)[\mathfrak{G}]) \to (Y, \mathfrak{G})$$

is a soft morphism. Applying $\widehat{\Phi}$ gives a soft morphism

$$f : (X, \Phi_X(P(f)[\mathfrak{G}])) \to (Y, \Phi_Y(\mathfrak{G})).$$

Hence

$$\Phi_X(P(f)[\mathfrak{G}]) \leq_{\mathrm{H}} P(f)[\Phi_Y(\mathfrak{G})].$$

Thus Φ is lax reindexing-compatible. □

Remark 3.3 (Cartesian morphisms and strict compatibility)**.** The lax condition in Theorem 3.2 is necessary and sufficient for the assignment $\widehat{\Phi}$ to be an endofunctor over the base category. Preservation of the displayed cartesian lifts is stronger. The cartesian lift of $f : X \to Y$ at $(Y, \mathfrak{G})$ has source $(X, P(f)[\mathfrak{G}])$; after applying $\widehat{\Phi}$ it becomes the arrow with source $(X, \Phi_X(P(f)[\mathfrak{G}]))$ and target $(Y, \Phi_Y(\mathfrak{G}))$. This is the chosen cartesian lift of f at $(Y, \Phi_Y(\mathfrak{G}))$ whenever $\Phi_X(P(f)[\mathfrak{G}]) = P(f)[\Phi_Y(\mathfrak{G})]$, namely whenever Φ is strictly reindexing-compatible. Thus lax compatibility gives functoriality, while strict compatibility gives preservation of the canonical cartesian change-of-base arrows, up to equality or preorder-equivalence in the fibres.

Corollary 3.4. *A soft operation on representatives is Molodtsov-correct precisely when it factors through the quotient functor*

$$Q : \mathrm{RepSoft}_P(\mathcal{C}) \to \mathrm{Soft}_P(\mathcal{C}).$$

If, moreover, the induced operation is compatible with reindexing in the sense above, it defines an endofunctor on $\mathrm{Soft}_P(\mathcal{C})$ *over* $\mathcal{C}$*.*

Proof. Factoring through Q means that the result of the operation depends only on the image family $\mathrm{Im}\, S$ and not on the parameter set or the particular representative map. This is precisely Molodtsov correctness. The second assertion follows from the previous theorem. □

Proposition 3.5 (Downward-closure operator)**.** *For each* $X \in \mathcal{C}$*, define*

$$\downarrow \mathfrak{F} = \{u \in P(X) : \exists v \in \mathfrak{F} \text{ with } u \leq v\}.$$

Then $\mathfrak{F} \mapsto \downarrow \mathfrak{F}$ *is a monotone idempotent operator on the fibre* $\mathscr{S}_P(X)$*. The family of operators* $\downarrow = \{\downarrow_X\}$ *is lax reindexing-compatible, and hence induces an idempotent endofunctor on* $\mathrm{Soft}_P(\mathcal{C})$*.*

Proof. Monotonicity and idempotence are immediate from the definition of downward closure. We verify the lax compatibility condition of Theorem 3.2. Let $f : X \to Y$ and let $\mathfrak{G} \subseteq P(Y)$. If

$$u \in \downarrow_X (P(f)[\mathfrak{G}]),$$

then $u \leq P(f)(v)$ for some $v \in \mathfrak{G}$. Since $v \in \downarrow_Y \mathfrak{G}$, the element $P(f)(v)$ belongs to $P(f)[\downarrow_Y \mathfrak{G}]$. Hence every $u \in \downarrow_X (P(f)[\mathfrak{G}])$ is below some element

of $P(f)[\downarrow_Y \mathfrak{G}]$, which is exactly

$$\downarrow_X (P(f)[\mathfrak{G}]) \leq_{\mathrm{H}} P(f)[\downarrow_Y \mathfrak{G}].$$

Thus the operator is lax reindexing-compatible. The induced endofunctor exists by Theorem 3.2, and its idempotence follows fibrewise. □

4 Pullbacks of doctrinal soft structures

We next prove that, under finite-meet hypotheses, pullbacks can be constructed in the total category over pullbacks in the base. The statement is intentionally formulated as a construction over base pullbacks rather than as a strict creation theorem: the projection remembers the underlying pullback square, while the soft family on the pullback object is the canonical meet-generated family determined by the two legs.

