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Commutative subdirectly irreducible semigroups

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Abstract. This paper aims to establish a structural characterization of subdirectly irreducible commutative semigroups. We unify the three classes of semigroups introduced by Schein into a single structure. Furthermore, we characterize the distinction between irreducible and subdirectly irreducible commutative semigroups.

1 Introduction and Preliminaries

Since Birkhoff's theorem [1, Chapter II, Theorem 8.6] establishes that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras of the same type, subdirectly irreducible semigroups have been extensively studied over the past six decades. Recall from [4] that a semigroup S is a subdirect product of the semigroups $\{S_i\}_{i\in I}$ if there exists an injective homomorphism $S \to \prod_{i\in I} S_i$ such that the induced projection $S \to S_i$ is surjective for all $i \in I$. A semigroup S is said to be subdirectly irreducible

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(respectively, finitely subdirectly irreducible or simply irreducible) if, whenever S is a subdirect product of a family (respectively, a finite family) of semigroups $\{S_i\}_{i\in I}$, the projection homomorphism π_j is an isomorphism onto S_j for at least one $j \in I$.

The study of such semigroups was initiated by G. Thierrin [14] and B. M. Schein [12], with the latter investigating their fundamental properties. In the existing literature on this subject, the notion of "disjunctive element" plays a key role. However, the existence of a disjunctive element is a very strong condition; for instance, a semigroup S with a disjunctive zero element is subdirectly irreducible if and only if it has a core K [11, Theorem 3.7], in which case the least nonidentical congruence is $\varepsilon^K = K \times K \cup \Delta_S$. Furthermore, the existence of a nonzero disjunctive element implies that the semigroup possesses a core, and any such element must belong to it.

In this paper, we aim to employ a weaker condition, namely the *weakly essential ideal*, to characterize subdirectly irreducible commutative semigroups. This approach provides a generalization of the notion of essential ideals.

In [11], subdirectly irreducible commutative semigroups are classified into three kinds. The first kind consists of subgroups of p^∞-groups (p is a prime), with or without an externally adjoined zero. The second kind comprises commutative semigroups with a nonzero annihilating disjunctive element, possibly possessing an external identity element. The third kind includes commutative monoids featuring a nontrivial zero divisor and a nonzero disjunctive element, where the set of all non-divisors of zero forms a group of the first kind. Semigroups of the third kind, in particular, represent extensions of groups belonging to the first kind. In this paper, we present necessary and sufficient conditions for such group extensions to be subdirectly irreducible. Indeed, we prove that a commutative semigroup S is subdirectly irreducible if and only if $S = G \sqcup T$, where G is empty or isomorphic to a subgroup of a p^∞-group, and T is empty or is an ideal of S. If T is nontrivial, it is a semigroup with a nontrivial annihilator K. This K is the core of S and is weakly essential therein (Theorem 3.1).

On the other hand, we prove that the gap between irreducible and subdirectly irreducible commutative semigroups lies in the existence of the core and the finiteness order of cancellative elements, if any. Moreover, the approach pursued in this paper provides a characterization of finitely generated commutative irreducible semigroups stated in [4]. In light of constructing finite commutative semigroups as certain permutation group extensions of nilpotent semigroups, as investigated in [9], the results presented in this paper may be employed to clarify the general construction of commutative subdirectly irreducible semigroups.

First, we recall the necessary ingredients for the subsequent arguments, which can mostly be found in [3, 8, 10, 11]. Throughout this paper, S will denote a semigroup. To every semigroup S we can associate the monoid S^1 with the identity element 1 adjoined if necessary. Indeed,

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Recall from [11] that for a subset H of a semigroup S, the equivalence $\mathscr{C}_H$ defined by

$$\mathscr{C}_H = \{(a, b) \in S \times S \mid (\forall x, y \in S^1)\, xay \in H \iff xby \in H\}$$

is a congruence on S. A subset H is called disjunctive if $\mathscr{C}_H = \Delta_S$. An element $s \in S$ is called a disjunctive element if $\{s\}$ is disjunctive ($\mathscr{C}_{\{s\}} = \Delta_S$). It can be verified that for a subset H, $\mathscr{C}_H$ is the greatest congruence for which H is saturated. Therefore, non-zero ideals of semigroups are not disjunctive, and hence the notion of disjunctivity is mainly employed for elements of a semigroup.

