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# Some categorical and algebraic results on (distributive) nearlattices

Luciano J. González

**Abstract.** The goal of this article is to determine whether the varieties of $N$-algebras (nearlattices) and $DN$-algebras (distributive nearlattices) satisfy certain categorical and algebraic properties. Specifically, we provide a characterization of the free $DN$-algebra freely generated by a finite set, and establish a connection between free $DN$-algebras and the free distributive lattice extensions of $DN$-algebras. We also prove that the variety of $DN$-algebras satisfies the amalgamation property but fails to satisfy the strong amalgamation property, whereas the category of $N$-algebras does not possess the amalgamation property. Finally, we show that, in the algebraic categories of $N$-algebras and $DN$-algebras, not every epimorphism is surjective.

## 1 Introduction and preliminaries

The concept of a *nearlattice* admits two equivalent definitions: as a join-semilattice satisfying certain properties, or as a ternary algebra holding some identities. Each approach presents distinct advantages and trade-offs. Treating nearlattices as join-semilattices is more intuitive, as it is similar to

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the well-established theory of lattices; however, under this framework, the class of nearlattices does not form a variety. Conversely, while defining nearlattices as ternary algebras ensures they form a variety, this representation is often less intuitive to work with. Fortunately, the isomorphism between these two descriptions allows us to transition between them, exploiting the strengths of both while circumventing their respective limitations. Nearlattices have been studied by several authors from different perspectives; see, for instance, [3, 5–7, 9, 11, 14–20, 22, 23, 26, 29] for a non-exhaustive list. This paper aims to establish or refute the categorical and algebraic properties outlined in the abstract for the variety of nearlattices viewed as ternary algebras.

We assume the reader is familiar with the elements of order and lattice theory ( [21]), universal algebra ( [4]), and category theory ( [27]). Standard notions and results from these areas that are not explicitly detailed here can be found in the respective references.

We begin with some basic definitions and notations. Let $\langle P, \leq\rangle$ be a poset. Given a subset $U \subseteq P$, we let the sets

$$[U) := \{x \in P : y \leq x, \text{ for some } y \in U\}$$

and

$$(U] := \{x \in P : x \leq y, \text{ for some } y \in U\}.$$

For a singleton $\{x\}$, we denote $[x)$ and $(x]$ instead $[\{x\})$ and $(\{x\}]$, respectively. A subset $U \subseteq P$ is called *upset* of $P$ if $U = [U)$, and dually $U$ is called *downset* if $U = (U]$. Let $\mathsf{Up}(P)$ be the set of all upsets of $P$ and $\mathsf{Do}(P)$ the set of all downsets of $P$.

Given a non-empty subset $X = \{a_1, \ldots, a_n\}$ of $P$, we write $\bigwedge_P X$ or $a_1 \wedge_P \cdots \wedge_P a_n$ to indicate that the infimum (or meet) of the set $X$ exists in $P$ and $\inf X = \bigwedge_P X = a_1 \wedge_P \cdots \wedge_P a_n$. When there is no danger of confusion, we omit the subscript in $\bigwedge X$ and $a_1 \wedge \cdots \wedge a_n$. An analogous notation is used for the supremum (or join) in $P$.

A *join-semilattice* is a semilattice $\langle A, \vee\rangle$ where the natural partial order $\leq$ on $A$ is defined as follows: $a \leq b$ if and only if $a \vee b = b$. A join-semilattice $\langle A, \vee\rangle$ is called *weakly distributive* if $a_1 \wedge \cdots \wedge a_n$ exists in $A$, then $(a_1 \vee a) \wedge \cdots \wedge (a_n \vee a)$ exists in $A$ for any $a \in A$ and equals $a \wedge (a_1 \vee \cdots \vee a_n)$ (see [10]). Notice that if $A$ is a weakly distributive join-semilattice and

$a_1 \wedge \cdots \wedge a_n$ and $b_1 \wedge \cdots \wedge b_k$ exist in $A$, then

$$(a_1 \wedge \cdots \wedge a_n) \vee (b_1 \wedge \cdots \wedge b_k) = \bigwedge_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} (a_i \vee b_j).$$

**Definition 1.1.** A join-semilattice $\langle A, \vee \rangle$ is called a *nearlattice* if for any two elements possessing a common upper bound have a supremum.

Notice that a join-semilattice $A$ is a nearlattice if and only if, for all $a \in A$, the principal upset $[a)$ is a lattice with respect to the partial order of $A$. Consequently, nearlattices constitute a natural generalization of lattices, as every lattice is evidently a nearlattice. Moreover, every finite join-semilattice is in fact a nearlattice. This definition also extends naturally to the distributive setting.

**Definition 1.2.** A nearlattice $\langle A, \vee \rangle$ is called *distributive* if it is weakly distributive.

We also note that a nearlattice $A$ is distributive if and only if, for any $a \in A$, the principal upset $[a)$ is a distributive lattice. An interesting subclass of distributive nearlattices consists of those in which every principal upset is a Boolean algebra. These structures were studied by Abbott in [2] under the name *semiboolean algebras*. The class of semiboolean algebras is in a one-to-one correspondence with the variety of implication algebras [1, 28].

We now present several examples to demonstrate that nearlattices arise naturally in various branches of mathematics.

**Example 1.3.** Let $X$ be a non-empty set. Let $\mathcal{P}(X)^* := \{Y \subseteq X : Y \neq \emptyset\}$. Then $\langle \mathcal{P}(X)^*, \cup \rangle$ is a semiboolean algebra.

**Example 1.4.** Let $P$ be a poset. Let $\mathsf{Up}(P)^* := \{U \in \mathsf{Up}(P) : U \neq \emptyset\}$. Then $\langle \mathsf{Up}(P)^*, \cup \rangle$ is a distributive nearlattice but it is not necessarily a semiboolean algebra.

**Example 1.5.** Let $G$ be a group. Let

$$\mathsf{Sub}(G)^* := \{H : H \text{ is subgroup of } G \text{ and } H \neq \{e\}\},$$

where $e$ is the neutral element of $G$. Then, $\langle \mathsf{Sub}(G)^*, \veebar \rangle$ is a nearlattice not necessarily distributive, where $H_1 \veebar H_2$ is the subgroup generated by $H_1 \cup H_2$.

**Example 1.6.** Let $X$ be a non-empty set. Let $\mathcal{P}(X)^*_\omega := \{Y \subseteq X : 0 < |Y| < \omega\}$. It is well-known (see [24]) that $\langle \mathcal{P}(X)^*_\omega, \cup\rangle$ is, up to isomorphism, the free semilattice freely generated by $X$ in the variety of semilattices. We notice that $\langle \mathcal{P}(X)^*_\omega, \cup\rangle$ is in particular a distributive nearlattice.

**Remark 1.7.** As previously mentioned, the class of nearlattices is not a variety, nor is the class of distributive nearlattices. In particular, they are not closed under subalgebras. Example 1.6 implies that the variety of semilattices, expressed over the language $\mathcal{L}_\vee := \{\vee\}$ of type (2), is generated by the class of distributive nearlattices, as well as by the class of nearlattices.

We now turn our attention to the characterization of nearlattices as ternary algebras.

**Definition 1.8.** An algebra $\langle A, m\rangle$ of type (3) is called an *N-algebra* if satisfies the following identities.

(P1) $m(x, y, x) = x$,

(P2) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$.

This axiomatization can be found in [3]. A more descriptive axiomatization can be found in [9]. Let us denote by $\mathbb{NA}$ the variety of $N$-algebras.

**Definition 1.9.** An $N$-algebra $\langle A, m\rangle$ is said to be *distributive* if it satisfies either of the following two equivalent identities:

(P3) $m(x, m(y, y, z), w) = m(m(x, y, w), m(x, y, w), m(x, z, w))$;

(P4) $m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w)$.

We call a distributive $N$-algebra simply a *DN-algebra*.

We denote by $\mathbb{DN}$ the variety of $DN$-algebras.

**Example 1.10.** Let $\langle L, \wedge, \vee\rangle$ be a (distributive) lattice. Then, defining $m(a, b, c) = (a \vee c) \wedge (b \vee c)$, it follows that $\langle L, m\rangle$ is a ($DN$-algebra) $N$-algebra.

**Theorem 1.11** ( [9]). (1) *If $\langle A, m\rangle$ is (a DN-algebra) an N-algebra, then the algebra $A_* = \langle A, \vee\rangle$, where*

$$x \vee y := m(x, x, y), \tag{1.1}$$

*is a (distributive) nearlattice.*

(2) *If $\langle S, \vee \rangle$ is a (distributive) nearlattice, then the algebra $S^* = \langle S, m \rangle$ with*

$$m(x, y, z) := (x \vee z) \wedge_z (y \vee z)$$

*(where $\wedge_z$ is the meet of $x \vee z$ and $y \vee z$ in $[z)$) is (a DN-algebra) an N-algebra.*

(3) *If $\langle A, m \rangle$ is (a DN-algebra) an N-algebra and $\langle S, \vee \rangle$ is a (distributive) nearlattice, then $(A_*)^* = A$ and $(S^*)_* = S$.*

From now on, let us consider $N$-algebras $\langle A, m \rangle$, where the join $\vee$ is defined as above: $x \vee y := m(x, x, y)$. Thus, $\langle A, \vee \rangle$ is a join-semilattice such that, for all $a, b, c \in A$, if $a \leq b, c$, then $b \wedge c$ exists in $A$.

