Categories and
General Algebraic Structures
with Applications

In press.

(u, v)-Absorbing primary hyperideals in multiplicative hyperrings

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Abstract. The present paper addresses the notion of (u, v)-absorbing primary hyperideals in commutative multiplicative hyperrings. A proper hyperideal P of A is said to be a (u, v)-absorbing primary hyperideal if $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1, \ldots, x_u \in A \backslash U(A)$ implies either $x_1 \circ \cdots \circ x_v \subseteq P$ or $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$. The study investigates the links and characteristics of (u, v)-absorbing primary hyperideals, presenting a general framework for realizing their importance in hyperring theory. We give some arguments and examples explaining the relationship between (u, v)-absorbing primary hyperideals and other famous classes of hyperideals, such as maximal, prime and primary hyperideals, despite being vastly different. Morevoer, we analyze how the (u, v)-absorbing primary hyperideals behave concerning images and inverse images of hyperring homomorphisms, quotients and localizations.

1 Introduction and Preliminaries

The notions of prime and primary ideals take an important place in ring theory. Many papers have been written on these concepts and their extensions.

Keywords: Primary hyperideal, maximal hyperideal, (u, v)-absorbing primary hyperideal.
Mathematics Subject Classification [2010]: 20N20, 16Y20.
Received: 20 December 2025, Accepted: 11 April 2026.
ISSN: Print 2345-5853, Online 2345-5861.

The research paradigm regarding generalizations of prime ideals shifted significantly in 2007, when Ayman Badawi unveiled the idea of 2-absorbing ideals in commutative rings [9]. Later, this concept was generalized by him, his collaborators, and many other researchers [7, 8, 10, 26]. A recent paper [18] is devoted to the study of notion of (u, v)-absorbing primary ideals. A proper ideal I of a commutative ring R refers to a (u, v)-absorbing primary ideal if $x_1 \cdots x_u \in P$ where $x_1, \ldots, x_u$ are nonunit elements in R, then $x_1 \cdots x_v \in I$ or $x_{v+1} \cdots x_u \in rad(I)$.

The idea of algebraic hyperstructures as a well established branch of classical algebraic theory goes back to Marty's research work [19] presented at the 8^{th} Congress of Scandinavian Mathematicians in 1934. So far, many mathematicians has studied on hyperstructures [17, 20, 22, 25]. The hyperrings as a class of algebraic hyperstructures were introduced and studied by many authors. In 1983, multiplicative hyperrings as a significant class of hyperrings were presented by Rota [23]. In these hyperrings, the addition is an operation and the multiplication is a hyperoperation. This type of hyperstructurs has been widely studied and investigated in [2–4, 11]. More exactly, a hyperoperation " $\circ$ " on a non-empty set A is a mapping from $A \times A$ into the family of all non-empty subsets of A denoted by$P^*(A)$. If " $\circ$ " is a hyperoperation on A, then we say that $(A, \circ)$ is a hypergroupoid [14]. For any given element $a \in A$ and subsets A_1 and A_2 of A, $A_1 \circ A_2 = \cup_{a_1 \in A_1, a_2 \in A_2} a_1 \circ a_2$, and $A_1 \circ a = A_1 \circ \{a\}$. The hypergroupoid $(A, \circ)$ is a semihypergroup if $\circ$ is associative that is $\cup_{a \in y \circ z} x \circ a = \cup_{b \in x \circ y} b \circ z$ for any $x, y, z \in A$. The semihypergroup A is a hypergroup if $A \circ a = a \circ A = A$ for any $a \in A$ [14]. Let $(A, \circ)$ be a semihypergroup. $\varnothing \neq B \subseteq A$ is a subhypergroup if $B \circ a = a \circ B = B$ for each $a \in B$ [14]. If **1.** $(A, +)$ is a commutative group, **2.** $(A, \circ)$ is a semihypergroup; **3.** $(-x) \circ y = x \circ (-y) = -(x \circ y)$ for any $x, y \in A$, **4.** $(y + z) \circ x \subseteq y \circ x + z \circ x$ and $x \circ (y + z) \subseteq x \circ y + x \circ z$ for any $x, y, z \in A$, **5.** $x \circ y = y \circ x$ for any $x, y \in A$, then the triple $(A, +, \circ)$ refers to a commutative multiplicative hyperring [14]. The multiplicative hyperring A is strongly distributive if in (4), the equality holds.

For each subset $\Phi \in P^\star(\mathbb{Z})$ where $(\mathbb{Z}, +, \cdot)$ is the ring of integers and $|\Phi| \geq 2$, there exists a multiplicative hyperring $(\mathbb{Z}_\Phi, +, \circ)$ such that $\mathbb{Z}_\Phi = \mathbb{Z}$ and $a \circ b = \{a.x.b \mid x \in \Phi\}$ for all $a, b \in \mathbb{Z}_\Phi$ [12]. An element e in A is considered as an identity element if $a \in e \circ a$ for every $a \in A$ [2]. An element x in A is called unit, if there exists y in A such that $e \in y \circ x$. We denote the set of all unit elements in A by $U(A)$ [2]. Furthermore, a hyperring A is a hyperfield if each non-zero element in A is unit. $\varnothing \neq B \subseteq A$ refers to a hyperideal if **1.** $x - y \in B$ for any $x, y \in B$, **2.** $r \circ x \subseteq B$ for any $x \in B$ and $r \in A$ [14]. A proper hyperideal B of A is a prime hyperideal if whenever $x, y \in A$ and $x \circ y \subseteq B$, then $x \in B$ or $y \in B$ [12]. For any given hyperideal B of A, the prime radical of B, denoted by $rad(B)$, is the intersection of all prime hyperideals of A containing B. If the multiplicative hyperring A has no prime hyperideal containing B, then we define $rad(B) = A$ [12]. We say that a proper hyperideal B of A is a primary hyperideal if whenever $x, y \in A$ and $x \circ y \subseteq B$, then $x \in B$ or $y \in rad(B)$ [12]. A hyperideal B of A is said to be a $\mathcal{C}$-hyperideal if $(a_1 \circ a_2 \circ \cdots \circ a_n) \cap B \neq \varnothing$ for all $a_1, \ldots, a_n \in A$ and $n \in \mathbb{N}$ imply $a_1 \circ a_2 \circ \cdots \circ a_n \subseteq B$. Notice that in a multiplicative hyperring A, we have $\{a \in A \mid a^n \subseteq B \text{ for some } n \in \mathbb{N}\} \subseteq rad(A)$. Proposition 3.2 in [12] shows that the equality holds if B is a $\mathcal{C}$-hyperideal of A. Furthermore, a hyperideal B of A is a strong $\mathcal{C}$-hyperideal if $\sum_{i=1}^{n}(\bigcirc_{j=1}^{k_i} x_{ij}) \cap B \neq \varnothing$ where $\bigcirc_{j=1}^{k_i} x_{ij} = x_{i1} \circ \cdots \circ x_{ik_i}$, $x_{i1}, \ldots, x_{ik_i} \in A$ and $k_i, n \in \mathbb{N}$, then $\sum_{i=1}^{n}(\bigcirc_{j=1}^{k_i} x_{ij}) \subseteq B$. For more details you can see [13]. A proper hyperideal B in A is maximal if for any hyperideal M of A with $B \subset M \subseteq A$, then $M = A$ [2]. We denote the intersection of all maximal hyperideals of A by $J(A)$. Also, the multiplicative hyperring A is local if it has just one maximal hyperideal [2]. For any give hyperideals B_1 and B_2 of A, we define $(B_2 : B_1) = \{a \in A \mid a \circ B_1 \subseteq B_2\}$ [2]. A multiplicative hyperring A with identity e is hyperdomain if $0 \in x \circ y$ for $x, y \in A$, then $x = 0$ or $y = 0$.

