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Outer measures on σ-frames and the structure of measurable elements

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Abstract. We present a pointfree generalization of outer measures by replacing $\mathcal{P}(X)$ with a bounded distributive lattice L, studying functions $\mu : L \to [0, \infty]$ satisfying lattice-theoretic versions of outer measure axioms (null-preserving, monotone, σ-subadditive). For such an outer measure, we introduce μ-measurable elements via a generalized Carathéodory condition and show that they form a sublattice closed under complements. When L is a Boolean σ-frame, the collection $\mathcal{M}_\mu$ of measurable elements is itself a Boolean σ-frame; counterexamples demonstrate the necessity of the Boolean hypothesis for closure under countable joins. Moreover, for any σ-frame L (without requiring the Boolean condition), the restriction of μ to $\mathcal{M}_\mu$ is σ-additive. We also examine the categorical behavior of the assignment $\mu \mapsto \mathcal{M}_\mu$, which yields a natural transformation between suitable functors, and pose an open question: whether every Boolean σ-subframe of L arises as $\mathcal{M}_\mu$ for some outer measure μ. This work extends classical measure theory to pointfree settings, revealing fundamental connections between order-theoretic, measure-theoretic, and categorical structures.

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1 Introduction

It is well known that an outer measure is classically defined as a function $\mu : \mathcal{P}(X) \to [0, \infty]$ satisfying countable subadditivity, monotonicity, and null emptiness. In this paper, we introduce a *pointfree* generalization of outer measures by replacing the power set $\mathcal{P}(X)$ with an arbitrary σ-frame L. Specifically, we study functions $\mu : L \to [0, \infty]$ satisfying appropriate lattice-theoretic analogues of the classical outer measure axioms. Recall from [20] that a measure on a σ-complete lattice L (L has countable joins) is defined as a map $\mu\colon L \to [0, \infty]$ satisfying:

1. $\mu(\perp_L) = 0$,
2. monotonicity: $x \leq y \Rightarrow \mu(x) \leq \mu(y)$,
3. modularity: $\mu(x) + \mu(y) = \mu(x \vee y) + \mu(x \wedge y)$,
4. σ-continuity: $\mu\big(\bigvee_{i\in\mathbb{N}} x_i\big) = \sup_{i\in\mathbb{N}} \mu(x_i)$ for every ascending chain $(x_i)_{i\in\mathbb{N}}$ in L.

In the literature, these functions are typically called σ-continuous valuations [11, 20]. Building on this definition, [8] introduces pointfree Lebesgue integration on σ-locales by considering measures on $\mathcal{S}(L)$, where $\mathcal{S}(L)$ denotes the coframe of all σ-sublocales of L; note that $\mathcal{S}(L)$ is a σ-complete lattice. It is important to emphasize that, within pointfree topology, $\mathcal{S}(L)$ serves as the natural counterpart to $\mathcal{P}(X)$ in classical measure theory. For more applications of the coframe $\mathcal{S}(L)$ to real-function analogues, we refer to [7, 13, 14].

In this article, building upon Definition 3.1 and inspired by the measure-theoretic framework of [20], we take a general bounded distributive lattice L as our starting point, replacing the classical powerset $\mathcal{P}(X)$. We define an outer measure simply as a function $\mu\colon L \to [0, \infty]$ satisfying the three basic axioms of null-preservation, monotonicity, and σ-subadditivity. Additional structure, such as requiring L to be a σ-frame or even a Boolean σ-frame, is then introduced only when necessary to obtain stronger results. This stepwise approach makes transparent the precise role each condition plays, thereby providing a pointfree generalization of the classical outer measure theory developed in [2, 10, 15].

Our main contributions are:

- A systematic definition of *μ-measurable elements* in a bounded distributive lattice L, generalizing the classical Carathéodory measurability condition. We prove (Lemma 3.7) that the collection of all measurable elements forms a sublattice closed under complements.
- A fundamental result (Theorem 3.9) showing that the restriction of μ to the σ-frame $\mathcal{M}_\mu$ of measurable elements is σ-additive.
- When L is a Boolean σ-frame, we establish (Proposition 3.12) that the measurable elements form a Boolean σ-frame.
- A remark (Remark 3.13) demonstrating, through counterexamples, that the Boolean hypothesis in Proposition 3.12 is essential for $\mathcal{M}_\mu$ to be closed under countable joins.
- Question 3.15 asks whether every Boolean σ-subframe $H \subseteq L$ arises as $\mathcal{M}_\mu$ for some outer measure μ on L, a problem that remains open.
- The categorical behavior of the function $\omega_L \colon \mathrm{Out}(L) \to \mathbf{Boo}(L)$ (defined by $\mu \mapsto \mathcal{M}_\mu$) is examined in Remark 3.18.

This work bridges order-theoretic and measure-theoretic perspectives, extending classical results while maintaining constructive rigor. Our approach reveals new connections between lattice theory and measurable structures.

