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Quasi pseudo equality algebras (BCK-algebras)

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Abstract. In this paper, by using the notions of "quasi" and "pseudo" in logical algebras, we introduce two generalizations of equality algebras. A commutative generalization of equality algebras, is called quasi-equality algebras and a non-commutative generalization of quasi-equality algebras is called quasi-pseudo equality algebras. Then we investigate some of their properties. In addition, according to [22] and knowing the relation between equality algebras and BCK(C)-meet-semilattices, we generalize the concepts of BCK-algebras to quasi-BCK-algebras and pseudo BCK-algebras to quasi-pseudo BCK-algebras, too. The related properties and the relation between different kinds of quasi-(pseudo) BCK-algebras are investigated. Moreover, we investigate the category of quasi-(pseudo) equality algebras and quasi-(pseudo) BCK-algebras and we show that they are equivalent.

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1 Introduction

Logical systems are established algebra of truth values, which includes, a set of basic connectives. Each of the connectives correspond with a basic operation of the underlying algebra. This enables to define semantics by assigning truth values to formulas so that each truth value can be computed using algebraic operations of the chosen algebra.

In [19], Henkin developed a formalism of type theory with equality (identity) representing equality between various kinds of objects as a sole basic connective. Fuzzy type theory (FTT) has been developed by V. Novák [28], as a fuzzy logic of higher order, the fuzzy version of the classical type theory of the classical logic of higher order. Since the first algebraic models for the set of truth values of FTT are residuated lattices, their basic operations are $\wedge$ (meet), $\vee$ (join), $\odot$ (multiplication) and $\rightarrow$ (residuum). In fuzzy logic the last operation is a semantic interpretation of the implication, while the logical equivalence is interpreted by the biresiduum $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$, which is a derived operation. Thus, we face a conceptual discrepancy, a basic connective is semantically interpreted by a derived operation. (For more information of equality algebras see [1–3, 5, 6, 13, 17, 18, 25, 27]). The corresponding algebra should express certain fundamental properties of fuzzy equality and, at the sametime, be able to serve as an algebra of truth values, i.e. it must be ordered and provide the implication. We need a specific algebra of truth values for the fuzzy type theory. The first version of such an algebra has been introduced by V. Novák [29, 30] under the name of *EQ-algebra* and a new concept of fuzzy type theory has been developed based on EQ-algebras [31]. The operation $\sim$ in EQ-algebras is a fuzzy equality and the implication $\rightarrow$ is defined by $x \rightarrow y = (x \wedge y) \sim x$, hence the tie between multiplication and residuation is weaker than in the case of residuated lattices. In this sense, EQ-algebras generalize the residuated lattices. As it was mentioned in [22], if the product operation in EQ-algebras is replaced by another binary operation smaller than or equal to the original product we still obtain an EQ-algebra, and this fact might make it difficult to obtain certain algebraic results. For this reason, a new structure was introduced by S. Jenei in [22], called *equality algebra* consisting of two binary operations meet and equivalence, and constant 1. It was proved in [10, 11, 22] that any equality algebra has a corresponding BCK(C)-meet-semilattice and any BCK(D)-meet-semilattice (with distributivity property) has a corresponding equality algebra. Since equality algebras could also be candidates for a possible algebraic semantics for fuzzy type theory, their study is highly motivated. As a generalization of equality algebras, Jenei and Kóródi introduced in [23] a concept of *pseudo equality algebras* and proved that the pseudo equality algebras are term equivalent to pseudo BCK-meet-semilattices. In [10] a gap was found in the proof of this result and a counterexample was given as well as a correct version of it. Moreover, Dvurevcenskij and Zahiri showed in [12] that every pseudo equality algebra in the sense of [23] is an equality algebra and they defined and investigated a new concept of pseudo equality algebras and established a connection between pseudo equality algebras and a special class of pseudo BCK-meet-semilattices (pseudo BCK(pC)-meet-semilattices).

The main idea of this paper is inspired by the article [26]. The original motivation for their study arise in connection with quantum computation, namely in an attempt to provide a convenient abstraction of the algebra over the set of all density operators of the Hilbert space $\mathbb{C}^2$, endowed with a suitable stock of quantum logical gates. Then by using this notion, they defined quasi MV-algebras as a generalization of MV-algebras. After that other mathematician studied the notion of "quasi" on the other logical algebras such as BL-algebras and hoops and some related properties are investigated (see [7, 9]). For this reason, we decided to study the notion of "quasi" on equality algebras.

In this paper, we introduce a commutative generalization of equality algebras, called quasi-equality algebras and a non-commutative generalization of quasi-equality algebras called quasi pseudo equality algebras. We investigate some of their properties. Also, we generalize the concepts of BCK-algebras to quasi BCK-algebras and pseudo BCK-algebras to quasi pseudo BCK-algebras. Some related properties and relation between different kinds of quasi (pseudo) BCK-algebras are investigated. Moreover, we find that under which conditions, quasi pseudo equality algebras (quasi-equality algebras) are equivalent to quasi pseudo BCK-meet-semilattices (quasi BCK-meet-semilattices). We prove that under some conditions, the category of quasi (pseudo) equality algebras and quasi (pseudo) BCK-algebras are equivalent.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections and we shall not cite them every time they are used, when there are no ambiguity.

Definition 2.1. [22] An algebraic structure $\mathcal{E} = (E, \wedge, \sim, 1)$ of the type $(2, 2, 0)$ is called an *equality algebra*, if it satisfies the following conditions, for all $x, y, z \in E$:
(E1) $(E, \wedge, 1)$ is a meet-semilattice with top element 1,
(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$,
where $x \leq y$ if and only if $x \wedge y = x$.

Definition 2.2. [12] A *pseudo equality algebra* is an algebraic structure $\mathcal{E} = (E, \wedge, \sim, \backsim, 1)$ of the type (2, 2, 2, 0), which satisfies the following conditions, for all $x, y, z \in E$:
(F1) $(E, \wedge, 1)$ is a meet-semilattice with top element 1,
(F2) $x \sim x = 1 = x \backsim x$,
(F3) $x \sim 1 = x = 1 \backsim x$,
(F4) $x \leq y \leq z$ implies $x \sim z \leq y \sim z, x \sim z \leq x \sim y, z \backsim x \leq z \backsim y$ and $z \backsim x \leq y \backsim x$,
(F5) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$ and $x \backsim y \leq (x \wedge z) \backsim (y \wedge z)$,
(F6) $x \sim y \leq (z \sim x) \backsim (z \sim y)$ and $x \backsim y \leq (x \backsim z) \sim (y \backsim z)$,
(F7) $x \sim y \leq (x \sim z) \sim (y \sim z)$ and $x \backsim y \leq (z \backsim x) \backsim (z \backsim y)$.

Definition 2.3. [20] A *BCK-algebra* is an algebraic structure $(B, \to, 1)$ of the type (2, 0) which satisfies the following conditions, for any $x, y, z \in B$:
(BCK1) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(BCK2) $1 \to x = x$,
(BCK3) $x \to 1 = 1$,
(BCK4) $x \to y = 1$ and $y \to x = 1$ imply $x = y$.

Definition 2.4. [14, 21] A *pseudo BCK-algebra* is a structure $(B, \to, \rightsquigarrow, 1)$ of the type (2, 2, 0) which satisfies the following conditions, for any $x, y, z \in$

B:
(PBCK1) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$ and $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,
(PBCK2) $1 \to x = x$ and $1 \rightsquigarrow x = x$,
(PBCK3) $x \to 1 = 1$ and $x \rightsquigarrow 1 = 1$,
(PBCK4) $x \to y = 1 = y \to x$ ($x \rightsquigarrow y = 1 = y \rightsquigarrow x$) implies $x = y$. The partial order $\leq$ is defined by $x \leq y$ if and only if $x \to y = 1$ (if and only if $x \rightsquigarrow y = 1$), for any $x, y \in B$.

Category theory is a branch of mathematics that seeks to generalize all of mathematics in terms of categories, independent of what their objects and arrows represent. A *category* is a collection of "objects" that are linked by "arrows". A category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. A simple example is the category of sets, whose objects are sets and whose arrows are functions. Let C and D be two categories. A *functor* F from categories C to D is a mapping that associates each object X in C to an object $F(X)$ in D, associates each morphism $f : X \to Y$ in C to a morphism $F(f) : F(X) \to F(Y)$ in D such that the following two conditions hold: (i) $F(id_X) = id_{F(X)}$ for every object X in C, (ii) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in C. That is, functors must preserve identity morphisms and composition of morphisms. If there is a one-to-one corresponding between two categories, then we called it *categorical equivalences*. For more details see [4].