The concepts of prime and primary hyperideals have been extended to more nuanced concepts [15, 16, 21, 24]. These expansions have paved the way for further discovery into more complex types of hyperideals. Despite these developments, there is a need to explore other classes of hyperideals for deeper understanding of commutative multiplicatve hyperrings structure.

In this paper, we aim to introduce the notion (u, v)-absorbing primary hyperideals where $u, v \in \mathbb{Z}$ with $u > v$. Indeed, this definition presents a distinct viewpoint by including the radical of hyperideals in the definition of (u, v)-absorbing prime hyperideals proposed in [6]. Among many results in this paper, we give an example of a (u, v)-absorbing primary hyperideal that is not a (u, v)-absorbing prime hyperideal (Example 2.4). Although every (u, v)-absorbing primary hyperideal of A is a (w, v)-absorbing primary hyperideal of A for any $w \geq u$, Example 2.6 shows that the converse may fail.

In Theorem 2.7, we conclude that the radical of a (u,v)-absorbing primary $\mathcal{C}$-hyperideal is a prime hyperideal. We obtain that in a local multiplicative hyperring A with the maximal hyperideal M, $P \circ M$ is a (u,v)-absorbing primary hyperideal of A where P is a prime $\mathcal{C}$-hyperideal of A in Proposition 2.8. In Theorem 2.12, we show that if there exists a $(u+1,v+1)$-absorbing primary strong $\mathcal{C}$-hyperideal of A or a $(u+1,v)$-absorbing primary strong $\mathcal{C}$-hyperideal of A that is not a (u,v)-absorbing primary hyperideal, then A is a local multiplicative hyperring. Theorem 2.13 presents a case where (u,v)-absorbing primary hyperideals are primary hyperideals. In Proposition 2.15, it is shown that if $P_1,\dots,P_n$ are (u,v)-absorbing primary $\mathcal{C}$-hyperideals of A such that $rad(P_i)=rad(P_j)$ for all $i,j \in \{1,\dots,n\}$, then the intersection of the $P_i^,$s is a (u,v)-absorbing primary hyperideal. We give Example 2.16 to show the condition "$rad(P_i)=rad(P_j)$ for all $i,j \in \{1,\dots,n\}$" in Proposition 2.15 is crucial. Moreover, we investigate the stability of (u,v)-absorbing primary hyperideals in various hyperring-theoretic constructions.

Throughout this study, A denotes a commutative multiplicative hyperring with identity element 1.

2 (u,v)-absorbing primary hyperideals

A proper hyperideal of P of A is said to be 1-absorbing primary hyperideal if whenever $x,y,z \in A\backslash U(A)$ and $x \circ y \circ z \subseteq P$, then $x \circ y \subseteq P$ or $z \in rad(P)$ [5]. Now, we aim to generalize this concept to notion of (u,v)-absorbing primary hyperideals and give some fundamental theorems and examples about them. We begin with the definition.

Definition 2.1. Let P be a proper hyperideal of A and $u,v \in \mathbb{N}$ with $u>v$. P refers to a (u,v)-*absorbing primary hyperideal* if $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1,\dots,x_u \in A\backslash U(A)$, then either $x_1 \circ \cdots \circ x_v \subseteq P$ or $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$.

Example 2.2. (i) Assume that $(\mathbb{Z}[i],+,\cdot)$ is the Gaussian integers ring. Consider the multiplicative hyperring $(A_\Phi,+,\circ)$ where $A_\phi=\mathbb{Z}[i]$, $\Phi=\{-1,3\}$ and for any $a,b \in A_\Phi$, $a \circ b=\{a \cdot x \cdot b \mid x \in \Phi\}$. In the hyperring, $P=2\mathbb{Z}[i]=\{-2x-2yi, 6x+6yi \mid x,y \in \mathbb{Z}\}$ is a (u,v)-absorbing primary hyperideal of A_Φ for all $u,v \in \mathbb{N}$ with $u>v$.

(ii) Consider the ring of polynomials $\mathbb{Z}[x]$ where $(\mathbb{Z},+,\cdot)$ is the ring of integers. Let $A=\mathbb{Z}+3x\mathbb{Z}[x]$, $\alpha \circ \beta=\{2\alpha\beta, 4\alpha\beta\}$ for each $\alpha,\beta \in \mathbb{Z}$ and

$P = 3x\mathbb{Z}[x]$. In the hyperring, P^2 is a $(3,2)$-absorbing primary hyperideal of A.

Recall from [6] that a proper hyperideal P of A is a (u,v)-absorbing prime hyperideal if $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1, \ldots, x_u \in A \backslash U(A)$ implies either $x_1 \circ \cdots \circ x_v \subseteq P$ or $x_{v+1} \circ \cdots \circ x_u \subseteq P$.

Remark 2.3. Every (u,v)-absorbing prime hyperideal of A is a (u,v)-absorbing primary hyperideal of A.

The following example shows that the converse of Remark 2.3 may not be true, in general.

Example 2.4. Consider the multiplicative hyperring $(\mathbb{Z}_\Phi, +, \circ)$ where $\Phi = \{2,3\}$. In the hyperring, $P = \langle 12 \rangle$ is a $(4,2)$-absorbing primary hyperideal of A. However, it is not a $(4,2)$-absorbing prime hyperideal of A since $2^3 \circ 3 = \{192, 288, 432, 648\} \subseteq P$ but $2 \circ 2 = \{8, 12\} \nsubseteq P$ and $2 \circ 3 = \{12, 18\} \nsubseteq P$.

Remark 2.5. (i) Every (u,v)-absorbing primary hyperideal of A is a $(u+1, v+1)$-absorbing primary hyperideal of A.

(ii) Every (u,v)-absorbing primary hyperideal of A is a (w,v)-absorbing primary hyperideal of A for any $w \geq u$.

The next example is given to explain that the converse of 2.5 (ii) may not be always true.

Example 2.6. In Example 2.4, the $(4,2)$-absorbing prime hyperideal P is not $(3,2)$-absorbing primary since $2 \circ 2 \circ 3 = \{48, 72, 108\} \subseteq P$ while $2 \circ 2 = \{8, 12\} \notin P$, $2 \circ 3 = \{12, 18\} \nsubseteq P$ and $2, 3 \notin rad(P)$.

The following theorem shows that the radical of a (u,v)-absorbing primary $\mathcal{C}$-hyperideal of A is a prime hyperideal of A.