Note that for a semigroup S, the monocyclic congruence generated by the pair (a, b) is denoted by $\rho(a, b)$, where $x\rho(a, b)y$ if and only if $x = y$ or there exist $p_i, q_i \in \{a, b\}$ and $w_i, v_i \in S^1$ forming a chain of equalities:

$$\begin{aligned} x = v_1p_1w_1, \quad v_1q_1w_1 &= v_2p_2w_2, \\ v_2q_2w_2 &= v_3p_3w_3, \\ \vdots &= \vdots \\ v_nq_nw_n &= y, \end{aligned} \tag{1.1}$$

where $n \in \mathbb{N}$ is the length of the sequence.

Following the terminology in [6], a congruence ρ on a semigroup S is said to be essential if for any congruence σ on S, $\rho \cap \sigma = \Delta_S$ implies $\sigma = \Delta_S$. Accordingly, an ideal I of S is called essential if the induced Rees congruence

ε^I $(I \times I \cup \Delta_S)$ is essential. Equivalently, any semigroup homomorphism from S that is a monomorphism on I is itself a monomorphism on S.

On the other hand, it is routinely observed that an ideal I of a semigroup S is essential if and only if for any non-identity (monocyclic) congruence ρ, there exists $i \in I$ such that the ρ-class $[i]_\rho$ contains at least two distinct elements of I. Clearly, every essential ideal is non-zero. Following the approach in [10], a semigroup S is called uniform if every non-trivial ideal is essential in S. It is evident that every irreducible semigroup is uniform.

Definition 1.1. Let S be a semigroup and $A \subseteq S$. A is called weakly essential if for any distinct $a, b \in S$, there exist $s, t \in S^1$ such that $sat \neq sbt$ and at least one of sat or sbt belongs to A. An element $a \in S$ is called weakly essential if the set $\{a\}$ is weakly essential.

Note that the notion of weak essentiality is inherited by supersets in a semigroup. The following lemma justifies the terminology used for weak essentiality.

Lemma 1.2. *A subset A of a semigroup S is weakly essential if and only if for any non-identity (monocyclic) congruence ρ on S, $[a]_\rho$ has more than one element for some $a \in A$. Therefore, any essential ideal is weakly essential.*

Proof. **Necessity:** Let $\rho(a, b)$ be a monocyclic congruence on S for distinct elements a and b. Since A is weakly essential, there exist $s, t \in S^1$ such that $sat \neq sbt$ and at least one of sat or sbt belongs to A. Therefore, A satisfies the desired property.

Sufficiency: By way of contradiction suppose that A is not weakly essential. Then there exist distinct elements a and b in S such that for any $s, t \in S^1$, $sat = sbt$ or $\{sat, sbt\} \cap A = \emptyset$. Now take the monocyclic congruence $\rho(a, b)$ on S and fix $x \in A$. Suppose that $x\rho(a, b)y$ for some $y \neq x$ on S. Thus there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in S$, $w_1, \ldots, w_n, v_1, \ldots, v_n \in S^1$ where for every $i = 1, \ldots, n$, $(p_i, q_i) \in \{(a, b), (b, a)\}$, with the following sequence of equalities:

$$\begin{aligned} x = & v_1p_1w_1 \quad v_2q_2w_2 = v_3p_3w_3 \cdots \\ & v_1q_1w_1 = v_2p_2w_2 \qquad\qquad \cdots = y. \end{aligned}$$

Since $v_1p_1w_1 \in A$, $v_1p_1w_1 = v_1q_1w_1 = v_2p_2w_2$. Again the existence of $v_2p_2w_2$ in A implies that $v_1p_1w_1 = v_2p_2w_2 = v_2q_2w_2$. By an inductive process, it follows that

$$x = v_1 p_1 w_1 = v_1 q_1 w_1 = \cdots = v_n q_n w_n = y.$$

Therefore, for any $x \in A$, the class $[x]_{\rho(a,b)}$ is a singleton, a contradiction. Now, the second statement is a straightforward conclusion of the first one. $\square$

Note that the implication essential ideal $\Longrightarrow$ weakly essential ideal is strict, which is established in the next example.

Example 1.3. Let $S = \langle a \rangle$ be a cyclic semigroup generated by a, and let $x = a^m$ and $y = a^{m+n}$ be two distinct elements. It can be shown that $\rho(x, y) = \Delta_S \cup \{(a^r, a^s) \mid n \mid |r - s|, r, s \geq m\}$. On the other hand, the structure of S implies that any ideal is weakly essential. Suppose that a has index r and period m, and consider the monocyclic congruence $\rho(a, a^m)$. Then the class of any element in the kernel of S is a singleton, and hence it is not essential.