Definition 4.1. We say that $P : \mathcal{C}^{op} \to \mathbf{Pos}$ is *finite-meet fibred* if:

(a) each fibre $P(X)$ has finite meets, where a meet is the operation $\wedge$ in the fibre poset and generalizes intersection in the power-set case;

(b) for every morphism $f : X \to Y$, the map $P(f) : P(Y) \to P(X)$ preserves finite meets.

Theorem 4.2 (Pullbacks over base pullbacks)**.** *Assume that $\mathcal{C}$ has pullbacks and that*

$$P : \mathcal{C}^{op} \to \mathbf{Pos}$$

is finite-meet fibred. Then $\mathrm{Soft}_P(\mathcal{C})$ *has pullbacks whose underlying objects are pullbacks in $\mathcal{C}$. More precisely, if*

$$f : (X, \mathfrak{F}) \to (Z, \mathfrak{H}), \qquad g : (Y, \mathfrak{G}) \to (Z, \mathfrak{H})$$

are soft morphisms and $W = X \times_Z Y$ is a pullback in $\mathcal{C}$ with projections $p_X : W \to X$ and $p_Y : W \to Y$, so that

$$\begin{array}{ccc} W & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is a pullback square in the base category. Then the pullback in $\mathrm{Soft}_P(\mathcal{C})$ *is given by*

$$(W, \mathfrak{K}),$$

where

$$\mathfrak{K} = \{\, P(p_X)(u) \wedge P(p_Y)(v) : u \in \mathfrak{F},\ v \in \mathfrak{G} \,\}.$$

Under π*, the above pullback square is sent to the chosen pullback square in* $\mathcal{C}$*.*

Proof. First, the projection $p_X : W \to X$ is a soft morphism because for every element

$$k = P(p_X)(u) \wedge P(p_Y)(v) \in \mathfrak{K}$$

we have

$$k \leq P(p_X)(u),$$

where $u \in \mathfrak{F}$. Hence

$$\mathfrak{K} \leq_{\mathrm{H}} P(p_X)[\mathfrak{F}].$$

Similarly,

$$\mathfrak{K} \leq_{\mathrm{H}} P(p_Y)[\mathfrak{G}].$$

Thus both projections are soft morphisms. Since their underlying maps form a pullback cone in $\mathcal{C}$, the equality

$$f \circ p_X = g \circ p_Y$$

holds in $\mathrm{Soft}_P(\mathcal{C})$ as well, because morphisms in $\mathrm{Soft}_P(\mathcal{C})$ are determined by their underlying morphisms in $\mathcal{C}$.

Now let $(T, \mathfrak{L})$ be a doctrinal soft object with soft morphisms

$$a : (T, \mathfrak{L}) \to (X, \mathfrak{F}), \qquad b : (T, \mathfrak{L}) \to (Y, \mathfrak{G})$$

such that

$$f \circ a = g \circ b$$

in $\mathrm{Soft}_P(\mathcal{C})$, equivalently in $\mathcal{C}$. Since W is the pullback in $\mathcal{C}$, there is a unique morphism

$$c : T \to W$$

with

$$p_X \circ c = a, \qquad p_Y \circ c = b.$$

It remains to prove that c is a soft morphism.