**1.1 Some notations** A *homomorphism* between algebras $\langle A, m_A \rangle$ and $\langle B, m_B \rangle$ of type (3) is simply an algebraic homomorphism $h \colon A \to B$, that is, for all $a_1, a_2, a_3 \in A$, $h(m_A(a_1, a_2, a_3) = m_B(h(a_1), h(a_2), h(a_3))$. It is know that a map $h \colon A \to B$ between $N$-algebras $\langle A, m_A \rangle$ and $\langle B, m_B \rangle$ is a homomorphism if and only if $h$ preserve finite joins and existing finite meets.

Let $\langle A, m \rangle$ be an algebra of type (3). For each natural number $n$ we define inductively, for all $a_1, \ldots, a_n, b \in A$, an element $m^{n-1}(a_1, \ldots, a_n, b)$ as follows:

- $m^0(a_1, b) := m(a_1, a_1, b)$ and
- for $n > 1$, $m^{n-1}(a_1, \ldots, a_n, b) := m(m^{n-2}(a_1, \ldots, a_{n-1}, b), a_n, b)$.

In particular, for a nearlattice $\langle A, m \rangle$, we obtain that $m^0(a_1, b) = a_1 \vee b$ and $m^1(a_1, a_2, b) = m(a_1, a_2, b)$. The following proposition follows directly by induction and thus we omit its proof.

**Proposition 1.12.** *Let $\langle A, m_A \rangle$ and $\langle B, m_B \rangle$ be algebras of type* (3) *and let $h \colon A \to B$ be a homomorphism. Then*

$$h(m_A^{n-1}(a_1, \ldots, a_n, b)) = m_B^{n-1}(h(a_1), \ldots, h(a_n), h(b)),$$

*for all $a_1, \ldots, a_n, b \in A$.*

**Proposition 1.13.** ([14]) *Let $\langle A, m \rangle$ be a nearlattice and let $a_1, \ldots, a_{n+1}$, $a, b \in A$. Then:*

(1) $m^{n-1}(a_1, \dots, a_n, b) = (a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$.

(2) $m^{n-1}(a_1, \dots, a_n, b) \leq a_i \vee b$, *for all* $1 \leq i \leq n$.

(3) $m^{n-1}(a_1, \dots, a_n, b) = m^{n-1}(a_{\sigma(1)}, \dots, a_{\sigma(n)}, b)$ *for every permutation* $\sigma$ *of* $\{1, \dots, n\}$.

**1.2 Discrete representation for finite $DN$-algebras (distributive nearlattices)** In order to achieve one of the main goals of this article, we need to consider a discrete representation for finite $DN$-algebras. The results in this section correspond to those given in [20] (see also [16]). We refer the reader to [20] for further details.

**Definition 1.14.** Let $A$ be a $DN$-algebra. An element $a \in A$ is said to be *irreducible* if for all $b, c \in A$ such that $b \wedge c$ exists in $A$, $a = b \wedge c$ implies $a = b$ or $a = c$.

Let us denote by $\mathrm{Irr}(A)$ the set of all irreducible elements of $A$, which is ordered by the induced order of $A$. The following characterizations of irreducible elements is straightforward, and we omit the proof.

**Proposition 1.15.** *Let $A$ be a $DN$-algebra and $a \in A$. Then, the following are equivalent.*

(1) *$a$ is irreducible.*

(2) *If $b \wedge c$ exists in $A$ and $b \wedge c \leq a$, then $b \leq a$ or $c \leq a$.*

(3) *If $m^{k-1}(a_1, \dots, a_k, b) \leq a$, then $a_i \vee b \leq a$ for some $i = 1, \dots, k$.*

**Proposition 1.16.** *Let $A$ be a $DN$-algebra. Then, for every $a \in A$, $a = \bigwedge\{x \in \mathrm{Irr} : a \leq x\}$.*

Let $A$ be a finite $DN$-algebra. Let $\gamma_A \colon \mathsf{Do}(\mathrm{Irr}(A)) \to \{0, 1\}$ be defined as follows: for every $U \in \mathsf{Do}(\mathrm{Irr}(A))$,

$$\gamma_A(U) = 1 \iff \bigwedge (\mathrm{Irr}(A) \setminus U) \text{ exists in } A.$$

**Proposition 1.17.** *The map $\gamma_A \colon \mathsf{Do}(\mathrm{Irr}(A)) \to \{0, 1\}$ satisfies the following properties:*

(1) $\gamma_A(\mathrm{Irr}(A)) = 1$.

(2) $\gamma_A(\mathrm{Irr}(A) \setminus [x)) = 1$, *for all* $x \in \mathrm{Irr}(A)$.

(3) *For all* $U, V \in \mathsf{Do}(\mathrm{Irr}(A))$, *if* $U \subseteq V$, *then* $\gamma_A(U) \leq \gamma_A(V)$.

Let $\mathcal{A} := \{U \in \mathsf{Do}(\mathrm{Irr}(A)) : \gamma_A(U) = 1\}$. By the previous proposition, it is easy observe that $\mathcal{A}$ is closed under union, and if $W \in \mathcal{A}$, $U, V \in \mathsf{Do}(\mathrm{Irr}(A))$ and $W \subseteq U \cap V$, then $U \cap V \in \mathcal{A}$. Hence, $\langle \mathcal{A}, \cup \rangle$ is a distributive nearlattice. Therefore, $\langle \mathcal{A}, m_{\mathcal{A}} \rangle$, where $m_{\mathcal{A}}(U, V, W) = (U \cup W) \cap (V \cup W)$, is a $DN$-algebra.

Let $\sigma \colon A \to \mathcal{A}$ be defined as follows: for every $a \in A$, $\sigma(a) = \{x \in \mathrm{Irr}(A) : a \nleq x\}$.

**Theorem 1.18** (Discrete representation)**.** *Let $A$ be a finite $DN$-algebra. Then, $\langle \mathcal{A}, m_{\mathcal{A}} \rangle$ is a finite $DN$-algebra and the map $\sigma \colon A \to \mathcal{A}$ is an isomorphism.*

## 2 Free $DN$-algebras generated by a finite set

In this section, we describe the free $DN$-algebras generated by a finite set. This characterization is similar to the one established for distributive lattices (see, for example, [21, p. 126]). We begin by recalling the well-known characterization from universal algebra (see [4]) for the free $DN$-algebra generated by a set $X$.

Let $\mathcal{L}_m := \{m\}$ be an algebraic language of type (3). We consider the term $\vee$ defined by: $x \vee y := m(x, x, y)$, with $x$ and $y$ variables. Recall that $\mathbb{DN}$ denotes the variety of $DN$-algebras over the language $\mathcal{L}_m$. Let $X$ be a non-empty set. Let $\mathrm{Term}(X)$ be the *term algebra* of type $\mathcal{L}_m$ over $X$. Let $A$ be a $DN$-algebra. Let $\mathrm{Hom}(\mathrm{Term}(X), A)$ be the set of all algebraic homomorphisms $h \colon \mathrm{Term}(X) \to A$. Let $t, s \in \mathrm{Term}(X)$. Let

$$\begin{aligned} \mathbb{DN} \models t \approx s \iff & \text{for all } A \in \mathbb{DN}, \text{ and for all } h \in \mathrm{Hom}(\mathrm{Term}(X), A), \\ & h(t) = h(s) \end{aligned}$$

and

$$\begin{aligned} \mathbb{DN} \models t \preccurlyeq s \iff & \text{for all } A \in \mathbb{DN}, \text{ and for all } h \in \mathrm{Hom}(\mathrm{Term}(X), A), \\ & h(t) \leq h(s) \end{aligned}$$

Let $\equiv$ be the congruence relation defined on term algebra $\mathrm{Term}(X)$ as follows: for all $t, s \in \mathrm{Term}(X)$,

$$t \equiv s \iff \mathbb{DN} \models t \approx s.$$

For simplicity, we shall denote $t/{\equiv}$ by $[t]$, for all $t \in \mathrm{Term}(X)$. That is, $[t] := t/{\equiv}$. Then, the quotient

$$\mathrm{Free}(\overline{X}) = \mathrm{Term}(X)/{\equiv} = \{[t] : t \in \mathrm{Term}(X)\}$$

with the ternary operation $m_F$ defined on $\mathrm{Free}(\overline{X})$ as $m_F([t_1],[t_2],[t_3]) = [m(t_1,t_2,t_3)]$ is, up to isomorphism, the *free DN-algebra generated by* $\overline{X} = \{[x] : x \in X\}$, where $|\overline{X}| = |X|$. Our goal is to obtain a characterization of $\mathrm{Free}(\overline{X})$ when $X$ is finite. It is known that the variety of $DN$-algebras is locally finite (see [8]). Thus, we know that $\mathrm{Free}(\overline{X})$ is finite.