3 Quasi-equality algebras and quasi BCK-algebras

In this section, we introduce a commutative generalization of equality algebras, called quasi-equality algebras. We investigate their properties and generalize the concepts of BCK-algebras to quasi BCK-algebras. Also, we find that under which conditions, quasi-equality algebras are equivalent to quasi BCK-meet-semilattices.

Definition 3.1. An algebraic structure $\mathcal{E} = (E, \wedge, \sim, 1)$ of the type $(2, 2, 0)$ is called a *quasi-equality algebra*, if it satisfies the following conditions, for all $x, y, z \in E$:
(QE1) $(E, \wedge, 1)$ is a meet-semilattice with top element 1,

(QE2) $x \sim y = y \sim x$,
(QE3) $x \sim x = 1$,
(QE4) $x \sim y = 1$ implies $y \leq x$,
(QE5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(QE6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(QE7) $x \sim y \leq (x \sim z) \sim (y \sim z)$,
where $x \leq y$ if and only if $x \wedge y = x$.

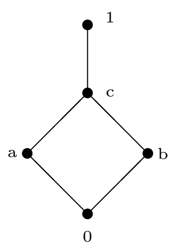
The operation $\wedge$ is called *meet (infimum)* and $\sim$ is called an *equality operation*. Also, other two operations are defined as follows and called *implication* and *equivalence operation*, respectively:

$$x \to y = (x \wedge y) \sim x \tag{I}$$

$$x \leftrightarrow y = (x \to y) \wedge (y \to x) \tag{II}$$

Remark 3.2. According to the above definition, a quasi-equality algebra is a generalization of an equality algebra. It should be pointed out that the commutativity of the operation $\sim$ in a quasi-equality algebra may not imply $x \sim 1 = x$, for any x (See Example 3.3). Hence, a quasi-equality algebra $(E, \wedge, \sim, 1)$ is an equality algebra if and only if $x \sim 1 = x$, for any $x \in E$. A quasi-equality algebra $(E, \wedge, \sim, 1)$ is called *proper* if it's not an equality algebra or equivalently there exists $x \in E$ such that $x \sim 1 \neq x$.

Example 3.3. (i) Let $(E = \{0, a, b, c, 1\}, \leq)$ be a partial order set with the following Hasse diagram. Clearly, $(E, \leq)$ is a lattice. Define the binary operation $\sim$ on E as follows,



$\sim$	0	a	b	c	1
0	1	c	a	a	a
a	c	1	a	a	a
b	a	a	1	c	c
c	a	a	c	1	c
1	a	a	c	c	1

Then $(E, \wedge, \sim, 1)$ is a quasi-equality algebra which is proper, since $0 \sim 1 = a \neq 0$.
(ii) Let $E = (-\infty, 1]$. Define, for any $x, y \in E$, $x \wedge y = \min\{x, y\}$ and

$x \sim y = 1- \mid x - y \mid$. By routine calculations, we can see that $(E, \wedge, \sim, 1)$ is a quasi-equality algebra.
(iii) Let $(E_1, \wedge_1, \sim_1, 1_1)$ and $(E_2, \wedge_2, \sim_2, 1_2)$ be two quasi-equality algebras. Then the Cartesian product $E := E_1 \times E_2$ with the following operations forms a quasi-equality algebra.

$$\begin{aligned}(x_1, x_2) \wedge (y_1, y_2) &:= (x_1 \wedge y_1, x_2 \wedge y_2), \quad \text{for all } x_1, y_1 \in E_1 \text{ and } x_2, y_2 \in E_2\\ (x_1, x_2) \sim (y_1, y_2) &:= (x_1 \sim_1 y_1, x_2 \sim_2 y_2), \quad \text{for all } x_1, y_1 \in E_1 \text{ and } x_2, y_2 \in E_2\\ 1 &:= (1_1, 1_2)\end{aligned}$$

The following proposition provides some properties of quasi-equality algebras.

Proposition 3.4. *Let $(E, \wedge, \sim, 1)$ be a quasi-equality algebra. Then the following properties hold, for all $x, y, z \in E$:*
(i) $x \sim y \leq x \leftrightarrow y \leq y \to x$,
(ii) $x \leq y$ *if and only if* $x \to y = 1$,
(iii) $x \to x = x \to 1 = 1$,
(iv) $x \leq y$ *implies* $z \to x \leq z \to y$,
(v) $y \leq z$ *implies* $z \to x \leq y \to x$,
(vi) $x \to y \leq (y \to z) \to (x \to z)$,
(vii) $x \to y = x \to (x \wedge y)$,
(viii) $x \to y \leq (x \wedge z) \to (y \wedge z)$,
(ix) $x \sim y \leq (z \to x) \sim (z \to y)$,
(x) $x \to y \leq (z \to x) \to (z \to y)$,
(xi) *if* $x \sim y = 1$, *then* $x = y$,
(xii) $x \sim 1 \leq (x \sim 1) \sim 1$,
(xiii) $y \leq x$, *then* $x \to y = x \leftrightarrow y = x \sim y$.

Proof. (i) By (QE6),

$$x \sim y \leq (x \wedge y) \sim (y \wedge y) = (x \wedge y) \sim y = y \to x.$$

By (QE6) and (QE2),

$$x \sim y \leq (x \wedge x) \sim (x \wedge y) = (x \wedge y) \sim x = x \to y.$$

Thus, $x \sim y \leq x \leftrightarrow y$.

(ii) Let $x \leq y$. Then $x \to y = (x \wedge y) \sim x = x \sim x = 1$. Conversely, $x \to y = 1$ implies $(x \wedge y) \sim x = 1$. By (QE4), $x \leq x \wedge y$, and so $x \leq y$.

(iii) It is straightforward consequence of (ii).

(iv) Since $x \wedge y \wedge z \leq y \wedge z \leq z$, by (QE5), we have $(x \wedge y \wedge z) \sim z \leq (y \wedge z) \sim z$. Since $x \leq y$, $(x \wedge z) \sim z \leq (z \wedge y) \sim z$, and so $z \to x \leq z \to y$.

(v) By (QE6), $(x \wedge z) \sim z \leq (x \wedge z \wedge y) \sim (z \wedge y)$. Since $y \leq z$, we get

$$z \to x = (x \wedge z) \sim z \leq (x \wedge y) \sim y = y \to x.$$

(vi) By (QE6), $(y \wedge z) \sim y \leq (x \wedge y \wedge z) \sim (x \wedge y)$. Then by (v), we have

$$\begin{aligned}&((x \wedge y \wedge z) \sim (x \wedge y)) \to ((x \wedge y \wedge z) \sim x)\\ &\quad \leq ((y \wedge z) \sim y) \to ((x \wedge y \wedge z) \sim x). \qquad (3.1)\end{aligned}$$

Since $x \wedge y \wedge z \leq x \wedge z \leq x$, (QE5) implies that $(x \wedge y \wedge z) \sim x \leq (x \wedge z) \sim x$. Then by (iv),

$$((y \wedge z) \sim y) \to ((x \wedge y \wedge z) \sim x) \leq ((y \wedge z) \sim y) \to ((x \wedge z) \sim x). \qquad (3.2)$$

Hence, by (3.1) and (3.2), we have

$$((x \wedge y \wedge z) \sim (x \wedge y)) \to ((x \wedge y \wedge z) \sim x) \leq ((y \wedge z) \sim y) \to ((x \wedge z) \sim x). \qquad (3.3)$$

Moreover, by (QE7) and (QE2),

$$\begin{aligned}x \to y &= (x \wedge y) \sim x\\ &\leq ((x \wedge y \wedge z) \sim (x \wedge y)) \sim ((x \wedge y \wedge z) \sim x), \quad \text{by (QE7)}\\ &\leq ((x \wedge y \wedge z) \sim (x \wedge y)) \to ((x \wedge y \wedge z) \sim x), \quad \text{by (i)}\\ &\leq ((y \wedge z) \sim y) \to ((x \wedge z) \sim x) \quad \text{by (3.3)}\\ &= (y \to z) \to (x \to z).\end{aligned}$$

(vii) $x \to (x \wedge y) = (x \wedge y \wedge x) \sim x = (x \wedge y) \sim x = x \to y$.