The following class of congruences is an extended version of the one stated in [4]. For a semigroup S, any subset A induces a congruence $\underline{A}$ defined by $x\underline{A}y$ if and only if, for all $s, t \in S^1$, $sxt \in A \iff syt \in A$, and $sxt \in A \implies sxt = syt$. Now, it can be readily checked that A is weakly essential if and only if $\underline{A}$ is the identity congruence.

The next proposition states that the existence of a disjunctive element implies that all nonzero ideals are weakly essential; thus, the concept of a disjunctive subset is stronger than that of a weakly essential subset.

Proposition 1.4. *Let S be a semigroup with a nonempty disjunctive subset H. Then H is weakly essential. If S contains a disjunctive zero element, then any subset of S containing the zero element, in particular any ideal, is weakly essential. If S contains a nonzero disjunctive element, then S has a core K, and any subset of S containing K, in particular any nonzero ideal, is weakly essential.*

Proof. To prove the first statement, suppose by way of contradiction that H is not weakly essential. Then there exist distinct elements $a, b \in S$ such that for any $s, t \in S^1$, $sat = sbt$ or $\{sat, sbt\} \cap H = \emptyset$. This implies that $(a, b) \in \mathscr{C}_H$, a contradiction. To prove the second statement, take

$H = \{0\}$ and note that the property of being weakly essential is inherited by supersets. For the third statement, suppose h is a nonzero disjunctive element, which implies the existence of a core K containing h (see [11]). If A is a subset of S containing K, then $h \in A$. If we assume A is not weakly essential, we reach a contradiction as $(a, b) \in \mathscr{C}_{\{h\}} = \Delta_S$. $\square$

To illustrate Proposition 1.4, every subdirectly irreducible semigroup has at least two distinct disjunctive elements [11, Theorem 3.3]; hence, any nonzero ideal of a subdirectly irreducible semigroup is weakly essential. On the other hand, if S is a rectangular band that is neither a left zero nor a right zero semigroup, with $|S| \geq 6$, then S is simple and contains a core; thus, any ideal is weakly essential, despite having no disjunctive elements. To prove the last claim, without loss of generality, assume $S = I \times \Lambda$ with $|I| \geq 3$ and $|\Lambda| \geq 2$, and let $g = (i, \lambda)$. Given $i', i'' \in I$ such that $i' \neq i, i'' \neq i$ and $i' \neq i''$, and choosing $\lambda' \neq \lambda$, one can readily check that $(i', \lambda')\mathscr{C}_{\{g\}}(i'', \lambda')$, implying that g is not disjunctive. Consequently, the existence of nonzero disjunctive elements implies that S has a core and any nonzero ideal is weakly essential, although the converse does not hold. We observe the following strict implications:

$$\text{disjunctive} \Longrightarrow (\text{weakly essential} + \text{existence of the core}).$$

The following lemma is a direct consequence for semigroups with nontrivial annihilators.

Lemma 1.5. *Let S be a semigroup with a nontrivial annihilator that has a core K. Then there exists $m \in S$ such that $K = \{0, m\}$ is primitive and is the annihilator of S.*

2 Commutative irreducible semigroups

In this section, we present some results on commutative irreducible semigroups, which in the finitely generated case were characterized in [4]. We employ our approach to characterize commutative finitely generated irreducible semigroups, ultimately leading to the identification of the gap between commutative irreducible and subdirectly irreducible semigroups.

First, we recall the necessary preliminaries.

For any commutative nilsemigroup S, since the equality $m = mu$ ($m, u \in S$) implies $m = 0$, the relation $x \leq y$ defined by $x = uy$ for some $u \in S^1$ is a partial order on S. A nonzero element m is said to be minimal if there is no element strictly between 0 and m. Therefore, if S is finitely generated (and thus finite), the set of minimal elements of S, denoted by M, together with the zero element, denoted by $\overline{M}$, forms the annihilator of S.

