Categories and
General Algebraic Structures
with Applications

WWW.CGASA.ir

In press.

Some lower separation axioms in LG-topologies

Ali R. Alibabad, Mehdi Badie*, and Mohammad Hossein Hajian

Abstract. We generalize the separation axioms T_0, T_1 and T_2 to LG-spaces and show that the most important results about these concepts can be extended in LG-spaces.

1 Introduction

One of the objectives of topological studies is the investigation of non-metric spaces, which are structures that lack algebraic tools. Undoubtedly, algebra provides a powerful framework for the study of such structures. As is well known, a topological space can be regarded as a frame. From this perspective, topological spaces may be studied using algebraic methods. Two approaches have been developed to study topological spaces within the framework of frame theory. In both approaches, topology is examined independently of points; hence, these are referred to as point-free topologies.

---

* Corresponding author

Keywords: LG-topology, lower separation axioms, topoframe, Boolean frame, continuous, product, subspace.
Mathematics Subject Classification [2010]:Primary 06D22, Secondary 54A05, 54D10.
Received: 10 February 2026, Accepted: 25 May 2026.
ISSN: Print 2345-5853, Online 2345-5861.

In the first approach, which has a longer history and has been more extensively explored, a topology is considered purely as a frame. This viewpoint was initiated in [7] and [15], and subsequently extended by some authors such as C.H. Dowker, D. Papert, J. Isbell, B. Banaschewski, etc. In the second approach, given that a topology is a subframe of the power set frame, it is studied as a subframe of a frame. The initial idea of this method was introduced in [17] and [18], and was first extended and formally presented in [1] under the name LG-topology and fundamental notions such as interior, closure, subspace, product space, and related concepts were introduced. Continuity and its associated notions were later developed and studied in [2] and [6]. Similar to the LG-space structure, another structure is introduced in [9] and [10]. This structure is called a topoframe. Topoframes are studied in several articles. In particular, in [22], the separation axioms of topoframes are studied. The difference between LG-spaces and topoframes is that in a topoframe every open element has a complement but in an LG-topology need not have a complement.

In this article, we aim to investigate basic separation axioms T_0, T_1, T_2, and related topics in LG-spaces.

In the next section, we recall certain preliminary notions, many of which have already been introduced in previous studies. We also review specific propositions that are required in the present work. In Section 3, two prerequisites for the separation axioms T_0, T_1, and T_2 are presented and examined, together with some equivalent conditions. In fact, the necessary requirement for an LG-topology to satisfy one of the separation axioms introduced in the subsequent sections is that these prerequisites hold.

Section 4 introduces the separation axiom T_0, provides its equivalent conditions, and demonstrates that the concept, as defined, extends the corresponding axiom in classical topology. Section 5 is devoted in a similar manner to the separation axiom T_1. We present certain equivalent conditions for T_1 in the context of topology and investigate how these conditions are related within LG-topology. Section 6 introduces the separation axiom T_2, and once again we show that this axiom extends the classical T_2 separation axiom in topology.

Finally, in the last section, we study the relationship between the separation axioms under consideration and the notions of subspace and product space. We then determine which of the discussed properties remain invariant under various mappings.

2 Preliminary

Two elements a and b of an ordered set are called parallel, denoted by $a \parallel b$, whenever $a \nleqslant b$ and $b \nleqslant a$. We say a and b are disjoint or orthogonal if $a \wedge b = 0$. The set of all disjoint elements from a is denoted by $a^{\perp}$. If S is an ordered set, T is a subset of S, and $x \in S$, then the hull (respectively, core) of x with respect to T, denoted by $h_T(x)$ (respectively, $c_T(x)$), is $\{t \in T : x \leqslant t\}$ (respectively, $\{t \in T : t \leqslant x\}$). Recall that a subset T of a lattice L is called a meet-dense (respectively, join-dense) subset of L if $x = \bigwedge h_T(x)$ (respectively, $x = \bigvee c_T(x)$) for every $x \in L$. A lattice is called complete if each subset of the lattice has the supremum. Thus each complete lattice has the greatest element 1 and the smallest element 0. A frame F is a complete lattice in which for each $a \in F$ and $\{b_i\}_{i\in I}$ of elements of F we have $a \wedge \big(\bigvee_{i\in I} b_i\big) = \bigvee_{i\in I}(a \wedge b_i)$. For every element a of a frame F the pseudocomplement of a, denoted by a^*, is $\bigvee a^{\perp}$. An element b of a frame F is called the complement of $a \in F$, if $a \vee b = 1$ and $a \wedge b = 0$. Clearly, the complement of an element a, if it exists, is unique and is denoted by a^c. If the complement of a exists, it is readily seen that $a^c = a^*$. If each element of a frame has a complement, then the frame is called a Boolean frame. A subset of a frame is called a subframe of the frame if it is closed under the finite meet and the arbitrary join.