Theorem 2.7. *Let $u, v \in \mathbb{N}$ with $u > v$. If P is a (u,v)-absorbing primary $\mathcal{C}$-hyperideal of A, then $rad(P)$ is a prime hyperideal of A.*

Proof. Let P be a (u,v)-absorbing primary hyperideal of A and $a \circ b \subseteq rad(P)$ for $a, b \in A$. Let us assume that $a, b \in A \backslash U(A)$. Then we get $a^n \circ b^n \subseteq P$ for some $n \in \mathbb{N}$. It follows that $a^{v-1} \circ a^n \circ b^{u-v-1} \circ b^n \subseteq P$. Take any $x \in a^n$ and $y \in b^n$. Let $x, y \in U(A)$. Then there exist $x', y' \in A$ such that $1 \in x \circ x'$ and $1 \in y \circ y'$. Hence $1 \in a^n \circ x' = a \circ (a^{n-1} \circ x')$ and $1 \in b^n \circ x' = b \circ (b^{n-1} \circ y')$. Therefore we have $1 \in a \circ p$ for some $p \in a^{n-1} \circ x'$ and $1 \in b \circ q$ for some $q \in b^{n-1} \circ y'$. This means that $a, b \in U(A)$, a contradiction. Thus $x, y \notin U(A)$. Now, put $x_1 = \cdots = x_{v-1} = a$, $x_v = x$, $x_{v+1} = \cdots = x_{u-1} = b$ and $x_u = y$. This implies that $a^{v-1} \circ x = x_1 \circ \cdots \circ x_{v-1} \circ x_v \subseteq P$ or $b^{u-v-1} \circ y = x_{v+1} \circ \cdots \circ x_{u-1} \circ x_u \subseteq rad(P)$ as P is a (u, v)-absorbing primary hyperideal of A. Since P is a $\mathcal{C}$-hyperideal of A, we conclude that $a^{v+n-1} \subseteq P$ which means $a \in rad(P)$ or $b^{u-v+n-1} \subseteq rad(P)$ which implies $b \in rad(P)$. Consequently, $rad(P)$ is a prime hyperideal of A. □

In the following proposition, we analyze when a product of two hyperideals is a (u, v)-absorbing primary hyperideal.

Proposition 2.8. *Let every hyperideal of A be $\mathcal{C}$-hyperideal and $u, v \in \mathbb{N}$ with $u > v$. If M is the only maximal hyperideal of A and P is a prime hyperideal of A, then $P \circ M$ is a (u, v)-absorbing primary hyperideal of A.*

Proof. It can be seen that the idea is true in a similar manner to the proof of Theorem 2.8 in [18]. □

Next, we investigate when $(P : x)$ is a (u, v)-absorbing primary hyperideal of A.

Proposition 2.9. *Let P be a $(u+1, v+1)$-absorbing primary hyperideal of A where $u, v \in \mathbb{N}$ with $u > v$ and $x \in A \backslash (P \cup U(A))$. Then $(P : x)$ is a (u, v)-absorbing primary hyperideal of A.*

Proof. It is seen to be true by using an argument similar to that in the proof of Theorem 2.7 in [18]. □

The following lemma is needed in the proof of our next result.

Lemma 2.10. *Let P be a strong $\mathcal{C}$-hyperideal of A. Then so is $rad(P)$.*

Proof. Let P be a strong $\mathcal{C}$-hyperideal of A and $\sum_{i=1}^{n}(\bigcirc_{j=1}^{k_i} x_{ij}) \cap rad(P) \neq \varnothing$ for $x_{ij} \in A$ and $k_i, n \in \mathbb{N}$. This means that there exists $x \in \sum_{i=1}^{n}(\bigcirc_{j=1}^{k_i} x_{ij}) \cap rad(P)$. Then we conclude that $x^t \subseteq P$ for some $t \in \mathbb{N}$. Since P is a strong $\mathcal{C}$-hyperideal of A and $(\sum_{i=1}^{n}(\bigcirc_{j=1}^{k_i} x_{ij}))^t \cap P \neq \varnothing$, we get $(\sum_{i=1}^{n}(\bigcirc_{j=1}^{k_i} x_{ij}))^t \subseteq P$ and so $\sum_{i=1}^{n}(\bigcirc_{j=1}^{k_i} x_{ij}) \subseteq rad(P)$, as needed. □

Let $a \in P$ where P is a strong $\mathcal{C}$-hyperideal of A. We next handle the case when $a+1$ is a nonunit of A.

Theorem 2.11. *Let P be a strong $\mathcal{C}$-hyperideal of A such that $a+1 \in A \backslash U(A)$ for some $a \in P$ and $u, v \in \mathbb{N}$ with $u > v$. Then P is a (u,v)-absorbing primary hyperideal of A if and only if $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1, \ldots, x_u \in A$ implies that $x_1 \circ \cdots \circ x_v \subseteq P$ or $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$.*

Proof. ($\Longrightarrow$) Let P is a (u,v)-absorbing primary hyperideal of A and $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1, \ldots, x_u \in A$. Put $t = \min\{i \mid 1 \leq i \leq u,\ x_i \notin U(A)\}$. Clearly, we have $x_{v+1} \circ \cdots \circ x_u \subseteq P \subseteq rad(P)$ if $t \geq v+1$. Assume that $t \leq v$. Put $X = \{i \mid 1 \leq i \leq v, x_i \notin U(A)\}$ and $Y = \{i \mid v+1 \leq i \leq u, x_i \notin U(A)\}$. We suppose that $card(X) = m$ and $card(Y) = n$. Therefore we get $(\bigcirc_{k \in X} x_k \circ (a+1)^{v-m}) \circ (\bigcirc_{k \in Y} x_k \circ (a+1)^{u-v-n}) \subseteq P$. Since P is a (u,v)-absorbing primary hyperideal of A, we conclude that $\bigcirc_{k \in X} x_k \circ (a+1)^{v-m} \subseteq P$ or $\bigcirc_{k \in Y} x_k \circ (a+1)^{u-v-n} \subseteq rad(P)$. On the other hand, we have $a \circ I + 1 \subseteq (a+1)^{v-m} = \sum_{j=0}^{v-m} \binom{v-m}{j} a^{v-m-j} \circ 1^j$ for some $I \subseteq A$ and $a \circ J + 1 \subseteq (a+1)^{u-v-n} = \sum_{j=0}^{u-v-n} \binom{u-v-n}{j} a^{u-v-n-j} \circ 1^j$ for some $J \subseteq A$. Then we obtain $\bigcirc_{k \in X} x_k \circ (a \circ I + 1) \subseteq P$ or $\bigcirc_{k \in Y} x_k \circ (a \circ J + 1) \subseteq rad(P)$. Since $rad(P)$ is a strong $\mathcal{C}$-hyperideal of A by Lemma 2.10 and $a \in P$, we get $\bigcirc_{k \in X} x_k \subseteq P$ or $\bigcirc_{k \in Y} x_k \subseteq rad(P)$ and so $x_1 \circ \cdots \circ x_v \subseteq P$ or $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$.