Let $\ell \in \mathfrak{L}$. Since a is soft, there exists $u \in \mathfrak{F}$ such that

$$\ell \leq P(a)(u).$$

Since b is soft, there exists $v \in \mathfrak{G}$ such that

$$\ell \leq P(b)(v).$$

Therefore

$$\ell \leq P(a)(u) \wedge P(b)(v).$$

Using $a = p_X \circ c$ and $b = p_Y \circ c$, and using preservation of finite meets by $P(c)$, we obtain

$$\begin{aligned} P(a)(u) \wedge P(b)(v) &= P(c)(P(p_X)(u)) \wedge P(c)(P(p_Y)(v)) \\ &= P(c)\big(P(p_X)(u) \wedge P(p_Y)(v)\big). \end{aligned}$$

The element

$$P(p_X)(u) \wedge P(p_Y)(v)$$

belongs to $\mathfrak{K}$. Hence

$$\mathfrak{L} \leq_{\mathrm{H}} P(c)[\mathfrak{K}],$$

so c is a soft morphism. Its uniqueness follows from uniqueness in the base pullback. Therefore $(W, \mathfrak{K})$ satisfies the pullback universal property in $\mathrm{Soft}_P(\mathcal{C})$. Under π, this pullback square is exactly the chosen pullback square in $\mathcal{C}$. □

Remark 4.3. The compatibility family $\mathfrak{H}$ over Z is not explicitly used in the formula for $\mathfrak{K}$ because the universal object over the base pullback must simultaneously satisfy the two source-side constraints imposed by $\mathfrak{F}$ and $\mathfrak{G}$. The family $\mathfrak{H}$ is nevertheless essential: it is the common codomain datum that makes f and g soft morphisms into the same object $(Z, \mathfrak{H})$, and hence makes the pullback problem well-posed in $\mathrm{Soft}_P(\mathcal{C})$.

5 Internal and external approximation in poset fibres

Molodtsov introduced internal and external approximations for soft sets and proved that, for finite nontrivial families of subsets, internal and external equivalences are determined by minimal and maximal elements [16]. We now formulate this at the level of arbitrary finite poset fibres.

Throughout this section, let L be a poset with bottom element $\bot$ and top element $\top$.

Definition 5.1. Following Molodtsov's finite-family convention for internal and external approximations [16, definitions of approximation equivalence], a finite family $\mathfrak{F} \subseteq L$ is called *nontrivial* if it contains at least one element different from $\bot$ and $\top$ when such distinction is relevant. Put $\mathfrak{F}^\circ = \mathfrak{F} \setminus \{\bot\}$ and $\mathfrak{F}_\circ = \mathfrak{F} \setminus \{\top\}$. The set of non-bottom minimal elements of $\mathfrak{F}$ is $\mathrm{Min}(\mathfrak{F}) = \{u \in \mathfrak{F}^\circ : \text{ there is no } u' \in \mathfrak{F}^\circ \text{ with } u' < u\}$, and the set of non-top maximal elements of $\mathfrak{F}$ is $\mathrm{Max}(\mathfrak{F}) = \{u \in \mathfrak{F}_\circ : \text{ there is no } u' \in \mathfrak{F}_\circ \text{ with } u < u'\}$.

Remark 5.2. Molodtsov's finite-family statements are usually formulated for nontrivial families. The results below remain valid with the empty antichain allowed; the nontriviality condition is only needed if one wants to exclude degenerate representatives such as $\{\bot\}$ or $\{\top\}$ from consideration.

Definition 5.3. For finite families $\mathfrak{F}, \mathfrak{G} \subseteq L$, define the internal and external approximation relations, in the sense of Molodtsov's internal/external comparison of finite soft families [16, definitions of internal and external approximation], as follows. Write $\mathfrak{F} \preceq_{\mathrm{int}} \mathfrak{G}$ if for every $g \in \mathfrak{G}^\circ$ there exists $f \in \mathfrak{F}^\circ$ such that $f \leq g$. Dually, write $\mathfrak{F} \preceq_{\mathrm{ext}} \mathfrak{G}$ if for every $g \in \mathfrak{G}_\circ$ there exists $f \in \mathfrak{F}_\circ$ such that $g \leq f$. The induced internal, external and weak equivalences are defined by $\mathfrak{F} \approx_{\mathrm{int}} \mathfrak{G}$ iff both $\mathfrak{F} \preceq_{\mathrm{int}} \mathfrak{G}$ and $\mathfrak{G} \preceq_{\mathrm{int}} \mathfrak{F}$ hold, $\mathfrak{F} \approx_{\mathrm{ext}} \mathfrak{G}$ iff both $\mathfrak{F} \preceq_{\mathrm{ext}} \mathfrak{G}$ and $\mathfrak{G} \preceq_{\mathrm{ext}} \mathfrak{F}$ hold, and $\mathfrak{F} \approx_{\mathrm{w}} \mathfrak{G}$ iff both $\mathfrak{F} \approx_{\mathrm{int}} \mathfrak{G}$ and $\mathfrak{F} \approx_{\mathrm{ext}} \mathfrak{G}$ hold.