Let F be a frame. We say $\tau \subseteq F$ is a lattice generalized topology (briefly, LG-topology) if τ is a subframe of F. Then (F, τ) is called a lattice generalized space (briefly, LG-space). Often, we use the symbol F_τ instead of (F, τ). Clearly, if $\tau \subseteq P(X)$ is a topology on X, then τ is an LG-topology, $(P(X), \tau)$ is an LG-space. A frame F is said to be spatial if there exists a topological space (X, τ) such that F is isomorphic to the lattice of open sets of X. Every element of an LG-topology $\tau \subseteq F$ is called open and every element of $\tau^* = \{t^* : t \in \tau\}$ is called a closed element of F. It is readily seen that for each subset $\mathcal{F}$ of τ^*, $\bigwedge \mathcal{F} \in \tau^*$. For every element a of F, the interior and closure of a are defined $a^\circ = \bigvee\{t : t \leqslant a\}$ and $\overline{a} = \bigwedge\{f \in \tau^* : a \leqslant f\}$, respectively. Suppose that F_τ is an LG-space, then F_τ is called a topoframe if each element of τ has a complement in F. Clearly, if (X, τ) is a topological space then $P(X)_\tau$ is a topoframe.

If F_τ is an LG-space and $a \in F$, then (F_a, τ_a) is an LG-space, in which $F_a = {\downarrow}a = \{x \in F : x \leqslant a\}$ and $\tau_a = \{t \wedge a : t \in \tau\}$. If $\{(F_i, \tau_i)\}_{i\in I}$ is a family of LG-spaces then $\tau_p = \{(t_i)_{i\in I} : \forall i \in I \quad t_i \in \tau_i \quad \text{and} \quad t_i = 1 \text{ for all except finitely many } i \in I\}$ and $\tau_b = \{(t_i)_{i\in I} : \forall i \in I \quad t_i \in \tau_i\}$ are subframes of $F = \prod_{i\in I} F_i$. (F, τ_p) and (F, τ_b) are called the product and box LG-space

Definition 2.1. An LG-space F_τ is called Boolean LG-space whenever F is a Boolean frame.

Clearly, every Boolean LG-space is a topoframe, and it is easy to provide an example that shows not every topoframe is necessarily a Boolean LG-space.

We can conclude from [12, §2.6 Corollary] that for any frame τ there is a frame F such that F_τ is a Boolean LG-space.

Now suppose that F_τ is a topoframe, $BF = \{a \in F : a^{**} = a\} = F^*$ and for each $A \subseteq BF$,

$$\bigvee_{BF} A = \Big(\bigvee_{F} A\Big)^{**} \quad , \quad \bigwedge_{BF} A = \bigwedge_{F} A.$$

Then, by [9, Proposition 5.2 and Corollary 5.3], BF_τ is a Boolean LG-space.

Suppose that F_1 and F_2 are two partially ordered sets. We say $g : F_2 \to F_1$ is a right adjoint of $f : F_1 \to F_2$ if

$$f(a) \leqslant b \quad \Leftrightarrow \quad a \leqslant g(b).$$

In this case, f is called a left adjoint of g and we denote f by g^* and g by f_*. We say f is arbitrary join (respectively, meet) preserving if $f\big(\bigvee A\big) = \bigvee f(A)$ (respectively, $f\big(\bigwedge A\big) = \bigwedge f(A)$), for all $A \subseteq F_1$. We recall the following facts that one can show easily.

(a) f is left adjoint of g if and only if $fg \leqslant I_{F_2}$ and $I_{F_1} \leqslant gf$.

(b) If f is a left adjoint map, then $ff_*f = f$ and $f_*ff_* = f_*$.

(c) If F_1 and F_2 are bounded lattices and f is left adjoint, then $f(0) = 0$ and $f_*(1) = 1$.

(d) If F_1 and F_2 are complete lattices, then f (respectively, g) is left adjoint (respectively, right adjoint) if and only if it preserves arbitrary joins (respectively, meets).

(e) If f is left adjoint, then the following are equivalent.

  (i) f is onto.
  (ii) $ff_* = I_{F_2}$.
  (iii) f_* is one-to-one.