(viii) By (QE6),

$$\begin{aligned}x \to y = (x \wedge y) \sim x &\leq (x \wedge y \wedge z) \sim (x \wedge z)\\ &= ((x \wedge z) \wedge (y \wedge z)) \sim (x \wedge z)\\ &= (x \wedge z) \to (y \wedge z).\end{aligned}$$

(ix) By (QE6) and (QE7),

$$x \sim y \leq (x \wedge z) \sim (y \wedge z) \leq ((x \wedge z) \sim z) \sim ((y \wedge z) \sim z)$$

$$= (z \to x) \sim (z \to y).$$

(x) By (ix), (i) and (iv),

$$\begin{aligned} x \to y = (x \wedge y) \sim x &\leq (z \to (x \wedge y)) \sim (z \to x) \\ &\leq (z \to x) \to (z \to (x \wedge y)) \leq (z \to x) \to (z \to y). \end{aligned}$$

(xi) If $x \sim y = 1$, then by (QE2), we have $y \sim x = 1$. Due to (QE4) we conclude that $x \leq y \leq x$, that is, $x = y$.

(xii) Let $x \in E$. By (QE7) and (QE3) we get that $x \sim 1 \leq (x \sim 1) \sim (1 \sim 1) = (x \sim 1) \sim 1$.

(xiii) Let $y \leq x$. Then $x \to y = x \sim (x \wedge y) = x \sim y$. Since $y \to x = 1$, clearly $x \to y = x \leftrightarrow y = x \sim y$. □

In the following, we generalize the concepts of BCK-algebras to quasi BCK-algebras. Also, we find that under which conditions, quasi-equality algebras are equivalent to quasi BCK-meet-semilattices.

Definition 3.5. A *quasi BCK-algebra* is a structure $(B, \to, 1)$ of the type $(2, 0)$ which satisfies the following conditions, for any $x, y, z \in B$:
(QB1) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(QB2) $x \to x = 1$ and $x \to 1 = 1$,
(QB3) $x \to y = 1 = y \to x$ implies $x = y$.

For any quasi BCK-algebra, we can define a relation "$\leq$" by $x \leq y$ if and only if $x \to y = 1$. In the following, we prove that $\leq$ is a partially order relation on B.

Let $x \in B$. Then by (QB2), we have $x \to x = 1$ and $x \to 1 = 1$, and so $x \leq x$ and $x \leq 1$, thus $\leq$ is reflexive and 1 is the greatest element of B. Consider $x, y \in B$ such that $x \leq y$ and $y \leq x$. So by (QB3), $x = y$, thus $\leq$ is anti-symmetric. Now, if $x \leq y$ and $y \leq z$, then $x \to y = 1$ and $y \to z = 1$. Then by (QB1) we have,

$$1 = (x \to y) \to ((y \to z) \to (x \to z)) = 1 \to (1 \to (x \to z)).$$

Thus, $1 \to (x \to z) \geq 1$, and so $1 \to (x \to z) = 1$. Again, we have $x \to z \geq 1$, and so $x \to z = 1$. Hence, $\leq$ is transitivity. Therefore, $(B, \leq)$ is a poset.

A quasi BCK-meet-semilattice is an algebra $(B, \wedge, \rightarrow, 1)$ of the type (2,2,0) such that $(B, \rightarrow, 1)$ is a quasi BCK-algebra and $(B, \wedge)$ is a meet-semilattice. The equivalence operation $\leftrightarrow$ of B is defined by $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$.

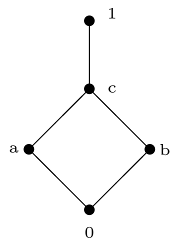
A quasi BCK-meet-semilattice $(B, \wedge, \rightarrow, 1)$ is called a quasi BCK-meet-semilattice with the *contraction* condition, or in short *quasi BCK(C)-meet-semilattice*, if for any $x, y, z \in B$, it satisfies in the condition (C).

$$x \rightarrow y \leq (x \wedge z) \rightarrow (y \wedge z). \qquad (C).$$

A quasi BCK-meet-semilattice $(B, \wedge, \rightarrow, 1)$ is called a quasi BCK-meet-semilattice with the *distributivity* condition, or in short *BCK(D)-meet-semilattice*, if for any $x, y \in B$, it satisfies in the condition (D).

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z). \qquad (D).$$

Example 3.6. (i) Let $B = (\{0, a, b, c, 1\}, \leq)$ be a poset with the following Hasse diagram. Define the operation $\rightarrow$ on B as follows,



$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	c	1	1
b	a	a	1	1	1
c	a	a	c	1	1
1	a	a	c	c	1

Then $(B, \wedge, \rightarrow, 1)$ is a quasi BCK(C)-meet-semilattice.
(ii) Let $B = [0, 1] \times [0, 1]$ and define the operation $\rightarrow$ on B as follows:

$$(x, y) \rightarrow (z, w) = (\max\{0, x - z\}, 0)\,, \quad 0 = (0, 0) \quad \text{and} \quad 1 = (1, 0),$$

for any $(x, y), (z, w) \in B$. Then $(B, \rightarrow, 0, 1)$ is a quasi BCK(D)-meet-semilattice.

Remark 3.7. According to Definition 3.5, quasi BCK-algebras and pseudo BCK-algebras are two generalizations of BCK-algebras. In quasi BCK-algebras, $1 \rightarrow x \neq x$, in general. So, clearly every quasi BCK-algebra is not a BCK-algebra or a pseudo BCK-algebra. The following counterexample medicates that not every pseudo BCK-algebra is a quasi BCK-algebra.

Example 3.8. Let $(B = \{0, a, b, c, 1\}, \leq)$ be a poset. Define the operations $\to$ and $\rightsquigarrow$ on B as follows:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(B, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra which is not a quasi BCK-algebra. Since $1 \to b = b \nleq 0 = (b \to 0) \to (1 \to 0)$, B does not satisfy (QB1).

Note: We said that an algebraic structure $(X, \to, \leq)$ has *Transitivity Condition* if for any $x, y, z \in X$,

$$x \to y \leq (z \to x) \to (z \to y).$$

Proposition 3.9. *Let $(B, \to, 1)$ be a quasi-BCK-semilattice. Then $x \leq y$ implies $y \to z \leq x \to z$.*

Proof. Since $(B, \leq)$ is a poset, 1 is the greatest element of B and $x \leq y$, by (QB1), the proof is clear. □

Proposition 3.10. *Let $(B, \wedge, \to, 1)$ be a quasi $BCK(C)$-meet-semilattice with the Transitivity Condition. Then for all $x, y, z \in E$,*

(i) *if $y \leq z$, then $x \to y \leq x \to z$.*

(ii) $x \to (x \wedge y) = x \to y$.

Proof. (i) By Transitivity Condition the proof is clear.

(ii) By condition (C), $x \to y \leq (x \wedge x) \to (x \wedge y) = x \to (x \wedge y)$. Since $x \wedge y \leq y$, by (i), $x \to (x \wedge y) \leq x \to y$. Thus, $x \to (x \wedge y) = x \to y$. □

Theorem 3.11. *Any quasi-BCK(D)-meet-semilattice with Transitivity Condition is a quasi-BCK(C)-meet-semilattice.*

Proof. Let $(B, \wedge, \to, 1)$ be a quasi-BCK(D)-meet-semilattice. Since $z \wedge x \leq x$, we get

$$x \to y \leq (z \wedge x) \to y \qquad \text{by Proposition 3.9}$$

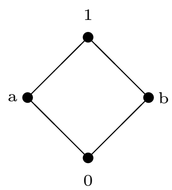
$$= ((z \wedge x) \to (z \wedge x)) \wedge ((z \wedge x) \to y)$$
$$= (z \wedge x) \to (z \wedge x \wedge y) \qquad \text{by (D)}$$
$$= (z \wedge x) \to ((z \wedge x) \wedge (z \wedge y))$$
$$= ((z \wedge x) \to (z \wedge x)) \wedge ((z \wedge x) \to (z \wedge y)) \qquad \text{by (D)}$$
$$= (z \wedge x) \to (z \wedge y)$$

Thus condition (C) holds. Hence, $(B, \wedge, \to, 1)$ is a quasi-BCK(C)-meet-semilattice □

Definition 3.12. Let $\mathcal{B} = (B, \to, 1)$ be a quasi-BCK-algebra. Then, $\mathcal{B}$ is called *prelinear*, if 1 is the unique upper bound of the set $\{x \to y, y \to x\}$, for all $x, y \in B$,

Notation. We can define a prelinear quasi-equality algebra similar to Definition 3.12. Prelinearity does not necessitate the presence of a join operation in E.