A subelementary semigroup is a commutative semigroup $S = N \cup C$ which is the disjoint union of an ideal N, which is a nilsemigroup, and a subsemigroup C in which every element is cancellative in S. Note that in such semigroups, the zero element of N is the zero element of S, and the identity element of C, if any, is the identity element of S. If $C = G$ is a group, S is called elementary. For any commutative semigroup S with the subsemigroup of cancellative elements C, we can construct the fraction semigroup induced by C, denoted by $C^{-1}S$ (the localization of S at C), consisting all fractions $\dfrac{a}{c}$, for $a \in S, c \in C^1$, where $\dfrac{a}{c} = \dfrac{b}{d}$ if $ad = bc$, in which S can be embedded therein by the embedding $s \to \dfrac{s}{1}$. Thus, any subelementary semigroup S can be extended to an elementary semigroup, namely $C^{-1}S = C^{-1}N \cup C^{-1}C$, with the group part $C^{-1}C$ and the nilsemigroup part $C^{-1}N$. An elementary semigroup $S = N \cup G$ is called homogeneous if, for any nonzero element $x \in S$ and $g \in G$, the equality $gx = x$ implies $g = 1$ (G acts regularly on $S\backslash 0$). A subelementary semigroup S is called homogeneous if its elementary extension $C^{-1}S$ is homogeneous; equivalently, for any nonzero element $x \in S$ and $c, d \in C$, $cx = dx$ implies $c = d$. On the other hand, the relation $\equiv$ on S, defined by $a \equiv b$ if $ca = db$ for some $c, d \in C^1$, is a congruence, and the class $a \in S$ modulo $\equiv$ is called the orbit of a; it can be routinely verified that the orbit of a is $(a C^{-1}C) \cap S$. Regarding the structure of subelementary semigroups and their fraction semigroups, we observe that for $a \in S$ and $c \in C$, $\dfrac{a}{c} \in C^{-1}N, C^{-1}\overline{M},$ or $C^{-1}C$ if and only if $a \in N, \overline{M},$ or C, respectively. Following the proof of Lemma 1.2 in [4], for a subelementary semigroup S, N is called *weakly irreducible* if the set of minimal elements of N is weakly essential. For more details on the aforementioned preliminaries, see [5].

In [4], finitely generated commutative irreducible semigroups are classified into cancellative, nil, and subelementary semigroups, with their characterizations provided in the following results. Moreover, Corollary 2.5 is a generalization of this classification.

Proposition 2.1. [4, Proposition 2.5] *Let $S \neq \{1\}$ be a cancellative, finitely generated commutative semigroup. Then S is irreducible if and only if it is isomorphic to $\mathbb{Z}$, $\mathbb{Z}(p^n)$ (with p prime), or to a subsemigroup of $\mathbb{N}$.*

Proposition 2.2. [4, Proposition 3.1] *A commutative nilsemigroup N of finite height is irreducible if and only if it is weakly irreducible and has only one minimal element.*

Theorem 2.3. [4, Theorem 3.6] *Let $S = N \cup C$ be a finitely generated subelementary semigroup, with $N \neq 0$ and C nontrivial. Then S is irreducible if and only if C is irreducible, N is weakly irreducible, the minimal elements of N form an orbit, and S is homogeneous.*

Besides, in [11], it is implicitly established that any subdirectly irreducible commutative periodic semigroup is classified into these three classes of semigroups. In the next proposition, regarding the implications

$$\text{subdirectly irreducible} \Longrightarrow \text{irreducible} \Longrightarrow \text{uniform}$$

we prove that a larger class of semigroups is also classified into these three classes. To unify these, we call a semigroup S *hyperelementary (hypersubelementary)* if it is a commutative monoid (semigroup) which is nil, a group (cancellative), or elementary (subelementary). Thus, a hyperelementary (hypersubelementary) semigroup S is a disjoint union $S = G \sqcup N$ ($S = C \sqcup N$), where N is either void or an ideal which is a nilsemigroup, and G (C) is either void or a subgroup (subsemigroup) in which every element is cancellative in S.

Following [13], for any element x in a commutative semigroup S, the *annihilator congruence* on S, defined by $\{(a,b) \mid xa = xb\}$, is denoted by C_x. Note that for an element x, the sequence of annihilator congruences $C_x \subseteq C_{x^2} \subseteq \cdots \subseteq C_{x^n} \subseteq \cdots$, is stationary if and only if $C_x \cap \varepsilon^{(x^nS)} = \Delta_S$ for some $n \in \mathbb{N}$. The following proposition states that this stationary condition leads to the nilpotency of an element in commutative uniform semigroups.