(f) If f is left adjoint, then the following are equivalent.

  (i) f is one-to-one.
  (ii) $f_*f = I_{F_1}$.
  (iii) f_* is onto.

A left adjoint map f is called the RL-adjoint whenever f_* is a left-adjoint map. This notion has been first introduced in [18] as GOH and then is called RL-adjoint in [2]. It is clear that if F_1 and F_2 are complete lattices, then f is RL-adjoint if and only if f_* preserves arbitrary joins.

Lemma 2.2. *Assume that F_1 and F_2 are lattices and $f : F_1 \to F_2$ is an RL-adjoint map. Then the following statements hold:*

(a) *$f_{**} \leqslant f$ if and only if f is onto.*

(b) *$f \leqslant f_{**}$ if and only if f is one-to-one.*

(c) *$f = f_{**}$ if and only if f is an order isomorphism.*

Proof. (a)$\Rightarrow$. Suppose that $y \in F_2$. Taking $x = f_*(y)$, we show that $f(x) = y$. Since $f_*(y) = x$, it follows that $f(x) \leq y$. On the other hand, we can write

$$f_*(y) = x \quad \Rightarrow \quad y \leq f_{**}(x) \leq f(x)$$

Hence $y = f(x)$.

(a)$\Leftarrow$. Suppose that $a \in F_1$ and $A_a = \{y \in F_2 : f_*(y) \leqslant a\}$. We know that $f_{**}(a) = \max(A_a)$. Thus, it is enough to show that $f(a)$ is an upper bound of A_a. Assume that $y \in A_a$. By the assumption, $x \in F_1$ exists such that $f(x) = y$. Then

$$f(x) \leqslant y \quad \Rightarrow \quad x \leqslant f_*(y) \leqslant a \quad \Rightarrow \quad y = f(x) \leqslant f(a).$$

(b)$\Rightarrow$. Suppose that $f(a) = f(b)$. By the hypothesis, we have $f(a) = f(b) \leqslant f_{**}(b)$. Thus $f_*f(a) \leqslant b$ and since $a \leqslant f_*f(a)$, it follows that $a \leqslant b$. Similarly, $b \leqslant a$ and therefore $a = b$.

(b)$\Leftarrow$. Suppose that $a \in F_1$. Since f is one-to-one, we have $f_* f(a) = a$ and so $f(a) \leq f_{**}(a)$.

(c). It is an immediate consequence of parts (a) and (b). $\square$

Let (F_1, τ_1) and (F_2, τ_2) be two LG-spaces. An arbitrary join preserving map $f : F_1 \to F_2$ is called o-continuous (respectively, c-continuous) if $f_*(t) \in \tau_1$ (respectively, $f_*(t^*) \in \tau_1^*$), for all $t \in \tau_2$. The o-continuous map is called weakly continuous and OLG in [2] and [6], respectively, and c-continuous is called CLG in [6]. Also, f is called open (respectively, closed) if $f(t) \in \tau_2$ (respectively, $f(t^*) \in \tau_2^*$).

We conclude this section by presenting the following unproven proposition.

Proposition 2.3. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two topoframes and* $f : F_1 \to F_2$ *is an RL-adjoint map. Then* f *is o-continuous if and only if* f *is c-continuous.*

3 Pre-separation and strongly pre-separation axioms

In this section, we introduce and study the conditions that are necessary for separation axioms.

Definition 3.1. Suppose that F is a frame.

(1) We say F has the primary separation property (abbreviated, ps-property) whenever for every two nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge y = 0$.

(2) We say F has the strong primary separation property (abbreviated, sps-property) whenever for every two nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge y = 0$ and either $x \nleqslant a \wedge b$ or $y \nleqslant a \wedge b$.

In the following proposition, the conditions equivalent to the ps-property and sps-property are presented.

Proposition 3.2. *Suppose that* F *is a frame. The following statements hold:*

(a) *F has the ps-property if and only if for each elements $0 < a < b$ in F, there exist nonzero elements x and y in F such that $x \leqslant a$, $y \leqslant b$ and $x \wedge y = 0$.*

(b) *The following statements are equivalent*

  (i) *F has the sps-property.*

  (ii) *For each nonzero elements $a < b$ in F, there exist $x, y \in F$ such that $0 < x \leqslant a$, $0 < y \leqslant b$, $y \nleqslant a$, and $x \wedge y = 0$.*

  (iii) *For each nonzero elements $b \nleqslant a$ in F, there exist $x, y \in F$ such that $0 < x \leqslant a$, $0 < y \leqslant b$, $y \nleqslant a$, and $x \wedge y = 0$.*

Proof. (a)$\Rightarrow$. This is evident.