($\Longleftarrow$) It is obvious. $\square$

Theorem 2.12. *Let $u, v \in \mathbb{N}$ with $u > v$. If there exists a $(u+1, v+1)$-absorbing primary strong $\mathcal{C}$-hyperideal of A or a $(u+1, v)$-absorbing primary strong $\mathcal{C}$-hyperideal of A that is not a (u,v)-absorbing primary hyperideal, then A is a local multiplicative hyperring.*

Proof. Assume that P is a strong $\mathcal{C}$-hyperideal of A that is not a (u,v)-absorbing primary hyperideal. This means that $x_1 \circ \cdots \circ x_u \subseteq P$ for some $x_1, \ldots, x_u \in A \backslash U(A)$ but $x_1 \circ \cdots \circ x_v \nsubseteq P$ and $x_{v+1} \circ \cdots \circ x_u \nsubseteq rad(P)$. Let P be a $(u+1, v+1)$-absorbing primary strong $\mathcal{C}$-hyperideal of A. Take any $x \in A \backslash U(A)$. Since $x \circ x_1 \circ \cdots \circ x_u \subseteq P$ and $x_{v+1} \circ \cdots \circ x_u \nsubseteq rad(P)$, we conclude that $x \circ x_1 \circ \cdots \circ x_v \subseteq P$. Take any $y \in U(A)$. We show that $x + y \in U(A)$. Let $x + y \notin U(A)$. From $(x+y) \circ x_1 \circ \cdots \circ x_u \subseteq P$, it follows that $(x+y) \circ x_1 \circ \cdots \circ x_v \subseteq P$ because $x_{v+1} \circ \cdots \circ x_u \nsubseteq rad(P)$. Since P is a strong $\mathcal{C}$-hyperideal and $(x \circ x_1 \circ \cdots \circ x_v) + (y \circ x_1 \circ \cdots \circ x_v) \cap P \neq \varnothing$, we obtain

$(x \circ x_1 \circ \cdots \circ x_v) + (y \circ x_1 \circ \cdots \circ x_v) \subseteq P$. Since $x \circ x_1 \circ \cdots \circ x_v \subseteq P$, we have $y \circ x_1 \circ \cdots \circ x_v \subseteq P$ which implies $x_1 \circ \cdots \circ x_v \subseteq P$ which is impossible. Hence $x + y \in U(A)$ which implies A is a local multiplicative hyperring by Lemma 2.6 in [16]. Now, suppose that P is a $(u+1, v)$-absorbing primary strong $\mathcal{C}$-hyperideal of A, $x \in A \backslash U(A)$ and $y \in U(A)$. Again, we show that $x + y \in U(A)$ and so we are done by Lemma 2.6 in [16]. Assume that $x + y \notin U(A)$. From $x_1 \circ \cdots \circ x_u \circ (x+y) \subseteq P$, it follows that $x_{v+1} \circ \cdots \circ x_u \circ (x+y) \subseteq rad(P)$ as P is a $(u+1, v)$-absorbing primary hyperideal of A and $x_1 \circ \cdots \circ x_v \nsubseteq P$. Since $(x_{v+1} \circ \cdots \circ x_u \circ x) + (x_{v+1} \circ \cdots \circ x_u \circ y) \cap rad(P) \neq \varnothing$, we obtain $(x_{v+1} \circ \cdots \circ x_u \circ x) + (x_{v+1} \circ \cdots \circ x_u \circ y) \subseteq rad(P)$ by Lemma 2.10. This implies that $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$ as $x_{v+1} \circ \cdots \circ x_u \circ x \subseteq rad(P)$ and $y \in U(A)$. This is a contradiction and so $x + y \in U(A)$. □

The next result gives a case where (u, v)-absorbing primary hyperideals are primary hyperideals.

Theorem 2.13. *Let P be a strong $\mathcal{C}$-hyperideal of A such that A is not a local multiplicative hyperring and $u, v \in \mathbb{N}$ with $u > v \geq 2$. Then P is a (u, v)-absorbing primary hyperideal of A if and only if P is a primary hyperideal of A.*

Proof. ($\Longrightarrow$) Assume that A is not a local multiplicative hyperring and P is a (u, v)-absorbing primary hyperideal of A. Therefore P is a $(u-1, v-1)$-absorbing primary hyperideal of A by Theorem 2.12. Therefore we conclude that P is a $(u-v+1, 1)$-absorbing primary hyperideal of A. Now, let $a \circ b \subseteq P$ for $a, b \in A$ but $a \notin P$. So $a \circ b^{u-v} \subseteq P$. Since P is a $(u-v+1, 1)$-absorbing primary hyperideal and $a \notin P$, we obtain $b^{u-v} \subseteq rad(P)$. Take any $x \in b^{u-v}$. Then $x^n \subseteq P$ for some $n \in \mathbb{N}$. Since $b^{n(u-v)} \cap P \neq \varnothing$, we have $b^{n(u-v)} \subseteq P$ which means $b \in rad(P)$. Thus P is a primary hyperideal of A.

($\Longleftarrow$) It is obvious. □

Theorem 2.14. *Let P be a strong $\mathcal{C}$-hyperideal of A and $u, v \in \mathbb{N}$ with $u > v$. If P is a $(u+1, v)$-absorbing primary hyperideal of A, then P is a (u, v)-absorbing primary hyperideal of A or A is a local multiplicative hyperring with maximal hyperideal M such that $rad(P) = M$.*

Proof. Let P be a $(u+1, v)$-absorbing primary hyperideal of A. Assume that P is not a (u, v)-absorbing primary hyperideal of A. Then there exist

$x_1,\ldots,x_u \in A\backslash U(A)$ such that $x_1 \circ \cdots \circ x_u \subseteq P$, $x_1 \circ \cdots \circ x_v \nsubseteq P$ and $x_{v+1} \circ \cdots \circ x_u \nsubseteq rad(P)$. On the other hand, we conclude that A is a local multiplicative hyperring by Theorem 2.12 as P is a $(u+1,v)$-absorbing primary hyperideal of A that is not a (u,v)-absorbing primary hyperideal of A. Let M be the only maximal hyperideal of A and $a \in M$. Since P is a $(u+1,v)$-absorbing primary hyperideal of A, $x_1 \circ \cdots \circ x_u \circ a \subseteq P$ and $x_1 \circ \cdots \circ x_v \nsubseteq P$, we get $x_{v+1} \circ \cdots \circ x_u \circ a \subseteq rad(P)$. By Theorem 2.7, $rad(P)$ is a prime hyperideal of A. Let $x \in x_{v+1} \circ \cdots \circ x_u$. If $x \in rad(P)$, then $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$, a contradiction. Then $x \notin rad(P)$. Since $x \circ a \subseteq rad(P)$, we get $a \in rad(P)$ which shows $rad(P) = M$, as required. □

Proposition 2.15. *Assume that $P_1,\ldots,P_n$ are (u,v)-absorbing primary $\mathcal{C}$-hyperideals of A where $u,v \in \mathbb{N}$ with $u > v$. If $rad(P_i) = rad(P_j)$ for all $i,j \in \{1,\ldots,n\}$, then $\cap_{i=1}^n P_i$ is (u,v)-absorbing primary hyperideal of A.*

Proof. Let $P_1,\ldots,P_n$ be (u,v)-absorbing primary $\mathcal{C}$-hyperideals of A. Then $rad(P_i)$ is a prime hyperideal of A for all $i \in \{1,\ldots,n\}$ by Theorem 2.7. Let us assume $rad(P_i) = Q$ for all $i \in \{1,\ldots,n\}$ where Q is a prime hyperideal of A. Put $P = \cap_{i=1}^n P_i$. Let $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1,\ldots,x_u \in A\backslash U(A)$ such that $x_1 \circ \cdots \circ x_v \nsubseteq P$. Then there exists $t \in \{1,\ldots,n\}$ such that $x_1 \circ \cdots \circ x_v \nsubseteq P_t$. Since P_t is a (u,v)-absorbing primary hyperideals of A and $x_1 \circ \cdots \circ x_u \subseteq P_t$, we conclude that $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P_t) = Q = rad(P)$ by Proposition 3.3 in [12]. Thus $P = \cap_{i=1}^n P_i$ is a (u,v)-absorbing primary hyperideal of A. □

Note that the conditions "$rad(P_i) = rad(P_j)$ for all $i,j \in \{1,\ldots,n\}$" in Proposition 2.15 can not be ignored.

