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Generating EQ-algebras from equality algebras with L-condition

Zhe Jie Duan, Xiao Long Xin*, and Rajab Ali Borzooei

Abstract. Inspired by W. Rump's foundational work on L-algebras and self-similarity, we introduce a new class of algebras called L-equality algebras by adding an implicative condition to equality algebras. Within this framework, we define self-similar L-equality algebras and show that they are equivalent to divisible residuated EQ-algebras satisfying a certain condition, via a naturally induced product operation. We induce filters as L-algebras, that is L-filters, and filters as equality-algebras, that is e-filters, and their prime versions. We verify L-filters and e-filters coincide, but prime L-filters and prime e-filters are not equivalent. To establish the reticulations from a bounded L-equality algebra E to a bounded lattice L, we first induce and study some new operations $\oplus$ and $\otimes$ on E, by which we introduce the concept of reticulations in an axiomatic manner. By utilizing the prime L-filters on E, we establish the prime L-spectrum of E, and by using the reticulation, we illustrate that the prime L-spectrum of E is homeomorphic to spectrum of L. Finally, we discuss the relationship between reticulation on L-equality algebras and reticulation on EQ-algebras, where the concept of reticulation on EQ-algebras is introduced by [X. Zhang, The spectra and reticulation

* Corresponding author
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of EQ-algebras, Soft Computing]. By utilizing the relationship between L-equality algebras and EQ-algebras as presented in our article, we conclude that these reticulations are consistent with each other.

1 Introduction

It is well known that the development of mathematical logic can be taken as two basic directions. The first one directs that the fundamental connective is implication and the basic inference rule is modus ponens (from A and $A \Rightarrow B$ infer B). The second one directs that the basic connective is logical equivalence (taken as an equality between truth values) and the basic inference rules are equanimity (from A and $A \equiv B$ infer B) and Leibniz (from $A \equiv B$ infer $C[p := A] \equiv C[p := B]$) ones. The connective of equality (equivalence) seems to be a more essential connective than implication (see [6]). In [16], the author constructs foundations of mathematics using equality in higher-order logic (type theory). Moreover, the formal proofs can be more effectively formed in an equational style. Various arguments in favor of such approach have been given in [15], namely that the equational style makes it possible to develop and present calculations in a rigorous manner, without complexity and detail overwhelming. Hence, proofs in this style are relatively easy to construct.

Equality algebras were initially introduced by Jenei in [17], motivated by EQ-algebra [20]. It has two basic operations, a meet operation and an equivalence operation, and a constant 1. It was proved in [7, 8, 17] that any equality algebra has a corresponding BCK-meet-semilattice and any $BCK(D)$-meet-semilattice (with distributivity property) has a corresponding equality algebra. In [28], the authors study the relation among equality algebras and other logical algebras such as residuated lattices, MTL-algebra, BL-algebra, MV-algebra, Boolean-algebra, EQ-algebra and hoop-algebra. Some properties of these algebras were mentioned in [4, 5]. EQ-logics are based on a special algebra (EQ-algebra) of truth values. It was introduced in [10, 11] and can be taken as a special class of fuzzy logics that differ from residuated fuzzy logics [12, 13]. Motivated by [10] we introduce a new formal logic, called equality propositional logic. The reticulation of a ring was defined by [23] for commutative rings, and it was extended to non-commutative rings in [3]. As for the algebras of fuzzy logics, the reticulation of EQ-algebras was introduced by [29].

The motivation of this paper arises from the following two aspects: (1) It has long been known that equality algebras are the product-free reducts of EQ-algebras, but a natural question is how an equality algebra can be converted into an EQ-algebra. (2) Can we follow the ideas of W. Rump's foundational work on L-algebras and self-similarity by first enriching equality algebras with an implication (L) operation, and then inducing a product operation via self-similarity, thereby verifying whether an equality algebra can be transformed into an EQ-algebra? The motivation for introducing L-equality algebras stems from the relationship between equality algebras and EQ-algebras. Although every EQ-algebra determines an equality algebra by forgetting the product operation, the converse does not hold in general. This naturally raises the question of identifying conditions under which an equality algebra admits a compatible product operation and can therefore be related to an EQ-algebra. Condition (L) provides such a criterion. It defines a class of equality algebras that is sufficiently rich to recover important algebraic properties associated with EQ-algebras while preserving the original equality-based structure. This makes L-equality algebras a natural setting for studying the interaction between equality operations, residuation, and related algebraic constructions.

2 Preliminaries

First, we recall some definitions and results which will be used in the following sections.

Definition 2.1. [14] An algebra $(E, \wedge, \cdot, \sim, 1)$ of type (2,2,2,0) is called an EQ-algebra if it satisfies the following conditions: for all $a, b, c, d \in E$,

($Eq1$) $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1,

($Eq2$) $(E, \cdot, 1)$ is a monoid and $\cdot$ is isotone in both arguments w.r.t $a \leq b$, then $a \cdot c \leq b \cdot c$, for every $c \in E$.

($Eq3$) $a \sim a = 1$, *(reflexivity axiom)*

($Eq4$) $((a \wedge b) \sim c) \cdot (d \sim a) \leq c \sim (d \wedge b)$, (*substitution axiom*)

($Eq5$) $(a \sim b) \cdot (c \sim d) \leq (a \sim c) \sim (b \sim d)$, (*congruence axiom*)

($Eq6$) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$, (*monotonicity axiom*)

($Eq7$) $a \cdot b \leq a \sim b$. (*boundedness axiom*)

Definition 2.2. [14] Let E be an EQ-algebra. Then E is called a

(i) residuated EQ-algebra, if for all $x, y, z \in E$,

$$(x \cdot y) \wedge z = x \cdot y \Longleftrightarrow x \wedge ((y \wedge z) \sim y) = x.$$

(ii) lattice-ordered EQ-algebra, if it has a lattice reduct.

(iii) lattice EQ-algebra (an ℓEQ-algebra), if it is a lattice-ordered EQ-algebra in which the following substitution axiom holds: for all $a, b, c, d \in E$,

$$((a \vee b) \sim c) \cdot (d \sim a) \leq ((d \vee b) \sim c).$$

Obviously, $(E, \leq)$ is a partial order, where $a \leq b$ if and only if $a \wedge b = a$. We put $a \to b = a \sim (a \wedge b)$, for all $a, b \in E$.

Proposition 2.3. [20] *Let E be an ℓEQ-algebra. Then for any $a, b, c, d \in E$, the following properties hold:*

(i) $a \to b = a \sim (a \wedge b) = (a \vee b) \sim b = (a \vee b) \to b$,

(ii) $(a \to b) \leq (a \vee c) \to (b \vee c)$,

(iii) $(a \to c) \cdot (b \to c) \leq (a \vee b) \to c$.

Definition 2.4. [25] An EQ-algebra E is called divisible, if for all $a, b \in E$,

$$a \wedge b = a \cdot (a \to b).$$

Definition 2.5. [17] An equality algebra is an algebra $\mathcal{E} = \langle E, \sim, \wedge, 1\rangle$ of type (2,2,0) such that the following axioms are fulfilled: for all $a, b, c \in E$,

(e1) $\langle E, \wedge, 1\rangle$ is a commutative idempotent integral monoid (i.e. $\wedge$-semilattice with the top element 1),

(e2) $a \sim b = b \sim a$,

(e3) $a \sim a = 1$,

(e4) $a \sim 1 = a$,

(e5) $a \leq b \leq c$ implies $a \sim c \leq b \sim c$ and $a \sim c \leq a \sim b$,

(e6) $a \sim b \leq (a \wedge c) \sim (b \wedge c)$,

(e7) $a \sim b \leq (a \sim c) \sim (b \sim c)$.

The operation $\wedge$ is called meet (infimum) and $\sim$ is an equality operation. We write $a \leq b$ (or $b \geq a$) iff $a \wedge b = a$, as usual. Define the following two derived operations, $\to$ implication and $\leftrightarrow$ equivalence operation of the equality algebra $\mathcal{E}$ by

$$a \to b = a \sim (a \wedge b) \quad , \quad a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

In what follows, we agree that $\sim$ and $\to$ have higher priority than $\wedge$.

Proposition 2.6. [17] *Let $\mathcal{E}$ be an equality algebra and consider the following statements: for all $a, b, c \in E$,*

(e5_a) $a \sim (a \wedge b \wedge c) \leq a \sim (a \wedge b)$,

(e5_b) $a \sim (a \wedge b) \leq (a \wedge c) \sim (a \wedge b \wedge c)$,

(e$5_{a'}$) $a \to (b \wedge c) \leq (a \to b)$,

(e$5_{b'}$) $a \to b \leq (a \wedge c) \to b$,

We have that (e5) is equivalent to $\{(e5_a), (e5_b)\}$ which in turn is equivalent to $\{(e5_{a'}), (e5_{b'})\}$.