Example 3.13. Let $(B = \{0, a, b, 1\}, \leq)$ be a lattice with the following Hasse diagram. Define the operation $\to$ on B as follows,



$\to$	0	a	b	1
0	1	1	1	1
a	a	1	a	1
b	a	a	1	1
1	a	a	a	1

Then $(B, \to, 1)$ is a prelinear quasi-BCK-algebra.

Theorem 3.14. *If $(B, \wedge, \to, 1)$ is a prelinear quasi-BCK(C)-meet-semilattice with Transitivity Condition, then B is a prelinear quasi-BCK(D)-meet-semilattice.*

Proof. We have to prove that for any $x, y, z \in B$, $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$. For this, since $y \wedge z \leq y, z$, by Proposition 3.10(i), $x \to (y \wedge z) \leq x \to y, x \to z$, and so $x \to (y \wedge z) \leq (x \to y) \wedge (x \to z)$. For converse, we have

$$y \to z$$

$$
\begin{aligned}
&= y \to (y \wedge z) \quad \text{by Proposition 3.10(ii)}\\
&\leq (x \to y) \to (x \to (y \wedge z)) \quad \text{by Transitivity Condition}\\
&\leq [(x \to y) \wedge (x \to z)] \to (x \to (y \wedge z))\,. \quad \text{by Proposition 3.9}
\end{aligned}
$$

Similarly, we have $z \to y \leq [(x \to y) \wedge (x \to z)] \to (x \to (y \wedge z))$. By assumption, B is prelinear, so

$$1 = \sup\{y \to z, z \to y\} \leq [(x \to y) \wedge (x \to z)] \to (x \to (y \wedge z))\,.$$

Thus $[(x \to y) \wedge (x \to z)] \to (x \to (y \wedge z)) = 1$, and so

$$(x \to y) \wedge (x \to z) \leq x \to (y \wedge z).$$

Hence, the condition (D) holds. Therefore, B is a prelinear quasi-BCK(D)-meet-semilattice. □

Remark 3.15. Let B be a quasi-BCK(C)-meet-semilattice as Example 3.6(i). Routine calculations show that B is a quasi-BCK(D)-algebra with the Transitivity Condition, but B is not prelinear. Because, $(a \to b) \vee (b \to a) = c \vee a = c \neq 1$. Hence, the prelinear condition in previous proposition is necessary.

Proposition 3.16. *Let $(E, \wedge, \sim, 1)$ be a prelinear quasi-equality algebra. Then for all $x, y, z, w \in E$, the following statements hold:*
(i) $x \leftrightarrow y = x \sim y$,
(ii) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Proof. (i) Let $x, y \in E$. Then

$$
\begin{aligned}
x \to y &= x \sim (x \wedge y) \quad \text{by definition}\\
&= (x \wedge y) \sim x \quad \text{by (QE2)}\\
&\leq ((x \wedge y) \sim y) \sim (x \sim y) \quad \text{by (QE7)}\\
&= (y \to x) \sim (x \sim y) \quad \text{by definition}\\
&\leq (y \to x) \to (x \sim y) \quad \text{by Proposition 3.4(i)} \qquad (3.4)
\end{aligned}
$$

On the other side, we have

$$(y \to x) \to (x \sim y)$$

$$\leq ((x \to y) \wedge (y \to x)) \to (x \sim y) \quad \text{by Proposition 3.4(v)}$$
$$= (x \leftrightarrow y) \to (x \sim y) \quad \text{by definition of the operation } \leftrightarrow \tag{3.5}$$

Then by (3.4) and (3.5) we have $x \to y \leq (x \leftrightarrow y) \to (x \sim y)$. By the similar way, we have $y \to x \leq (x \leftrightarrow y) \to (x \sim y)$. Since E is prelinear, we have

$$1 = \sup\{x \to y, y \to x\} \leq (x \leftrightarrow y) \to (x \sim y).$$

Thus $(x \leftrightarrow y) \to (x \sim y) = 1$, and so $x \leftrightarrow y \leq x \sim y$. By Proposition 3.4(i), $x \sim y \leq x \leftrightarrow y$. Therefore, $x \leftrightarrow y = x \sim y$.

(ii) Let $x, y, z \in E$. Then

$$y \to z = y \to (y \wedge z) \quad \text{by Proposition 3.4(vii)}$$
$$\leq (x \to y) \to (x \to (y \wedge z)) \quad \text{by Proposition 3.4(x)}$$
$$\leq ((x \to y) \wedge (x \to z)) \to (x \to (y \wedge z)) \quad \text{by Proposition 3.4(v)}$$

By the similar way,

$$z \to y \leq ((x \to y) \wedge (x \to z)) \to (x \to (y \wedge z)).$$

Since E is prelinear, we have $\sup\{z \to y, y \to z\} = 1$, and so

$$((x \to y) \wedge (x \to z)) \to (x \to (y \wedge z)) = 1.$$

Thus,

$$(x \to y) \wedge (x \to z) \leq x \to (y \wedge z).$$

By Proposition 3.4(iv), the proof of other side is clear. □

The following theorem provides a connection of quasi-equality algebras with the class of quasi-BCK-meet-semilattices.

Theorem 3.17. (i) *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a quasi-equality algebra. Then $\Psi(\mathcal{E}) = (E, \wedge, \to, 1)$ is a quasi-BCK(C)-meet-semilattice with the Transitivity Condition, where $x \to y = (x \wedge y) \sim x$, for any $x, y \in E$.*

(ii) *Let $\mathcal{B} = (B, \wedge, \to, 1)$ be a quasi-BCK(D)-meet-semilattice with the Transitivity Condition. Then $\Phi(\mathcal{B}) = (B, \wedge, \leftrightarrow, 1)$ is a quasi-equality algebra, where $\leftrightarrow$ denotes the equivalence operation of $\mathcal{B}$. Moreover, the implication of $\Phi(\mathcal{B})$ coincides with $\to$, that is, we have, $x \to y = x \leftrightarrow (x \wedge y)$.*

Proof. (i) Let $(E, \wedge, \sim, 1)$ be a quasi-equality algebra. By (QE6), $y \sim (y \wedge z) \leq (y \wedge x) \sim (y \wedge z \wedge x)$. Then by Proposition 3.4(v),

$$((x \wedge y) \sim (x \wedge y \wedge z)) \to (x \sim (x \wedge z \wedge y)) \leq (y \sim (y \wedge z)) \to (x \sim (x \wedge z \wedge y)). \tag{3.6}$$

Moreover, we have

$$\begin{aligned}
& x \to y \\
&= (x \wedge y) \sim x \quad \text{by definition of the operation } \to \\
&= x \sim (x \wedge y) \quad \text{by (QE2)} \\
&\leq (x \sim (x \wedge y \wedge z)) \sim ((x \wedge y) \sim (x \wedge y \wedge z)) \quad \text{by (QE7)} \\
&\leq ((x \wedge y) \sim (x \wedge y \wedge z)) \to (x \sim (x \wedge y \wedge z)) \quad \text{by Proposition 3.4(i) and (QE2)} \\
&\leq (y \sim (y \wedge z)) \to (x \sim (x \wedge z \wedge y)) \quad \text{by (3.6)} \\
&= ((y \wedge z) \sim y) \to ((x \wedge z \wedge y) \sim x) \quad \text{by (QE2)} \\
&= (y \to z) \to (x \to (z \wedge y)) \quad \text{by definition of the operation } \to \\
&\leq (y \to z) \to (x \to z) \quad \text{by Proposition 3.4(iv).}
\end{aligned}$$

Hence, it shows (QB1). By Proposition 3.4(iii) and (ii), respectively, (QB2) and (QB3) hold. Hence, by Proposition 3.4(viii), (v), (iv) and (x), $(E, \wedge, \to, 1)$ is a quasi-BCK(C)-meet-semilattice satisfying Transitivity Condition.