2 Preliminaries

A *lattice* is a partially ordered set $(L, \leq)$ in which every pair of elements has both a supremum (join) and an infimum (meet). A lattice L is called *distributive* if it satisfies the following distributive laws:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in L$. Furthermore, in this paper by a *bounded lattice*, we mean a lattice that contains both a top element (greatest element) and a bottom element (least element). We denote the top element and the bottom element of L by $\top$ and $\bot$ respectively. An element $a \in L$ is called *complemented* if there exists $b \in L$ such that $a \wedge b = \bot$ and $a \vee b = \top$. In this case, b is called the complement of a and is usually denoted by a^c.

For further reading on distributive lattices and related concepts, we recommend the following fundamental references: Grätzer's comprehensive treatise [12] provides the foundational theory of lattices, while Davey and Priestley [9] offers an accessible introduction to lattice theory and ordered structures. For specialized treatment of distributive lattices, the monograph by Balbes and Dwinger [4] is particularly valuable. Those interested in the topological aspects may consult Johnstone's work [16] on Stone spaces and their connection to distributive lattices.

We recall some definitions and results from the literature on frames and σ-frames. The foundation of this paper is the σ-frames. For further information, see [18] on frame-theoretic concepts and [5, 6] on σ-frames. Also, we briefly review outer measures; for more details, see [2].

A *frame* (resp. σ-frame) is a complete (resp. σ-complete) lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$ (resp. countable $S \subseteq L$). Note that σ-frames are a generalization of frames.

An **outer measure** on a set X is a function $\mu : \mathcal{P}(X) \to [0, \infty]$ satisfying:

- $\mu(\emptyset) = 0$,
- $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$,
- $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$ for any countable collection $\{A_n\}$.

A set $E \subseteq X$ is **measurable** with respect to μ if it satisfies the Carathéodory condition:

$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c) \quad \text{for all } A \subseteq X.$$

For complete details, we refer the reader to [2].

3 Outer measure on a lattice

In [20] (Definition 2.1), the notion of a measure on a σ-complete lattice is introduced. Moreover, Definition 2.2 of the same work establishes the property of *countable additivity* for $[0,\infty]$-valued functions on lattices. In our current work, while drawing upon classical terminology from [2] and maintaining maximal consistency with [20], we introduce the following definition:

Definition 3.1. Let L be a bounded distributive lattice and $\mathcal{S} \subseteq L$ be a subset of L. Let $\mu : \mathcal{S} \to [0,\infty]$ be a function. We call μ:

(i) *Null-preserving*: $\perp \in \mathcal{S}$ and $\mu(\perp) = 0$.

(ii) *Monotone*: If $a \leq b$ then $\mu(a) \leq \mu(b)$.

(iii) σ-*Subadditive*: For every sequence $\{a_n\}_{n\in\mathbb{N}}$ in L, if $\bigvee\limits_{n=1}^{\infty} a_n$ exists in $\mathcal{S}$, then $\mu\Big(\bigvee\limits_{n=1}^{\infty} a_n\Big) \leq \sum\limits_{n=1}^{\infty} \mu(a_n)$.

(iv) σ-*Additive* if for every sequence $\{a_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements (i.e., $a_i \wedge a_j = 0$ for $i \neq j$) such that $\bigvee\limits_{n=1}^{\infty} a_n$ exists in $\mathcal{S}$,

$$\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n).$$

(v) *Subadditive*: For every $a, b \in \mathcal{S}$, if $a \vee b \in \mathcal{S}$, $\mu(a \vee b) \leq \mu(a) + \mu(b)$.

(vi) *Subtractive*: For every $a, b \in \mathcal{S}$ with $b \leq a$ where b is complemented and $\mu(b) < \infty$, if $a \wedge b^c \in \mathcal{S}$, then

$$\mu(a \wedge b^c) = \mu(a) - \mu(b).$$

(vii) *Additive*: For every disjoint $a, b \in \mathcal{S}$, if $a \vee b \in \mathcal{S}$, then $\mu(a \vee b) = \mu(a) + \mu(b)$.

(viii) *Modularity*: For every $a, b \in \mathcal{S}$, if $a \vee b, a \wedge b \in \mathcal{S}$, then $\mu(a) + \mu(b) = \mu(a \vee b) + \mu(a \wedge b)$.

A function $\mu\colon \mathcal{S} \to [0,\infty]$ is called an *outer measure* if it satisfies properties (i)-(iii) (Null-preserving, Monotone, and σ-subadditive).

The following proposition establishes the relationship between the properties of modularity, additivity, and subadditivity.