Example 2.16. Consider the multiplicative hyperring $(\mathbb{Z}_\Phi, +, \circ)$ where $\Phi = \{2,4\}$. The hyperideals $\langle 3\rangle, \langle 5\rangle$ and $\langle 7\rangle$ are $(3,2)$-absorbing primary $\mathcal{C}$-hyperideals of A but $rad(\langle 3\rangle) \neq rad(\langle 5\rangle) \neq rad(\langle 7\rangle)$ and $\langle 150\rangle = \langle 3\rangle \cap \langle 5\rangle \cap \langle 7\rangle$ is not $(3,2)$-absorbing primary hyperideals of A.

Now, we give a a characterization of $(v+1,v)$-absorbing primary hyperideals of A.

Theorem 2.17. *Let P be a $\mathcal{C}$-hyperideal of A and $u,v \in \mathbb{N}$ with $u > v$. Then the followings are equivalent.*

(i) *P is a $(v+1,v)$-absorbing primary hyperideal of A.*

(ii) *If* $a_1 \circ \cdots \circ a_v \nsubseteq P$ *for all* $a_1, \ldots, a_v \in A \backslash U(A)$*, then* $(P : a_1 \circ \cdots \circ a_v) \subseteq rad(P)$*.*

(iii) *If* $a_1 \circ \cdots \circ a_v \circ Q \subseteq P$ *for some hyperideal* Q *of* A *and* $a_1, \ldots, a_v \in A \backslash U(A)$*, then* $a_1 \circ \cdots \circ a_v \subseteq P$ *or* $Q \subseteq rad(P)$*.*

(iv) *If* $P_1 \circ \cdots \circ P_v \circ P_{v+1} \subseteq P$ *for proper hyperideals* $P_1, \ldots, P_{v+1}$ *of* A*, then* $P_1 \circ \cdots \circ P_v \subseteq P$ *or* $P_{v+1} \subseteq rad(P)$*.*

Proof. (i) $\Longrightarrow$ (ii) Assume that P is a $(v+1, v)$-absorbing primary hyperideal of A and $a_1 \circ \cdots \circ a_v \nsubseteq P$ for all $a_1, \ldots, a_v \in A \backslash U(A)$. Let $b \in (P : a_1 \circ \cdots \circ a_v)$. So $a_1 \circ \cdots \circ a_v \circ b \subseteq P$. Assume that $b \in U(A)$. Then we have $a_1 \circ \cdots \circ a_v \subseteq a_1 \circ \cdots \circ a_v \circ 1 \subseteq a_1 \circ \cdots \circ a_v \circ b \circ b^{-1} \subseteq P$, a contradiction. Then $b \notin U(A)$. By the hypothesis, we conclude that $b \in rad(P)$ as $a_1 \circ \cdots \circ a_v \nsubseteq P$. Thus we have $(P : a_1 \circ \cdots \circ a_v) \subseteq rad(P)$.

(ii) $\Longrightarrow$ (iii) Assume that $a_1 \circ \cdots \circ a_v \circ Q \subseteq P$ for some hyperideal Q of A and $a_1, \ldots, a_v \in A \backslash U(A)$ such that $a_1 \circ \cdots \circ a_v \nsubseteq P$. From $a_1 \circ \cdots \circ a_v \circ Q \subseteq P$, it follows that $Q \subseteq (P : a_1 \circ \cdots \circ a_v)$. Thus, we conclude that $Q \subseteq rad(P)$ by (ii).

(iii) $\Longrightarrow$ (iv) Let $P_1 \circ \cdots \circ P_v \circ P_{v+1} \subseteq P$ for proper hyperideals $P_1, \ldots, P_v$, P_{v+1} of A such that $P_1 \circ \cdots \circ P_v \nsubseteq P$. This implies that $a_1 \circ \cdots \circ a_v \nsubseteq P$ for some $a_1 \in P_1, \ldots, a_v \in P_v$. From $a_1 \circ \cdots \circ a_v \circ P_{v+1} \subseteq P$ it follows that $P_{v+1} \subseteq rad(P)$ by (iii).

(iv) $\Longrightarrow$ (i) Let $a_1 \circ \cdots \circ a_v \circ a_{v+1} \subseteq P$ for $a_1, \ldots, a_v, a_{v+1} \in A \backslash U(A)$. Then we have $\langle a_1 \rangle \circ \cdots \circ \langle a_v \rangle \circ \langle a_{v+1} \rangle \subseteq \langle a_1 \circ \cdots \circ a_v \circ a_{v+1} \rangle \subseteq P$ by Proposition 2.15 in [12]. By (iv), we conclude that $\langle a_1 \rangle \circ \cdots \circ \langle a_v \rangle \subseteq P$ or $\langle a_{v+1} \rangle \subseteq rad(P)$. This implies that $a_1 \circ \cdots \circ a_v \subseteq P$ or $a_{v+1} \in rad(P)$. Consequently, P is a $(v+1, v)$-absorbing primary hyperideal of A. $\square$

Assume that A is a hyperdomain with quotient hyperfield F. Recall from [16] that a hyperdomain A is said to be a Dedekind hyperdomain if every nonzero proper hyperideal of A is invertible, that is, for every nonzero proper hyperideal P of A, $P \circ P^{-1} = A$ where $P^{-1} = \{x \in F \mid x \circ P \subseteq A\}$. Now, we examine when a $\mathcal{C}$-hyperideal in a Dedekind hyperdomain is a (u, v)-absorbing primary hyperideal.

Theorem 2.18. *Let* A *be a Dedekind hyperdomain,* P *a* $\mathcal{C}$*-hyperideal of* A *and* $u, v \in \mathbb{N}$ *with* $u > v$*. Then* P *is a* (u, v)*-absorbing primary hyperideal of* A *if and only if* $rad(P)$ *is a prime hyperideal of* A*.*

Proof. ($\Longrightarrow$) Since P is a (u, v)-absorbing primary hyperideal of A, the calim follows from 2.7.