Proposition 2.7. [4, 17, 28] *Let $\mathcal{E} = \langle E, \sim, \wedge, 1\rangle$ be an equality algebra. Then the followings hold: for all $a, b, c \in E$,*

(i) $a \sim b \leq a \leftrightarrow b \leq a \to b$,

(ii) $a \leq (a \sim b) \sim b$,

(iii) $a \to b = 1$ *if and only if* $a \leq b$,

((iv) $1 \to a = a$, $a \to 1 = 1$, $a \to a = 1$,

(v) $a \leq b \to a$,

(vi) $a \leq (a \to b) \to b$,

(vii) $a \to (b \to c) = b \to (a \to c)$,

(viii) $b \leq a$ *implies* $a \leftrightarrow b = a \to b = a \sim b$,

(ix) $a \leq b$ *implies* $b \to c \leq a \to c$, $c \to a \leq c \to b$,

(x) $b \leq a \sim (a \wedge b)$, $a \sim b \leq a \sim (a \wedge b)$,

Definition 2.8. [28] An equality algebra $\mathcal{E} = \langle E, \sim, \wedge, 1\rangle$ is bounded if there exists an element $0 \in E$ such that $0 \leq a$, for all $a \in E$. In a bounded equality algebra $\mathcal{E}$, we define the negation "$'$" on $\mathcal{E}$ by $x' = x \to 0 = x \sim 0$, for all $x \in E$. If $(x')' = x$, for all $x \in E$, then the bounded equality algebra E is called involutive.

Definition 2.9. [21] An L-algebra is an algebra $(L, \to, 1)$ of type (2, 0) satisfying

($L1$) $x \to x = x \to 1 = 1$, $1 \to x = x$,

($L2$) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,

($L3$) $x \to y = y \to x = 1$ implies $x = y$,

for all $x, y, z \in L$.

Condition ($L1$) states that 1 is a logical unit. Note that a logical unit is always unique. There is partial ordering by

$(L4)$ $x \leq y \Leftrightarrow x \rightarrow y = 1$.

Definition 2.10. [21] Let $(L, \rightarrow, 1)$ be an L-algebra. We call $F \subseteq L$ an filter of L, if it satisfies the following conditions: for all $x, y \in L$,

$(F0)$ $1 \in F$,

$(F1)$ $x, x \rightarrow y \in F$ imply $y \in F$,

$(F2)$ $x \in F$ implies $(x \rightarrow y) \rightarrow y \in F$,

$(F3)$ $x \in F$ implies $y \rightarrow x$, $y \rightarrow (x \rightarrow y) \in F$.

Definition 2.11. [21] A monoid $(H, \cdot)$ with an additional binary operation "$\rightarrow$" is called a left hoop if the following are satisfied: for $a, b, c \in H$,

(H_1) $a \rightarrow a = 1$,

(H_2) $(a \cdot b) \rightarrow c = a \rightarrow (b \rightarrow c)$,

(H_3) $(a \rightarrow b) \cdot a = (b \rightarrow a) \cdot b$.

For all elements $x, y \in H$, we define $x \leq y$ if and only if $x \rightarrow y = 1$. Then the relation $\leq$ becomes a partial order. Thus any hoop is a partially ordered set. As to the order, we see that any hoop is a meet-semilattice, that is,

$$x \wedge y = x \odot (x \rightarrow y).$$

Proposition 2.12. [21] *Let $(H, \cdot, \rightarrow, 1)$ be a hoop. Then, the following properties hold: for all $x, y, z \in H$,*

(i) $x \leq y \rightarrow x$,

(ii) $x \cdot y \leq x, y$,

(iii) $x \leq y \implies z \rightarrow x \leq z \rightarrow y$,

(iv) $x \leq y \implies y \rightarrow z \leq x \rightarrow z$,

(v) $x \cdot (y \wedge z) \leq (x \cdot y) \wedge (x \cdot z)$,

(vi) $(x \rightarrow y) \cdot (y \rightarrow z) \leq (x \rightarrow z)$,

(vii) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

Proposition 2.13. [21] *A left hoop $(H, \rightarrow, \cdot, 1)$ gives rise to an L-algebra $(H, \rightarrow)$.*

For an element x of an L-algebra $(X, \rightarrow)$, we define the downset

$$\downarrow x := \{y \in X | y \leq x\}.$$

Definition 2.14. [18] We call an L-algebra $(X, \rightarrow)$ self-similar if for each $x \in X$, the left multiplication $y \mapsto x \rightarrow y$ induces a bijection $\downarrow x \tilde{\rightarrow} X$.

Proposition 2.15. [21] *Every self-similar L-algebra is a left hoop.*

Proposition 2.16. [21] *For a left hoop H, and $a \in H$, the following are equivalent:*

(i) *The map $b \mapsto (b \cdot a)$ is injective,*
(ii) *The map $b \mapsto (a \to b)$ is surjective,*
(iii) *$a \to (b \cdot a) = b$ for all $b \in H$,*
(iv) *$(b \cdot a) \to (c \cdot a) = b \to c$ for all $b, c \in H$,*
(vi) *$a \to (b \cdot c) = ((c \to a) \to b) \cdot (a \to c)$ holds for all $b, c \in H$.*

Definition 2.17. [18] A non-empty subset F of a lattice L is called filter of L if

$(F1)$ $a, b \in F \Rightarrow a \wedge b \in F$,
$(F2)$ $a \in F, l \in L \Rightarrow a \vee l \in F$.

Definition 2.18. [18] A proper filter F of lattice L is called a prime filter, if $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Proposition 2.19. [26] *Let $E = (E, \sim, \wedge, 1)$ be a bounded L-algebra. Then for all $a, b, c \in E$, we have the following properties:*

(i) *$a \sqcup b \geq b$, $a \oplus b \geq b''$,*
(ii) *$a \sqcap b \leq a''$,*
(iii) *$a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$,*
(iv) *$a \odot b \leq b''$,*
(v)*$a \sqcup b \leq a \oplus b$.*
(vi) *$a \leq a \oplus b$, $b \leq a \oplus b$,*
(vii) *$a \sqcap b \leq b''$, $a \sqcup b \geq a$,*
(viii) *$a \odot b \leq a''$,*
(ix) *$a \odot (a \to b) \leq a'', b''$,*
(x) *$a \odot b \leq a \sqcap b$,*
(xi) *If $a \vee b$ exists, then $a \vee b \leq a \oplus b$,*

Corollary 2.20. [19] *Let E be an equality algebra with join. Then we have, for $x, y \in A$*

$$\langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle.$$

Definition 2.21. [29] Let E be an lEQ-algebra, L be a bounded distributive lattice and $\lambda : E \to L$ be a map. An e-reticulation of E is a pair (L, λ), which satisfies the following conditions:

(i) $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$, for any $a, b \in E$;
(ii) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$, for any $a, b \in E$;
(iii) $\lambda(0) = 0, \lambda(1) = 1$;
(iv) λ is isotone and surjective;
(v) $\lambda(a) \leq \lambda(b)$ if and only if $(\exists n \in N+)$ such that an $a^n \leq b$.

Lemma 2.22. [29] *Let (L, λ) be an e-reticulation, $\lambda(a) = \lambda(b)$ if and only if $< a >=< b >$, for any $a, b \in E$.*

3 L-algebras, equality algebras and EQ-algebras

In this chapter, L-equality algebras are introduced, that is, given an equality algebra $(E, \sim, \wedge, 1)$, we induce a binary operation $\to$, such that it satisfies the L-equation

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z). \quad (L)$$

Then we may conclude that the following holds. Furthermore, the process of generating EQ-algebras from them via self-similarity is explored. This lays a foundation for the domain-related research in Sections 4 and 5.

In general, an equality algebra may not satisfy the L-equation, which can be seen from the following example.

Example 3.1. Let $(E, \leq)$ be a lattice that diagram is below, where $E = \{0, a, b, c, d, 1\}$. Define the operation $\sim$ and induce the operation $\to$ on E as follows.

By routine calculations, we can see that $(E, \wedge, \sim, 0, 1)$ is an equality algebra. Since

$$(c \to b) \to (c \to a) = 1 \neq d = (b \to c) \to (b \to a),$$

E does not satisfy (L).

Definition 3.2. An equality algebra E is called the *L-equality algebra* if E satisfies the condition (L).

Example 3.3. Let $E = \{0, a, b, 1\}$ be a lattice whose diagram is listed figure 2. Define the operation $\sim$ and induce the operation $\to$ on E as table 3 and table 4.

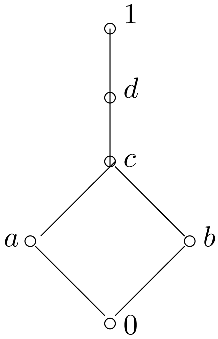

FIGURE 1

Table 1

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

Table 2

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	1	d	1

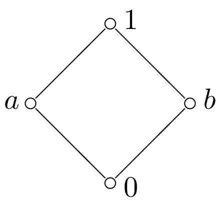

FIGURE 2

Table 3

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

Table 4

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then by routine calculations, we can see that $(E, \wedge, \sim, 1)$ is an equality algebra. Moreover, we can check that ε is an L-equality algebra.