Proposition 2.4. *A commutative uniform semigroup S is hypersubelementary if and only if for any $x \in S$, $C_x \cap \varepsilon^{(x^nS)} = \Delta_S$ for some $n \in \mathbb{N}$.*

Proof. Since any element of a hypersubelementary semigroup is either nil or cancellative, we only need to prove the sufficiency. Suppose $x \in S$ is not cancellative. Then for some $n \in \mathbb{N}$, $C_x \cap \varepsilon^{(x^n S)} = \Delta_S$; furthermore, the uniformity of S implies $|x^n S| = 1$, which means x^{n+1} is a zero element. Therefore, x is nilpotent, and S is a hypersubelementary semigroup. □

As a consequence of this argument, we obtain the following corollary:

Corollary 2.5. *Any commutative periodic* (*finitely generated*) *uniform semigroup is hyperelementary* (*hypersubelementary*)*.*

Proof. Let x be a non-nilpotent element in a commutative periodic uniform semigroup with index m and period r. Then $x^r x^m = x^m$. By [10, Corollary 3.15], S has a zero element and $x^m = 0$, or S has an identity element and $x^r = 1$. Since x is not nilpotent, the second case must occur. Moreover, if S is finitely generated, then by Rédei's theorem [8], the sequence of annihilator congruences $\{C_{x^n}\}_{n\in\mathbb{N}}$ is stationary, implying $C_x \cap \varepsilon^{(x^n S)} = \Delta_S$ for some n (see the proof of [4, Proposition 2.2]). □

Note that for any uniform monoid S, S is the disjoint union of a subgroup G and an ideal I by [10, Proposition 3.13], where the group part is clearly uniform. Besides, it is implicitly shown in [11] that the group part of any commutative subdirectly irreducible monoid is subdirectly irreducible. In other words, for commutative monoids, the notions of uniform and subdirectly irreducible are transferred from the monoid to its group part. The next proposition states that the same result holds for commutative irreducible monoids.

Proposition 2.6. *The group part of any commutative irreducible monoid is irreducible and is thus isomorphic to a subgroup of a p^∞-group for some prime p, or to an additive subgroup of the rational numbers.*

Proof. Considering the order-preserving correspondence between the congruences of a group and its normal subgroups, suppose N and H are two nontrivial subgroups of G, and let ρ_N and ρ_H be the congruences on S defined by $x\rho_N y$ if $x \in yN$, and $x\rho_H y$ if $x \in yH$. Since ρ_N and ρ_H are nonidentical, there exist two distinct elements $a, b \in S$ such that $a(\rho_N \cap \rho_H)b$, which implies $a = bn$ and $a = bh$ for some $n \in N$ and $h \in H$. The condition $a \neq b$ necessitates that $b \neq 0$ and $n, h \neq 1$. By [10, Corollary 3.15],

$b = bhn^{-1}$ implies $n = h$. Therefore, $1 \neq n \in N \cap H$, proving that G is irreducible. Now, since any irreducible abelian group is a uniform $\mathbb{Z}$-module with an indecomposable injective envelope, and since any indecomposable injective (divisible) abelian $\mathbb{Z}$-module is isomorphic to a p^∞-group (for a prime p) or to the additive group of rational numbers $\mathbb{Q}$ (see [2, Chapter IV]), it follows that G is isomorphic to a subgroup of a p^∞-group or an additive subgroup of $\mathbb{Q}$. □

To establish the results in general, we define $C^{-1}S = S^1$ for any commutative semigroup S with no cancellative elements, which is indeed the fraction semigroup of S^1 by the cancellative subsemigroup $\{1\}$. The next two propositions present the approach we are taking to reach our main target.

Proposition 2.7. *Let $S = N \cup C$ be a finitely generated subelementary semigroup, with $N \neq 0$ and C nontrivial. Then S is irreducible if and only if in its elementary fraction semigroup, $C^{-1}S$, the group part $C^{-1}C$ is irreducible and $C^{-1}\overline{M}$ is the core of $C^{-1}S$, which is weakly essential in $C^{-1}S$.*

Proof. Following [4, Proposition 2.3], $C^{-1}C$ is irreducible if and only if C is irreducible. Thereby, in light of Theorem 2.3, we only need to prove that the last two claimed conditions are equivalent to the conditions stated therein.