($\Longleftarrow$) Let $rad(P)$ be a prime hyperideal of A. By the hypothesis, $rad(P)$ is maximal. Assume that $x \circ y \subseteq P$ and $y \notin rad(P)$. Since $rad(P)$ is maximal hyperideal of A, then $\langle y, rad(P)\rangle = A$. Then there exists $a \in rad(P)$ and $s \in A$ such that $1 \in y \circ s + a$. So, we obtain $1 = r + a$ for some $r \in y \circ s$. Since $a \in rad(P)$, there exists $n \in \mathbb{N}$ such that $a^n \subseteq P$. Therefore we have $1 \in (r + a)^n \subseteq \sum_{j=0}^{n-1} \binom{n}{j} r^{n-j} \circ a^j + a^n$. If follows that $x \in x \circ 1 \subseteq x \circ (\sum_{j=0}^{n-1} \binom{n}{j} r^{n-j} \circ a^j + a^n) \subseteq \sum_{j=0}^{n-1} \binom{n}{j} x \circ r^{n-j} \circ a^j + x \circ a^n \subseteq \sum_{j=0}^{n-1} \binom{n}{j} x \circ (y \circ s)^{n-j} \circ a^j + x \circ a^n \subseteq P$ which shows P is a primary hyperideal of A. Thus we conclude that P is a (u, v)-absorbing primary hyperideal of A. $\square$

Recall from [16] that a hyperring is divided if for each prime hyperideal Q of A, we have $Q \subseteq \langle a\rangle$ for every $a \in A\backslash Q$. The following theorem shows that a $\mathcal{C}$-hyperideal of a divided multiplicative hyperring A is a (u, v)-absorbing primary hyperideal if and only if it is a primary hyperideal.

Theorem 2.19. *Let A be a divided multiplicative hyperring, P a $\mathcal{C}$-hyperideal of A and $u, v \in \mathbb{N}$ with $u > v$. Then P is a (u, v)-absorbing primary hyperideal of A if and only if P is a primary hyperideal of A.*

Proof. ($\Longrightarrow$) Let P be a (u, v)-absorbing primary hyperideal of A and $a \circ b \subseteq P$ for $a, b \in A\backslash U(A)$ but $b \notin rad(P)$. Since P is a (u, v)-absorbing primary hyperideal of A, $rad(P)$ is a prime hyperideal of A by Theorem 2.7. This implies that $a \in rad(P)$ as $a \circ b \subseteq rad(P)$ and $b \notin rad(P)$. Take any $c \in b^{v-1}$. Since A is a divided multiplicative hyperring and $c \notin rad(P)$, we get $rad(P) \subseteq \langle c\rangle$. Since $a \in rad(P)$, there exists $r \in A$ such that $a \in c \circ r$. Let $r \in U(A)$. Then $a \circ r^{-1} \subseteq c \circ r \circ r^{-1}$. Since $rad(P)$ is a $\mathcal{C}$-hyperideal of A and $c \circ r \circ r^{-1} \cap rad(P) \neq \varnothing$, we obtain $c \in c \circ 1 \subseteq c \circ r \circ r^{-1} \subseteq rad(P)$, a contradiction. Hence Let $r \notin U(A)$. Since P is a (u, v)-absorbing primary hyperideal of A and $a \circ b^{u-v} \subseteq b^{v-1} \circ r \circ b^{u-v} \subseteq P$, we conclude that $b^{v-1} \circ r \subseteq P$ and so $a \in P$. Consequently, P is a primary hyperideal of A.

($\Longleftarrow$) It is straightforward. $\square$

Recall from [14] that a mapping η from the commutative multiplicative hyperring $(A_1, +_1, \circ_1)$ into the commutative multiplicative hyperring

$(A_2, +_2, \circ_2)$ is a hyperring good homomorphism if $\eta(x +_1 y) = \eta(x) +_2 \eta(y)$ and $\eta(x \circ_1 y) = \eta(x) \circ_2 \eta(y)$ for every $x, y \in A_1$.

Theorem 2.20. *Suppose that A_1 and A_2 are two commutative multiplicative hyperrings such that the mapping η from A_1 into A_2 is a hyperring good homomorphism, $\eta(x) \notin U(A_2)$ for each $x \in A_1 \backslash U(A_1)$ and $u, v \in \mathbb{N}$ with $u > v$. Then the followings are satisfied:*

(i) *If P_2 is a (u, v)-absorbing primary $\mathcal{C}$-hyperideal of A_2, then $\eta^{-1}(P_2)$ is a (u, v)-absorbing primary $\mathcal{C}$-hyperideal of A_1.*

(ii) *If P_1 is a is a (u, v)-absorbing primary $\mathcal{C}$-hyperideal of A_1 with $Ker(\eta) \subseteq P_1$ and η is surjective, then $\eta(P_1)$ is a (u, v)-absorbing primary $\mathcal{C}$-hyperideal of A_2.*

Proof. (i) Assume that P_2 is a (u, v)-absorbing primary $\mathcal{C}$-hyperideal of A_2. Now, let $x_1 \circ_1 \cdots \circ_1 x_u \subseteq \eta^{-1}(P_2)$ for $x_1, \ldots, x_u \in A_1 \backslash U(A_1)$. Then we have $\eta(x_1 \circ_1 \cdots \circ_1 x_u) = \eta(x_1) \circ_2 \cdots \circ_2 \eta(x_u) \subseteq P_2$ as η is a good homomorphism. Since P_2 is a (u, v)-absorbing primary hyperideal of A_2 and $\eta(x_1), \ldots, \eta(a_u) \notin U(A_2)$, we get $\eta(x_1 \circ_1 \cdots \circ_1 x_v) = \eta(x_1) \circ_2 \cdots \circ_2 \eta(x_v) \subseteq P_2$ which means $x_1 \circ_1 \cdots \circ_1 x_v \subseteq \eta^{-1}(P_2)$ or $\eta(x_{v+1} \circ_1 \cdots \circ_1 x_u) = \eta(x_{v+1}) \circ_2 \cdots \circ_2 \eta(x_u) \subseteq rad(P_2)$ which implies $x_{v+1} \circ_1 \cdots \circ_1 x_u \subseteq \eta^{-1}(rad(P_2)) \subseteq rad(\eta^{-1}(P_2))$. Hence, $\eta^{-1}(P_2)$ is a (u, v)-absorbing primary hyperideal of A_1.

(ii) Suppose that P_1 is a (u, v)-absorbing primary $\mathcal{C}$-hyperideal of A_1 such that $Ker(\eta) \subseteq P_1$ and η is surjective. Now, assume that $y_1 \circ_2 \cdots \circ_2 y_u \subseteq \eta(P_1)$ for $y_1, \ldots, y_u \in A_2 \backslash U(A_2)$. Then there exists $x_i \in A_1 \backslash U(A_1)$ with $\eta(x_i) = y_i$ for every $i \in \{1, \cdots, , u\}$ as η is surjective. Hence, we have $\eta(x_1 \circ_1 \cdots \circ_1 x_u) = \eta(x_1) \circ_2 \cdots \circ_2 \eta(x_u) \subseteq \eta(P_1)$. Now, take any $p \in x_1 \circ_1 \cdots \circ_1 x_u$. Then we get $\eta(p) \in \eta(x_1 \circ_1 \cdots \circ_1 x_u) \subseteq \eta(P_1)$ and so there exists $q \in P_1$ such that $\eta(p) = \eta(q)$. Then we get $\eta(p - q) = 0$ which means $p - q \in Ker(\eta) \subseteq P_1$ and so $p \in P_1$. Therefore we conclude that $x_1 \circ_1 \cdots \circ_1 x_u \subseteq P_1$ as P_1 is a $\mathcal{C}$-hyperideal. Since P_1 is a (u, v)-absorbing primary hyperideal of A_1, we obtain $x_1 \circ_1 \cdots \circ_1 x_v \subseteq P_1$ or $x_{v+1} \circ_1 \cdots \circ_1 x_u \subseteq rad(P_1)$. This implies that $y_1 \circ_2 \cdots \circ_2 y_v = \eta(x_1 \circ_1 \cdots \circ_1 x_v) \subseteq \eta(P_1)$ or $y_{v+1} \circ_2 \cdots \circ_2 y_u = \eta(x_{v+1} \circ_1 \cdots \circ_1 x_u) \subseteq \eta(rad(P_1)) = rad(\eta(P_1))$. Thus, $\eta(P_1)$ is a (u, v)-absorbing primary hyperideal of A_2. $\square$

Now, we have the following result.