Lemma 3.4. *Let E be an L-equality algebra. Then, we have the following properties:*
(i) *$x = y$ if and only if $x \to z = y \to z$, for all $z \in E$,*
(ii) *$(x \wedge y) \sim z \leq (x \sim t) \to (z \sim (y \wedge t))$, for all $x, y, z \in E$.*

Proof. Proof steps are similar to those of L-algebras. □

For any L-equality algebra $(E, \sim, \wedge, 1)$, the reduct $(E, \to, 1)$ of E is an L-algebra.

Definition 3.5. We call an L-equality algebra $(E, \sim, \wedge, 1)$ self-similar if E is a *self-similar L-algebra.*

Proposition 3.6. *Let E be a self-similar L-equality algebra, that is, the restriction of $c \mapsto b \to c$ to $\downarrow b$ is a bijection $\downarrow b \tilde{\to} E$, for all $a, b \in E$. We define product "$\cdot$" by the following conditions:*

$$a \cdot b \leq b \quad , \quad b \to (a \cdot b) = a. \quad (M)$$

Then we can check the product "$\cdot$" is uniquely determined and associated.

Proof. Let E be a self-similar L-equality algebra, and let $a, b \in E$. Then $a \cdot b$ is uniquely determined by (M).

We first show that this product is associative. For $a, b, c \in E$, we have $a \cdot (b \cdot c) \leq b \cdot c \leq c$ and $(a \cdot b) \cdot c \leq c$. Therefore, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ is equivalent to

$$c \to (a \cdot (b \cdot c)) = c \to ((a \cdot b) \cdot c).$$

i.e. $c \to (a \cdot (b \cdot c)) = a \cdot b$. Since $a \cdot (b \cdot c) \leq (b \cdot c) \leq c$, we have $c \to (a \cdot (b \cdot c)) \leq c \to (b \cdot c) = b$ by Definition 2.5 $(e5)$, and hence

$$c \to (a \cdot (b \cdot c)) \leq c \to (b \cdot c) = b,$$

by Proposition 2.7 (ix). This shows that $c \to (a \cdot (b \cdot c)) \in \downarrow b$. Hence $c \to (a \cdot (b \cdot c)) = a \cdot b$ is equivalent to $b \to (c \to (a \cdot (b \cdot c))) = a$. Now we have the following:

$$b \to (c \to (a \cdot (b \cdot c)))$$

$$
\begin{aligned}
&= (c \to (b \cdot c)) \to (c \to (a \cdot (b \cdot c))), \quad \text{by}(M) \\
&= ((b \cdot c) \to c) \to ((b \cdot c) \to (a \cdot (b \cdot c))), \quad \text{by (L)} \\
&= 1 \to a, \quad \text{by } (M) \text{ and Proposition 2.7 (iii))} \\
&= a \;\; \text{by Proposition 2.7 (iv))}
\end{aligned}
$$

□

Theorem 3.7. *Let $(E, \wedge, \to, 1)$ be a self-similar L-equality algebra. We then generate the operation "$\cdot$" by virtue of condition (M) from Property* 2.7 *such that the following hold for all $a, b, c \in E$:*

(i) $(E, \cdot, 1)$ *is a monoid,*

(ii) $(E, \sim, \cdot, 1)$ *satisfies the following identity:*

$$(a \cdot b) \to c = a \to (b \to c), \tag{A}$$

(iii) $(E, \sim, \cdot, 1)$ *satisfies the following identity:*

$$(a \to b) \cdot a = (b \to a) \cdot b, \tag{H}$$

(iv) *The product $''\cdot''$ is isotone in both arguments:*
If $a \leq b$, then $a \cdot c \leq b \cdot c$ for every $c \in E$,

(v) *$a \cdot b \leq c$ if and only if $a \leq b \to c$,*

(vi) $(a \sim b) \cdot (b \sim c) \leq a \sim c$,

(vii) $(E, \to, 1)$ *satisfies the following condition:*
For every $a \in E$, the map $b \mapsto (a \to b)$ is surjective. (S)

Proof. Proof steps are similar to those of L-algebras. □

Example 3.8. Let $(E, \to, \wedge, 1) = (R^-, (y - x) \wedge 0, min, 0)$ and

$$x \sim y = (x \to y) \wedge (y \to x).$$

It is easy to verify that the left multiplication map $L_x : E \to E, y \mapsto x \to y$, restricted to the down-set $\downarrow x = \{z \in |z \leq x\}$ is a bijection onto E, and $(E, \to, \wedge, 1)$ is a self-similar L-equivalent algebra. Moreover, it satisfies all seven properties stated in Proposition 3.7.

Theorem 3.9. *Every self-similar L-equality algebra E can be regarded as a residuated divisible EQ-algebra satisfying condition (S).*

Proof. It follows from Definition 2.5 $(e1)$ and $(e3)$ that $(Eq1)$ and $(Eq3)$ hold. By parts (i) and (iv) of Theorem 3.7, condition $(Eq2)$ is satisfied.

From Proposition 2.6 ($e5_a$), we obtain ($Eq6$). For ($Eq4$), using Lemma 3.4 (ii), we have

$$(a \wedge b) \sim c \leq (a \sim t) \to (c \sim (b \wedge t)).$$

By Theorem 3.7(v), It follows that

$$((a \wedge b) \sim c) \cdot (a \sim t) \leq c \sim (t \wedge b).$$

For ($Eq5$), by (e7) and (Eq2) and Theorem 3.7 (iv), we have:

$$\begin{aligned}(a \sim b) \cdot (c \sim d) &\leq ((a \sim c) \sim (b \sim c)) \cdot ((c \sim b) \sim (d \sim b)) \\ &\leq (a \sim c) \sim (b \sim d)\end{aligned}$$

For ($Eq7$), by Proposition 2.7 (ii), (viii) and by (M), we have

$$\begin{aligned}(a \cdot b) &\leq (a \cdot b) \sim b \\ &= b \sim (b \sim (a \cdot b)) = b \sim (b \to (a \cdot b)) = b \sim a \\ &= a \sim b\end{aligned}$$

Combining the above, we conclude that E is an EQ-algebra. By (ii) and (iii) of Theorem 3.7, E is residuated and divisible. □

Definition 3.10. Let H be a left hoop. If H satisfies one of the conditions in Proposition 2.16, then it is called a *self-similar left hoop.*

Example 3.11. Let $H = N$, $a \cdot b = a + b$, $1 = 0$ and

$$a \to b = max\{0, b - a\} = \begin{cases} b - a, & \text{if } b \geq a \\ 0, & \text{others.} \end{cases}$$

Then we can obtain that $(H, \cdot, \to, 1)$ is a self-similar left hoop.

Proposition 3.12. *Let $(E, \wedge, \cdot, \sim, 1)$ be a residuated divisible EQ-algebra E with the condition (S), then $(E, \cdot, \to, 1)$ can be seen a self-similar left hoop.*

Proof. Let $(E, \wedge, \cdot, \sim, 1)$ be a residuated divisible EQ-algebra and suppose condition (S) holds. First, we need to prove $(E, \cdot, \to, 1)$ is a left hoop, where we define $x \to y = x \sim (x \wedge y)$. It is easy to see that

$$x \to x = x \sim (x \wedge x) = x \sim x = 1.$$

By Definition 2.2, (i.e., $(x \cdot y) \wedge z = x \cdot y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$), we can conclude that $x \cdot y \leq z$ if and only if $x \leq y \to z$, by Definition 2.4, (i.e., $x \wedge y = x \cdot (x \to y)$, we have

$$x \cdot (x \to y) = x \wedge y = y \wedge x = y \cdot (y \to x)).$$

Therefore, $(E, \cdot, \to, 1)$ is a left hoop.

For any $z \in E$, let $z \leq x$ and $z \leq y$, then we have $z \to x = 1$ and $z \to y = 1$. By proposition 2.12(vii), we obtain

$$z \to (x \wedge y) = (z \to x) \wedge (z \to y) = 1.$$

Hence $z \leq x \wedge y = x \cdot (x \to y)$.

Finally, since the condition (S) holds (i.e., for any $x \in E$, the map $y \mapsto (x \to y)$ is surjective), by Definition 3.5, $(E, \cdot, \to, 1)$ is a self-similar left hoop. $\square$

Proposition 3.13. *Let $(E, \cdot, \to, 1)$ be a self-similar left hoop, then $(E, \sim, \wedge, 1)$ is a self-similar L-equality algebra, where $x \wedge y = (x \to y) \cdot x$, $x \sim y = (x \to y) \wedge (y \to x)$, for any $x, y \in E$.*

Proof. It is straightforward to check that $(E, \wedge, 1)$ is a meet-semilattice with top element 1, and thus $(e1)$ holds. By definition of $x \wedge y$ and $x \sim y$, for any $x, y \in E$, we get the following implications:

$$\begin{gathered} x \sim y = (x \to y) \wedge (y \to x) = (y \to x) \wedge (x \to y) = y \sim x, \\ x \sim x = (x \to x) \wedge (x \to x) = 1, \\ x \sim 1 = (x \to 1) \wedge (1 \to x) = 1 \wedge x = x, \end{gathered}$$

Then we prove that $(e2), (e3), (e4)$ holds true.