We start noting that, for all $t, s \in \mathrm{Term}(X)$, we have

$$[t] \leq [s] \quad \text{in } \mathrm{Free}(\overline{X}) \iff \mathbb{DN} \models t \preccurlyeq s.$$

From now on, let $X = \{x_1, \ldots, x_n\}$ be a finite non-empty set of variables. Let $\mathbf{n} := \{1, \ldots, n\}$. Let $\mathcal{P}(\mathbf{n})^* := \{J \subseteq \mathbf{n} : J \neq \emptyset \text{ and } J \neq \mathbf{n}\}$. Consider $\mathcal{P}(\mathbf{n})^*$ ordered by the set inclusion $\subseteq$. Let

$$\mathcal{A} = \{V \in \mathsf{Do}(\mathcal{P}(\mathbf{n})^*) : \exists i \in \mathbf{n} \text{ such that } \mathbf{n} \setminus \{i\} \in V\}.$$

It follows that $\mathcal{A}$ is closed under unions, and if $U, V, W \in \mathcal{A}$ and $W \subseteq U \cap V$, then $U \cap V \in \mathcal{A}$. Hence, $\langle \mathcal{A}, m_{\mathcal{A}} \rangle$ is a $DN$-algebra where $m_{\mathcal{A}}(U,V,W) = (U \cup W) \cap (V \cup W)$. We shall show that for every $n$, $\langle \mathcal{A}, m_{\mathcal{A}} \rangle$ is the free $DN$-algebra over $n$ generators. That is, we shall prove:

**Theorem 2.1.** $\mathrm{Free}(\overline{X}) \cong \{U \in \mathsf{Do}(\mathcal{P}(\mathbf{n})^*) : \exists i \in \mathbf{n}$ *such that* $\mathbf{n} \setminus \{i\} \in U\}$.

We proceed to prove several auxiliary results. We begin by introducing two syntactic notations. Consider a countable set $Y = \{y_i : i = 1, 2, \ldots,\}$ of variables. We define recursively, for every integer $k \geq 1$ and variables $y_1, \ldots, y_k, x$, the term $m^{k-1}(y_1, \ldots, y_k, x)$ as follows:

- $m^0(y_1, x) := m(y_1, y_1, x)$
- for $k > 1$, $m^{k-1}(y_1, \ldots, y_k, x) := m(m^{k-2}(y_1, \ldots, y_{k-1}, x), y_k, x)$.

We define also recursively the term $y_1 \vee \cdots \vee y_{k+1}$ for all $k = 1, 2, \ldots$, as follows:

$$y_1 \vee \cdots \vee y_{k+1} := m(y_1 \vee \cdots \vee y_k, y_1 \vee \cdots \vee y_k, y_{k+1}).$$

For instance, $y_1 \vee y_2 = m(y_1, y_1, y_2)$ (which coincide the previous definition of the binary term $\vee$), and $y_1 \vee y_2 \vee y_3 = m(y_1 \vee y_2, y_1 \vee y_2, y_3) = m(m(y_1, y_1, y_2), m(y_1, y_1, y_2), y_3)$.

For every non-empty $J \subseteq \mathbf{n}$, we define the term $t_J \in \mathrm{Term}(X)$ as follows: if $J = \{i_1, \ldots, i_k\}$ such that $i_1 < \cdots < i_k$, then

$$t_J := x_{i_1} \vee \cdots \vee x_{i_k}.$$

Let $A \in \mathbb{DN}$ and $h\colon \mathrm{Term} \to A$ be a homomorphism. Then, for every $J = \{i_1, \ldots, i_k\} \subseteq \mathbf{n}$ non-empty, we have that $h(t_J) = h(x_{i_1}) \vee \cdots \vee h(x_{i_k})$, that is, $h(t_J)$ is the suprumum of the set $\{h(x_{i_1}), \ldots, h(x_{i_k})\}$ in $A$. We also have, for all variables $y_1, \ldots, y_k, x \in X$, that

$$h(m^{k-1}(y_1, \ldots, y_k, x)) = m_A^{k-1}(h(y_1), \ldots, h(y_k), h(x)).$$

**Proposition 2.2.** *Let $J, K \subseteq \mathbf{n}$ be non-empty. Then, $[t_J] \leq [t_K]$ if and only if $J \subseteq K$.*

*Proof.* ($\Leftarrow$) It is trivial.

($\Rightarrow$) Suppose that $J \nsubseteq K$. Let $j \in J$ be such that $j \notin K$. Let $C_2 \in \mathbb{DN}$ be the two-element $DN$-algebra. That is, $C_2 = \{0, 1\}$ with $0 < 1$. Consider the homomorphism $h\colon \mathrm{Term} \to C_2$ defined by: $h(x_j) = 1$ and $h(x_i) = 0$ for all $i \neq j$ Thus, we obtain that $C_2 \not\models t_J \preccurlyeq t_K$. That is, $[t_J] \nleq [t_K]$. □

**Proposition 2.3.** *Let $J, K \subseteq$ Free be non-empty. Then $t_J \vee t_K \equiv t_{J\cup K}$.*

*Proof.* We need to show that $\mathbb{DN} \models t_J \vee t_K \approx t_{J\cup K}$. Let $A \in \mathbb{DN}$ and $h\colon Term(X) \to A$ be a homomorphism. Then

$$\begin{aligned} h(t_J \vee t_K) = h(t_J) \vee h(t_K) &= \bigvee\{h(x_j) : j \in J\} \vee \bigvee\{h(x_k) : k \in K\} \\ &= \bigvee\{h(x_i) : i \in J \cup K\} = h(t_{J\cup K}). \end{aligned}$$

Hence $\mathbb{DN} \models t_J \vee t_K \approx t_{J\cup K}$. □

The following proposition is highly technical; nevertheless, it is crucial for the subsequent development.

**Proposition 2.4.** *For every term $t \in \mathrm{Term}(X)$, there exist non-empty subsets $J_1, \dots, J_k \subseteq \mathbf{n}$ and $x \in X$ such that $t \equiv m^{k-1}(t_{J_1}, \dots, t_{J_k}, x)$.*

*Proof.* We proceed by induction on the complexity of the terms.

• Let $t := x_i \in X$. By (P1), it follows that $x_i \equiv m(x_i, x_i, x_i)$.

• Let $t := m(t_1, t_2, t_3)$ and assume that $t_1, t_2, t_3$ are of the form required. That is, for $\ell = 1, 2, 3$, there exist non-empty subsets $J_1^\ell, \dots, J_{k_1}^\ell \subseteq \mathbf{n}$ and $i_\ell \in \mathbf{n}$ such that

$$t_\ell \equiv m^{k_\ell - 1}\left(t_{J_1^\ell}, \dots, t_{J_{k_\ell}^\ell}, x_{i_\ell}\right).$$

For every $\alpha \in \{1, \dots, k_1\}$, $\beta \in \{1, \dots, k_2\}$ and $\gamma \in \{1, \dots, k_3\}$, let

$$\begin{aligned} J_{\alpha\gamma}^{13} &:= J_\alpha^1 \cup J_\gamma^3 \cup \{i_1\} \\ J_{\beta\gamma}^{23} &:= J_\beta^2 \cup J_\gamma^3 \cup \{i_2\}. \end{aligned} \tag{2.1}$$

We define the term $s$ as follows:

$$s := m^{(k_1+k_2)k_3-1}\left(t_{J_{11}^{13}}, \dots, t_{J_{\alpha\gamma}^{13}}, \dots, t_{J_{k_1k_3}^{13}}, t_{J_{11}^{23}}, \dots, t_{J_{\beta\gamma}^{23}}, \dots, t_{J_{k_2k_3}^{23}}, x_{i_3}\right).$$

Let us show that $t \equiv s$. Let $A \in \mathbb{DN}$ and $h \in \mathrm{Hom}(\mathrm{Term}, A)$. Let

$$\begin{aligned} h\left(t_{J_\alpha^1}\right) &= a_\alpha && \text{for } 1 \leq \alpha \ \leq k_1 \\ h\left(t_{J_\beta^2}\right) &= b_\beta && \text{for } 1 \leq \beta \ \leq k_1 \\ h\left(t_{J_\gamma^3}\right) &= c_\gamma && \text{for } 1 \leq \gamma \ \leq k_1 \end{aligned}$$

and $h(x_{i_1}) = u$, $h(x_{i_2}) = v$ and $h(x_{i_3}) = w$.