(ii) Clearly, $(B, \wedge, 1)$ is a meet-semilattice with top element 1. Let $x \in B$. Then by (QB2) we have

$$x \leftrightarrow x = (x \to x) \wedge (x \to x) = 1 \wedge 1 = 1.$$

Clearly, $x \leftrightarrow y = y \leftrightarrow x$. Hence, (QE2) and (QE3) hold. Moreover, if $x \leftrightarrow y = 1$, then $(x \to y) \wedge (y \to x) = 1$. Thus, $x \to y = 1 = y \to x$, and so $x = y$. Hence, (QE4) holds. Let $x, y, z \in B$. We get,

$$\begin{aligned}
& (x \wedge z) \leftrightarrow (y \wedge z) \\
&= ((x \wedge z) \to (y \wedge z)) \wedge ((y \wedge z) \to (x \wedge z)) \\
&= ((x \wedge z) \to y) \wedge ((x \wedge z) \to z) \wedge ((y \wedge z) \to x) \wedge ((y \wedge z) \to z) \quad \text{by (D)} \\
&= ((x \wedge z) \to y) \wedge ((y \wedge z) \to x) \\
&\geq (x \to y) \wedge (y \to x) \quad \text{by Proposition 3.9} \\
&= x \leftrightarrow y.
\end{aligned}$$

In addition, by Transitivity Condition, we have

$$\begin{aligned}
&(y \leftrightarrow z) \leftrightarrow (x \leftrightarrow z) \\
&= [((y \to z) \wedge (z \to y)) \to ((x \to z) \wedge (z \to x))] \\
&\quad \wedge [((z \to x) \wedge (x \to z)) \to ((y \to z) \wedge (z \to y))] \\
&= [((y \to z) \wedge (z \to y)) \to (x \to z)] \wedge [((y \to z) \wedge (z \to y)) \to (z \to x)] \\
&\quad \wedge [((z \to x) \wedge (x \to z)) \to (y \to z)] \wedge [((z \to x) \wedge (x \to z)) \to (z \to y)] \qquad \text{by (D)} \\
&\geq ((y \to z) \to (x \to z)) \wedge ((z \to y) \to (z \to x)) \\
&\quad \wedge ((x \to z) \to (y \to z)) \wedge ((z \to x) \to (z \to y)) \\
&\geq (y \to x) \wedge (x \to y) \\
&= x \leftrightarrow y.
\end{aligned}$$

Hence, (QE7) holds. The proof of the property (QE5) is similar, but we refrained from giving it because the proof is too long. Therefore, $\Phi(\mathcal{B}) = (B, \wedge, \leftrightarrow, 1)$ is a quasi-equality algebra. □

Let Pre($\mathcal{E}$) be the category of all prelinear quasi-equality algebras whose objects are prelinear quasi-equality algebras and whose morphisms are homomorphisms between them. The category whose objects are prelinear quasi-BCK(D)-meet-semilattices with the Transitivity Condition and whose morphisms are homomorphisms of them is called the category of prelinear quasi-BCK(D)-meet-semilattices and is denoted by $Pre(\mathcal{B})$. By Theorems 3.17 and 3.14, $\Psi : Pre(\mathcal{E}) \to Pre(\mathcal{B})$ and $\Phi : Pre(\mathcal{B}) \to Pre(\mathcal{E})$ are functors. In the next theorem provides a relation between these functors.

Theorem 3.18. *$Pre(\mathcal{E})$ and $Pre(\mathcal{B})$ are categorically equivalent.*

Proof. It is sufficient to show that the class of prelinear quasi-BCK(D)-meet-semilattices with the Transitivity Condition, that is Obj(Pre($\mathcal{B}$)), and the class of perlinear quasi-equality algebras, that is Obj(Pre($\mathcal{E}$)), are term equivalent. Let $(B, \wedge, \to, 1)$ be a prelinear quasi-BCK(D)-meet-semilattice with the Transitivity Condition. Then by Theorem 3.17, we get

$$\Psi(\Phi(B, \wedge, \to, 1)) = \Psi(B, \wedge, \leftrightarrow, 1) = (B, \wedge, \to, 1),$$

where $x \to y = x \leftrightarrow (x \wedge y)$. Thus, the map $\Psi : Obj(Pre(\mathcal{E})) \to Obj(Pre(\mathcal{B}))$ is onto. Also, let $(E, \wedge, \sim, 1)$ be a prelinear quasi-equality algebra. By Theorem 3.17(i), $\Psi(E, \wedge, \sim, 1) = (E, \wedge, \to, 1)$ is a prelinear quasi-BCK(C)-meet-semilattice with the Transitivity Condition. By Theorem 3.14, $(E, \wedge, \rightarrow, 1)$ is a prelinear quasi-BCK(D)-meet-semilattice. So, by Theorem 3.17 and Proposition 3.16(i), we have

$$\Phi(\Psi(E, \wedge, \sim, 1)) = \Phi(E, \wedge, \rightarrow, 1) = (E, \wedge, \leftrightarrow, 1) = (E, \wedge, \sim, 1).$$

If $(E, \wedge, +, 1)$ and $(E, \wedge, \sim, 1)$ are two prelinear quasi-equality algebras such that, $\Psi(E, \wedge, +, 1) = \Psi(E, \wedge, \sim, 1)$, then

$$(E, \wedge, \sim, 1) = \Phi(\Psi(E, \wedge, \sim, 1)) = \Phi(\Psi(E, \wedge, +, 1)) = (E, \wedge, +, 1).$$

Therefore, Ψ is a one-to-one map. □

4 Quasi-pseudo equality algebras and quasi-pseudo BCK-algebras

We introduce a non-commutative generalization of quasi (pseudo) equality algebras, called quasi-pseudo equality algebras. We investigate some of their properties and generalize the concepts of pseudo BCK-algebras to quasi-pseudo BCK-algebras. Also, we find that under which conditions, quasi-pseudo equality algebras are equivalent to quasi-pseudo BCK-meet-semilattices.

Definition 4.1. An algebraic structure $\mathcal{E} = (E, \wedge, \sim, \backsim, 1)$ of type $(2, 2, 2, 0)$ is called a *quasi-pseudo equality algebra*, if for all $x, y, z \in E$, it satisfies the following conditions:
(QPE1) $(E, \wedge, 1)$ is a meet-semilattice with top element 1,
(QPE2) $x \sim x = x \backsim x = 1$,
(QPE3) $x \sim 1 = 1 \backsim x$,
(QPE4) $x \sim y = 1$ or $y \backsim x = 1$ implies $y \leq x$,
(QPE5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$, $x \sim z \leq x \sim y$, $z \backsim x \leq z \backsim y$ and $z \backsim x \leq y \backsim x$,
(QPE6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$ and $x \backsim y \leq (x \wedge z) \backsim (y \wedge z)$,
(QPE7) $x \sim y \leq (z \sim x) \backsim (z \sim y)$ and $x \backsim y \leq (x \backsim z) \sim (y \backsim z)$,
(QPE8) $x \sim y \leq (x \sim z) \sim (y \sim z)$ and $x \backsim y \leq (z \backsim x) \backsim (z \backsim y)$,
where $x \leq y$ if and only if $x \wedge y = x$. Two *implications* on $\mathcal{E}$ are defined as follows:

$$x \rightarrow y = (x \wedge y) \sim x$$

$$x \rightsquigarrow y = x \backsim (x \wedge y).$$

for any $x, y \in E$.

Note: Obviously, any equality algebra is a quasi-pseudo equality algebra. Moreover, any quasi-equality algebra is a quasi-pseudo equality algebra. Conversely, if $(E, \wedge, \sim, \backsim, 1)$ is a quasi-pseudo equality algebra such that the operation $\sim$ is commutative, then $(E, \wedge, \sim, 1)$ is a quasi-equality algebra. On the other hand, quasi-pseudo equality algebras are also the generalization of pseudo equality algebras. Pseudo equality algebras which are the non-commutative generalizations of equality algebras were defined by A. Dvurevcenskij and O. Zahiri in [12].

We consider a quasi-pseudo equality algebra as identical to a pseudo equality algebra, except for (E4), it means that 1 needs not be an identity element for $\sim$ and $\backsim$. So, it is clear that a quasi-pseudo equality algebra is a pseudo equality algebra if and only if it satisfies $x \sim 1 = x = 1 \backsim x$.