Proposition 3.2. *Let L be a bounded distributive lattice, $\mathcal{S}$ be a sublattice of L and $\mu\colon \mathcal{S} \to [0,\infty]$ be a null-preserving function.*

(i) *Modularity implies both additivity and subadditivity.*

(ii) *Additivity implies subtractivity.*

(iii) *If μ is additive and monotone, and if $a, b \in \mathcal{S}$ are complemented in L with $(a \wedge b)^c \in \mathcal{S}$, then*

$$\mu(a) + \mu(b) = \mu(a \vee b) + \mu(a \wedge b).$$

(iv) *If $\mathcal{S} = BL$ (the Boolean subalgebra of all complemented elements of L), and if μ is monotone, then μ is modular.*

Proof. (i) To prove subadditivity:

$$\mu(a \vee b) \leq \mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b).$$

For additivity, assume $a \wedge b = \perp$. Then we have:

$$\mu(a \vee b) + 0 = \mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b).$$

(ii) Let μ be an additive function and $a, b \in \mathcal{S}$ with $b \leq a$ where b is complemented and $\mu(b) < \infty$. Since $a \wedge b^c$ and b are disjoint, by the additivity of μ and distributivity of L:

$$\mu(a) = \mu(a \vee b) = \mu((a \wedge b^c) \vee b) = \mu(a \wedge b^c) + \mu(b),$$

Given that $\mu(b) < \infty$, we deduce that:

$$\mu(a \wedge b^c) = \mu(a) - \mu(b).$$

To prove (iii), first note that if $\mu(a \wedge b) = \infty$, then by the monotonicity of μ, we have $\mu(a) = \infty$, $\mu(b) = \infty$, and $\mu(a \vee b) = \infty$. Consequently, the equation $\mu(a) + \mu(b) = \mu(a \vee b) + \mu(a \wedge b)$ holds trivially, as both sides are infinite. So, we can assume that $\mu(a \wedge b) < \infty$. Hence, by the distributivity of L, we have the decomposition:

$$a \vee b = (a \wedge (a \wedge b)^c) \vee (b \wedge (a \wedge b)^c) \vee (a \wedge b).$$

Since the elements $(a \wedge (a \wedge b)^c)$, $(b \wedge (a \wedge b)^c)$, and $(a \wedge b)$ are pairwise disjoint, by additivity we have:

$$\begin{aligned} \mu(a \vee b) &= \mu\left(a \wedge (a \wedge b)^c\right) + \mu\left(b \wedge (a \wedge b)^c\right) + \mu(a \wedge b) \\ &= \mu(a) - \mu(a \wedge b) + \mu(b) - \mu(a \wedge b) + \mu(a \wedge b) \\ &= \mu(a) + \mu(b) - \mu(a \wedge b), \end{aligned}$$

and consequently, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$, which establishes the required modularity property. (iv) This follows immediately from (iii) above. $\square$

Definition 3.3. Let L be a bounded distributive lattice. A complemented element $m \in L$ is called *μ-measurable* if for every $a \in L$,

$$\mu(a) = \mu(a \wedge m) + \mu(a \wedge m^c).$$

Lemma 3.4. *Let L be a bounded distributive lattice and $\mu\colon L \to [0, \infty]$ be an outer measure. A complemented element $m \in L$ is μ-measurable if and only if for every $a \in L$,*

$$\mu(a) \geq \mu(a \wedge m) + \mu(a \wedge m^c).$$

Proof. The forward implication ($\Rightarrow$) follows directly from the definition of a measurable element. Therefore, it suffices to prove the converse implication ($\Leftarrow$). Since L is distributive, we have the decomposition

$$a = (a \wedge m) \vee (a \wedge m^c).$$

By the subadditivity of μ, it follows that

$$\mu(a) = \mu\big((a \wedge m) \vee (a \wedge m^c)\big) \leq \mu(a \wedge m) + \mu(a \wedge m^c) \leq \mu(a).$$

Consequently, we obtain the equality

$$\mu(a) = \mu(a \wedge m) + \mu(a \wedge m^c).$$

This establishes the μ-measurability of m. $\square$

Definition 3.5. A complemented element $a \in L$ is said to be a *null element* if $\mu(a) = 0$.

Proposition 3.6. *Let L be a bounded distributive lattice and $\mu\colon L \to [0, \infty]$ be an outer measure. Every null element of L is μ-measurable.*

Proof. Let $m \in L$ be a null element (i.e., $\mu(m) = 0$). For any $a \in L$, by subadditivity and monotonicity of μ:

$$\mu(a) \leq \mu(a \wedge m) + \mu(a \wedge m^c) \leq \mu(m) + \mu(a) = \mu(a).$$

The equality $\mu(a) = \mu(a \wedge m) + \mu(a \wedge m^c)$ follows, proving the μ-measurability of m. $\square$

Let $\mu : L \to [0, \infty]$ be an outer measure. The set of all μ-measurable elements of L (which are, by definition, complemented in L) is denoted by $\mathcal{M}_\mu$.