Then,

$$h(t_1) = m_A^{k_1-1}(a_1, \dots, a_{k_1}, u) = \bigwedge_{1 \leq \alpha k_1} (a_\alpha \vee u)$$

$$h(t_2) = m_A^{k_2-1}(b_1, \dots, b_{k_2}, v) = \bigwedge_{1 \leq \beta \leq k_2} (b_\beta \vee v)$$

$$h(t_3) = m_A^{k_3-1}(c_1, \ldots, c_{k_3}, w) = \bigwedge_{1 \leq \gamma \leq k_3} (c_\gamma \vee w).$$

Hence

$$\begin{aligned} h(t) &= m_A(h(t_1), h(t_2), h(t_3)) = (h(t_1) \vee h(t_3)) \wedge (h(t_2) \vee h(t_3)) \\ h(t) &= \left[ \bigwedge_{1 \leq \alpha k_1} (a_\alpha \vee u) \vee \bigwedge_{1 \leq \gamma \leq k_3} (c_\gamma \vee w) \right] \wedge \\ &\quad \wedge \left[ \bigwedge_{1 \leq \beta \leq k_2} (b_\beta \vee v) \vee \bigwedge_{1 \leq \gamma \leq k_3} (c_\gamma \vee w) \right] \\ h(t) &= \bigwedge_{\substack{a \leq \alpha \leq k_1 \\ 1 \leq \gamma \leq k_3}} (a_\alpha \vee c_\gamma \vee u \vee w) \wedge \bigwedge_{\substack{1 \leq \beta \leq k_2 \\ 1 \leq \gamma \leq k_3}} (b_\beta \vee c_\gamma \vee v \vee u). \end{aligned} \tag{2.2}$$

On the other hand, by (2.1) and Proposition 2.3, we have that for all $1 \leq \alpha \leq k_1$, $1 \leq \beta \leq k_2$, and $1 \leq \gamma \leq k_3$,

$$t_{J^{13}_{\alpha\gamma}} \equiv t_{J^1_\alpha} \vee t_{J^3_\gamma} \vee x_{i_1} \quad \text{and} \quad t_{J^{23}_{\beta\gamma}} \equiv t_{J^2_\beta} \vee t_{T^3_\gamma} \vee x_{i_2}.$$

Thus

$$h\left(t_{J^{13}_{\alpha\gamma}}\right) = a_\alpha \vee c_\gamma \vee u \quad \text{and} \quad h\left(t_{J^{23}_{\beta\gamma}}\right) = b_\beta \vee c_\gamma \vee v.$$

Hence

$$\begin{aligned} h(s) = m_A^{(k_1+k_2)k_3-1} &\left( h\left(t_{J^{13}_{11}}\right), \ldots, h\left(t_{J^{13}_{\alpha\gamma}}\right), \ldots, h\left(t_{J^{13}_{k_1k_3}}\right), \right. \\ &\left. h\left(t_{J^{23}_{11}}\right), \ldots, h\left(t_{J^{23}_{\beta\gamma}}\right), \ldots, h\left(t_{J^{23}_{k_2k_3}}\right), x_{i_3} \right) \\ m_A^{(k_1+k_2)k_3-1}&((a_1 \vee c_1 \vee u), \ldots, (a_\alpha \vee c_\gamma \vee u), \ldots, (a_{k_1} \vee c_{k_3} \vee u), \\ &(b_1 \vee c_1 \vee v), \ldots, (b_\beta \vee c_\gamma \vee v), \ldots, (b_{k_2} \vee c_{k_3} \vee v), w) \\ &= \bigwedge_{\substack{1 \leq \alpha \leq k_1 \\ 1 \leq \gamma \leq k_3}} (a_\alpha \vee c_\gamma \vee u \vee w) \wedge \bigwedge_{\substack{1 \leq \beta \leq k_2 \\ 1 \leq \gamma \leq k_3}} (b_\beta \vee c_\gamma \vee v \vee w) \\ &= h(t). \end{aligned}$$

□

Now we find the irreducible elements of the $DN$-algebra $\mathrm{Free}(\overline{X})$.

**Proposition 2.5.** $\operatorname{Irr}\left(\operatorname{Free}(\overline{X})\right) = \{[t_J] : J \in \mathcal{P}(\mathbf{n})^*\}$.

*Proof.* Let $J \in \mathcal{P}(\mathbf{n})^*$. We show that $[t_J]$ is an irreducible element of $\operatorname{Free}(\overline{X})$. Let $t_1, t_2 \in \operatorname{Term}(X)$ and suppose that $[t_J] = [t_1] \wedge [t_2]$. Notice that the previous implies that $[t_J] = ([t_1] \wedge [t_2]) \vee [t_J] = ([t_1] \vee [t_J]) \wedge ([t_2] \vee [t_J]) = [m(t_1, t_2, t_J)]$. Hence

$$\mathbb{DN} \models t_J \approx m(t_1, t_2, t_J). \tag{2.3}$$

By Proposition 2.4, there exist $J_\alpha, K_\beta \subseteq \mathbf{n}$ non-empty for $1 \leq \alpha \leq k_1$ and $1 \leq \beta \leq k_2$, and $x_{i_1}, x_{i_2} \in X$ such that

$$t_1 \equiv m^{k_1-1}(t_{J_1}, \ldots, t_{J_{k_1}}, x_{i_1}) \qquad \text{and} \qquad t_2 \equiv m^{k_2-1}(t_{K_1}, \ldots, t_{K_{k_2}}, x_{i_2}).$$

Suppose that $[t_J] < [t_1]$ and $[t_J] < [t_2]$. Then, we have that

$$[t_J] < [m^{k_1-1}(t_{J_1}, \ldots, t_{J_{k_1}}, x_{i_1})] \leq [t_{J_\alpha}] \vee [x_{i_1}] = [t_{J_\alpha \cup \{i_1\}}]$$

for all $1 \leq \alpha \leq k_1$ and

$$[t_J] < [m^{k_2-1}(t_{K_1}, \ldots, t_{K_{k_1}}, x_{i_2})] \leq [t_{K_\beta}] \vee [x_{i_2}] = [t_{K_\beta \cup \{i_2\}}]$$

for all $1 \leq \beta \leq k_2$. Now, by Proposition 2.2, we obtain that for all $1 \leq \alpha \leq k_1$ and $1 \leq \beta \leq k_2$,

$$J \subset J_\alpha \cup \{i_1\} \qquad \text{and} \qquad J \subset K_\beta \cup \{i_2\}.$$

Let $h$ be a homomorphism from $\operatorname{Term}(X)$ into the two-element $DN$-algebra $C_2 = \{0, 1\}$ such that: $h(x_i) = 0$ for all $i \in J$, and $h(x_i) = 1$ for all $i \notin J$. Then, $h(t_J) = 0$,

$$h(t_1) = h(m^{k_1-1}(t_{J_1}, \ldots, t_{J_{k_1}}, x_{i_1})) = \bigwedge_{1 \leq \alpha \leq k_1} (h(t_{J_\alpha}) \vee h(x_{i_1})) = 1$$

and

$$h(t_2) = h(m^{k_2-1}(t_{K_1}, \ldots, t_{K_{k_2}}, x_{i_2})) = \bigwedge_{1 \leq \beta \leq k_2} (h(t_{K_\beta}) \vee h(x_{i_2})) = 1.$$

Hence, $h(m(t_1, t_2, t_J)) = 1$. That is, we obtained that $h(t_J) = 0$ and $h(m(t_1, t_2, t_J)) = 1$, which is a contradiction from (2.3). Therefore, $[t_J]$ is an irreducible element of $\operatorname{Free}(\overline{X})$.

Let us show now that every irreducible element of $\mathrm{Free}(\overline{X})$ is of the form $[t_J]$ for some $J \in \mathcal{P}(\mathbf{n})^*$. Let $t \in \mathrm{Term}(X)$ be such that $[t] \in \mathrm{Irr}(\mathrm{Free}(\overline{X}))$. By Proposition 2.4, there exist non-empty $J_1, \dots, J_k \subseteq \mathbf{n}$ and $j \in \mathbf{n}$ such that $t \equiv m^{k-1}(t_{J_1}, \dots, t_{J_k}, x_j)$. Then,

$$[t] = [m^{k-1}(t_{J_1}, \dots, t_{J_k}, x_j)] = \bigwedge_{1 \leq \alpha \leq k} ([t_{J_\alpha}] \vee [x_i]) = \bigwedge_{1 \leq \alpha \leq k} [t_{J_\alpha \cup \{j\}}].$$

Thus, by the irreducibility of $[t]$, $[t] = [t_{J_\alpha \cup \{j\}}]$ for some $1 \leq \alpha \leq k$. $\square$

**Proposition 2.6.** *Let $J_1, \dots, J_k \in \mathcal{P}(\mathbf{n})^*$. Then, $[t_{J_1}] \wedge \cdots \wedge [t_{J_k}]$ exists in* $\mathrm{Free}(\overline{X})$ *if and only if $J_1 \cap \cdots \cap J_k \neq \emptyset$.*

*Proof.* ($\Rightarrow$) Assume that $[t_{J_1}] \wedge \cdots \wedge [t_{J_k}]$ exists in $\mathrm{Free}(\overline{X})$. By Proposition 2.4, let $L_1, \dots, L_p \subseteq \mathbf{n}$ be non-empty and $j \in \mathbf{n}$ such that $[t_{J_1}] \wedge \cdots \wedge [t_{J_k}] = [m^{p-1}(t_{L_1}, \dots, t_{L_p}, x_j]$. Thus, $[m^{p-1}(t_{L_1}, \dots, t_{L_p}, x_j] \leq [t_{J_\alpha}]$, for all $1 \leq \alpha \leq k$. Then, for every $1 \leq \alpha \leq k$, by the irreducibility of $[t_{J_\alpha}]$ we obtain that

$$[t_{L_{\beta_\alpha} \cup \{j\}}] = [t_{L_{\beta_\alpha}}] \vee [x_j] \leq [t_{J\alpha}]$$

for some $1 \leq \beta_\alpha \leq p$. Hence, for every $1 \leq \alpha \leq k$, there exists $1 \leq \beta_\alpha \leq p$ such that $L_{\beta_\alpha} \cup \{j\} \subseteq J_\alpha$. Therefore, $j \in J_1 \cap \cdots \cap J_k$.