Necessity: Suppose that N is weakly irreducible, the set of minimal elements of N forms an orbit, and S is homogeneous. To prove that $C^{-1}\overline{M}$ is the core of $C^{-1}S$, we show that $C^{-1}\overline{M}$ is a 0-minimal ideal which meets any nonzero ideal of $C^{-1}S$. Take two elements $m, n \in M$. Since M is an orbit, $cm = dn$ for some $c, d \in C$, which implies that $m C^{-1}S = n C^{-1}S$; consequently, $C^{-1}\overline{M}$ is a 0-minimal ideal of $C^{-1}S$. Now, take an arbitrary nonzero element x in $C^{-1}S$. If x is in the group part, clearly $C^{-1}\overline{M} \subseteq x C^{-1}S = C^{-1}S$. Otherwise, $x = \dfrac{n}{c}$ for some $n \in N, c \in C$. Since $x \neq 0$, $n \neq 0$ and there exists $m \in M$ such that $m = nu$ for some $u \in N^1$. Since $n C^{-1}S$ meets $C^{-1}\overline{M}$, $C^{-1}\overline{M} \subseteq n C^{-1}S$. Thus, $C^{-1}\overline{M}$ is the core of $C^{-1}S$.

To prove the weak essentiality of $C^{-1}\overline{M}$ in $C^{-1}S$, take two distinct elements $x = \dfrac{a}{c}, y = \dfrac{b}{d}$ in $C^{-1}S$ for $a, b \in S$ and $c, d \in C$. Three cases may occur:

Case 1: If $a, b \in N$, since N is weakly irreducible and $ad \neq bc$, there exists $u \in N^1$ such that $adu \neq bcu$ and $adu \in M$ or $bcu \in M$. Without loss of generality, suppose $adu \in M$ (which implies $au \in M$). Thus $xu \in C^{-1}\overline{M}$ and $yu = \dfrac{bu}{d} \neq \dfrac{au}{c} = xu$.

Case 2: If $a, b \in C$, then $ad \neq bc$ and S being homogeneous implies that $adm \neq bcm$ for some $m \in M$. This necessitates that $xm, ym \in C^{-1}\overline{M}$ and $xm \neq ym$.

Case 3: If $a \in C$ and $b \in N$, for a fixed element $m \in M$, $xm \in C^{-1}M$ and $ym = 0$ since m annihilates N. The case $a \in N$ and $b \in C$ follows analogously.

Sufficiency: First, we prove that N is weakly irreducible. Take two distinct elements x, y in N. It can be readily checked that N fulfills the weak essentiality condition if one element belongs to $\overline{M}$. Thus, we assume $x, y \in N \backslash \overline{M}$. Since $C^{-1}\overline{M}$ is weakly essential, there exist $a \in S, c \in C$ such that $\dfrac{xa}{c} \neq \dfrac{ya}{c}$ and $xa \in \overline{M}$ or $ya \in \overline{M}$. One may assume $xa \in \overline{M}$. Since $x \notin M$, $a \in N$.

To prove M is an orbit, take distinct $m, n \in M$. Since $C^{-1}\overline{M}$ is the core, $m C^{-1}S = n C^{-1}S = C^{-1}\overline{M}$, so $n = \dfrac{c}{d}m$ for some $c \in S, d \in C$, implying $cm = dn$. Now $dn \in M$ implies $c \in C$.

Finally, suppose $cm = dm$ for some $m \in M, c, d \in C$ and $c \neq d$. Considering the pair $(1, \dfrac{c}{d})$, the weak essentiality of $C^{-1}\overline{M}$ implies that $1\dfrac{a}{b} \neq \dfrac{c}{d}\dfrac{a}{b}$ for some $a \in S, b \in C$, and either $\dfrac{a}{b} \in C^{-1}\overline{M}$ or $\dfrac{c}{d}\dfrac{a}{b} \in C^{-1}\overline{M}$. Therefore, $adb \neq acb$ and $a \in M$ or $ac \in M$. Since $C^{-1}\overline{M}$ is 0-minimal, there exist $e \in S, f \in C$ such that $\dfrac{ae}{f} = m$, or $ae = mf$. Since $mf \in M$ and $a \in M$, we get $e \in C$; hence $cmf = cae \neq dae = dmf$, a contradiction. $\square$

Note that if S is a nilsemigroup, taking $C = \{1\}$ in S^1, $C^{-1}I = I$ for any ideal I of S. Therefore, the next proposition follows implicitly from Proposition 2.2.