Lemma 5.4. *If $\mathfrak{F} \subseteq L$ is finite and $u \in \mathfrak{F}^\circ$, then there exists $m \in \mathrm{Min}(\mathfrak{F})$ such that $m \leq u$. Dually, if $u \in \mathfrak{F}_\circ$, then there exists $m \in \mathrm{Max}(\mathfrak{F})$ such that $u \leq m$.*

Proof. Consider the finite nonempty set $\{v \in \mathfrak{F}^\circ : v \leq u\}$. It contains u, hence has a minimal element m. This m belongs to $\mathrm{Min}(\mathfrak{F})$ and satisfies $m \leq u$. The maximal statement is dual. □

Theorem 5.5 (Antichain representation of approximation equivalence)**.** *Let L be a poset with bottom and top, and let $\mathfrak{F}, \mathfrak{G} \subseteq L$ be finite families. Then*

$$\mathfrak{F} \approx_{\mathrm{int}} \mathfrak{G} \quad \Longleftrightarrow \quad \mathrm{Min}(\mathfrak{F}) = \mathrm{Min}(\mathfrak{G}),$$

and

$$\mathfrak{F} \approx_{\mathrm{ext}} \mathfrak{G} \quad \Longleftrightarrow \quad \mathrm{Max}(\mathfrak{F}) = \mathrm{Max}(\mathfrak{G}).$$

Consequently,

$$\mathfrak{F} \approx_{\mathrm{w}} \mathfrak{G} \quad \Longleftrightarrow \quad \mathrm{Min}(\mathfrak{F}) = \mathrm{Min}(\mathfrak{G}) \text{ and } \mathrm{Max}(\mathfrak{F}) = \mathrm{Max}(\mathfrak{G}).$$

Proof. We prove the internal statement; the external statement is dual.

Assume first that $\mathrm{Min}(\mathfrak{F}) = \mathrm{Min}(\mathfrak{G})$. Let $g \in \mathfrak{G}^{\circ}$. By the previous lemma, there exists $m \in \mathrm{Min}(\mathfrak{G})$ with $m \leq g$. Since $\mathrm{Min}(\mathfrak{F}) = \mathrm{Min}(\mathfrak{G})$, we have $m \in \mathfrak{F}^{\circ}$. Thus $\mathfrak{F} \preceq_{\mathrm{int}} \mathfrak{G}$. The reverse approximation is identical, so $\mathfrak{F} \approx_{\mathrm{int}} \mathfrak{G}$.

Conversely, assume $\mathfrak{F} \approx_{\mathrm{int}} \mathfrak{G}$. Let $m \in \mathrm{Min}(\mathfrak{F})$. Since $\mathfrak{G} \preceq_{\mathrm{int}} \mathfrak{F}$, there exists $g \in \mathfrak{G}^{\circ}$ with $g \leq m$. By the lemma, there exists $n \in \mathrm{Min}(\mathfrak{G})$ with $n \leq g$. Since $\mathfrak{F} \preceq_{\mathrm{int}} \mathfrak{G}$, there exists $f \in \mathfrak{F}^{\circ}$ with $f \leq n$. Hence $f \leq n \leq g \leq m$. Because m is minimal in $\mathfrak{F}^{\circ}$ and $f \in \mathfrak{F}^{\circ}$, it follows that $f = m$, and the chain forces $m = n = g = f$. In particular, $m = n \in \mathrm{Min}(\mathfrak{G})$, so $\mathrm{Min}(\mathfrak{F}) \subseteq \mathrm{Min}(\mathfrak{G})$. The reverse inclusion follows symmetrically. Therefore $\mathrm{Min}(\mathfrak{F}) = \mathrm{Min}(\mathfrak{G})$. The weak equivalence statement follows immediately from the internal and external characterizations. □