($\Leftarrow$) Suppose $J_1 \cap \cdots \cap J_k \neq \emptyset$. Let $J := J_1 \cap \cdots \cap J_k$. Since $J \subseteq J_\alpha$ for all $1 \leq \alpha \leq k$, it follows that $[t_J] \leq [t_{J_\alpha}]$ for all $1 \leq \alpha \leq k$. Hence, there exists the infimum $[t_{J_1}] \wedge \cdots \wedge [t_{J_k}]$ in $\mathrm{Free}(\overline{X})$. $\square$

The last step in proving Theorem 2.1 is to apply the discrete representation to the finite $DN$-algebra $\mathrm{Free}(\overline{X})$.

Let $\gamma \colon \mathsf{Do}(\mathrm{Irr}(\mathrm{Free}(\overline{X}))) \to \{0,1\}$ be defined as follows: for every $U \in \mathsf{Do}(\mathrm{Irr}(\mathrm{Free}(\overline{X})))$,

$$\gamma(U) = 1 \iff \bigwedge \big(\mathrm{Irr}(\mathrm{Free}(\overline{X})) \setminus U\big) \text{ exists in } \mathrm{Free}(\overline{X}).$$

Let $\mathcal{A}_F := \{U \in \mathsf{Do}(\mathrm{Irr}(\mathrm{Free}(\overline{X}))) : \gamma(U) = 1\}$. Then, by Theorem 1.18, $\mathrm{Free}(\overline{X}) \cong \mathcal{A}_F$. Recall that the ternary operation $m_{\mathcal{A}_F}$ over $\mathcal{A}_F$ is defined by: $m_{\mathcal{A}_F}(U,V,W) = (U \cup W) \cap (V \cup W)$, for all $U, V, W \in \mathcal{A}_F$.

We are now in a position to prove Theorem 2.1.

*Proof of Theorem* 2.1. By Propositions 2.5 and 2.2, we obtain that the map $\Phi\colon \mathcal{P}(\mathbf{n})^* \to \mathrm{Irr}(\mathrm{Free}(\overline{X}))$ defined as follows: $\Phi(J) = [t_J]$ for all $J \in \mathcal{P}(\mathbf{n})^*$, is an order-isomorphism. Thus, the map $\widehat{\Phi}\colon \mathsf{Do}(\mathcal{P}(\mathbf{n})^*) \to \mathsf{Do}(\mathrm{Irr}(\mathrm{Free}(\overline{X})))$ defined by $\widehat{\Phi}(V) = \{\Phi(J) : J \in V\}$ is a lattice-isomorphism.

Let $\widehat{\gamma}\colon \mathsf{Do}(\mathcal{P}(\mathbf{n})^*) \to \{0,1\}$ be defined by: for every $V \in \mathsf{Do}(\mathcal{P}(\mathbf{n})^*)$, $\widehat{\gamma}(V) = \gamma(\widehat{\Phi}(V))$. Hence, $\langle \mathcal{A}, m_{\mathcal{A}}\rangle$ defined by

$$\mathcal{A} = \{V \in \mathsf{Do}(\mathcal{P}(\mathbf{n})^*) : \widehat{\gamma}(V) = 1\}$$

and

$$m_{\mathcal{A}}(V_1, V_2, V_3) = (V_1 \cup V_3) \cap (V_2 \cup V_3)$$

is a $DN$-algebra and $\mathcal{A}_F \cong \mathcal{A}$. Hence $\mathrm{Free}(\overline{X}) \cong \mathcal{A}$. Finally, let $V \in \mathsf{Do}(\mathcal{P}(\mathbf{n})^*)$. Then

$$\begin{aligned}
V \in \mathcal{A} &\iff \widehat{\gamma}(V) = 1 \\
&\iff \gamma(\widehat{\Phi}(V)) = 1 \\
&\iff \bigwedge \left(\mathrm{Irr}(\mathrm{Free}(\overline{X})) \setminus \widehat{\Phi}(V)\right) \text{ exists in } \mathrm{Free}(\overline{X}) \\
&\overset{\text{Prop. 2.6}}{\iff} \bigcap\{J \in \mathcal{P}(\mathbf{n})^* : J \notin V\} \neq \emptyset \\
&\iff \exists i \in \mathbf{n} \text{ such that } \mathbf{n} \setminus \{i\} \in V.
\end{aligned}$$

Hence, $\mathrm{Free}(\overline{X}) \cong \mathcal{A} = \{V \in \mathsf{Do}(\mathcal{P}(\mathbf{n})^*) : \exists i \in \mathbf{n} \text{ such that } \mathbf{n}\setminus\{i\} \in V\}$. $\square$

## 3 Free distributive lattice extension and amalgamation property

In this section, we establish a connection between the free distributive lattice extension of $DN$-algebras and the free $DN$-algebras. We show that the variety of $DN$-algebras has the amalgamation property but does not have the strong amalgamation property, whereas the variety of $N$-algebras does not possess the amalgamation property. Lastly, we show that in both the varieties of $N$-algebras and $DN$-algebras, not every epimorphism is surjective.

We begin by recalling the free distributive lattice extension of $DN$-algebras. First, we need the following terminology. Let $\langle A, m_A\rangle$ be a $N$-algebra and let $\langle L, \wedge, \vee\rangle$ be a lattice. We say that a map $h\colon A \to L$ is an *N-homomorphism* if for all $a_1, a_2, a_3 \in A$, $h(m_A(a_1, a_2, a_3)) = (h(a_1) \vee h(a_3)) \wedge (h(a_2) \vee h(a_3))$. It follows straightforwardly that a map $h\colon A \to L$ is an N-homomorphism if and only if $h$ preserves finite joins and existent finite meets. We say that $h\colon A \to L$ is an *N-embedding* if it is an injective N-homomorphism. Notice that every N-embedding $h\colon A \to L$ satisfies that: $a \leq b$ if and only if $h(a) \leq h(b)$.

**Definition 3.1.** Let $\langle A, m\rangle$ be a $DN$-algebra. A *free distributive lattice extension of* $A$ is a pair $\langle L, \alpha_L\rangle$ such that $L$ is a distributive lattice and $\alpha_L\colon A \to L$ is an N-embedding satisfying the following universal mapping property: if $D$ is a distributive lattice and $f\colon A \to D$ is an N-homomorphism, then there exists a unique lattice homomorphism $\widehat{f}\colon L \to D$ such that $\widehat{f} \circ \alpha_L = f$.

In [10], the authors show that a free distributive lattice extension exists for every $DN$-algebra. By categorical arguments, this extension is known to be unique up to isomorphism. Given a $DN$-algebra $A$, let us denote its free distributive lattice extension by $\mathrm{DL}_\alpha(A)$. It is also shown in [10] that $\mathrm{DL}_\alpha(A)$ is the distributive lattice of finitely generated filters. Subsequently, in [6], the authors established the existence of $\mathrm{DL}_\alpha(A)$ from a topological perspective.

Let $X$ be a set. Let us a denote by $\mathrm{Free}_{\mathbb{DN}}(X)$ the free $DN$-algebra generated by $X$. Let $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ be the free distributive lattice extension of the $DN$-algebra $\mathrm{Free}_{\mathbb{DN}}(X)$. Lastly, let $\mathrm{Free}_{\mathbb{DL}}(\alpha[X])$ be, up to isomorphism, the free distributive lattice generated by the set $\alpha[X]$.

**Theorem 3.2.** *For every set $X$,* $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X)) \cong \mathrm{Free}_{\mathbb{DL}}(\alpha[X])$.