Proposition 2.8. *Suppose S is a finitely generated commutative nilsemigroup. S is irreducible if and only if $C^{-1}S$ has a core $C^{-1}\overline{M}$ which is weakly essential in S.*

Regarding the terminology used in this paper, the fraction semigroups of finitely generated commutative irreducible semigroups are hyperelementary. The next theorem identifies the interrelation between these two classes of semigroups.

Theorem 2.9. *A finitely generated commutative semigroup S is irreducible if and only if $C^{-1}S$ is a hyperelementary monoid in which $C^{-1}C$ is irreducible and, if the nil part is nonempty, $C^{-1}\overline{M}$ is the core of $C^{-1}S$ and is weakly essential therein.*

3 Commutative subdirectly irreducible semigroups

In this section, we present the main result of the paper, which provides a structural characterization of commutative subdirectly irreducible semigroups. Furthermore, we unify all three classes of such semigroups into a general form. Ultimately, we discuss the gap between commutative irreducible and subdirectly irreducible semigroups. The following theorem characterizes commutative subdirectly irreducible semigroups.

Theorem 3.1. *A commutative semigroup S is subdirectly irreducible if and only if $S = G \sqcup T$, where G is either void or isomorphic to a subgroup of a p^∞-group (p is a prime), and T is either void or an ideal of S which, if nontrivial, is a semigroup with a nontrivial annihilator K that is the core of S and is weakly essential.*

Proof. **Necessity.** Let S be a commutative subdirectly irreducible semigroup with core K. Two cases arise.

Case 1: If K is globally idempotent, in light of [11, Theorem 5.1], S is isomorphic to a subgroup of a p^∞-group, potentially with an adjoined zero. In this case, S follows the desired structure where T is void or trivial.

Case 2: If K is nilpotent, two subcases occur.

Subcase 1: If S lacks an identity element and k is a nonzero element of K, then kS is an ideal of S contained in K. If $kS = K$, then $kl = k$ for some $l \in S$, which [11, Theorem 3.5] implies the existence of an identity element, a contradiction. Thus, $kS = 0$, and S has a nontrivial annihilator. Lemma 1.5 implies that for some $k \in S$, $K = \{0, k\}$ is primitive and is the annihilator of S. In view of Lemma 1.2, S possesses the desired structure where G is void.

Subcase 2: If S has an identity element 1, by [10, Proposition 3.13], [7, Theorem 3.5], and [11, Theorem 5.1], $S = G \sqcup T$, where G is isomorphic to a subgroup of a p^∞-group (for a prime p) and T is an ideal. If G is trivial, in view of [11, Theorem 3.6], T is a subdirectly irreducible commutative semigroup with no identity element, which satisfies the conditions of Subcase 1. If G is nontrivial, it is isomorphic to a subgroup of a p^∞-group with H as the least nontrivial subgroup of order p. We prove that the core of S, $K \subseteq T$, is the nontrivial annihilator of T, which is weakly essential by Lemma 1.2. For a nonzero $k \in K$, $kT = K$ would imply $kl = k$ for some $l \in T$, hence l is the identity of S, a contradiction. Thus, K is contained in the annihilator of T. Now, [7, Lemma 3.11] implies that K is the annihilator of T.

Sufficiency: Three cases occur.

Case 1: If T is void or trivial, S is subdirectly irreducible by [11, Theorem 5.1].

Case 2: If G is void or trivial, suppose K is the annihilator of T, the core of S, and is weakly essential. By Lemma 1.5, $K = \{0, k\}$ for some $k \in T$. Let $x, y \in S$ be distinct. Our assumption implies there exists $s \in S$ such that $xs \neq ys$ and $xs \in K$ or $ys \in K$. Without loss of generality, suppose $xs \in K$. If $ys \in K$, then $(xs, ys) \in \varepsilon^K$, implying $\varepsilon^K \subseteq \rho(x, y)$. If $ys \notin K$, then $(ys)t = k$ for some $t \in T$, so $(k, 0) = ((ys)t, (xs)t) \in \varepsilon^K$. Thus, $\varepsilon^K \subseteq \rho(x, y)$, implying ε^K is the least non-identical congruence.