For all $x, y, z \in E$, if $x \leq y \leq z$, we have

$$x \sim z = (x \to z) \wedge (z \to x) = z \to x$$

and

$$y \sim z = (y \to z) \wedge (z \to y) = z \to y.$$

It can be obtained $x \sim z \leq y \sim z$ by Proposition 2.12(iii). Similarly, This $x \sim z \leq x \sim z$ holds. Hence $(e5)$ holds.

For all $x, y, z \in E$, since

$$(x \to y) \wedge (y \to x) \leq y \to x,$$

and by Definition 2.2, we get $((x \to y) \wedge (y \to x)) \cdot y \leq x$.

Also, by Proposition 2.12(v), we have the following implications:

$$\begin{aligned}
&(x \sim y) \cdot (y \wedge z) \\
&= ((x \to y) \wedge (y \to x)) \cdot (y \wedge z) \\
&\leq (((x \to y) \wedge (y \to x)) \cdot y) \wedge (((x \to y) \wedge (y \to x)) \cdot z) \\
&\leq x \wedge z.
\end{aligned}$$

Moreover, by residuated property, we have $(x \sim y) \leq (y \wedge z) \to (x \wedge z)$. Similarly, we get $(x \sim y) \leq (x \wedge z) \to (y \wedge z)$, and by definition of $\sim$,

$$(x \sim y) \leq ((y \wedge z) \to (x \wedge z)) \wedge ((x \wedge z) \to (y \wedge z)).$$

Hence $(e6)$ holds.

For all $x, y, z \in E$, by Proposition2.12(vi), we have;

$$\begin{aligned}
&(x \sim y) \cdot (x \sim z) \\
&= ((x \to y) \wedge (y \to x)) \cdot ((x \to z) \wedge (z \to x)) \\
&\leq (((x \to y) \wedge (y \to x)) \cdot (x \to z))) \wedge (((x \to y) \wedge (y \to x)) \cdot (z \to x))) \\
&\leq ((y \to x) \cdot (x \to z)) \wedge ((x \to y) \cdot (z \to x)) \\
&\leq ((y \to z) \wedge (z \to y)) \\
&= y \sim z.
\end{aligned}$$

Hence by residuated property, we obtain $(x \sim y) \leq (x \sim z) \to (y \sim z)$. Similarly, we have $(x \sim y) \leq (y \sim z) \to (x \sim z)$. Hence, by definition of $\sim$, $(e7)$ holds.

Now, by (H_2) and (H_3) of Definition 2.11, we have

$$\begin{aligned}
(x \to y) \to (x \to z) &= (x \to y) \cdot x \to z \\
&= (y \to x) \cdot y \to z \\
&= (y \to x) \to (y \to z).
\end{aligned}$$

Therefore, $(E, \sim, \wedge, 1)$ is an L-equality algebra.

Since $(E, \cdot, \rightarrow, 1)$ is a self-similar left hoop, we need prove that for any $x, y \in E$, the map $y \mapsto x \rightarrow y$ is surjective from $\downarrow x$ to E. First, we prove that $x \rightarrow y = x \rightarrow (x \wedge y)$, for all $x, y \in E$. It is easy to see that $x \rightarrow (x \wedge y) \leq x \rightarrow y$. Conversely, from $(x \rightarrow y) \cdot x = x \wedge y$, we get $x \rightarrow y \leq x \rightarrow (x \wedge y)$. Hence $x \rightarrow y = x \rightarrow (x \wedge y)$. Moreover, we want to prove $f(y) = x \rightarrow y$ is a surjective from $\downarrow x$ to E. For any $y \in E$, there is $z = y \cdot x$ such that $f(y \cdot x) = x \rightarrow y \cdot x = y$, so we get the map $y \mapsto (x \rightarrow y)$ is surjective from $\downarrow x$ to E. Secondly, we check that the map $y \mapsto (x \rightarrow y)$ is injective from $\downarrow x$ to E. Let $y_1, y_2 \in \downarrow x$ and $x \rightarrow y_1 = x \rightarrow y_2$. Then $(x \rightarrow y_1) \cdot x = (x \rightarrow y_2) \cdot x$ and so $x \wedge y_1 = x \wedge y_2$. It follows $y_1 = y_2$ that is the map $y \mapsto (x \rightarrow y)$ is injective from $\downarrow x$ to E.

Above all of this, $(E, \sim, \wedge, 1)$ is a self-similar L-equality algebra. □

Theorem 3.14. *Let $(E, \sim, \wedge, 1)$ be an equality algebra, then the following are equivalent:*

(i) $(E, \cdot, \rightarrow, 1)$ *is a self-similar left hoop,*

(ii) $(E, \sim, \wedge, 1)$ *is a self-similar L-equality algebra,*

(iii) $(E, \wedge, \cdot, \sim, 1)$ *is a residuated divisible EQ-algebra E with the condition (S),*

where the multiplication "$\cdot$" is defined by (M).

Proof. Based on Theorems 3.9, 3.12 and 3.13, the conclusion is self-evident. □

Example 3.15. Let $E = (-\infty, 1]$. Define $a \wedge b = min\{a, b\}$ and $a \sim b = 1 - |a - b|$ for all $a, b \in E$. Then $(E, \wedge, \sim, 1)$ is a self-similar L-equality algebra, where $\rightarrow$ is induced by $a \rightarrow b = a \sim (a \wedge b)$.

By virtue of the definitions of operation $\sim$ and $\wedge$, and in light of Definition 2.5, we can prove that $(E, \wedge, \sim, 1)$ is an equality algebra. And then discuss the magnitude relationship of $a, b, c \in E$ and verify its satisfaction of the L-equation

$$(a \rightarrow b) \rightarrow (a \rightarrow c) = (b \rightarrow a) \rightarrow (b \rightarrow c).$$

For $b \in E$, consider the left multiplication f, defined by $f(a) = b \rightarrow a$ for all $a \in \downarrow b$. We claim that f is a bijection from $\downarrow b$ to E.Firstly we prove that f is surjection. For any $x \in (-\infty, 1]$, let $f(a) = x$ with $a \in \downarrow b$. Then we have $a \leq b$ and $b \rightarrow a = b \sim a = x$ by Proposition 2.7(viii). Therefore

$1-|b-a| = 1-(b-a) = x$ and hence $a = b+x-1 \leq 1$. It follows that there exists $a \in E$ such that $f(a) = x$, which implies that f is a surjection. Then we check that f is injective. Let $f(x) = f(y)$ for any $x, y \in \downarrow b$. Then $b \to x = b \to y$ and thus $1-|b-x| = 1-|b-y|$. Since $x, y \leq b$, we get $1-(b-x) = 1-(b-y)$, and thus $x = y$. This shows that f is injective.

Combining the above arguments we get that f is a bijection from $\downarrow b$ to E. This means that E is a self-similar L-equality algebra.

Based on the above theorem, now provide a diagram of the relationships between L-equivalent algebras and other algebraic structures.

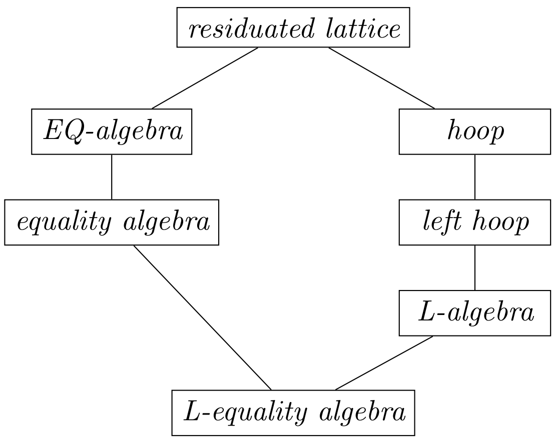

FIGURE 3: EQ-algebra and other algebra structures

4 The e-prime filters and L-prime filters of L-equality algebras

As is well known, an L-equality algebra has two aspects, which are both an L-algebra and an equality algebra. To study filters of L-equality algebra, we introduce two types of filters: filters of equality algebra and filters of L-algebra, and introduce two types of prime filters: prime L-filters and prime e-filters. We can check the two types of filters are equivalent, but the two types of prime filters are not equivalent.We refer to filters of equality algebra and filters of L algebra as e-filters and L-filters for short, and further conduct research on L-prime filters and e-prime filters.

Proposition 4.1. *Let $(E, \wedge, \sim, 1)$ be an L-equality algebra. Then $(L, \rightarrow, 1)$ is a CKL-algebra.*

Proof. It follows from Proposition 2.7 (v) and (vii). $\square$

The following examples illustrate that a CKL-algebra need not be an L-equality algebra.

Example 4.2. Let $A = [0, 1]$ equipped with the Łukasiewicz implication $x \rightarrow y = \min\{1,, 1 - x + y\}$. It is known that $\boldsymbol{A} = (A, \rightarrow, 1)$ is an CKL-algebra, but it is not an equality algebra. Since take $x = 0.4$, $y = 0.1$. Compute step by step: $x \rightarrow y = \min\{1,, 1 - 0.4 + 0.1\} = 0.7$,

$$x \sim (y \wedge x) = 0.4 \sim (0.4 \wedge 0.1) = 0.4 \sim 0.1 \neq 0.7 = x \rightarrow y.$$

Definition 4.3. Let $(E, \wedge, \sim, 1)$ be an L-equality algebra, and $\varnothing \neq F \subseteq E$. Then

(i) F is called *e-filter* of E, if for all $x, y \in E$, $x \in F$ and $x \leq y$ implies $y \in F$; moreover, $x \in F$ and $x \sim y \in F$ implies $y \in F$.