Lemma 3.7. *Let L be a bounded distributive lattice and $\mu : L \to [0, \infty]$ be an outer measure. Then $\mathcal{M}_\mu$ is closed under complements, finite joins, and finite meets.*

Proof. It is clear that $\mathcal{M}_\mu$ is closed under complements. Now, let $m_1, m_2 \in \mathcal{M}_\mu$ and consider $m = m_1 \vee m_2$. By the distributivity of L, we can write:

$$m = m_1 \vee (m_2 \wedge m_1^c).$$

For any $a \in L$, the following inequalities hold:

$$\begin{aligned} \mu(a \wedge m) &\leq \mu(a \wedge m_1) + \mu(a \wedge m_1^c \wedge m_2), \\ \mu(a \wedge m^c) &= \mu(a \wedge m_1^c \wedge m_2^c). \end{aligned}$$

Hence, we obtain the following inequalities:

$$\begin{aligned} \mu(a \wedge m) + \mu(a \wedge m^c) &\leq [\mu(a \wedge m_1) + \mu((a \wedge m_1^c) \wedge m_2)] + \mu((a \wedge m_1^c) \wedge m_2^c) \\ &= \mu(a \wedge m_1) + [\mu((a \wedge m_1^c) \wedge m_2) + \mu((a \wedge m_1^c) \wedge m_2^c)] \\ &= \mu(a \wedge m_1) + \mu(a \wedge m_1^c) \quad (\text{since } m_2 \in \mathcal{M}_\mu) \\ &= \mu(a) \quad (\text{since } m_1 \in \mathcal{M}_\mu). \end{aligned}$$

By Lemma 3.4, this implies that m is μ-measurable. For meets, observe that: $m_1 \wedge m_2 = (m_1^c \vee m_2^c)^c \in \mathcal{M}_\mu$, since $\mathcal{M}_\mu$ is closed under finite joins and complements. It completes the proof. □

Lemma 3.8. *Let L be a bounded distributive lattice, and let $\mu \colon L \to [0, \infty]$ be an outer measure. For any finite collection of pairwise disjoint μ-measurable elements $\{m_i\}_{i=1}^k$ and any $a \in L$, we have:*

$$\mu\left(\bigvee_{i=1}^{k} (a \wedge m_i)\right) = \sum_{i=1}^{k} \mu(a \wedge m_i).$$

Proof. We prove the lemma by induction on k. The case $k = 1$ is trivial. Now assume the result holds for some $k \geq 1$, and let $m_1, m_2, \ldots, m_{k+1}$ be pairwise disjoint μ-measurable elements. We compute:

$$\begin{aligned}
\mu\left(\bigvee_{i=1}^{k+1}(a\wedge m_i)\right) &= \mu\left(a\wedge\bigvee_{i=1}^{k+1} m_i\right)\\
&= \mu\left(a\wedge(\bigvee_{i=1}^{k+1} m_i)\wedge m_{k+1}\right)+\mu\left(a\wedge(\bigvee_{i=1}^{k+1} m_i)\wedge m_{k+1}^c\right)\\
&= \mu\left(a\wedge m_{k+1}\right)+\mu\left(a\wedge\bigvee_{i=1}^{k} m_i\right)\\
&= \mu(a\wedge m_{k+1})+\sum_{i=1}^{k}\mu(a\wedge m_i)\quad\text{(by the induction hypothesis)}\\
&= \sum_{i=1}^{k+1}\mu(a\wedge m_i).
\end{aligned}$$

Thus the equality holds by induction. □

Theorem 3.9. *Let L be a σ-frame and $\mu\colon L\to[0,\infty]$ be an outer measure. Then μ is σ-additive on $\mathcal{M}_\mu$.*

Proof. Let $\{m_k\}$ be a pairwise disjoint sequence of $\mathcal{M}_\mu$. Put $m=\bigvee_{k=1}^{\infty} m_k$. To prove σ-additivity on $\mathcal{M}_\mu$, assume that $m\in\mathcal{M}_\mu$. Since μ is σ-subadditive, we have

$$\mu(m)\leq\sum_{k=1}^{\infty}\mu(m_k).$$

For the reverse inequality, Lemma 3.8 implies that for every $k\geq 1$:

$$\sum_{i=1}^{k}\mu(m_i)=\sum_{i=1}^{k}\mu(m\wedge m_i)=\mu\left(m\wedge\bigvee_{i=1}^{k} m_i\right)\leq\mu(m).$$

Taking the limit as $k\to\infty$ yields:

$$\sum_{i=1}^{\infty}\mu(m_i)\leq\mu(m).$$

Therefore, we conclude:

$$\mu(m)=\sum_{i=1}^{\infty}\mu(m_i),$$

which completes the proof. □

Corollary 3.10. *Let L be a σ-frame and $\mu\colon L \to [0,\infty]$ an outer measure. Then μ is subtractive on $\mathcal{M}_\mu$.*