Remark 5.6. The assumption that L has both $\bot$ and $\top$ is a convenient way to match Molodtsov's original nontrivial finite-family convention. If no bottom or top element is distinguished, the same antichain argument works by replacing $\mathfrak{F}^{\circ}$ and $\mathfrak{F}_{\circ}$ simply by $\mathfrak{F}$ and using the ordinary sets of minimal and maximal elements. If only a bottom element is present, the internal statement uses $\mathfrak{F} \setminus \{\bot\}$ while the external statement uses ordinary maximal elements; the dual convention applies when only a top element is present.

Corollary 5.7. *For finite families in a poset fibre $P(X)$, every internal-equivalence class has a unique minimal antichain representative* $\mathrm{Min}(\mathfrak{F})$*, and every external-equivalence class has a unique maximal antichain representative* $\mathrm{Max}(\mathfrak{F})$*.*

6 Examples

Example 6.1 (Ordinary soft sets)**.** This example recovers Molodtsov's original soft sets after passing from a representative $S : A \to 2^X$ to its image family [15, 16]. Let $\mathcal{C} = \mathbf{Set}$ and let $P = P_{\mathrm{pow}}$, where $P_{\mathrm{pow}}(X) = 2^X$. A doctrinal soft structure is a family $\mathfrak{F} \subseteq 2^X$. A soft morphism

$$f : (X, \mathfrak{F}) \to (Y, \mathfrak{G})$$

is a function $f : X \to Y$ such that for every $U \in \mathfrak{F}$ there exists $V \in \mathfrak{G}$ with

$$U \subseteq f^{-1}(V).$$

This is equivalent to $f(U) \subseteq V$. Thus the morphism sends every selected approximate description of X into a selected approximate description of Y.

Example 6.2 (Subobject doctrine)**.** This is the standard subobject doctrine of categorical logic [11, 12]. Let $\mathcal{C}$ be a category with pullbacks and let

$$P(X) = \mathrm{Sub}_{\mathcal{C}}(X).$$

A doctrinal soft object is a family of subobjects of X. A morphism

$$f : (X, \mathfrak{F}) \to (Y, \mathfrak{G})$$

requires every selected subobject of X to be contained in the pullback along f of some selected subobject of Y.

Example 6.3 (Groups)**.** Soft groups were introduced by Aktas and Cagman using Molodtsov's soft sets [2]. In the present framework, let $\mathcal{C} = \mathbf{Grp}$ and let $P(G)$ be the poset of subgroups of G. For a homomorphism $\varphi : G \to H$, reindexing is inverse image of subgroups:

$$P(\varphi)(K) = \varphi^{-1}(K).$$

A doctrinal soft group is a family of subgroups $\mathfrak{F} \subseteq P(G)$. A morphism

$$\varphi : (G, \mathfrak{F}) \to (H, \mathfrak{G})$$

satisfies: for every $U \in \mathfrak{F}$ there exists $V \in \mathfrak{G}$ such that

$$U \leq \varphi^{-1}(V),$$

equivalently $\varphi(U) \leq V$.

Example 6.4 (Modules)**.** Soft modules and related algebraic variants are obtained by replacing subsets with parameterized submodules; see, for example, Atagün and Sezgin's treatment of soft substructures of rings, fields and modules [4]. Let R be a ring, $\mathcal{C} = R\text{-}\mathbf{Mod}$, and let $P(M)$ be the poset of submodules of M. A doctrinal soft module is a family of submodules of M. A morphism is an R-linear map $f : M \to N$ such that every chosen submodule of M is mapped into a chosen submodule of N.

Example 6.5 (Relations)**.** Soft relations and functions have been investigated in the ordinary soft-set literature [5]. Let $\mathcal{C} = \mathbf{Set}$ and let $P(X) = 2^{X\times X}$ be the poset of binary relations on X. If $f : X \to Y$, define

$$P(f)(R) = \{(x, x') \in X \times X : (f(x), f(x')) \in R\}.$$

A doctrinal soft relation on X is a family of binary relations on X. A soft morphism transports every selected relation on X into the inverse relational image of a selected relation on Y.