(ii) F is called an *L-filter* of E, if for all $x, y \in E$,

(LF_1) $1 \in F$;

(LF_2) $x, x \rightarrow y \in F$ imply $y \in F$;

(LF_3) $x \in F$ imply $(x \rightarrow y) \rightarrow y \in F$;

(LF_4) $x \in F$ imply $y \rightarrow x, y \rightarrow (x \rightarrow y) \in F$.

Remark 4.4. Generally, L-filters are filters of L-algebras, e-filters are filters of equality algebras. In fact, in L-equality algebras, L-filters and e-filters are coincident.

Proposition 4.5. *Let $(E, \wedge, \sim, 1)$ be an L-equality algebra, and $\varnothing \neq F \subseteq E$. Then F is an e-filter of E if and only if F is an L-filter of E.*

Proof. Let F be an e-filter of L-equality algebra E. Then (LF_1) and (LF_2) are obviously true. Suppose $x \in F$, by Propersition 2.7 (v) and (vi), it means that $x \leq y \rightarrow x$ and $x \leq (x \rightarrow y) \rightarrow y$. Then, (LF_3) and (LF_4) are valid.

Conversely, let F is an L-filter of L-equality algebra E, also $x \in F$ and $x \leq y$. Then we have $x \rightarrow y = 1 \in F$, and so $y \in F$. Suppose $x \in F$ and $x \sim y \in F$, by Propersition 2.7 (x), $x \sim y \leq x \sim (x \wedge y) = x \rightarrow y$. Then $(x \sim y) \rightarrow (x \rightarrow y) = 1 \in F$, Hence, $x \rightarrow y \in F$, and hence $y \in F$. $\square$

We call e-filters and L-filters of L-equality algebra E as filters of E. And the set of all filters of E is denoted by $F(E)$.

Definition 4.6. Let $(E, \wedge, \sim, 1)$ be an L-equality algebras, and P be a proper filter of E.

(i) P is called a *prime e-filter* of E, if for all $x, y \in E, x \to y \in P$ or $y \to x \in P$.

(ii) P is called a *prime L-filter* of E, if for all $J \in F(E)$, $J \subseteq P$ or $J \to P \subseteq P$ where $J \to P = max\{H \in F(E) | H \cap J \subseteq P\}$.

Proposition 4.7. *Let $(E, \wedge, \sim, 1)$ be an L-equality algebra, and P be a filter of E. If P is a prime e-filter of E, then P is a prime L-filter of E.*

Proof. Assume that P is a prime e-filter of E, and J is a filter of E. If $J \nsubseteq P$, then there is $x \in J \setminus P$. Moreover, for any $y \in D \doteq J \to P$, we will prove that $y \in P$.

(i) Let $y \to x \in P$. From $x \leq y \to x$ and $x \in J$, we get $y \to x \in J$, and hence, $\langle y \to x \rangle \cap J = \langle y \to x \rangle \subseteq P$. By the definition of D, we have $\langle y \to x \rangle \subseteq D$. and then $y \to x \in D$. But $y \in D$ implies $x \in D$. From $x \in J$, we get $x \in D \cap J \subseteq P$, it is contradictory.

(ii) From above argments, we get $x \to y \in P$. From $y \in D$, we get $x \to y \in D$. Moreover, $(x \to y) \to y \in D$. By use of Proposition 2.7 (vi), then $x \leq (x \to y) \to y$, we have $(x \to y) \to y \in J$. Hence $(x \to y) \to y \in J \cap D \subseteq P$.

It follows that $(x \to y) \to y \in P$. By assumption, $x \to y \in P$, and hence, $y \in P$. $\square$

Remark 4.8. In general, the inverse of Proposition 4.7 does not be hold, from the following example,

Example 4.9. Let $E = \{0, e, f, g, 1\}$ be a lattice. Define the operation $\sim$ and induce the operation $\to$ on E as Table 5 and 6.

Let D be a nonempty subset of an L-equality algebra E. The smallest filter of E including D is said to be the filter generated by D and is marked by $\langle D \rangle$

Theorem 4.10. *Let A be an L-equality algebra and P be a proper filter of A. Then the following assertions are equivalent:*

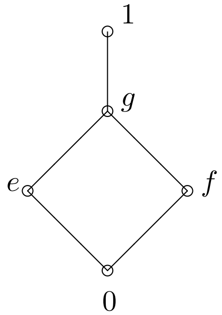

FIGURE 4

Table 5

$\sim$	0	e	f	g	1
0	1	f	e	0	0
e	f	1	0	e	e
f	e	0	1	1	f
g	0	e	1	1	g
1	0	e	f	g	1

Table 6

$\to$	0	e	f	g	1
0	1	1	1	1	1
e	f	1	f	1	1
f	e	e	1	1	1
g	0	e	f	1	1
1	0	e	f	g	1

(i) *P is a prime L-filter,*
(ii) *for all $J, K \in F(A)$, $J \cap K \subseteq P$ implies $J \subseteq P$ or $K \subseteq P$,*
(iii) *for all $a, b \in A$, $\langle a\rangle \cap \langle b\rangle \subseteq P$ implies $a \in P$ or $b \in P$.*

Proof. (i) $\Rightarrow$ (ii) For all $J, K \in F(A)$ and $J \cap K \subseteq P$, the following two cases are discussed:
Case 1. Assume $J \subseteq P$. Then (ii) is true.
Case 2. Assume $J \nsubseteq P$. By Definition 4.6(ii), then $J \to P \subseteq P$. Since $J \to P = max\{H | J \cap H \subseteq P\}$ and $J \cap K \subseteq P$, thus $J \cap (J \to P) \subseteq P$ and $K \in \{H | J \cap H \subseteq P\}$. Hence $K \subset J \to P$. Since $J \to P \subseteq P$, we get $K \subseteq P$.
(ii) $\Rightarrow$ (i) Let $J \in F(A)$. We will prove $J \subseteq P$ or $J \to P \subseteq P$.
If $J \subseteq P$, we are done. If $J \nsubseteq P$, we prove $J \to P \subseteq P$ in the following. Let $K = J \to P$. Since $K \in \{H | J \cap H \subseteq P\}$, then $J \cap K \subseteq P$. Note that $J \nsubseteq P$, by (ii), we get $K \subseteq P$, that is, $J \to P \subseteq P$. Thus P is prime L-filter.
(ii) $\Rightarrow$ (iii) Assume (ii) holds. Let $a, b \in A$ be such that $\langle a\rangle \cap \langle b\rangle \subseteq P$. By (ii), we have $\langle a\rangle \subseteq P$ or $\langle b\rangle \subseteq P$. Since $a \in \langle a\rangle$ and $b \in \langle b\rangle$, we get $a \in P$ or $b \in P$.
(iii) $\Rightarrow$ (ii) Assume $J \cap K \subseteq P$ and $J \nsubseteq P$. Then there exists $s \in J \setminus P$, this means $\langle s\rangle \subseteq J$. Then for any $t \in K$, $\langle t\rangle \subseteq K$, and so, we have $\langle s\rangle \cap \langle t\rangle \subseteq J \cap K \subseteq P$. By (iii), $s \in P$ or $t \in P$. But $s \notin P$, hence $t \in P$. Thus we get $K \subseteq P$. Then (ii) is established. $\square$

Proposition 4.11. *Let A be an L-equality algebra with join and P be a proper filter of A. Then the following are equivalent:*

(i) *P is a prime L-fliter of A,*
(ii) *For all $a, b \in A$, $a \vee b \in P$ implies $a \in P$ or $b \in P$.*

Proof. (i) $\Rightarrow$ (ii) Let $P \in PF(A)$ and $a, b \in P$. If $a \vee b \in P$, then $\langle a \vee b \rangle \subseteq P$. From Corollary 2.20, $\langle a \rangle \cap \langle b \rangle = \langle a \vee b \rangle \subseteq P$. By Theorem 4.10, we have $a \in P$ or $b \in P$.

(ii) $\Rightarrow$ (i) Let (ii) hold and $a, b \in P$ such that $\langle a \rangle \cap \langle b \rangle \subseteq P$. By Corollary 2.20, we get $a \vee b \in \langle a \vee b \rangle = \langle a \rangle \cap \langle b \rangle \subseteq P$, that is, $a \vee b \in P$. From (ii), we have $a \in P$ or $b \in P$. By Theorem 4.10, P is prime L-filter. □

5 The reticulation of L-equality algebras

General algebra introduces the concept of reticulation based on the binary operations of $\oplus$ and $\odot$, such as MV-algebras and BL-algebra. However, L-equality algebras only have the operations of $\to$ and $\sim$, so we need to induce three binary operations: $\oplus$, $\odot$ and $\sqcup$, thereby defining the structure of reticulation on L-equality algebras.