Proof. This follows immediately from Theorem 3.9 and Proposition 3.2(ii). □

Remark 3.11. The σ-additivity of a function, as given in Definition 3.1(iv), is a conditional property: the equality $\mu(\bigvee\limits_n a_n) = \sum\limits_n \mu(a_n)$ is required only for those disjoint sequences whose join $\bigvee\limits_n a_n$ already lies within the domain $\mathcal{S}$. Theorem 3.9 applies this definition to the restriction $\mu|_{\mathcal{M}_\mu}$. Its proof explicitly assumes that the join $m = \bigvee\limits_k m_k$ of the given disjoint sequence in $\mathcal{M}_\mu$ is itself in $\mathcal{M}_\mu$, which is precisely the condition to apply the definition. Consequently, Corollary 3.10 also relies only on this conditional property and does not assert the general closure of $\mathcal{M}_\mu$ under countable joins. Therefore, Theorem 3.9 and Corollary 3.10 do not logically require the assumption that $\mathcal{M}_\mu$ is closed under countable joins. Consequently, their validity does not depend on Proposition 3.12 (in which, under the Boolean hypothesis, we prove that $\mathcal{M}_\mu$ is closed under countable joins and hence a Boolean σ-frame). Hence, the Boolean hypothesis is not required for either theorem or corollary.

Proposition 3.12. *Let L be a Boolean σ-frame equipped with an outer measure $\mu\colon L \to [0,\infty]$. Then the set $\mathcal{M}_\mu$ of all μ-measurable elements is closed under both countable joins and countable meets.*

Proof. Let $\{m_i\}_{i\in\mathbb{N}}$ be a countable subset of $\mathcal{M}_\mu$, and let $m = \bigvee\limits_{i=1}^{\infty} m_i$. We construct a sequence of pairwise disjoint elements t_i as follows:

- Set $t_1 = m_1$
- For $i \geq 1$, define $t_{i+1} = m_{i+1} \wedge \left(\bigvee\limits_{j=1}^{i} m_j\right)^c$.

By Lemma 3.7, each t_i belongs to $\mathcal{M}_\mu$, and the t_i are pairwise disjoint by construction. We now prove by induction that for all $k \geq 1$:

$$\bigvee_{i=1}^{k} t_i = \bigvee_{i=1}^{k} m_i$$

For the case ($k = 1$) the equality holds trivially. Assume the equality holds for some $k \geq 1$. Then:

$$\begin{aligned}\bigvee_{i=1}^{k+1} t_i &= \left(\bigvee_{i=1}^{k} t_i\right) \vee t_{k+1} \\ &= \left(\bigvee_{i=1}^{k} m_i\right) \vee \left(m_{k+1} \wedge \left(\bigvee_{i=1}^{k} m_i\right)^c\right) \\ &= \bigvee_{i=1}^{k+1} m_i\end{aligned}$$

Thus, we have $m = \bigvee_{i=1}^{\infty} t_i$. For each $k \geq 1$, define the partial join $s_k = \bigvee_{i=1}^{k} t_i$. By Lemma 3.7, each s_k is μ-measurable.

Now, let $a \in L$. Applying Lemma 3.8 to the pairwise disjoint measurable family $(t_i)_{i=1}^k$ yields:

$$\mu(a \wedge s_k) = \mu\left(\bigvee_{i=1}^{k} (a \wedge t_i)\right) = \sum_{i=1}^{k} \mu(a \wedge t_i).$$

Hence, for every $k \geq 1$,

$$\begin{aligned}\mu(a) &= \mu(a \wedge s_k) + \mu(a \wedge s_k^c) \\ &= \left(\sum_{i=1}^{k} \mu(a \wedge t_i)\right) + \mu(a \wedge s_k^c) \\ &\geq \left(\sum_{i=1}^{k} \mu(a \wedge t_i)\right) + \mu(a \wedge m^c).\end{aligned}$$

Taking the limit as $k \to \infty$ and using the σ-frame property of L, we obtain:

$$\begin{aligned}\mu(a) &\geq \left(\sum_{i=1}^{\infty} \mu(a \wedge t_i)\right) + \mu(a \wedge m^c) \\ &\geq \mu\left(\bigvee_{i=1}^{\infty} (a \wedge t_i)\right) + \mu(a \wedge m^c) \\ &\geq \mu\left(a \wedge \bigvee_{i=1}^{\infty} t_i\right) + \mu(a \wedge m^c) \\ &= \mu(a \wedge m) + \mu(a \wedge m^c).\end{aligned}$$

Thus, m is μ-measurable by Lemma 3.4. Therefore, $\mathcal{M}_\mu$ is closed under countable joins. Moreover, using the Boolean σ-frame structure of L, we show that $\mathcal{M}_\mu$ is also closed under countable meets. For this, let $\{a_i\}_{i\in\mathbb{N}}$ be a countable family in $\mathcal{M}_\mu$. Since L is a Boolean σ-frame, each a_i has a complement a_i^c, and the countable join $\bigvee_{i\in\mathbb{N}} a_i^c$ exists in L and belongs to $\mathcal{M}_\mu$, because $\mathcal{M}_\mu$ is closed under countable joins and complements. Define

$$b := \left(\bigvee_{i\in\mathbb{N}} a_i^c\right)^c \in \mathcal{M}_\mu.$$

We claim that $b = \bigwedge_{i\in\mathbb{N}} a_i$. To verify this:

- For each $i \in \mathbb{N}$, we have $a_i^c \leq \bigvee_{i\in\mathbb{N}} a_i^c$, and taking complements yields $b = \left(\bigvee_{i\in\mathbb{N}} a_i^c\right)^c \leq a_i^{cc} = a_i$. Hence b is a lower bound for the family.
- Suppose $d \in L$ satisfies $d \leq a_i$ for all $i \in \mathbb{N}$. Then $a_i^c \leq d^c$ for each i, and therefore $\bigvee_{i\in\mathbb{N}} a_i^c \leq d^c$. Taking complements again gives $d \leq \left(\bigvee_{i\in\mathbb{N}} a_i^c\right)^c = b$.

Thus b satisfies the universal property of the meet, so $b = \bigwedge_{i\in\mathbb{N}} a_i$. Since $b \in \mathcal{M}_\mu$, we conclude that $\mathcal{M}_\mu$ is closed under countable meets. □

Remark 3.13. The Boolean hypothesis in Proposition 3.12 is essential. Without it, the set of measurable elements may fail to be closed under countable joins, as the following examples illustrate.

1. **Example: The Zero Measure.** Let $\zeta : L \to [0, \infty]$ be the zero measure (i.e., $\zeta(a) = 0$ for all $a \in L$). Let L be *any* σ-frame in which the complemented elements are not closed under countable joins (for instance, the σ-frame L constructed in Example 2 below has this property). Under the zero measure ζ, every complemented element of L is trivially ζ-measurable. However, taking a countable family of complemented (hence ζ-measurable) elements whose join exists in L but is not complemented yields an immediate counterexample to the conclusion of Proposition 3.12.

2. **Example: The Cardinality Measure.** Consider the specific σ-frame:

$$L = \{A \subseteq \mathbb{N} \cup \{\infty\} \mid \infty \in A \implies A \text{ is cofinite}\}.$$

The partial order on L is given by set inclusion, and the join (respectively, meet) of any countable family in L is its union (respectively, intersection), which again lies in L. Equip it with the *cardinality measure* $\mu(a) = |a|$. For any singleton $m = \{n\} \subset \mathbb{N}$, its complement $m^c = (\mathbb{N} \setminus \{n\}) \cup \{\infty\}$ is in L. For every $a \in L$, the sets $a \cap m$ and $a \cap m^c$ are disjoint and their union is a. By the additivity of cardinality over disjoint sets, we have:

$$\mu(a) = |a| = |a \cap m| + |a \cap m^c| = \mu(a \cap m) + \mu(a \cap m^c).$$

Therefore, each singleton $\{n\}$ is μ-measurable. Their countable join is $\bigcup_{n \in \mathbb{N}} \{n\} = \mathbb{N}$, which is an element of L but is *not* complemented. Consequently, it cannot be μ-measurable, providing a concrete counterexample.

Both examples underscore that the Boolean structure (guaranteeing that joins of complemented elements remain complemented) is indispensable for the validity of the proposition.

Corollary 3.14. *Let L be a Boolean σ-frame and $\mu\colon L \to [0, \infty]$ be an outer measure. Then $\mathcal{M}_\mu$ forms a Boolean sub-σ-frame of L.*

Proof. This follows immediately from Lemma 3.7 and Proposition 3.12. $\square$

We now raise the following question, which we do not answer in this paper:

Open Question 3.15. *Let L be a Boolean σ-frame and $H \subseteq L$ be a Boolean σ-subframe. Does there exist an outer measure $\mu\colon L \to [0, \infty]$ such that $\mathcal{M}_\mu = H$?*

Example 3.16. To illustrate the possible relationships between $\mathcal{M}_\mu$ and Boolean subalgebras of L, we consider two extreme cases:

1. Let H be the smallest Boolean subalgebra $\{\perp, \top\}$. We can construct the following trivial outer measure:

   Define $\delta\colon L \to [0, \infty]$ by:

$$\delta(a) = \begin{cases} 0, & a = \perp \\ 1, & a \neq \perp \end{cases}$$

   for each $a \in L$.

   Trivially, the only δ-measurable elements in L are $\perp$ and $\top$. Thus we obtain $\mathcal{M}_\delta = \{\perp, \top\}$ for this special case.

2. Let $H = L$. Now, consider the zero function $\zeta : L \to [0, \infty]$ given by $\zeta(a) = 0$ for all $a \in L$. Clearly, every complement element is ζ-measurable. Therefore, $\mathcal{M}_\zeta = H = L$.

In the following proposition we show that outer measures are preserved under σ-frame maps, and that the collection of measurable elements is preserved under surjective σ-frame maps.