*Proof.* Let us show that the distributive lattice $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ is freely generated by $\alpha[X]$. Since $\mathrm{Free}_{\mathbb{DN}}(X)$ is generated by the set $X$ as a $DN$-algebra, it follows by Propositions 2.4 and 1.13 that, for all $a \in \mathrm{Free}_{\mathbb{DN}}(X)$, there exist finite sets $X_1, \ldots, X_k \subseteq X$ and $x \in X$ such that

$$a = m^{k-1}(\bigvee X_1, \ldots, \bigvee X_k, x) = \bigwedge_{1\leq i\leq k} (\bigvee X_i \vee x) = \bigwedge_{1\leq i\leq k} \bigvee (X_i \cup \{x\})\,.$$

First, we prove that $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ is generated by $\alpha[X]$. It is known that $\mathrm{Free}_{\mathbb{DN}}(X)$ is finitely meet-dense on $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ (see [10]), that is, for every element $u \in \mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$, there are $a_1, \ldots, a_n \in \mathrm{Free}_{\mathbb{DN}}(X)$ such that $u = \alpha(a_1) \wedge \cdots \wedge \alpha(a_n)$. Let $u \in \mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ and $a_1, \ldots, a_n \in \mathrm{Free}_{\mathbb{DN}}(X)$ be such that $u = \alpha(a_1) \wedge \cdots \wedge \alpha(a_n)$. For every $i = 1, \ldots, n$, let finite $X_1^i, \ldots, X_{k_i}^i \subseteq X$ be such that

$$a_i = \bigwedge_{1 \leq j \leq k_i} \left( \bigvee_{1 \leq j \leq k_i} X_j \right)$$

Then, since $\alpha$ preserves finite joins and existent finite meets, it follows that for every $i = 1, \ldots, n$,

$$\alpha(a_i) = \bigwedge_{1 \leq j \leq k_i} \left( \bigvee_{1 \leq j \leq k_i} \alpha[X_j] \right)$$

Hence, $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ is generated by the set $\alpha[X]$. Now let us show that $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ has the universal mapping property for $\mathbb{DL}$ over $\alpha[X]$. Let $D$ be a distributive lattice and $f \colon \alpha[X] \to D$ a map. Let $f \circ \alpha \colon X \to D$ be the map $(f \circ \alpha)(x) = f(\alpha(x))$, for every $x \in X$. Since $D$ is in particular a $DN$-algebra and $\mathrm{Free}_{\mathbb{DN}}(X)$ is freely generated by the set $X$, it follows that there exists an N-homomorphism $\widehat{f \circ \alpha} \colon\ \colon \mathrm{Free}_{\mathbb{DN}}(X) \to D$ extending the map $f \circ \alpha$, that is, $\widehat{f \circ \alpha}(x) = (f \circ \alpha)(x)$, for all $x \in X$. Now, since $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ is the free distributive lattice extension of the $DN$-algebra $\mathrm{Free}_{\mathbb{DN}}(X)$, there exists a lattice homomorphism $\widehat{f} \colon \mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X)) \to D$ such that $\widehat{f \circ \alpha} = \widehat{f} \circ \alpha$. Let $x \in X$. Then, $\widehat{f}(\alpha(x)) = \widehat{f \circ \alpha}(x) = (f \circ \alpha)(x) = f(\alpha(x))$. Hence, $\widehat{f}$ is an extension of $f$. We have proved that the distributive lattice $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X))$ is freely generated by $\alpha[X]$. Therefore, $\mathrm{DL}_\alpha(\mathrm{Free}_{\mathbb{DN}}(X)) \cong \mathrm{Free}_{\mathbb{DL}}(\alpha[X])$. $\square$

We next turn our attention to the amalgamation property. First, we present a useful remark.

**Remark 3.3.** Let $A$ and $B$ be $DN$-algebras, and let $\mathrm{DL}_{\alpha_A}(A)$ and $\mathrm{DL}_{\alpha_B}(B)$ its free distributive lattice extensions, respectively. If $h \colon A \to B$ is an embedding, then it follows that there exists a unique lattice embedding $\widehat{h} \colon \mathrm{DL}_{\alpha_A}(A) \to \mathrm{DL}_{\alpha_B}(B)$ such that $\alpha_B \circ h = \widehat{h} \circ \alpha_A$.

**Proposition 3.4.** *The variety $\mathbb{DN}$ of $DN$-algebras has the amalgamation property, but it has not the strong amalgamation property.*

$$\begin{tikzcd}
& B_1 \arrow[rr, hook, "\alpha_{B_1}"] & & \mathrm{DL}(B_1) \arrow[dr, hook, dashed, "\beta_1"] & \\
A \arrow[ur, hook, "h_1"] \arrow[rr, hook, "\alpha_A"] \arrow[dr, hook, "h_2"'] & & \mathrm{DL}(A) \arrow[ur, hook, dashed, "\widehat{h_1}"] \arrow[dr, hook, dashed, "\widehat{h_2}"'] & & D \\
& B_2 \arrow[rr, hook, "\alpha_{B_2}"'] & & \mathrm{DL}(B_2) \arrow[ur, hook, dashed, "\beta_2"'] &
\end{tikzcd}$$

Figure 1

*Proof.* Let $A, B_1, B_2$ be $DN$-algebras, and let $h_1\colon A \to B_1$ and $h_2\colon A \to B_2$ be embeddings. By the previous remark, and since the variety of distributive lattices satisfies the amalgamation property, we obtain the commutative diagram shown in Figure 1, where $D$ is a distributive lattice and $\widehat{h}_1, \widehat{h}_2, \beta_1, \beta_2$ are lattice embeddings. In particular, because $D$ is a $DN$-algebra, the compositions $\gamma_1 := \beta_1 \circ \alpha_{B_1}\colon B_1 \to D$ and $\gamma_2 := \beta_2 \circ \alpha_{B_2}\colon B_2 \to D$ are also embeddings. The commutativity of the diagram then implies that $(\gamma_1 \circ h_1)(a) = (\gamma_2 \circ h_2)(a)$ for all $a \in A$. This establishes the amalgamation property for $\mathbb{DN}$.

We now show that the variety of $DN$-algebras does not have the strong amalgamation property. The argument to show this is similar to the one given in the distributive lattice setting (see, for instance [13]). Let $A = \{0, a, b, 1\}$ and $B = \{0, a, c, 1\}$ be two four-element Boolean lattices with 0 as the bottom element and 1 as the top element. Then, $S = A \cap B = \{0, a, 1\}$ is a three-element chain. We consider $A$, $B$, and $S$ as $DN$-algebras (since they are distributive lattices). Suppose that $C$ is a $DN$-algebra containing $A$ and $B$ as subalgebras, with $S$ being a subalgebra of both $A$ and $B$. Notice that the interval $[0, 1]_C$ is a bounded distributive lattice such that $a, b, c \in [0, 1]_C$, where both $b$ and $c$ are complements of $a$ in $[0, 1]_C$. Thus, it follows that $b = c$, which implies $S \subsetneq A \cap B$ in $C$. Therefore, the variety of $DN$-algebras does not satisfy the strong amalgamation property. $\square$

We now focus on showing that the amalgamation property fails in the variety $\mathbb{NA}$ of $N$-algebras. To this end, let us consider the presentation of Dean's solution [12] to the word problem for lattices given by Lakser [25].

We aim to be as self-contained as possible without becoming overly dense. Consequently, the finer details, which can be found in [25], are left to the reader.

Let $Q$ be a poset. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets of non-empty finite subsets of $Q$ such that $\bigvee X$ and $\bigwedge Y$ exist in $Q$ for each $X \in \mathcal{X}$ and $I \in \mathcal{Y}$. If $L$ is a lattice, a map $\varphi\colon Q \to L$ is called a $(\mathcal{X}, \mathcal{Y})$-*morphism* if $\varphi$ is an order-preserving map such that $\varphi\left(\bigvee X\right) = \bigvee \varphi(X)$ and $\varphi\left(\bigwedge Y\right) = \bigwedge \varphi(Y)$ for each $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. If, in addition, $\varphi$ is an order-embedding, then we say that $\varphi$ is a $(\mathcal{X}, \mathcal{Y})$-*embedding*.

**Theorem 3.5.** ([12]) *Let $Q$ be a poset and $\mathcal{X}$ and $\mathcal{Y}$ as before. Then, it follows that there exist a lattice* $\mathrm{FL}(Q, \mathcal{X}, \mathcal{Y})$ *and a $(\mathcal{X}, \mathcal{Y})$-embedding $\eta\colon Q \to$* $\mathrm{FL}(Q, \mathcal{X}, \mathcal{Y})$ *such that* $\mathrm{FL}(Q, \mathcal{X}, \mathcal{Y})$ *is generated as a lattice by $\eta(Q)$ and the universal mapping property holds: for each lattice $M$ and $(\mathcal{X}, \mathcal{Y})$-morphism $\varphi\colon Q \to M$, there is a lattice homomorphism $\widehat{\varphi}\colon$* $\mathrm{FL}(Q, \mathcal{X}, \mathcal{Y}) \to M$ *with $\varphi = \widehat{\varphi} \circ \eta$.*

By the usual abstract general nonsense, the pair $(\eta, \mathrm{FL}(Q, \mathcal{X}, \mathcal{Y}))$ is unique up to isomorphism. Next, we recall the characterization of $\mathrm{FL}(Q, \mathcal{X}, \mathcal{Y})$ given in [25]. This characterization allows us to show that if $\varphi$ in the previous theorem is a $(\mathcal{X}, \mathcal{Y})$-embedding, $\widehat{\varphi}$ is not necessarily a lattice embedding. We will use this fact to show that the variety of $N$-algebras does not satisfy the amalgamation property.