Let E be a bounded L-equality algebra. Define the binary operations $\oplus, \odot, \sqcup, \sqcap$ on E as follows:

$$a \oplus b = a' \to b \quad , \quad a \odot b = (a \to b')'$$
$$a \sqcup b = (a \to b) \to b \quad , \quad a \sqcap b = ((b' \to a') \to a')'$$

for all $a, b \in E$.

Proposition 5.1. *Let E be a bounded L-equality algebra. Then for all $a, b \in E$, we have the following properties:*
(i) $a \oplus 0 = a''$, $0 \oplus a = a$,
(ii) $a \odot 0 = 0$, $0 \odot a = 0$,
(iii) $a \odot 1 = a''$, $1 \odot a = a''$,
(iv) $a \oplus a' = 1$, $a' \odot a = 0$,
(v) $a \leq b$ *implies* $a \sqcup b = b$,
(vi) $1 \sqcup a = a \sqcup 1 = 1$, $0 \sqcup a = a$, $a \sqcup 0 = a''$,
(vii) $(a \to b) \odot a = (b \to a) \odot b$.

Proof. (i) Note that $a \oplus 0 = a' \to 0 = a''$ and $0 \oplus a = 0' \to a = 1 \to a = a$.

(ii) By (i), we have

$$a \odot 0 = (a \to 0')' = (a \to 1)' = 1' = 0$$

and $0 \odot a = (0 \to a')' = 1' = 0$.

(iii) It is clear that

$$a \odot 1 = (a \to 1')' = (a \to 0)' = a''$$

and $1 \odot a = (1 \to a')' = a''$.

(iv) Note that $a \oplus a' = a' \to a' = 1$, and $a' \odot a = (a' \to a')' = 1' = 0$.

(v) Let $a \leq b$. Then we get

$$a \sqcup b = (a \to b) \to b = 1 \to b = b.$$

(vi) It follows from (6) that $a \sqcup 1 = 1$ and $0 \sqcup a = a$. Note that $1 \sqcup a = (1 \to a) \to a = a \to a = 1$ and $a \sqcup 0 = (a \to 0) \to 0 = a''$.

(vii) By (L) in definition 3.2,

$$\begin{aligned}(a \to b) \odot a &= ((a \to b) \to a')' \\ &= ((a \to b) \to (a \to 0))' \\ &= ((b \to a) \to (b \to 0))' \\ &= ((b \to a) \to b')' \\ &= (b \to a) \odot b.\end{aligned}$$

□

In this section, we firstly introduce the reticulations of L-equality algebras by use of induced operations.

Definition 5.2. Let E be a bounded L-equality algebras, L be a bounded lattice and $\lambda : E \to L$ be a map. The pair (L, λ) (or λ, by simplicity) is called an *L-reticulation* of E if it satisfies the following conditions:

(R1) λ is isotone and surjective,

(R2) $\lambda(a \oplus b) = \lambda(a) \vee \lambda(b)$, for any $a, b \in X$,

(R3) $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$, for any $a, b \in X$,

(R4) $\lambda(\mathrm{i}) = 1$,

(R5) $\lambda(a) \leq \lambda(b)$ if and only if $\exists n \in \mathbb{N}^+$ such that $a^n \leq b$, where $a^n = a \odot a \odot \cdots \odot a$ (n times).

Example 5.3. Consider the bounded L-equality algebras $L = (\{0, a, b, 1\}, \to, 0, 1)$ given in the Example 3.3. Consider the lattice $(L, \leq)$, where $L = \{0, x, y, 1\}$ with order $0 \leq x, y \leq 1$. Define a map λ by $\lambda(0) = 0, \lambda(a) = x, \lambda(b) = y, \lambda(\mathrm{i}) = 1$, then we can verify that λ is an L-reticulation of E.

Proposition 5.4. *Let E be a bounded L-equality algebra and F be a filter of E. Then we have the following properties:*

(i) *If $a, b \in F$, then $a \odot b \in F$,*

(ii) *If $a, b \in F$, then $a \odot (a \to b) \in F$.*

Proof. (i) Let $a, b \in F$, by Proposition 2.7 (v), we have

$$\begin{aligned} a \to (b \to (a \odot b)) &= a \to (b \to (a \to b')') \\ &= a \to ((a \to b') \to b') \\ &= (a \to b') \to (a \to b') \\ &= 1 \end{aligned}$$

Since F is a filter of E, then $a \odot b \in F$.

(ii) Let $a, b \in F$, by Proposition 2.7 (v) and F is a filter of E, we get $a \to b \in F$. by Proposition 2.7 (vii), it follows that

$$\begin{aligned} a \to ((a \to b) \to (a \to (a \to b)')') &= a \to ((a \to (a \to b)') \to (a \to b)') \\ &= (a \to (a \to b)') \to (a \to (a \to b)') \\ &= 1 \in F. \end{aligned}$$

Thus $a \odot (a \to b) \in F$. □

Now, we give some characterizations of bounded L-equality algebras becoming left hoops, by use of the induced operations.

Theorem 5.5. *Let E be a bounded involutory L-equality algebra. Then the following statements are equivalent:*

(i) *$(E, \odot, \to, 1)$ is a left hoop,*

(ii) *E satisfies the conditions (G),*

(iii) *E satisfies the conditions (A).*

Proof. (i) $\Rightarrow$ (ii) Let $(E, \odot, \to, 1)$ be a left hoop. Then (H_1) implies $(a \odot b) \odot c' = a \odot (b \odot c')$.

(ii) $\Leftrightarrow$ (iii) Let A be a bounded involutory L-equality algebra with (G). Then,

$$\begin{aligned} &(a \odot b) \odot c' = a \odot (b \odot c') \\ \Leftrightarrow\ &((a \to b')' \to c'')' = (a \to (b \to c'')'')' \end{aligned}$$

$$\Leftrightarrow ((a \to b')' \to c)' = (a \to (b \to c))'$$
$$\Leftrightarrow (a \to b')' \to c = a \to (b \to c)$$
$$\Leftrightarrow (a \odot b) \to c = a \to (b \to c).$$

(iii) $\Rightarrow$ (i) Let E satisfy (A). then (H_2) holds. Since E is an equality algebra, H_1 is clearly, By Proposition 5.1(vii), we get $(H3)$. □

Proposition 5.6. *Let E be a bounded L-equality algebra and (L, λ) be an L-reticulation of E. Then for all $a, b \in E$,*
(i) $\lambda(0) = 0$,
(ii) $\lambda(a'') = \lambda(a)$,
(iii) $\lambda(a \oplus b) = \lambda(b \oplus a)$, $\lambda(a \odot b) = \lambda(b \odot a)$,
(iv) $\lambda(a) \vee \lambda(a') = 1$, $\lambda(a) \wedge \lambda(a') = 0$,
(v) $L = \lambda(A)$ *is a complemented lattice, and* $(\lambda(a))' = \lambda(a')$,
(vi) $\lambda(a \sqcup b) = \lambda(a) \vee \lambda(b)$, *for all* $a, b \in A$,
(vii) *If for* $a, b \in A$, $a \vee b$ *exists, then* $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$,
(viii) $\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$, *for all* $a, b \in A$,
(ix) $\lambda(a'' \wedge b'') = \lambda(a) \wedge \lambda(b)$.

Proof. (i) By use of Proposition 5.1 (ii) , we get $\lambda(0) = \lambda(a \odot 0) = \lambda(a) \wedge \lambda(0)$, and hence, $\lambda(0) \le \lambda(a)$ for any $a \in A$. Since λ is surjective, then $\lambda(0) = 0$.

(ii) By Proposition 5.1 (i) and (R2), we have

$$\lambda(a'') = \lambda(a \oplus 0) = \lambda(a) \vee \lambda(0) = \lambda(a) \vee 0 = \lambda(a).$$

(iii) It follows from $(R2)$ and $(R3)$.

(iv) It follows from Proposition 5.1 (iv) that $\lambda(a) \vee \lambda(a') = \lambda(a \oplus a') = \lambda(\mathrm{i}) = 1$ and

$$\lambda(a) \wedge \lambda(a') = \lambda(a \odot a') = \lambda(a' \odot a) = \lambda(0) = 0.$$

(v) By Proposition 5.6 (iv), we have $(\lambda(a))' = \lambda(a')$. Since λ is surjective, we get $L = \lambda(A)$ is a complemented lattice.

(vi)By Proposition 2.19(v), we have $a \sqcup b \le a \oplus b$, and thus,

$$\lambda(a \sqcup b) \le \lambda(a \oplus b) = \lambda(a) \vee \lambda(b)$$

by $(R1)$ and $(R2)$. From Propositions 2.19 (i) and (vii), we get $\lambda(a \sqcup b) \ge \lambda(a), \lambda(b)$, and so, $\lambda(a \sqcup b) \ge \lambda(a) \vee \lambda(b)$. Combining the above results, we get $\lambda(a \sqcup b) = \lambda(a) \vee \lambda(b)$.