Proposition 3.17. *Let $f\colon L \to M$ be a σ-frame map, and $\mu\colon M \to [0, \infty]$ be an outer measure. Then:*

(i) *$\mu \circ f\colon L \to [0, \infty]$ is an outer measure.*

(ii) *If f is surjective, then $f[\mathcal{M}_{\mu\circ f}] = \mathcal{M}_\mu$.*

Proof. (i) The composition $\mu \circ f$ is clearly null-preserving and monotone. To verify σ-subadditivity, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in L. We have:

$$\begin{aligned} \mu \circ f\left(\bigvee_{n=1}^{\infty} a_n\right) &= \mu\left(\bigvee_{n=1}^{\infty} f(a_n)\right) \\ &\leq \sum_{n=1}^{\infty} \mu(f(a_n)) \\ &= \sum_{n=1}^{\infty} \mu \circ f(a_n) \end{aligned}$$

(ii) Assume f is surjective. First, let $m \in \mathcal{M}_{\mu\circ f} \subseteq L$ and take $b = f(a) \in M$. Then:

$$\begin{aligned} \mu(b \wedge f(m)^c) + \mu(b \wedge f(m)) &= \mu(f(a) \wedge f(m)^c) + \mu(f(a) \wedge f(m)) \\ &= \mu(f(a \wedge m^c)) + \mu(f(a \wedge m)) \\ &= \mu \circ f(a \wedge m^c) + \mu \circ f(a \wedge m) \\ &= \mu \circ f(a) \\ &= \mu(b) \end{aligned}$$

This shows $f(m) \in \mathcal{M}_\mu$.

Conversely, suppose $f(m) \in \mathcal{M}_\mu$. For any $a \in L$:

$$\begin{aligned} \mu \circ f(a \wedge m^c) + \mu \circ f(a \wedge m) &= \mu(f(a \wedge m^c)) + \mu(f(a \wedge m)) \\ &= \mu(f(a) \wedge f(m)^c) + \mu(f(a) \wedge f(m)) \\ &= \mu(f(a)) \\ &= \mu \circ f(a) \end{aligned}$$

Hence $m \in \mathcal{M}_{\mu \circ f}$, completing the proof. □

Remark 3.18. Let $\mathrm{Out}(L)$ be the set of all outer measures on L, and $\mathbf{Boo}(L)$ be the set of all Boolean subalgebras of L. By Corollary 3.14, the assignment $\mu \mapsto \mathcal{M}_\mu$ defines a function $\omega_L \colon \mathrm{Out}(L) \to \mathbf{Boo}(L)$. Note that Question 3.15 asks whether the map ω_L is surjective. In any case, this function is not injective. To see this, let $k \neq 1$ be any fixed positive real number. Given $\mu \in \mathrm{Out}(L)$, define $k\mu \colon L \to [0, \infty]$ by $(k\mu)(a) = k \cdot \mu(a)$. Then:

- $k\mu$ is clearly an outer measure, and
- $\mathcal{M}_\mu = \mathcal{M}_{k\mu}$.

Thus $\omega_L(\mu) = \omega_L(k\mu)$, proving that ω_L is not injective.

Although ω_L does not preserve order, it exhibits interesting categorical behavior. For technical terms from category theory we refer to [1]. Consider the category $\mathbf{Boo\sigma FrmSur}$ of Boolean σ-frames with surjective σ-frame maps. We define two functors:

1. The contravariant functor $\mathbf{Out} \colon \mathbf{Boo\sigma FrmSur} \to \mathbf{Set}$ given on objects by $L \mapsto \mathrm{Out}(L)$, and on morphisms by $f \mapsto \mathrm{Out}(f)$, where for any σ-frame map $f \colon L \to M$, the function $\mathrm{Out}(f) \colon \mathrm{Out}(M) \to \mathrm{Out}(L)$ is defined by $\mathrm{Out}(f)(\mu) = \mu \circ f$.
2. The covariant functor $\mathbf{Boo} \colon \mathbf{Boo\sigma FrmSur} \to \mathbf{Set}$ given on objects by $L \mapsto \mathbf{Boo}(L)$, and on morphisms by $f \mapsto \mathbf{Boo}(f)$, where for any σ-frame map $f \colon L \to M$, the function $\mathbf{Boo}(f) \colon \mathbf{Boo}(L) \to \mathbf{Boo}(M)$ is defined by $\mathbf{Boo}(f)(\mathcal{B}) = f[\mathcal{B}] = \{f(x) : x \in \mathcal{B}\}$.