Example 4.2. (i) Let $E = \{0, a, b, 1\}$ be a chain. Define the operations $\sim$ and $\backsim$ on E as follows,

$\sim$	0	a	b	1
0	1	a	a	a
a	a	1	b	b
b	a	b	1	b
1	a	b	1	1

$\backsim$	0	a	b	1
0	1	a	a	a
a	a	1	b	b
b	a	b	1	1
1	a	b	b	1

Then $(E, \wedge, \sim, \backsim, 1)$ is a quasi-pseudo equality algebra.

(ii) Let $(G, ., ^{-1}, e, \leq)$ be an ℓ-group written multiplicatively with an inversion $^{-1}$ and the identity element e, equipped with a lattice order $\leq$ such that $x \leq y$ entails $zxw \leq zyw$, for all $x, y, z, w \in G$. We denote $G^- = \{g \in G : g \leq e\}$ the negative cone of G. If we endow the negative cone G^- with two binary operations $x \sim y = (xy^{-1}) \wedge e$, $x \backsim y = (x^{-1}y) \wedge e$, then $(G^-, \wedge, \sim, \backsim, e)$ is a pseudo equality algebra [12]. By [12, Proposition 2.8(iii)], $x \sim y = e$ and $y \backsim x = e$ imply $y \leq x$, for any $x \in G^-$. So, $(G^-, \wedge, \sim, \backsim, e)$ is a quasi-pseudo equality algebra satisfies in condition $x \sim e = x = e \backsim x$, for any $x \in G^-$. But the operations $\sim$ and $\backsim$ are not commutative.

(iii) Let $(E_1, \wedge_1, \sim_1, \backsim_1, 1_1)$ and $(E_2, \wedge_2, \sim_2, \backsim_2, 1_2)$ be two quasi-pseudo equality algebras. Denote $E := E_1 \times E_2 = \{(x_1,\ x_2) \mid x_1 \in E_1,\ x_2 \in E_2\}$ and for all $(x_1, x_2),\ (y_1, y_2) \in E$, define the operations $\wedge$, $\sim$, $\backsim$ and 1 as follows:

$$\begin{aligned}(x_1, x_2) \wedge (y_1, y_2) &:= (x_1 \wedge y_1, x_2 \wedge y_2)\\ (x_1, x_2) \sim (y_1, y_2) &:= (x_1 \sim_1 y_1, x_2 \sim_2 y_2)\\ (x_1, x_2) \backsim (y_1, y_2) &:= (x_1 \backsim_1 y_1, x_2 \backsim_2 y_2)\\ 1 &:= (1_1, 1_2)\end{aligned}$$

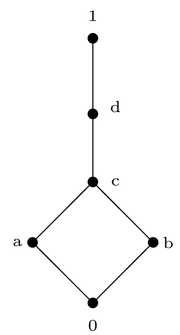
Then $(E, \wedge, \sim, \backsim, 1)$ is a quasi-pseudo equality algebra.

Remark 4.3. It should be pointed out that the commutativity of the operations $\sim$ and $\backsim$ in a quasi-pseudo equality algebra may not imply $x \sim 1 = x = 1 \backsim x$, for any x (See Example 4.6). Also, if a quasi-pseudo equality algebra is satisfying only in condition $x \sim 1 = x = 1 \backsim x$, for any x, then the operations $\sim$ and $\backsim$ may not be commutative. Actually, if a quasi-pseudo equality algebra is satisfying only in condition $x \sim 1 = x = 1 \backsim x$, for any x, then it is a pseudo equality algebra that the operations $\sim$ and $\backsim$ are not commutative, in general. (See Example 4.2(ii))

Remark 4.4. [12] If $(E, \wedge, \sim, \backsim, 1)$ is a pseudo equality algebra such that $\sim$ and $\backsim$ are commutative binary operations on E, then $\sim = \backsim$ and $(E, \wedge, \sim, 1)$ is an equality algebra.

Remark 4.5. Remark 4.4 is not true for quasi-pseudo equality algebras, in general.

Example 4.6. Let $(E = \{0, a, b, c, d, 1\}, \leq)$ be a lattice as the following diagram.



Define the operations $\sim$ and $\backsim$ on E as follows:

$\sim$	0	a	b	c	d	1
0	1	d	a	a	a	a
a	d	1	a	a	a	a
b	a	a	1	c	c	c
c	a	a	c	1	c	c
d	a	a	c	c	1	d
1	a	a	c	c	d	1

$\backsim$	0	a	b	c	d	1
0	1	c	a	a	a	a
a	c	1	a	a	a	a
b	a	a	1	c	c	c
c	a	a	c	1	c	c
d	a	a	c	c	1	d
1	a	a	c	c	d	1

Then $(E, \wedge, \sim, \backsim, 1)$ is a quasi-pseudo equality algebras.

As we see in Example 4.6, the operations $\sim$ and $\backsim$ are commutative, but $a \sim 0 = d \neq c = a \backsim 0$. So $\sim \neq \backsim$. Moreover, if we can define $a \backsim 0 = d = 0 \backsim a$, then $\sim$ and $\backsim$ are commutative binary operations on E and $\sim = \backsim$. By routine calculations, we can see that $(E, \wedge, \sim, \backsim, 1)$ is a quasi-equality algebra, but $(E, \wedge, \sim, 1)$ is not an equality algebra. Because, $1 \sim 0 = a \neq 0$ and $1 \sim b = c \neq b$.

Proposition 4.7. *If* $(E, \wedge, \sim, \backsim, 1)$ *is a quasi-pseudo equality algebra such that* $1 \sim x = x = x \backsim 1$*, for any* $x \in E$*, then* $(E, \wedge, \sim, 1)$ *is an equality algebra.*

Proof. Let $x, y \in E$. By (QPE7),

$$x \sim y \leq (1 \sim x) \backsim (1 \sim y) = x \backsim y$$

and

$$x \backsim y \leq (x \backsim 1) \sim (y \backsim 1) = x \sim y.$$

Then $\sim = \backsim$. By (QPE7), (QPE2) and our assumption, we have

$$x \sim y \leq (y \sim x) \backsim (y \sim y) = (y \sim x) \backsim 1 = y \sim x.$$

Similarly, $y \sim x \leq x \sim y$. Thus, the operation $\sim$ is commutative. In a similar way, we can show that $x \backsim y = y \backsim x$. Hence, $(E, \wedge, \sim, 1)$ is an equality algebra. □

The following proposition provides some properties of quasi-pseudo equality algebras.

Proposition 4.8. *Let* $(E, \wedge, \sim, \backsim, 1)$ *be a quasi-pseudo equality algebra. Then the following properties hold, for all* $x, y, z \in E$*:*

(i) $x \sim y \leq y \to x$ *and* $x \backsim y \leq x \rightsquigarrow y$,
(ii) $x \leq y$ *if and only if* $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$,
(iii) $x \to x = x \to 1 = x \rightsquigarrow x = x \rightsquigarrow 1 = 1$ *and* $1 \to x = 1 \rightsquigarrow x$,
(iv) $x \leq y$ *implies* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$,
(v) $y \leq z$ *implies* $z \to x \leq y \to x$ *and* $z \rightsquigarrow x \leq y \rightsquigarrow x$,
(vi) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$ *and* $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$,
(vii) $x \to y = x \to (x \wedge y)$ *and* $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y)$,
(viii) $x \to y \leq (x \wedge z) \to (y \wedge z)$ *and* $x \rightsquigarrow y \leq (x \wedge z) \rightsquigarrow (y \wedge z)$,
(ix) $x \sim y \leq (z \to x) \to (z \to y)$ *and* $x \backsim y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,
(x) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$.

Proof. We only present the proof of (vi).