Corollary 2.21. *Let the hyperideal P of A be a subset of the $\mathcal{C}$-hyperideal Q of A, $x+P \notin U(A/P)$ for all $x \in A \backslash U(A)$ and $u,v \in \mathbb{N}$ with $u>v$. Then Q is a (u,v)-absorbing primary hyperideal of A if and only if Q/P is a (u,v)-absorbing primary hyperideal of A/P.*

Proof. Consider the good epimorphism $\eta : A \longrightarrow A/P$ defined by $\eta(a) = a+P$. Now, the claim follows from Theorem 2.20. □

For any given multiplicative hyperring A, $M_m(A)$ denotes the set of all hypermatrices of A. Let $I=(I_{ij})_{m\times m}, J=(J_{ij})_{m\times m} \in P^\star(M_m(A))$. Then $I \subseteq J$ if and only if $I_{ij} \subseteq J_{ij}$ [2].

Theorem 2.22. *Let $u,v \in \mathbb{N}$ with $u>v$ and P be a hyperideal of A. If $M_m(P)$ is a (u,v)-absorbing primary $\mathcal{C}$-hyperideal of $M_m(A)$, then P is a (u,v)-absorbing primary $\mathcal{C}$-hyperideal of A.*

Proof. Let $x_1 \circ \cdots \circ x_n \cap P \neq \varnothing$ for $x_1,\ldots,x_n \in A$. Then we have

$$\begin{pmatrix} x_1 \circ \cdots \circ x_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cap M_m(P) \neq \varnothing$$

which means

$$\left(\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} x_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \right) \cap M_m(P) \neq \varnothing.$$

Since $M_m(P)$ is a $\mathcal{C}$-hyperideal of $M_m(A)$, we obtain

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} x_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_m(P).$$

and so

$$\begin{pmatrix} x_1 \circ \cdots \circ x_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_m(P).$$

This means $x_1 \circ \cdots \circ x_n \subseteq P$. Hence P is a $\mathcal{C}$-hyperideal of A. Now, let $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1, \ldots, x_u \in A \backslash U(A)$. Then we obtain

$$\begin{pmatrix} x_1 \circ \cdots \circ x_u & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_m(P).$$

Since $M_m(P)$ is a (u,v)-absorbing primary hyperideal of $M_m(A)$ and

$$\begin{pmatrix} x_1 \circ \cdots \circ x_u & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} x_u & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

we conclude that

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} x_v & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \circ \cdots \circ x_v & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_m(P)$$

or

$$\begin{pmatrix} x_{v+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} x_u & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_{v+1} \circ \cdots \circ x_u & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq rad(M_m(P)).$$

In the first possibility, we get $x_1 \circ \cdots \circ x_v \subseteq P$. In the second possibility, there exists $n \in \mathbb{N}$ such that

$$\begin{pmatrix} (x_{v+1} \circ \cdots \circ x_u)^n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} x_{v+1} \circ \cdots \circ x_u & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}^n \subseteq M_n(P)$$

which implies $(x_{v+1} \circ \cdots \circ x_u)^n \subseteq P$ and so $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$. Thus we conclude that P is a (u,v)-absorbing primary hyperideal of A. $\square$

A non-empty subset S of A containing 1 refers to a multiplicative closed subset (briefly, MCS) if S is closed under the hypermultiplication [2]. Consider the set $(A \times S/ \sim)$ of equivalence classes denoted by $S^{-1}A$ such that $(x,r) \sim (y,s)$ if and only if there exists $t \in S$ with $t \circ r \circ y = t \circ s \circ x$. The equivalence class of $(x,r) \in A \times S$ is denoted by $\frac{x}{r}$. The triple $(S^{-1}A, \oplus, \odot)$ is a commutative multiplicative hyperring where

$$\frac{x}{r} \oplus \frac{y}{s} = \frac{r \circ y + s \circ x}{r \circ s} = \{\frac{a+b}{c} \mid a \in r \circ y, b \in s \circ x, c \in r \circ s\}$$

$$\frac{x}{r} \odot \frac{y}{s} = \frac{x \circ y}{r \circ s} = \{\frac{a}{b} \mid a \in x \circ y, b \in r \circ s\}$$

The localization map $\pi : A \longrightarrow S^{-1}A$, defined by $a \mapsto \frac{a}{1}$, is a homomorphism of hyperrings. Furthermore, if I is a hyperideal of A, then $S^{-1}P$ is a hyperideal of $S^{-1}P$ [21]. Next, we discuss the relationship between (u,v)-absorbing primary hyperideals and their localizations.

Theorem 2.23. *Assume that P is a $\mathcal{C}$-hyperideal of A, S a MCS such that $P \cap S = \varnothing$ and $u, v \in \mathbb{N}$ with $u > v$. If P is a (u,v)-absorbing primary hyperideal of A, then $S^{-1}P$ is a $(u-1, v-1)$-absorbing primary hyperideal of $S^{-1}A$.*

Proof. Suppose that $\frac{x_1}{r_1} \odot \cdots \odot \frac{x_{u-1}}{r_{u-1}} = \frac{x_1 \circ \cdots \circ x_{u-1}}{r_1 \circ \cdots \circ r_{u-1}} \subseteq S^{-1}P$ for $r_1, \ldots, r_{u-1} \in S$ and $x_1, \ldots, x_{u-1} \in A \backslash U(A)$ but $\frac{x_1}{r_1} \odot \cdots \odot \frac{x_{v-1}}{r_{v-1}} \nsubseteq S^{-1}Q$. Take any $r \in r_1 \circ \cdots \circ r_{u-1}$ and $x \in x_1 \circ \cdots \circ x_{u-1}$. Therefore $\frac{x}{r} \in \frac{x_1 \circ \cdots \circ x_{u-1}}{r_1 \circ \cdots \circ r_{u-1}}$ and so $\frac{x}{r} = \frac{x'}{r'}$ for some $r' \in S$ and $x' \in P$. Then, we conclude that $t \circ x \circ r' = t \circ x' \circ r$ for some $t \in S$. This implies that $t \circ x \circ r' \subseteq P$. Since $x \in x_1 \circ \cdots \circ x_{u-1}$, we obtain $t \circ x \circ r' \subseteq t \circ x_1 \circ \cdots \circ x_{u-1} \circ r'$. Since P is a $\mathcal{C}$-hyperideal of A and $t \circ x_1 \circ \cdots \circ x_{u-1} \circ r' \cap P \neq \varnothing$, we get $t \circ x_1 \circ \cdots \circ x_{u-1} \circ r' \subseteq P$. Take any $w \in t \circ r'$. If $w \circ x_1 \circ \cdots \circ x_{v-1} \subseteq P$, then we get $\frac{x_1}{r_1} \odot \cdots \odot$