Let $Q$, $\mathcal{X}$ and $\mathcal{Y}$ as before. A subset (possibly empty) $I$ of $Q$ is said to be a $\mathcal{X}$-*ideal* if $I$ is a down-set and $X \subseteq I$ implies $\bigvee X \in I$ for each $X \in \mathcal{X}$. Dually, a subset (possibly empty) $F$ of $Q$ is said to be a $\mathcal{Y}$-*filter* if $F$ is an up-set and $Y \subseteq F$ implies $\bigwedge Y \in F$, for each $Y \in \mathcal{Y}$. Notice that the principal down-sets $(a]$ are $\mathcal{X}$-ideals and the principal up-sets $[a)$ are $\mathcal{Y}$-filters, for all $a \in Q$. Moreover, it is clear that the arbitrary intersections of $\mathcal{X}$-ideals and $\mathcal{Y}$-filters are, respectively, $\mathcal{X}$-ideals and $\mathcal{Y}$-filters. Hence, the set of all $\mathcal{X}$-ideals and the set of all $\mathcal{Y}$-filters are complete lattices.

Let $\mathrm{Term}(Q)$ be the term algebra over the algebraic language $\mathcal{L} = \{\wedge, \vee\}$ of type (2,2) with all variables being elements of $Q$. By recursion on the complexity of the term $t \in \mathrm{Term}(Q)$, we associate with each $t \in \mathrm{Term}(Q)$ an $\mathcal{X}$-ideal $\underline{t}$ and a $\mathcal{Y}$-filter $\overline{t}$:

1. If $t = q \in Q$, then $\underline{q} = (q]_Q$, and $\overline{q} = [q)_Q$.

2. $\underline{t \vee s} = \underline{t} \sqcup \underline{s}$, where $\sqcup$ is the join in the lattice of all $\mathcal{X}$-ideals, and $\overline{t \vee s} = \overline{t} \cap \overline{s}$.
3. $\underline{t \wedge s} = \underline{t} \cap \underline{s}$, and $\overline{t \wedge s} = \overline{t} \sqcup \overline{s}$, where $\sqcup$ is the join in the lattice of all $\mathcal{Y}$-filters.

We now define a binary relation $\preceq$ on $\mathrm{Term}(Q)$ as follows: for all $t, s \in \mathrm{Term}(Q)$, $t \preceq s$ if and only if it can be derived from the following rules:

1. $t = t_0 \wedge t_1$ with $t_0 \preceq s$ or $t_1 \preceq s$;
2. $t = t_0 \vee t_1$ with $t_0 \preceq s$ and $t_1 \preceq s$;
3. $s = s_0 \wedge s_1$ with $t \preceq s_0$ and $t \preceq s_1$;
4. $s = s_0 \vee s_1$ with $t \preceq s_0$ or $t \preceq s_1$;
5. $\overline{t} \cap \underline{s} \neq \emptyset$.

**Proposition 3.6.** ([25]) *The relation $\preceq$ on* $\mathrm{Term}(Q)$ *is a quasi-ordering extending the partial order $\leq$ in $Q$.*

Let $\equiv$ be the equivalence relation on $\mathrm{Term}(Q)$ defined as follows: $t \equiv s$ if and only if $t \preceq s$ and $s \preceq t$. We denote the equivalent class of any $t \in \mathrm{Term}(Q)$ by $\langle t \rangle$. Thus, there is a partial ordering $\sqsubseteq$ on $\mathrm{Term}(Q)/{\equiv}$ by setting $\langle t \rangle \sqsubseteq \langle s \rangle$ if and only if $t \preceq s$.

**Theorem 3.7.** ([25]) *From the above,* $\mathrm{Term}(Q)/{\equiv}$ *is a lattice, whereby $\langle t \rangle \vee \langle s \rangle = \langle t \vee s \rangle$ and $\langle t \rangle \wedge \langle s \rangle = \langle t \wedge s \rangle$, for all $t, s \in$* $\mathrm{Term}(Q)$*. The map $\eta \colon Q \to$* $\mathrm{Term}(Q)/{\equiv}$ *defined by $\eta(q) = \langle q \rangle$ is a $(\mathcal{X}, \mathcal{Y})$-morphism. Hence,* $\mathrm{Term}(Q)/{\equiv}$*, along with the mapping $\eta \colon Q \to$* $\mathrm{Term}(Q)/{\equiv}$*, is the free lattice* $\mathrm{FL}(Q, \mathcal{X}, \mathcal{Y})$.

Now, let us apply the previous theorem to the particular case where $Q$ is an $N$-algebra, $\mathcal{X}$ is the set of all non-empty finite subsets of $Q$, and $\mathcal{Y}$ is the set of all non-empty finite subsets $Y$ of $Q$ such that $\bigwedge Y$ exists in $Q$. Notice that, in this setting, a map $\varphi \colon Q \to L$, where $L$ is a lattice, is a $(\mathcal{X}, \mathcal{Y})$-morphism ($(\mathcal{X}, \mathcal{Y})$-embedding) if and only if $\varphi$ is an $N$-homomorphism ($N$-embedding).

We now show that if the map $\varphi$ in Theorem 3.5 is an $N$-embedding, then $\widehat{\varphi}$ is not necessarily a lattice embedding.

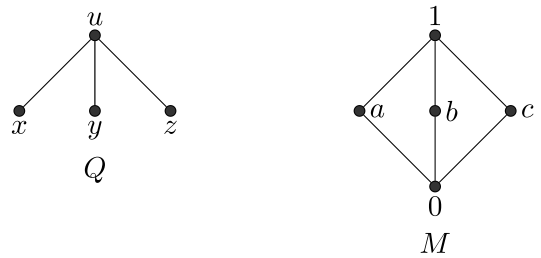


Figure 2: The $N$-algebra $Q$ and lattice $M$ of Example 3.8

$$\begin{tikzcd}
Q \arrow[rr, hook, "\varphi"] \arrow[d, hook, "\imath"] & & M \\
\mathrm{Term}(Q) \arrow[urr, "f"] \arrow[rr, "\pi"'] & & \mathrm{Term}(Q)/{\equiv} \arrow[u, dashed, "\widehat{\varphi}"']
\end{tikzcd}$$

Figure 3: $\widehat{\varphi}$ is not necessarily an embedding

**Example 3.8.** Let $Q$ be the $N$-algebra given in Figure 2. Let $\mathrm{Term}(Q)$ be the term algebra of type $\mathcal{L} = \{\wedge, \vee\}$ with variables in $Q$ and $\mathrm{Term}(Q)/{\equiv}$ the free lattice. Let $\imath\colon Q \to \mathrm{Term}(Q)$ be the identity map. Let $M$ be the lattice given in Figure 2, and $\varphi\colon Q \to M$ defined as follows: $\varphi(u) = 1$, $\varphi(x) = a$, $\varphi(y) = b$ and $\varphi(z) = c$. Notice that $\varphi$ is an N-embedding. Let now $f\colon \mathrm{Term}(Q) \to M$ be the extension map of $\varphi$, that is, $f(q) = \varphi(q)$, for each $q \in Q$, and $f(t \vee s) = f(t) \vee f(s)$ and $f(t \wedge s) = f(t) \wedge f(s)$, for all $t, s \in \mathrm{Term}(Q)$. Then, by Theorem 3.5 and following the proof given in [25, Theorem], the lattice homomorphism $\widehat{\varphi}\colon \mathrm{Term}(Q)/{\equiv} \to M$ is defined by $\widehat{\varphi}(\langle t\rangle) = f(t)$. Let $\pi\colon \mathrm{Term}(Q) \to \mathrm{Term}(Q)/{\equiv}$ be the canonical projection: $\pi(t) = \langle t\rangle$. Thus, $\eta = \pi \circ \imath\colon Q \to \mathrm{Term}(Q)/{\equiv}$. See the commutative diagram in Figure 3. Lastly, we show that $\widehat{\varphi}$ is not an embedding. Let $t := x \wedge y \in \mathrm{Term}(Q)$ and $z \in Q \subseteq \mathrm{Term}(Q)$. Thus, $\widehat{\varphi}(\langle t\rangle) = f(t) = f(x) \wedge f(y) = \varphi(x) \wedge \varphi(y) = a \wedge b = 0$, and $\widehat{\varphi}(\langle z\rangle) = f(z) = \varphi(z) = c$. Then $\widehat{\varphi}(\langle t\rangle) \leq \widehat{\varphi}(\langle z\rangle)$ (in $M$). However, by the definition of the quasi-ordering $\preceq$ in $\mathrm{Term}(Q)$, it follows that $t \npreceq z$. Then $\langle t\rangle \not\sqsubseteq \langle z\rangle$. Hence, $\widehat{\varphi}$ is not an embedding.