(a)$\Leftarrow$. Suppose that a and b are two distinct elements of F. If $a \wedge b = 0$, then, by taking $x = a$ and $y = b$, we are done. Now we suppose that $c = a \wedge b \neq 0$. Clearly, without loss of generality, we can assume that $c < a$. By the assumption, there are nonzero elements $x \leqslant a$ and $y \leqslant c \leqslant b$ such that $x \wedge y = 0$.

(b). The proof of this part closely parallels the proof of part (a). □

In the above proposition condition "$b \nleqslant a$" cannot replaced by $a \parallel b$. Also, we note that it is clear every frame with the sps-property has the ps-property, but the converse need not be true. The following example shows these facts.

Example 3.3. Let $F = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$. It is easy to see that F is a frame with the ps-property but does not have the sps-property. In addition, $a = \{1\}$ and $b = \{2\}$ are the only two nonzero parallel elements of F. If we take x = a and y = b, then we have $x \leqslant a$, $y \leqslant b$, $y \nleqslant a$, and $x \wedge y = 0$.

The following proposition and Proposition 7.2 show that the class of frames with ps-property is sufficiently large compared to the class of classical topologies. Of course, given that the product of spatial frames is not necessarily a spatial frame, these classes are strictly larger than topological Hausdorff spaces.

Recall that if (X, τ) is a topological space and $a, b \in X$, then we say a and b can be separated by open sets if there exist open sets U and V such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$.

Proposition 3.4. (a) *If (X, τ) is a Hausdorff topological space, then τ is a frame with the sps-property.*

(b) *Let (X, τ) be a T_0 topological space which has no any isolated point and*

$$A = \{(a, b) : a \text{ and } b \text{ can be separated by open sets}\}.$$

Then τ is a frame with the ps-property if and only if $\overline{A} = X \times X$.

Proof. (a) Suppose that (X, τ) is a Hausdorff topological space, $U, V \in \tau$, and $\emptyset \neq U \subset V$. Hence, there exist $x \in U$ and $y \in V \setminus U$ and so there are $W_1, W_2 \in \tau$ such that $x \in W_1 \subseteq U$, $y \in W_2 \subseteq V$ and $W_1 \cap W_2 = \emptyset$. Clearly, $W_2 \nsubseteq U$. Thus, Proposition 3.2(b) concludes that τ has the sps-property.

(b)$\Rightarrow$. Let $U_1 \times U_2$ be a nonempty open set in $X \times X$. First, assume $U_1 \neq U_2$. Then, by the assumption, there exist $W_1, W_2 \in \tau$ such that $\emptyset \neq W_1 \subseteq U_1$, $\emptyset \neq W_2 \subseteq U_2$, and $W_1 \cap W_2 = \emptyset$. Clearly, $\emptyset \neq W_1 \times W_2 \subseteq A$ and so $(U_1 \times U_2) \cap A \neq \emptyset$.

Now suppose that $U_1 = U_2$. Since X has no isolated point, there are distinct elements a and b in $U_1 = U_2$. Since X is T_0, without loss of generality, open set V exists such that $a \in V$ and $b \notin V$. Set $V_1 = U_1 \cap V \neq \emptyset$ and $V_2 = U_2$. Then V_1 and V_2 are distinct. Thus, similar to the first case, $(V_1 \times V_2) \cap A \neq \emptyset$ and therefore $(U_1 \times U_2) \cap A \neq \emptyset$.

(b)$\Leftarrow$. Let U and V be two distinct nonempty open sets in τ. By hypothesis, $(U \times V) \cap A \neq \emptyset$. Therefore, there exist $(x, y) \in U \times V$ such that x and y can be separate by open sets. Hence, there are $W_1, W_2 \in \tau$ such that $x \in W_1 \subseteq U$, $y \in W_2 \subseteq V$, and $W_1 \cap W_2 = \emptyset$. $\square$

Part (b) of Proposition 3.2 prompts the question: which frame F possesses the property that for every elements $0 \neq a < b$, there is $0 \neq y \leqslant b$ in F such that $y \wedge a = 0$, or equivalently, $a^* \wedge b \neq 0$? The following proposition answers this question.