Now, by Proposition 3.17, for any surjective σ-frame map $f \colon L \to M$, we have $f[\mathcal{M}_{\mu \circ f}] = \mathcal{M}_\mu$, which implies

$$\mathbf{Boo}(f) \circ \omega_L \circ \mathrm{Out}(f) = \omega_M.$$

This identity means that the following diagram commutes for every morphism f in $\mathbf{Boo}\sigma\mathbf{FrmSur}$:

$$\begin{tikzcd}
\mathrm{Out}(M) \arrow[r, "\omega_M"] \arrow[d, "\mathbf{Out}(f)"'] & \mathbf{Boo}(M) \\
\mathrm{Out}(L) \arrow[r, "\omega_L"'] & \mathbf{Boo}(L) \arrow[u, "\mathbf{Boo}(f)"']
\end{tikzcd}$$

Thus, despite the functors having opposite variance, the family $(\omega_L)_L$ satisfies the required compatibility condition and therefore constitutes a natural transformation from the contravariant functor **Out** to the covariant functor **Boo**. In categorical terms, one may view ω as a natural transformation between functors defined on $\mathbf{Boo}\sigma\mathbf{FrmSur}$ when **Out** is regarded as a functor on the opposite category, but the diagram above makes the relationship explicit.

Remark 3.19. The present work contributes to the ongoing development of measure theory in pointfree and categorical settings, a research programme that includes several related but distinct approaches. In particular, there exists a family of works, including [17] by Kříž–Pultr and [3] by Ball–Pultr, that develop a categorical approach to abstract σ-algebras (i.e., Boolean algebras with countable joins) and the Daniell integral. Specifically, [17] presents a categorical treatment of integration without points. The work of Simpson [20] investigates measures on σ-locales and their extension to σ-sublocales, with applications to randomness. More recently, Bernardes [8] develops Lebesgue integration for simple functions on σ-locales, building upon Simpson's framework. Pultr's lecture notes [19] provide a detailed exposition of Daniell's approach to the Lebesgue integral. While all these investigations share a common pointfree motivation, they differ fundamentally in their objectives and methodology.

Our work concentrates specifically on the structure of measurable elements arising from an outer measure on a σ-frame, following the classical Carathéodory approach. We begin with minimal assumptions (bounded distributive lattices) and progressively introduce additional structure, σ-frames and Boolean σ-frames, only when necessary to obtain stronger results. This stepwise, meticulous approach allows us to isolate precisely which conditions are required for each property of the measurable elements, thereby offering a complementary perspective that enriches the broader understanding of measure theory in pointfree settings. The resulting analysis provides insights that, together with the existing literature, contribute to a more comprehensive picture of the subject.

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References

[1] Adamek, J., Herrlich, H., and Strecker, G.E., "Abstract and Concrete Categories: The Joy of Cats", Dover Publications, 2009.

[2] Aliprantis, C.D. and Burkinshaw, O., "Principles of Real Analysis", 3rd ed., Academic Press, 1998.

[3] Ball, R.N. and Pultr, A., *Extending semilattices to frames using sites and coverages*, Math. Slovaca 64(3) (2014), 527-544.

[4] Balbes, R. and Dwinger, P., "Distributive Lattices", University of Missouri Press, 1974.

[5] Banaschewski, B. and Gilmour, C.R.A., *Rings of continuous functions on σ-frames*, Topology Appl. 158 (2011), 861-868.

[6] Banaschewski, B. and Gilmour, C.R.A., *Realcompact spaces and regular σ-frames*, Math. Proc. Cambridge Philos. Soc. 95(1) (1984), 33-60.

[7] Banaschewski, B., Gutiérrez García, J., and Picado, J., *Extended real functions in pointfree topology*, J. Pure Appl. Algebra 216 (2012), 905-922.

[8] Bernardes, R., *Lebesgue integration on σ-locales: simple functions*, Preprint, arXiv:2408.13911 [math.FA], 2024.

[9] Davey, B.A. and Priestley, H.A., "Introduction to Lattices and Order", 2nd ed., Cambridge University Press, 2002.

[10] Evans, L.C. and Gariepy, R., "Measure Theory and Fine Properties of Functions", Revised Edition, CRC Press, 2015.

[11] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M.W., and Scott, D.S., "Continuous Lattices and Domains", Encyclopedia of Mathematics and its Applications 93, Cambridge University Press, 2003.

[12] Grätzer, G., "Lattice Theory: Foundation", Birkhäuser, 2011.

[13] Gutiérrez García, J., and Picado, J., *Rings of real functions in pointfree topology*, Topology Appl. 158 (2011), 2264-2278.

[14] Gutiérrez García, J., Kubiak, T., and Picado, J., *Localic real functions: a general setting*, J. Pure Appl. Algebra 213 (2009), 1064-1074.

[15] Halmos, P.R., "Measure Theory", Graduate Texts in Mathematics, Springer-Verlag, 1988.

[16] Johnstone, P.T., "Stone Spaces", Cambridge University Press, 1982.

[17] Kříž, I. and Pultr, A., *Categorical geometry and integration without points*, Appl. Categ. Structures 22(1) (2014), 79-97.

[18] Picado, J. and Pultr, A., "Frames and Locales: Topology Without Points", Frontiers in Mathematics, Springer, 2012.

[19] Pultr, A., "Daniell's version of Lebesgue integral", Lecture Notes, University of Coimbra, 2010.

[20] Simpson, A., *Measure, randomness and sublocales*, Ann. Pure Appl. Logic 163(11) (2012), 1642-1659.

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