(vi) By (QPE6), $(y \wedge z) \sim y \leq (x \wedge y \wedge z) \sim (x \wedge y)$. Then by (v),

$$((x \wedge y \wedge z) \sim (x \wedge y)) \rightsquigarrow ((x \wedge y \wedge z) \sim x) \leq ((y \wedge z) \sim y) \rightsquigarrow ((x \wedge y \wedge z) \sim x). \tag{4.1}$$

Since $x \wedge y \wedge z \leq x \wedge z \leq x$, by (QPE5), $(x \wedge y \wedge z) \sim x \leq (x \wedge z) \sim x$. Then by (iv),

$$((y \wedge z) \sim y) \rightsquigarrow ((x \wedge y \wedge z) \sim x) \leq ((y \wedge z) \sim y) \rightsquigarrow ((x \wedge z) \sim x). \tag{4.2}$$

Hence, by (4.1) and (4.2) we have

$$((x \wedge y \wedge z) \sim (x \wedge y)) \rightsquigarrow ((x \wedge y \wedge z) \sim x) \leq ((y \wedge z) \sim y) \rightsquigarrow ((x \wedge z) \sim x). \tag{4.3}$$

Moreover, we have

$$\begin{aligned} x \to y &= (x \wedge y) \sim x \\ &\leq ((x \wedge y \wedge z) \sim (x \wedge y)) \backsim ((x \wedge y \wedge z) \sim x) \quad \text{by (QPE7)} \\ &\leq ((x \wedge y \wedge z) \sim (x \wedge y)) \rightsquigarrow ((x \wedge y \wedge z) \sim x) \quad \text{by (i)} \\ &\leq ((y \wedge z) \sim y) \rightsquigarrow ((x \wedge z) \sim x) \quad \text{by (4.3)} \\ &= (y \to z) \rightsquigarrow (x \to z). \end{aligned}$$

The proof of the other parts is similar to the proof of Proposition 3.4. □

In the following, similar to Section 3, we generalize the concepts of pseudo BCK-algebras to quasi-pseudo BCK-algebras. Then we find that under which conditions, quasi-pseudo equality algebras are equivalent to quasi-pseudo BCK-meet-semilattices.

Definition 4.9. A *quasi-pseudo BCK-algebra* is a structure $(B, \rightarrow, \rightsquigarrow, 1)$ of the type (2, 2, 0) which satisfies the following conditions, for any $x, y, z \in B$:
(QPB1) $(x \rightarrow y) \rightsquigarrow ((y \rightarrow z) \rightsquigarrow (x \rightarrow z)) = 1$ and $(x \rightsquigarrow y) \rightarrow ((y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)) = 1$,
(QPB2) $x \rightarrow 1 = x \rightarrow x = 1$ and $x \rightsquigarrow 1 = x \rightsquigarrow x = 1$,
(QPB3) $1 \rightarrow x = 1 \rightsquigarrow x$,
(QPB4) $x \rightarrow y = 1 = y \rightarrow x$ $(x \rightsquigarrow y = 1 = y \rightsquigarrow x)$ implies $x = y$.

For any quasi-pseudo BCK-algebra one can define a partially order relation $\leq$ by $x \leq y$ if and only if $x \rightarrow y = 1$ (if and only if $x \rightsquigarrow y = 1$).

A quasi-pseudo BCK-meet-semilattice is an algebra $(B, \wedge, \rightarrow, \rightsquigarrow, 1)$ of type (2, 2, 2, 0) such that $(B, \rightarrow, \rightsquigarrow, 1)$ is a quasi-pseudo BCK-algebra and $(B, \wedge)$ is a meet-semilattice.

A quasi-pseudo BCK-meet-semilattice $(B, \wedge, \rightarrow, \rightsquigarrow, 1)$ is called a *quasi-pseudo BCK(pC)-meet-semilattice*, if, for any $x, y, z \in B$, it satisfies in the conditions (pC).

$$x \rightarrow y \leq (x \wedge z) \rightarrow (y \wedge z) \quad \text{and} \quad x \rightsquigarrow y \leq (x \wedge z) \rightsquigarrow (y \wedge z) \qquad (pC)$$

A quasi-pseudo BCK-meet-semilattice $(B, \wedge, \rightarrow, \rightsquigarrow, 1)$ is called a *quasi-pseudo BCK(pD)-meet-semilattice*, if, for any $x, y, z \in B$, it satisfies in the conditions (pD).

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z) \text{ and } x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z) \ (pD)$$

Example 4.10. Let $(B = \{0, a, b, 1\}, \leq)$ be a chain where $0 < a < b < 1$. Define the operation $\rightarrow$ on E as follows,

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	b	1	1
1	a	b	b	1

Then $(B, \rightarrow, \rightsquigarrow, 1)$ is a quasi-pseudo BCK-algebra such that $x \rightsquigarrow y = x \rightarrow y$, for all $x, y \in B$.

Remark 4.11. According to Definition 4.9, quasi-pseudo BCK-algebras are generalization of quasi-BCK-algebras and pseudo BCK-algebras. In quasi-pseudo BCK-algebras, $1 \rightsquigarrow x = 1 \to x \neq x$, in general. So, not every quasi-pseudo BCK-algebra is a pseudo-BCK-algebra. The following counterexample medicates that not every quasi-pseudo BCK-algebra is a quasi-BCK-algebra.

Example 4.12. Let $B = \{0, a, b, 1\}$ be a chain where $0 < a < b < 1$. Define the operations $\to$ and $\rightsquigarrow$ on E as follows,

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	a	1	1
1	0	a	b	1

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(B, \to, \rightsquigarrow, 1)$ is a quasi-pseudo BCK-algebra (pseudo BCK-algebra). Since $1 \to b = b \nleq a = (b \to 0) \to (1 \to 0)$, B does not satisfy (QB1). Thus, $(B, \to, 1)$ is not a quasi-BCK-algebra.

A quasi-pseudo BCK-algebra $(B, \to, \rightsquigarrow, 1)$ is a pseudo-BCK-algebra if and only if it satisfies $1 \to x = x = 1 \rightsquigarrow x$. The following proposition shows that under which condition, a quasi-pseudo BCK-algebra is a quasi-BCK-algebra.

Proposition 4.13. *If* $(B, \to, \rightsquigarrow, 1)$ *is a quasi-pseudo BCK-algebra such that* $x \rightsquigarrow y \leq x \to y$*, for any* $x, y \in B$*. Then* $(B, \to, 1)$ *is a quasi-BCK-algebra.*

Proof. The proof is straightforward. □

In Example 4.12, we have $a \rightsquigarrow 0 = b \nleq a = a \to 0$. So, this example shows that the condition in previous proposition is necessary.

Proposition 4.14. *Let* $(B, \to, \rightsquigarrow, 1)$ *be a quasi-pseudo BCK-algebra. Then* $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$*.*

Proof. By (QPB1) and since $x \leq y$, the proof is clear. □

Note: A quasi-pseudo BCK-algebra $(B, \rightarrow, \rightsquigarrow, 1)$ satisfies condition (4.4), where for any $x, y, z \in B$,

$$x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), \quad \text{and} \quad x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y). \tag{4.4}$$

Proposition 4.15. *Let $(B, \wedge, \rightarrow, 1)$ be a quasi-pseudo BCK(pC)-meet-semilattice satisfying condition (4.4). Then for all $x, y, z \in E$,*
(i) *$x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$.*
(ii) *$x \rightarrow (x \wedge y) = x \rightarrow y$ and $x \rightsquigarrow (x \wedge y) = x \rightsquigarrow y$.*

Proof. The proof is similar to the proof of Proposition 3.10. □

Proposition 4.16. *Any quasi-pseudo BCK(pD)-meet-semilattice satisfying condition (4.4) is a quasi-pseudo BCK(pC)-meet-semilattice.*

Proof. The proof is similar to the proof of Theorem 3.11. □

The following theorem provides a connection of quasi-pseudo equality algebras with the class of quasi-pseudo BCK(pC)-meet-semilattices.

Theorem 4.17. (i) *Let $(E, \wedge, \sim, \backsim, 1)$ be a quasi-pseudo equality algebra. Then $(E, \wedge, \rightarrow, \rightsquigarrow, 1)$ is a quasi-pseudo BCK(pC)-meet-semilattice with (4.4) condition, where $x \rightarrow y = (x \wedge y) \sim x$ and $x \rightsquigarrow y = x \backsim (x \wedge y)$, for any $x, y \in E$.*

(ii) *Let $(B, \wedge, \rightarrow, \rightsquigarrow, 1)$ be a quasi-pseudo BCK(pC)-meet-semilattice with (4.4) condition. Then $(B, \wedge, \sim, \backsim, 1)$ is a quasi-pseudo equality algebra, where $x \sim y = y \rightarrow x$ and $x \backsim y = x \rightsquigarrow y$, for any $x, y \in B$.*

Proof. (i) The proof follows from Proposition 4.8(iii), (iv), (v), (vi), (viii) and (x).