(vii) Let $a \vee b$ exist. Then $a \vee b \leq a \sqcup b \leq a \oplus b$ by Proposition 2.19 (vi), and thus,

$$\lambda(a \vee b) \leq \lambda(a \sqcup b) \leq \lambda(a \oplus b) = \lambda(a) \vee \lambda(b)$$

by Propositions 2.19 (i), (iii) and $(R1)$. On the other hand, we have $\lambda(a \vee b) \geq \lambda(a), \lambda(b)$, and hence, $\lambda(a \vee b) \geq \lambda(a) \vee \lambda(b)$. Combining the above arguments, we get $\lambda(a \vee b) = \lambda(a \sqcup b) = \lambda(a) \vee \lambda(b)$.

(viii) From Proposition 2.19 (10), we get $a \odot b \leq a \sqcap b$, and hence, $\lambda(a) \wedge \lambda(b) = \lambda(a \odot b) \leq \lambda(a \sqcap b)$. On the other hand, by Propositions 2.19 (ii) and (vii), $a \sqcap b \leq a'', b''$, and thus,

$$\lambda(a \sqcap b) \leq \lambda(a'') \wedge \lambda(b'') = \lambda(a) \wedge \lambda(b)$$

by Proposition 5.6 (ii). Combining the above arguments, we get $\lambda(a \sqcap b) = \lambda(a) \wedge \lambda(b)$.

(ix) Assume that $a'' \wedge b''$ exists, then $a \odot b \leq a'' \wedge b''$ by Propositions 2.19 (iv) and (viii), and thus,

$$\lambda(a) \wedge \lambda(b) = \lambda(a \odot b) \leq \lambda(a'' \wedge b'').$$

Conversely, $\lambda(a'' \wedge b'') \leq \lambda(a''), \lambda(b'')$ implies

$$\lambda(a'' \wedge b'') \leq \lambda(a'') \wedge \lambda(b'') = \lambda(a) \wedge \lambda(b)$$

by Proposition 5.6 (ii). It follows that $\lambda(a'' \wedge b'') = \lambda(a) \wedge \lambda(b)$. □

In the following, we always assume that E is a bounded L-equality algebra, L is an lattice and (L, λ) is a reticulation of E, without the special indiction.

Proposition 5.7. *Let E be a involutory L-equality algebra. If F is a filter of L, then $\lambda^{-1}(F)$ is a filter of E.*

Proof. Since F is a filter of L, then $1 \in F \neq \varnothing$. And so, we have $1 = \lambda(\mathrm{i}) \in F$, which implies $1 \in \lambda^{-1}(F)$. Now, we prove that $\lambda^{-1}(F)$ is an upset of E. Let $a, b \in E$, and $a \in \lambda^{-1}(F)$, $a \leq b$. Since λ is isotone, then $\lambda(a) \leq \lambda(b)$. By $\lambda(a) \in F$ and F is a filter of L, we have $\lambda(b) \in F$, and so, $b \in \lambda^{-1}(F)$, this shows that $\lambda^{-1}(F)$ is an upset of E. Let $a, b \in \lambda^{-1}(F)$, then $\lambda(a)$, $\lambda(b) \in F$. Since F is a filter of L, we have $\lambda(a) \wedge \lambda(b) \in F$. By condition $(R3)$, we get $\lambda(a \odot b) \in F$, and then $a \odot b \in \lambda^{-1}(F)$ holds.

Suppose that $a, b \in E$, and $a, a \to b \in \lambda^{-1}(F)$. By the above argument, we get $a \odot (a \to b) \in \lambda^{-1}(F)$. Note that $\lambda^{-1}(F)$ is an upset of E and by Proposition 2.19 (xi), we have $b'' \in \lambda^{-1}(F)$, and thus, $\lambda(b) = \lambda(b'') \in F$. It follows that $b \in \lambda^{-1}(F)$. Let $a, b \in E$, and $a, a \sim b \in \lambda^{-1}(F)$. Then $\lambda(a), \lambda(a \sim b) \in F$, by Proposition 2.7 (i) and Proposition 2.19 (iii), we get $a \odot (a \sim b) \leq a \odot (a \to b)$. From Proposition 2.19 (xi), $a \odot (a \sim b) \leq b''$, and then $b \in \lambda^{-1}(F)$. By Definition 4.3, $\lambda^{-1}(F)$ is a filter of E. □

Proposition 5.8. *Let E be an L-equality algebra and λ be a reticulation of E. If F is a filter of E, then $\lambda(F)$ is a filter of L.*

Proof. Let F be a filter of E. Then $1 \in F$. Applying condition $(R1)$, it follows that $1 = \lambda(\mathrm{i}) \in \lambda(F)$, and so, $\lambda(F) \neq \varnothing$. Let $x, y \in \lambda(F)$, then there exist $a, b \in F$ such that $x = \lambda(a)$, $y = \lambda(b)$. Since F is a filter of E, we have $a \odot b \in F$. By condition $(R3)$, we get $x \wedge y = \lambda(a) \wedge \lambda(b) = \lambda(a \odot b) \in \lambda(F)$. Let $z \in L$ such that $x \leq z$. Since λ is surjective, there exists $c \in F$ such that $\lambda(c) = z$. Thus, $\lambda(a) \leq \lambda(c)$, by condition $(R5)$, there exists $n \in N^+$ such that $a^n \leq c$. Since $a \in F$, we have $a^n \in F$, and then, $c \in F$. Therefore, $z = \lambda(c) \in \lambda(F)$. □

Proposition 5.9. *Let E be a bounded L-equality algebra and P be a filter of E. If for all $a, b \in E$, $a \oplus b \in P$ implies $a \in P$ or $b \in P$, then P is a prime L- filter of E.*

Proof. Let $a, b \in E$ and $\langle a \rangle \wedge \langle b \rangle \subseteq P$. By Proposition 2.19 (vii), $a, b \leq a \oplus b$, since $a \in \langle a \rangle$ and $b \in \langle b \rangle$, thus, $a \oplus b \in \langle a \rangle$ and $a \oplus b \in \langle b \rangle$. It follows that $a \oplus b \in \langle a \rangle \wedge \langle b \rangle \subseteq P$. This shows that $a \oplus b \in P$. By the hypothesis, we have $a \in P$ or $b \in P$, which implies that P is a prime L-filter of E by Theorem 4.10. □

Example 5.10. From Example 4.9, we know that $P = \{1\}$ is a prime L-filter and $e \oplus f \in P$, but neither e nor f belongs to P. Therefore, $a \oplus b \in P \Rightarrow a \in P$ or $b \in P$ is a sufficient but not necessary condition.

Proposition 5.11. *If E is an L-equality algebra and P is a prime filter of L, then $\lambda^{-1}(P)$ is a prime L- filter of E.*

Proof. By Proposition 5.8, we get $\lambda^{-1}(P)$ is an filter of E. Since $P \neq L$ and the fact that λ is surjective, we have $\lambda^{-1}(P) \neq E$. Suppose that $a, b \in E$

such that $a \oplus b \in \lambda^{-1}(P)$. Applying condition $R(2)$, we get $\lambda(a) \vee \lambda(b) = \lambda(a \oplus b) \in P$. Since P is prime, by Definition 2.18, it follows that $\lambda(a) \in P$ or $\lambda(b) \in P$, so $a \in \lambda^{-1}(P)$ or $b \in \lambda^{-1}(P)$. Therefore, $\lambda^{-1}(P)$ is a prime L-filter of E by Proposition 5.9. □

Proposition 5.12. *Let E be an L-equality algebra and λ be an L-reticulation. Then the following statements hold:*

(i) *If F is a filter of E such that $\lambda(a) = \lambda(b)$ for some $a, b \in E$, then $a \in F$ if and only if $b \in F$,*

(ii) *$\lambda(a) = \lambda(b)$ if and only if $\langle a \rangle = \langle b \rangle$ for any $a, b \in E$,*

(iii) *If F is an filter of E, then $\lambda(a) \in \lambda(F)$ if and only if $a \in F$ for any $a \in E$,*

(iv) *If F be an filter of E, then $\lambda^{-1}\lambda(F) = F$.*

Proof. (i) Let $a, b \in E$ such that $\lambda(a) = \lambda(b)$. Suppose that $a \in F$, by the condition $(R5)$, since $\lambda(a) \leq \lambda(b)$, there exists $n \in \mathbb{N}^+$ such that $a^n \leq b$. By condition (A) and $a \in F$, $a^n \to b = a \to (a^{n-1} \to b) = 1 \in F$ implies $a^{n-1} \to b \in F$. Repeating the above argument, we conclude that, $a \to b \in F$, hence $b \in F$. Using $\lambda(b) \leq \lambda(a)$, one gets the proof of the converse implication.