Theorem 3.5. *For any frame F, the following statements are equivalent.*

(a) *If $b \nleqslant a$ are two nonzero elements of F, then $a^* \wedge b \neq 0$.*

(b) *For every nonzero elements $b \nleqslant a$ in F there is $0 \neq y \leqslant b$ such that $y \wedge a = 0$.*

(c) *For every nonzero elements $a < b$, there exists $0 \neq y \leqslant b$ such that $y \wedge a = 0$.*

(d) *For every* $a \in F$, *we have* $a = a^{**}$.

(e) *For every* $a \in F$, *if* $a^* = 0$, *then* $a = 1$.

(f) F *is Boolean.*

Proof. (a)⇒(b). Put $y = a^* \wedge b$. By hypothesis, we have $y \neq 0$ and clearly, $0 < y \leqslant b$ and $y \wedge a = 0$.

(b)⇒(c). This is clear.

(c)⇒(d). On the contrary, suppose that $a < a^{**}$. By the assumption, $0 \neq y \leqslant a^{**}$ exists such that $y \wedge a^{**} = 0$. So $y \leqslant a^* \wedge a^{**} = 0$, which is a contradiction.

(d)⇒(e). Suppose that $a^* = 0$. Thus, $a = a^{**} = 1$.

(e)⇒(f). For each $a \in F$,

$$(a \vee a^*)^* = a^* \wedge a^{**} = 0 \quad \Rightarrow \quad a \vee a^* = 1$$

Hence $a^* = a^c$. This shows that F is a Boolean algebra.

(f)⇒(a). On the contrary, assume that there exist $a, b \in F$ such that $b \nleqslant a$ and $a^* \wedge b = 0$, so $b \leqslant a^{**} = a^{cc} = a$, which is a contradiction. □

The following corollary is an immediate consequence of the above proposition.

Corollary 3.6. *Every Boolean frame has the sps-property.*

4 Generalizations of the separation axiom T_0 in LG-spaces

In this section, we extend the first separation axiom, namely the T_0 separation axiom, to LG-topologies.

Definition 4.1. Suppose that F_τ is an LG-space and a and b are two distinct nonzero elements of F.

(1) We write $aT_{01}b$ (respectively, $aT_{02}b$), if there exist nonzero elements $x \leqslant a$ and $y \leqslant b$ of F, and t in τ such that $x \leqslant t$ and $y \wedge t = 0$ (respectively, $y \leqslant t$ and $x \wedge t = 0$).

(2) We write aT_0b, if either $aT_{01}b$ or $aT_{02}b$.

(3) We say that F_τ is T_{01} (respectively, T_{02}) at a, if $aT_{01}c$ (respectively, $aT_{02}c$) for each other nonzero element c of F.

(4) We say that F_τ is T_0 at a, if a is aT_0b for each other nonzero element $b \in F$.

(5) We say that F_τ is T_0 if F_τ is T_0 at each nonzero element.

It is clear that if F has not the ps-property, then any LG-space F_τ is not T_0. Also, each $P(X)$ has the sps-property and there are some topological spaces which are not T_0, so there is a large class of LG-spaces having the sps-property which are not T_0.

Lemma 4.2. *Let F_τ be an LG-space with the ps-property and a nonzero element a in F. The following statements are equivalent.*

(a) *F is T_{01} at a.*

(b) *For every nonzero element b distinct from a, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge \overline{y} = 0$.*

(c) *For every nonzero element b disjoint from a, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge \overline{y} = 0$.*

Proof. (a)⇒(b). According to the assumption, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ in F and t in τ such that $x \leqslant t$ and $y \wedge t = 0$, so $y \leqslant t^*$, hence $x \wedge \overline{y} \leqslant t \wedge t^*$ and therefore $x \wedge \overline{y} = 0$.

(b)⇒(a). By the assumption, $x \nleqslant \overline{y}$, thus there is some $t \in \tau$ such that $y \leqslant t^*$ and $x \nleqslant t^*$. Set $x_1 = x \wedge t$, then $0 \neq x_1 \leqslant a$, $x_1 \leqslant t$ and $y \wedge t = 0$. It shows that aT_0b.

(a)⇒(c). This is clear.

(c)⇒(b). Since F has the ps-property, there are nonzero disjoint elements $a' \leqslant a$ and $b' \leqslant b$. By the assumption, there are $x \leqslant a' \leqslant a$ and $y \leqslant b' \leqslant b$ such that $x \wedge \overline{y} = 0$. □

Now, using the previous lemma, we state the conditions equivalent to the T_0 separation axiom.