Case 3: If G and T are both nontrivial, let H be the least nontrivial subgroup of G (order p) and let K be the weakly essential nontrivial annihilator of T. The zero element of T is the zero of S. Fix $k \in K \setminus \{0\}$ and consider the ideal $kS = kG \cup \{0\} \subseteq K$. Since K is the core, $kS = K$, and each nonzero element of K is of the form kg for some $g \in G$. This representation is unique: if $kg = k$ for $g \neq 1$, the weak essentiality of K implies that for $(1, g)$, there exists $s \in S$ such that $s \neq gs$ and $s \in K$, leading to a contradiction. Let ρ be the congruence: $a \rho b$ if $a = b$ or $a = kh_1, b = kh_2$ for some $h_1, h_2 \in G$ with $h_1 h_2^{-1} \in H$. We prove ρ is the least non-identical congruence. For $x \neq y$, weak essentiality implies there exists $s \in S$ such that $xs \neq ys$ and $xs \in K$ or $ys \in K$. Following the arguments in Case 2, we show $\rho \subseteq \rho(x, y)$. Finally, using the property of H as the least nontrivial subgroup, for $xs = kg_1, ys = kg_2$, we find n such that $(g_1 g_2^{-1})^n = h_1 h_2^{-1}$, ensuring $(kh_1, kh_2) \in \rho(x, y)$. $\square$

Let S be a commutative semigroup and C be the subsemigroup of cancellative elements. For $a, b \in S$ and $c, d \in C$, $\frac{a}{c}\frac{b}{d} = 0$ if and only if $ab = 0$. Hence, for a subelementary semigroup $S = N \cup C$, $\overline{M}$ is the annihilator of N if and only if $C^{-1}\overline{M}$ is the annihilator of $C^{-1}N$. Therefore, if a finitely generated subelementary semigroup $S = N \cup C$ is irreducible, the group part $C^{-1}C$ is irreducible and $C^{-1}\overline{M}$ is both the annihilator of $C^{-1}N$ and the core of $C^{-1}S$, which is weakly essential. Consequently, as an application of Theorems 3.1 and 2.9, it follows that S is subdirectly irreducible if and only if $C^{-1}C$ is subdirectly irreducible.

In view of the result in [4, Theorem 2.1], which establishes that every finitely generated commutative semigroup is a subdirect product of irreducible ones, commutative irreducible semigroups are of particular significance and warrant further investigation. Based on the above argument and Theorem 3.1, regarding the strict implication *subdirectly irreducible* $\Longrightarrow$ *irreducible* for commutative semigroups, we demonstrate that the necessary conditions for this implication are the existence of the core and the finiteness of the order of cancellative elements.

Theorem 3.2. *A commutative irreducible semigroup S is subdirectly irreducible if and only if S possesses the core and every cancellative element, if any, is of finite order.*

Proof. In light of Theorem 3.1, we only need to prove the sufficiency. Let K be the core of S, and assume every cancellative element has finite order. Two cases are considered.

Case 1: Suppose S has no cancellative elements. We aim to show that K is both the annihilator of S and weakly essential in S. Take a nonzero element $k \in K$. Since K is the core, $kS = K$, or S contains a zero element 0 such that $kS = \{0\}$. The former case implies $ks = k$ for some $s \in S$, leading to $s = 1$ or $k = 0$, which is a contradiction. Thus, $kS = \{0\}$, and K is the annihilator of S. If K contained two distinct nonzero elements k and k', then $\{0, k\}$ and $\{0, k'\}$ would be two distinct ideals, contradicting the irreducibility of S. Hence, $K = \{0, k\}$ for some $k \in S$. The result then follows from Lemma 1.2.

Case 2: Suppose S contains a cancellative element c. Since c is of finite order, c^n is an idempotent for some $n \in \mathbb{N}$. As any nonzero idempotent in a commutative irreducible semigroup is the identity, c must be invertible. The subsemigroup of cancellative elements forms an abelian irreducible group. Given the finite order property and Proposition 2.6, we have $S = G \sqcup T$, where G is isomorphic to a subgroup of a p^∞-group and T is an ideal of S. If T is empty or trivial, S is subdirectly irreducible. Otherwise, $K \subseteq T$. For any nonzero $k \in K$, $kT = K$ or $kT = \{0\}$. Similar to Case 1, K is the annihilator of T, which is weakly essential. □

As a consequence of the above theorem, we obtain the following corollary, where the second part is cited from [4].

Corollary 3.3. *For a commutative irreducible semigroup S, the following statements hold:*

(i) *If S is periodic or has no cancellative element, it is subdirectly irreducible if and only if it possesses the core.*

(ii) *If S is finitely generated, it is subdirectly irreducible if and only if all cancellative elements are of finite order.*

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