(ii) By condition $(R5)$, there exists $n \in \mathbb{N}^+$ such that $a^n \leq b$, then $b \in \langle a \rangle$, and so, $\langle b \rangle \subseteq \langle a \rangle$. Similarly, since $\lambda(b) \leq \lambda(a)$, we get $\langle a \rangle \subseteq \langle b \rangle$. Thus, $\langle a \rangle = \langle b \rangle$. Conversely, let $\langle a \rangle = \langle b \rangle$, then $b \in \langle a \rangle$, by the definition of generated filter, we get $a^n \leq b$. It follows that $\lambda(a^n) \leq \lambda(b)$. By $(R3)$, we have

$$\lambda(a) = \lambda(a) \wedge \lambda(a) \cdots \wedge \lambda(a) = \lambda(a \odot a \cdots \odot a) = \lambda(a^n) \leq \lambda(b).$$

Similarly, from $a \in \langle b \rangle$, we can get $\lambda(b) \leq \lambda(a)$. Therefore, $\lambda(a) = \lambda(b)$.

(iii) Suppose that F is a filter of E and $a \in E$. Let $\lambda(a) \in \lambda(F)$. By $(R1)$ we get λ is surjective, then there exists $b \in F$ such that $\lambda(a) = \lambda(b)$. Applying (i), we get $a \in F$. The converse implication is obvious.

(iv) By (iii) and condition $(R1)$, it is easy to see that $a \in F \Leftrightarrow \lambda(a) \in \lambda(F) \Leftrightarrow a \in \lambda^{-1}\lambda(F)$, which implies $\lambda^{-1}\lambda(F) = F$. □

Proposition 5.13. *If E is an L-equality algebra with join and P is a prime L-filter of E, then $\lambda(P)$ is a prime filter of L.*

Proof. According to Proposition 5.8, we get $\lambda(P)$ is a filter of L. Since $P \neq E$, there exists $a \in E$ such that $a \notin P$. By Propsition 5.12(iii), we get $\lambda(a) \notin \lambda(P)$, and so, $\lambda(P)$ is a proper filter of L. Suppose that $u, v \in L$ such that $u \vee v \in \lambda(P)$. Using condition $(R1)$, there exist $a, b \in E$ such that $\lambda(a) = u$, $\lambda(b) = v$. Thus, $\lambda(a) \vee \lambda(b) \in \lambda(P)$. By Proposition 5.6(vii), it follows that $\lambda(a \vee b) = \lambda(a) \vee \lambda(b) \in \lambda(P)$. With Proposition 5.12(iii)), we have $a \vee b \in P$. Since P is prime L-filter, then $a \in P$ or $b \in P$ by Proposition 4.11. Thus, $\lambda(a) \in \lambda(P)$ or $\lambda(b) \in \lambda(P)$, that is, $u \in \lambda(P)$ or $v \in \lambda(P)$. Therefore, $\lambda(P)$ is a prime filter of L. $\square$

Proposition 5.14. *Let E be an L-equality algebra with join. Then the map $\lambda^{-1} : Spec(L) \to Spec(E)$ is bijective.*

Proof. From Proposition 5.7, the map λ^{-1} is well-defined. Let $Q \in Spec(E)$. By Proposition 5.13 we have $\lambda(Q) \in Spec(L)$ and according to Propsition 5.12(iv), we get $\lambda^{-1}\lambda(Q) = Q$. Therefore, λ^{-1} is surjective. In the following we prove that λ^{-1} is injective. Let $I, J \in Spec(L)$ and $\lambda^{-1}(I) = \lambda^{-1}(J)$. We will prove that $I = J$. Let $a \in I$. Then from $(R1)$, there is $x \in E$ such that $\lambda(x) = a$. This means that $x \in \lambda^{-1}(I) = \lambda^{-1}(J)$, and hence, $\lambda(x) \in J$. Therefore, $a \in J$, that is, $I \subseteq J$. Similarly, we can prove that $J \subseteq I$, and so, $I = J$. It follows that λ^{-1} is injective, and so, λ^{-1} is a bijective map. $\square$

Proposition 5.15. *Let E be an L-equality algebra with join. The map $\lambda^{-1} : Spec(L) \to Spec(E)$ is continuous and open.*

Proof. Let $a \in E$ and $D(a) = \{P \in Spec(L) | a \notin P\}$ be the neighborhood of a. Then we get

$$\begin{aligned}(\lambda^{-1})^{-1}(D(a)) &= \{P \in Spec(L) | \lambda^{-1}(P) \in D(a)\} \\ &= \{P \in Spec(L) | a \notin \lambda^{-1}(P)\} \\ &= \{P \in Spec(L) | \lambda(a) \notin P\} \\ &= D(\lambda(a)).\end{aligned}$$

Thus, λ^{-1} is continuous. Let $u \in L$. Since λ is surjective, there exists $a \in x$ such that $\lambda(a) = u$. So

$$\begin{aligned}\lambda^{-1}(D(u)) &= \lambda^{-1}(D(\lambda(a))) \\ &= \{\lambda^{-1}(P) | P \in D(\lambda(a))\}\end{aligned}$$

$$
\begin{aligned}
&= \{\lambda^{-1}(P)|P \in Spec(L), \lambda(a) \notin P\} \\
&= \{\lambda^{-1}(P)|P \in Spec(L), a \notin \lambda^{-1}(P)\} \\
&= \{Q \in Spec(E)|a \notin Q\} \\
&= D(a).
\end{aligned}
$$

Therefore, λ^{-1} is open. □

Theorem 5.16. *Let A be an L-equality algebra with join and pair (L, λ) be an L-reticulation, then the map $\lambda^{-1} : Spec(L) \to Spec(E)$ is a homeomorphism of topological spaces.*

Proof. It directly follows from Propositions 5.14 and 5.15. □

Theorem 5.17. *Let E be a residuated divisible lEQ-algebra with condition (S), then the following conditions are equivalent:*
(i) (L, λ) *is an L-reticulation of E,*
(ii)(L, λ) *is an e-reticulation of E.*

Proof. (i) $\Rightarrow$ (ii) Let (L, λ) be an L-reticulation of E, from Poprosition 2.19(11) and $(R1)$, we can obtain $\lambda(a \vee b) \leq \lambda(a \oplus b)$, $\lambda(a) \leq \lambda(a \vee b)$ and $\lambda(b) \leq \lambda(a \vee b)$, thus

$$\lambda(a) \vee \lambda(b) \leq \lambda(a \vee b) \leq \lambda(a \oplus b) = \lambda(a) \vee \lambda(b),$$

which implies $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$.

(ii) $\Rightarrow$ (i) Let (L, λ) be an e-reticulation of E, then $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$ for any $a, b \in E$. It remains to show that $\lambda(a \vee b) = \lambda(a \oplus b)$. By Proposition 2.19(xi), we can obtain $< a \vee b > \subseteq < a \oplus b >$. Since $< a \oplus b >$ and $< a \vee b >$ are all filters of E, we have $a \oplus b \in < a \oplus b >$ and $a \vee b \in < a \vee b >$, by Proposition 2.19(xi), we obtain $a \oplus b \in < a \vee b >$, which implies $< a \oplus b > \subseteq < a \vee b >$. Therefore $< a \oplus b > = < a \vee b >$. By Lemma 2.22, we conclude that $\lambda(a \vee b) = \lambda(a \oplus b)$. □

6 Conclusion and future works

This paper introduced the class of L-equality algebras, namely equality algebras satisfying condition (L), and showed that this additional condition is sufficient to generate a product operation compatible with the original algebraic structure. As a consequence, every self-similar L-equality algebra determines a divisible residuated EQ-algebra satisfying condition (S), establishing the correspondence between self-similar L-equality algebras, self-similar left hoops, and this class of EQ-algebras.

The paper also examined the filter structure of L-equality algebras. While e-filters and L-filters were shown to coincide, the distinction between prime e-filters and prime L-filters indicates that the prime structure of these algebras is more delicate than that of ordinary equality algebras. In addition, a reticulation was constructed for bounded L-equality algebras, and it was proved that the associated prime spectrum is homeomorphic to the prime spectrum of the corresponding bounded distributive lattice. This construction extends the existing reticulation theory for EQ-algebras and confirms its compatibility with the present setting.

Several questions remain open. A natural continuation of this work is the development of L-equality logic, taking L-equality algebras as its algebraic semantics. Such a study should include the investigation of soundness, completeness, and algebraizability, together with a comparison between L-equality logic, L-logic, EQ-logic, and equality logic.

The topological aspects of L-equality algebras also deserve further attention. The reticulation established here suggests that spectral methods for distributive lattices may provide sheaf representations and, ultimately, global representation theorems for L-equality algebras. Whether these representations can be obtained in full generality remains an interesting problem.

Another direction is the study of congruences and decomposition theory. In particular, it would be desirable to characterize subdirectly irreducible L-equality algebras and to investigate decomposition techniques based on ordinal sums, following the corresponding developments for left hoops. Results in this direction would contribute to a better understanding of the internal structure of L-equality algebras and their relationship with other algebraic systems arising in fuzzy logic.

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Zhe Jie Duan *School of Mathematics and Statistics, Weinan Normal University, 714000, Weinan, China*
Email: 394695712@qq.com

Xiao Long Xin *Bussiness School, Xi'an International University, Xi'an 710048, China*
Email: xlxin@nwu.edu.cn

Rajab Ali Borzooei *Soft Computing Center, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran*

Email: borzooei@sbu.ac.ir