Example 6.6 (Topological predicates)**.** Soft topological spaces were introduced by Shabir and Naz as soft-set analogues of topological spaces [18]. A related doctrinal version is obtained as follows. Let $\mathcal{C} = \mathbf{Top}$ and let $P(X)$ be the poset of open subsets of X. For a continuous map $f : X \to Y$, reindexing is inverse image of open sets. A doctrinal soft topological predicate on X is a family of open subsets of X. A soft morphism requires every chosen open predicate on the domain to be contained in the inverse image of a chosen open predicate on the codomain.

7 Conclusion

This paper has proposed a categorical reconstruction of Molodtsov's equivalence principle for soft sets. The central point is that a parameterized map

$$S : A \to P(X)$$

should not be treated as the invariant soft structure itself. It is a representative whose mathematical content, in the sense of Molodtsov equivalence, is the image family

$$\operatorname{Im} S \subseteq P(X).$$

By replacing the classical power-set doctrine P_{pow}, given by $P_{\mathrm{pow}}(X) = 2^X$, with an arbitrary indexed poset

$$P : \mathcal{C}^{op} \to \mathbf{Pos},$$

we obtain a general theory of doctrinal soft structures.

The resulting category $\mathrm{Soft}_P(\mathcal{C})$ has a clean categorical form. It is the Grothendieck construction of the indexed preorder of fibrewise soft families. Its projection to the base category is a Grothendieck fibration, so soft structures vary functorially over the underlying category. Correct operations, in Molodtsov's sense, are precisely those operations that factor through the image-family quotient; when they are compatible with reindexing, they induce endofunctors on the category of doctrinal soft structures. Under finite-meet assumptions, pullbacks in the soft category are constructed over pullbacks in the base category, and the constructed pullback squares are sent by the projection to the chosen pullback squares in the base category. Finally, Molodtsov's internal and external approximation equivalences extend from families of subsets to arbitrary finite poset fibres, where their equivalence classes are represented by minimal and maximal antichains.

The main theorems also have distinct methodological roles. The Grothendieck representation theorem solves the problem of constructing a parameter-free category of soft structures without introducing ad hoc morphisms. The fibration theorem shows that change of base is canonical: a soft structure over a codomain can be pulled back along any base morphism. The Molodtsov quotient theorem gives a precise categorical form to the passage from parameterized soft sets to invariant image families. The correct-operation theorem provides a practical criterion for deciding whether a proposed soft operation is structurally meaningful rather than parameter-dependent. The pullback theorem supplies a finite-limit construction in the doctrinal category, while the antichain theorem gives canonical finite representatives for internal and external approximation classes.

This framework shows that soft set theory is not intrinsically restricted to ordinary subsets of a universe. Its invariant core is a parameter-free family of predicates in a doctrine. Ordinary soft sets arise from the power-set doctrine [15]; soft subgroups, soft submodules, soft relations, soft topological predicates and soft subobjects arise from other doctrines, in a way that

is compatible with earlier soft algebraic, relational and topological constructions [2, 4, 5, 18]. This categorical formulation clarifies which operations are structurally meaningful and which depend on accidental parameterizations.

Several directions remain open. First, one may replace indexed posets by indexed categories in order to treat genuinely structured fibres rather than posetal doctrines. Second, one may study enriched or quantale-valued doctrines to preserve degree information in fuzzy-soft settings, avoiding the loss of membership-level data observed by Molodtsov. Third, the fibration $\mathrm{Soft}_P(\mathcal{C}) \to \mathcal{C}$ suggests possible connections with sheaf-like soft structures, descent, stacks of local soft data and categorical decision systems. Finally, the antichain representation theorem indicates that finite doctrinal soft structures may admit efficient minimal representations, which could be useful in applications to knowledge spaces, formal concept analysis, approximation systems and multi-criteria decision models.

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