$\frac{x_{v-1}}{r_{v-1}} = \frac{x_1 \circ \cdots \circ x_{v-1}}{r_1 \circ \cdots \circ r_{v-1}} = \frac{w \circ x_1 \circ \cdots \circ x_{v-1}}{w \circ r_1 \circ \cdots \circ r_{v-1}} \subseteq S^{-1}P$, a contradiction. Since P is a (u,v)-absorbing primary hyperideal of A, $w \circ x_1 \circ \cdots \circ x_{u-1} \subseteq P$ and $w \circ x_1 \circ \cdots \circ x_{v-1} \nsubseteq P$, we have $x_v \circ \cdots \circ x_{u-1} \subseteq rad(P)$ which implies $\frac{x_v}{r_v} \odot \cdots \odot \frac{x_{u-1}}{r_{u-1}} = \frac{x_v \circ \cdots \circ x_{u-1}}{r_v \circ \cdots \circ r_{u-1}} \subseteq S^{-1}(rad(P)) = rad(S^{-1}P)$. Consequently, $S^{-1}P$ is a $(u-1, v-1)$-absorbing primary hyperideal of $S^{-1}A$. □

Theorem 2.24. *Let P be a $\mathcal{C}$-hyperideal of A, S be a MCS with $\Gamma \cap S = \varnothing$ where $\Gamma = \{a \in A \mid a \circ b \subseteq P$ for some $b \in A \backslash P\}$ and $u, v \in \mathbb{N}$ with $u > v$. If $S^{-1}P$ is a (u,v)-absorbing primary hyperideal of A, then P is a (u,v)-absorbing primary hyperideal of A.*

Proof. Assume that $x_1 \circ \cdots \circ x_u \subseteq P$ for $x_1, \ldots, x_u \in A \backslash U(A)$. Then we have $\frac{x_1 \circ \cdots \circ x_u}{1 \circ \cdots \circ 1} = \frac{x_1}{1} \odot \cdots \odot \frac{x_u}{1} \subseteq S^{-1}P$. Since $S^{-1}P$ is a (u,v)-absorbing primary hyperideal of A, we conclude that $\frac{x_1 \circ \cdots \circ x_v}{1 \circ \cdots \circ 1} = \frac{x_1}{1} \odot \cdots \odot \frac{x_v}{1} \subseteq S^{-1}P$ or $\frac{x_{v+1} \circ \cdots \circ x_u}{1 \circ \cdots \circ 1} = \frac{x_{v+1}}{1} \odot \cdots \odot \frac{x_u}{1} \subseteq rad(S^{-1}P) = S^{-1}rad(P)$. In the first case, we have $\frac{a}{1} \in S^{-1}P$ for some $a \in x_1 \circ \cdots \circ x_v$. Therefore there exists $p \in P$ and $t \in S$ such that $\frac{a}{1} = \frac{p}{t}$ and so $s \circ t \circ a = s \circ p \circ 1$ for some $s \in S$ which means $s \circ t \circ a \subseteq P$. Take any $r \in s \circ t$. Since $\Gamma \cap S = \varnothing$ and $r \circ a \subseteq P$, we have $a \in P$. Since P is a $\mathcal{C}$-hyperideal of A and $(x_1 \circ \cdots \circ x_v) \cap P \neq \varnothing$, we obtain $x_1 \circ \cdots \circ x_v \subseteq P$. In the second case, we get $\frac{b}{1} \in S^{-1}rad(P)$ for some $b \in x_{v+1} \circ \cdots \circ x_u$. Therefore there exists $q \in rad(P)$ and $t \in S$ such that $\frac{b}{1} = \frac{q}{t}$ and so $s \circ t \circ b = s \circ q \circ 1$ for some $s \in S$ which means $s \circ t \circ b \subseteq rad(P)$. Assume that $c \in r \circ b$ such that $r \in s \circ t$. Then there exists $n \in \mathbb{N}$ such that $c^n \subseteq P$. since P is a $\mathcal{C}$-hyperideal of A and $(r \circ x_{v+1} \circ \cdots \circ x_u)^n \cap P \neq \varnothing$, we get $(r \circ x_{v+1} \circ \cdots \circ x_u)^n \subseteq P$. Let $\alpha \in r^n$ and $\beta \in (x_{v+1} \circ \cdots \circ x_u)^n$. Since $\Gamma \cap S = \varnothing$, $\alpha \circ \beta \subseteq P$ and $\alpha \notin \Gamma$, we have $\beta \in P$ and so $(x_{v+1} \circ \cdots \circ x_u)^n \subseteq P$ which means $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$. Thus, P is a (u,v)-absorbing primary hyperideal of A. □

3 Conclusion

In this paper, we introduced and investigated an expansion of 1-absorbing primary hyperideals in multiplicative hyperrings called (u,v)-absorbing primary hyperideals where $u, v \in \mathbb{Z}$ with $u > v$. We gave several specific results explaining this new structure. We indicated that the concepts of (u,v)-absorbing prime hyperideals and (u,v)-absorbing primary hyperideals are different, although every (u, v)-absorbing prime hyperideal is a (u, v)-absorbing primary hyperideal. We analyzed when a product of two hyperideals is a (u, v)-absorbing primary hyperideal. Furthermore, it was examined when $(P : x)$ is a (u, v)-absorbing primary hyperideal. We presented a condition by which the intersection of some (u, v)-absorbing primary hyperideals is a (u, v)-absorbing primary hyperideal and then gave an example showing this condition is crucial. We concluded that a $\mathcal{C}$-hyperideal in a Dedekind hyperdomain is a (u, v)-absorbing primary hyperideal if and only if it's radical is a prime hyperideal. Also, we investigated when a $\mathcal{C}$-hyperideal in a divided multiplicative hyperring A is a (u, v)-absorbing primary hyperideal. Moreover, we studied the stability of (u, v)-prime hyperideals with respect to localization. The study can be continued for other classes of hyperstructures.

4 Future work

In [1], Akray and Anjuman proposed the notion of v-absorbing I-primary hyperideals in a multiplicative hyperring. As future work, we will study the concept (u, v)-absorbing I-primary hyperideal as a generalization of (u, v)-absorbing primary hyperideals.

Definition 4.1. Let P be a proper hyperideal of A and $u, v \in \mathbb{N}$ with $u > v$. For fixed proper hyperideal I of A, we say that P is a (u, v)-*absorbing* I-*primary hyperideal* if $x_1 \circ \cdots \circ x_u \subseteq P \backslash IP$ for $x_1, \ldots, x_u \in A \backslash U(A)$, then either $x_1 \circ \cdots \circ x_v \subseteq P$ or $x_{v+1} \circ \cdots \circ x_u \subseteq rad(P)$.

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