$$\begin{tikzcd}
Term(Q) \arrow[drr, "f"] & & & & \\
& & M \arrow[drr, dashed, hook, "\beta_1"] & & \\
Q \arrow[uu, "\imath"] \arrow[urr, hook, "\varphi"] \arrow[drr, hook, "\eta"'] & & & & A \\
& & \mathrm{FL}(Q) \arrow[uu, dashed, "\widehat{\varphi}"'] \arrow[urr, dashed, hook, "\beta_2"'] & &
\end{tikzcd}$$

Figure 4: Commuting diagram

**Proposition 3.9.** *The variety of $N$-algebras does not satisfy the amalgamation property.*

*Proof.* Suppose, for the sake of contradiction, that the variety of $N$-algebras has the amalgamation property. Let $Q$ be an $N$-algebra, $M$ a lattice and $\varphi\colon Q \to M$ be an N-embedding. By Theorem 3.5, there exists a unique lattice homomorphism $\widehat{\varphi}\colon \mathrm{FL}(Q) \to M$ such that $\varphi = \widehat{\varphi} \circ \eta$, where $\mathrm{FL}(Q) = \mathrm{FL}(Q, \mathcal{X}, \mathcal{Y})$ with $\mathcal{X}$ the set of finite non-empty subsets of $Q$ and $\mathcal{Y}$ the set of finite non-empty subsets $Y$ of $Q$ such that $\bigwedge Y$ exists in $Q$. Now let us show that $\widehat{\varphi}$ is an embedding.

To prove this, let us consider that $\mathrm{FL}(Q) = \mathrm{Term}(Q)/{\equiv}$, $\eta\colon Q \to \mathrm{FL}(Q)$ is given by $\eta(q) = \langle q \rangle$, $f\colon \mathrm{Term}(Q) \to M$ is the lattice extension of $\varphi$, and $\widehat{\varphi}(\langle t \rangle) = f(t)$, for all $t \in \mathrm{Term}(Q)$, see Figure 4.

By considering the lattices $M$ and $\mathrm{FL}(Q)$ as $N$-algebras, we see that $\varphi\colon Q \to M$ and $\eta\colon Q \to \mathrm{FL}(Q)$ are embeddings. Since we are assuming the amalgamation property for the variety of $N$-algebras, there exists an $N$-algebra $A$ and embeddings $\beta_1\colon M \to A$ and $\beta_2\colon \mathrm{FL}(Q) \to A$ such that $\beta_1 \circ \varphi = \beta_2 \circ \eta$, as shown in the commutative diagram of Figure 4.

<u>Claim:</u> $\beta_1 \circ \widehat{\varphi} = \beta_2$.

*Proof.* We proceed by induction on the complexity of the terms.

- If $t = q \in Q$, then $\beta_1(\widehat{\varphi}(\langle q \rangle)) = \beta_1(f(q)) = \beta_1(\varphi(q)) = (\beta_2 \circ \eta)(q) = \beta_2(\langle q \rangle)$.

- Let $t = t_1 * t_2$, with $* \in \{\wedge, \vee\}$. By induction hypothesis, and using that $\widehat{\varphi}$, $\beta_1$ and $\beta_2$ preserve finite joins and meets, we obtain that

$$\begin{aligned}(\beta_1 \circ \widehat{\varphi})(\langle t_1 * t_2 \rangle) &= (\beta_1 \circ \widehat{\varphi})(\langle t_1 \rangle * \langle t_2 \rangle) \\ &= \beta_1(\widehat{\varphi}(\langle t_1 \rangle)) * \beta_1(\widehat{\varphi}(\langle t_2 \rangle)) \\ &= \beta_2(\langle t_1 \rangle) * \beta_2(\langle t_2 \rangle) \\ &= \beta_2(\langle t_1 * t_2 \rangle). \end{aligned} \qquad \square$$

Now let $t, s \in \mathrm{Term}(Q)$ and suppose that $\widehat{\varphi}(\langle t \rangle) = \widehat{\varphi}(\langle s \rangle)$. Then, by the previous Claim, it follows that

$$\begin{aligned}\widehat{\varphi}(\langle t \rangle) = \widehat{\varphi}(\langle s \rangle) &\implies \beta_1(\widehat{\varphi}(\langle t \rangle)) = \beta_1(\widehat{\varphi}(\langle s \rangle)) \implies \\ &\implies \beta_2(\langle t \rangle) = \beta_2(\langle s \rangle) \implies \langle t \rangle = \langle s \rangle.\end{aligned}$$

Hence, we have proved that $\widehat{\varphi}\colon \mathrm{FL}(Q) \to M$ is a lattice embedding for each an $N$-embedding $\varphi\colon Q \to M$, which contradicts the results shown in Example 3.8. $\square$

We conclude by showing that, in the algebraic categories of $N$-algebras and $DN$-algebras, not every epimorphism is surjective.

**Example 3.10.** Let $A$ and $B$ be the $DN$-algebras shown in Figure 5, and let $h\colon A \to B$ be the homomorphism defined therein. Clearly, $h$ is not surjective. Let us show that $h$ is an epimorphism in the algebraic category of $DN$-algebras. Let $C$ be a $DN$-algebra and $g_1, g_2\colon B \to C$ two homomorphisms such that $g_1 \circ h = g_2 \circ h$. We need to prove that $g_1 = g_2$. It is enough to prove that $g_1(y) = g_2(y)$. Since $0 = x \wedge y$, it follows that $g_2(0) = g_1(0) = g_1(x \wedge y) = g_1(x) \wedge g_1(y)$. Then

$$\begin{aligned}g_2(y) &= g_2(0) \vee g_2(y) = (g_1(x) \wedge g_1(y)) \vee g_2(y) = (g_2(x) \wedge g_1(y)) \vee g_2(y) \\ &= (g_2(x) \vee g_2(y)) \wedge (g_1(y) \vee g_2(y)) \\ &= g_2(1) \wedge (g_1(y) \vee g_2(y)) \\ &= (g_1(y) \vee g_2(y)).\end{aligned}$$

Hence, $g_1(y) \leq g_2(y)$. Analogously, $g_2(y) \leq g_1(y)$. Thus, $g_1 = g_2$. Therefore, $h$ is an epimorphism.

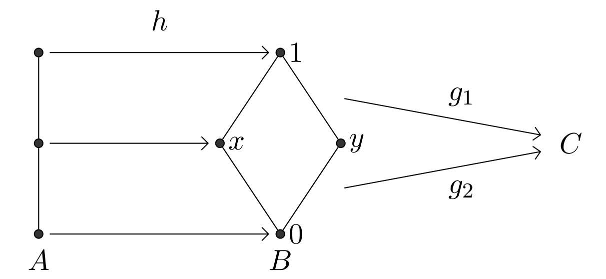


Figure 5: A non-surjective epimorphism in $\mathbb{DN}$

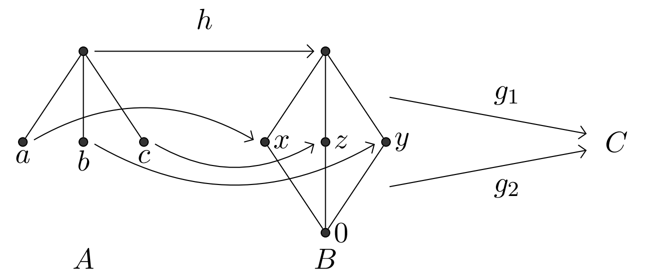


Figure 6: A non-surjective epimorphism in $\mathbb{NA}$

**Example 3.11.** Consider the $N$-algebras $A$ and $B$ and the homomorphism $h\colon A \to B$ shown in Figure 6. Clearly, $h$ is not surjective. We shall now demonstrate that $h$ is an epimorphism in the algebraic category of $N$-algebras. Let $C$ be an $N$-algebra and $g_1, g_2\colon B \to C$ homomorphisms such that $g_1 \circ h = g_2 \circ h$. To complete the proof, it is enough to show that $g_1(0) = g_2(0)$. Since $g_1$ and $g_2$ preserve finite meets whenever they exist, we have that

$$\begin{aligned} g_1(0) &= g_1(x \wedge y) = g_1(h(a) \wedge h(b)) = g_1(h(a)) \wedge g_1(h(b)) \\ &= g_2(h(a)) \wedge g_2(h(b)) = g_2(x) \wedge g_2(y) = g_2(x \wedge y) \\ &= g_2(0). \end{aligned}$$

Hence, $h$ is an epimorphism.

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***Luciano Javier González*** *Universidad Nacional de La Pampa, Facultad de Ciencias Exactas, Naturales, Santa Rosa, Argentina*

*Email: lucianogonzalez@exactas.unlpam.edu.ar*