(ii) Let $(B, \wedge, \rightarrow, \rightsquigarrow, 1)$ be a quasi-pseudo BCK(pC)-meet-semilattice with (4.4) condition. The verifying of (QPE1), (QPE2), (QPE3) and (QPE4) is straightforward. Let $x \leq y \leq z$. By (QPB1), $y \rightarrow z \leq (z \rightarrow x) \rightsquigarrow (y \rightarrow x)$. Since $y \rightarrow z = 1$, $(z \rightarrow x) \rightsquigarrow (y \rightarrow x) = 1$. Thus, $z \rightarrow x \leq y \rightarrow x$, and so $x \sim z \leq x \sim y$. The proof of the second part is similar. Hence, (QPE5) holds. Pseudo-contraction condition (pC) implies (QPE6) and (QPB1) implies (QPE7). Also, (QPE8) is straightforward by our assumption. Hence, $(B, \wedge, \sim, \backsim, 1)$ is a quasi-pseudo equality algebra. □

Note: By Theorem 4.17, if $(E, \wedge, \sim, \backsim, 1)$ is a quasi-pseudo equality algebra, then $F((E, \wedge, \sim, \backsim, 1)) := (E, \wedge, \rightarrow, \rightsquigarrow, 1)$ is a quasi-pseudo BCK(pC)-meet-semilattice, where $x \rightarrow y = (x \wedge y) \sim x$ and $x \rightsquigarrow y = x \backsim (x \wedge y)$, for any $x, y \in E$. Moreover, if $(B, \wedge, \rightarrow, \rightsquigarrow, 1)$ is a quasi-pseudo BCK(pC)-meet-semilattice with (4.4) condition, then $G((B, \wedge, \rightarrow, \rightsquigarrow, 1)) := (B, \wedge, \sim', \backsim', 1)$ is a quasi-pseudo equality algebra, where $x \sim' y = y \rightarrow x$ and $x \backsim' y = x \rightsquigarrow y$, for any $x, y \in B$.

The category whose objects are quasi-pseudo equality algebras and whose morphisms are homomorphisms of quasi-pseudo equality algebras is called the category of quasi-pseudo equality algebras and is denoted by $\mathcal{E}$. The category of quasi-pseudo BCK-meet-semilattices can be defined similarly. Let $\mathcal{B}$ be its subcategory whose objects are pseudo BCK(pC)-meet-semilattices with (4.4) condition. Then clearly, $F : \mathcal{E} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{E}$ are functors.

In the next theorem provides a relation between these functors and its proof is similar to the proof of Theorem 3.3 in [12].

Definition 4.18. A quasi-pseudo equality algebra $(E, \wedge, \sim, \backsim, 1)$ is called invariant if there exists a quasi-pseudo BCK(pC)-meet-semilattice $(B, \wedge, \mapsto, \rightarrowtail, 1)$ such that $\mathrm{G}(B, \wedge, \mapsto, \rightarrowtail, 1) = (E, \wedge, \sim, \backsim, 1)$.

Theorem 4.19. *Inv($\mathcal{E}$), the category of invariant quasi-pseudo equality algebras, and $\mathcal{B}$ are categorically equivalent.*

Proof. It is sufficient to show that the class of quasi-pseudo BCK(pC)-meet-semilattices with (4.4) condition, Obj($\mathcal{B}$), and the class of invariant quasi-pseudo equality algebras, Obj(Inv($\mathcal{E}$)), are term equivalent. First, we prove that a quasi-pseudo equality algebra $(E, \wedge, \sim, \backsim, 1)$ is invariant if and only if $G(F(E, \wedge, \sim, \backsim, 1)) = (E, \wedge, \sim, \backsim, 1)$. Let quasi-pseudo equality algebra $(E, \wedge, \sim, \backsim, 1)$ be invariant. Then there exists a quasi-pseudo BCK(pC)-meet-semilattice $(E, \wedge, \mapsto, \rightarrowtail, 1)$ such that $G(E, \wedge, \mapsto, \rightarrowtail, 1) = (E, \wedge, \sim, \backsim, 1)$. Consider $\sim''$ and $\backsim''$ be the binary operations induced by $\mapsto$ and $\rightarrowtail$ on the quasi-pseudo BCK(pC)-meet-semilattice $(E, \wedge, \mapsto, \rightarrowtail, 1)$. Then $G(E, \wedge, \mapsto, \rightarrowtail, 1) = (E, \wedge, \sim'', \backsim'', 1)$. Let $\rightarrow$ and $\rightsquigarrow$ be two derived operations on the quasi-pseudo equality algebra $(E, \wedge, \sim, \backsim, 1)$ and $\sim'$ and $\backsim'$ be the binary operations induced by $\rightarrow$ and $\rightsquigarrow$ on the quasi-pseudo BCK(pC)-meet-semilattice $\mathrm{F}(E, \wedge, \sim, \backsim, 1)$. By definition,

$$G(F(E, \wedge, \sim, \backsim, 1)) = G(E, \wedge, \rightarrow, \rightsquigarrow, 1) = (E, \wedge, \sim', \backsim', 1).$$

It suffices to show that $(E,\wedge,\sim',\backsim',1)=(E,\wedge,\sim'',\backsim'',1)$. By definition and Proposition 4.15,

$$x\sim'y=y\to x=(x\wedge y)\sim y=y\mapsto x=x\sim''y,$$

and

$$x\backsim'y=x\rightsquigarrow y=x\backsim(x\wedge y)=x\rightarrowtail y=x\backsim''y,$$

for any $x,y\in E$. Hence, $G(F(E,\wedge,\sim,\backsim,1))=(E,\wedge,\sim,\backsim,1)$. Now, if $(E,\wedge,+,-,1)$ and $(E,\wedge,\sim,\backsim,1)$ are two invariant quasi-pseudo equality algebras such that, $F(E,\wedge,+,-,1)=F(E,\wedge,\sim,\backsim,1)$, then

$$(E,\wedge,\sim,\backsim,1)=G(F(E,\wedge,\sim,\backsim,1))=G(F(E,\wedge,+,-,1))=(E,\wedge,+,-,1).$$

Therefore, the map $F:Obj(Inv(\mathcal{E}))\to Obj(\mathcal{B})$ is one-to-one. On the other side, we prove that for any quasi-pseudo BCK(pC)-meet-semilattice with (4.4) condition, $F(G(E,\wedge,\to,\rightsquigarrow,1))=(E,\wedge,\to,\rightsquigarrow,1)$. Let $\sim'$ and $\backsim'$ be the binary operations induced by $\to$ and $\rightsquigarrow$ on the quasi-pseudo BCK(pC)-meet-semilattice $(E,\wedge,\to,\rightsquigarrow,1)$ and $\to'$ and $\rightsquigarrow'$ be two derived operations on the quasi-pseudo equality algebra $G(E,\wedge,\to,\rightsquigarrow,1)=(E,\wedge,\sim',\backsim',1)$. By definition,

$$F(G(E,\wedge,\to,\rightsquigarrow,1))=F(E,\wedge,\sim',\backsim',1)=(E,\wedge,\to',\rightsquigarrow',1).$$

It suffices to show that $(E,\wedge,\to,\rightsquigarrow,1)=(E,\wedge,\to',\rightsquigarrow',1)$. Let $x,y\in E$. By Proposition 4.15,

$$x\to'y=(x\wedge y)\sim'x=x\to(x\wedge y)=x\to y$$

and

$$x\rightsquigarrow'y=x\backsim'(x\wedge y)=x\rightsquigarrow(x\wedge y)=x\rightsquigarrow y.$$

Therefore, $F(G(E,\wedge,\to,\rightsquigarrow,1))=(E,\wedge,\to,\rightsquigarrow,1)$, and so F is a onto map. □

5 Conclusions and future works

In this paper, a commutative and non-commutative generalization of equality algebras are defined and called quasi- and quasi-pseudo equality algebras. Then some of their properties and generalize the concepts of BCK-algebras as quasi and quasi pseudo BCK-algebras are investigated. Then the relation between them and the category of them are studied. In the future, we try to investigate some subalgebras of quasi and pseudo-quasi equality algebras, such as weak filters and the weak filters generated by some non-empty subsets. Then we will study some kinds of weak filters such as normal, prime and maximal weak filters of a quasi and pseudo-quasi-equality algebras. In continuing our study, we will present the topological properties of the set of all prime weak filters.

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