Theorem 4.3. *Suppose that F_τ is an LG-space. The following statements are equivalent.*

(a) *F_τ is T_0.*

(b) *For every pair distinct nonzero elements a and b in F there are t in τ and nonzero elements $x \leqslant a$ and $y \leqslant b$ in F such that $y \leqslant t$ and $x \wedge t = 0$ or $x \leqslant t$ and $y \wedge t = 0$.*

(c) *For every pair distinct nonzero elements* a *and* b *in* F *there are* $t \in \tau$ *and nonzero elements* $x \leqslant a$ *and* $y \leqslant b$ *such that* $b \wedge t \neq 0$ *and* $x \wedge t = 0$ *or* $a \wedge t \neq 0$ *and* $y \wedge t = 0$.

(d) *For every pair nonzero distinct elements* a *and* b *there are nonzero elements* $x \leqslant a$ *and* $y \leqslant b$ *in* F *such that* $\overline{x} \neq \overline{y}$.

(e) *For each pair nonzero distinct elements* a *and* b *of* F*, there are nonzero elements* $x \leqslant a$ *and* $y \leqslant b$*, and* $t \in \tau$ *such that either* $x \wedge t \neq 0$ *and* $b \wedge t^* \neq 0$ *or* $y \wedge t \neq 0$ *and* $a \wedge t^* \neq 0$.

Proof. (a)⇒(b). This is clear.

(b)⇒(c). By the assumption, either $0 \neq y \leqslant b \wedge t$ or $0 \neq x \leqslant a \wedge t$, which completes the proof.

(c)⇒(d). By the assumption, without loss of generality, suppose that there exist $t \in \tau$ and $0 \neq x \leqslant a$ such that $b \wedge t \neq 0$ and $x \wedge t = 0$. Taking $y = b \wedge t$, we conclude that $0 \neq y \leqslant b$ and $x \leqslant t^*$. Thus, $\overline{x} \leqslant t^*$ and so $y \wedge \overline{x} = 0$. Therefore, $\overline{x} \neq \overline{y}$.

(d)⇒(e). By hypothesis there exist $0 \neq x \leqslant a$ and $0 \neq y \leqslant b$ such that $\overline{x} \neq \overline{y}$. Without loss of generality, assume that $\overline{x} \nleqslant \overline{y}$. Thus, $x \nleqslant \overline{y}$ and so there exists $t \in \tau$ such that $y \leqslant t^*$ and $x \nleqslant t^*$. Therefore, $x \wedge t \neq 0$ and $0 \neq y \leqslant b \wedge t^*$.

(e)⇒(a). Suppose that a and b are two nonzero distinct elements in F. By hypothesis, without loss of generality, there exist $0 \neq x_1 \leqslant a$ and $t \in \tau$ such that $x_1 \wedge t \neq 0$ and $b \wedge t^* \neq 0$. Taking $x = x_1 \wedge t$ and $y = b \wedge t^*$, clearly, $0 \neq x \leqslant a$, $0 \neq y \leqslant b$, $x \leqslant t$, and $y \wedge t = 0$. Hence, $aT_{01}b$. □

In the above theorem, if F has the ps-property, then we can replace "distinctness" of the elements a and b with "disjointness" in items (b), (c), (d), and (e); also "distinctness" of the elements a and b can be replaced by the condition "$a < b$" in item (c).

In the following proposition, we show that the T_0 separation axiom in LG-topologies is a generalization of T_0 separation axiom in general topologies.

Corollary 4.4. *Let* (X, τ) *be a topological space. Then* (X, τ) *is a* T_0 *topological space if and only if* $\big(P(X), \tau\big)$ *is a* T_0 *as an LG-space.*

Proof. This is straightforward, by Theorem 4.3(d) and the fact that X is a T_0 topological space if and only if for every pair distinct elements x and y, $\overline{\{x\}} \neq \overline{\{y\}}$. $\square$

5 Generalizations of the separation axiom T_1 in LG-spaces

At this stage, we introduce the T_1 separation axiom.

Definition 5.1. Suppose F_τ is an LG-space and a and b are two distinct elements of F.

(1) We write aT_1b if there exist nonzero elements $x \leqslant a$ and $y \leqslant b$ in F and s and t in τ such that $x \leqslant s$, $y \leqslant t$, $s \wedge y = 0$ and $x \wedge t = 0$.

(2) We say F_τ is T_1 at a whenever aT_1b for all other nonzero elements $b \in F$.

(3) We say F_τ is T_1 if F_τ is T_1 at each nonzero element.

Clearly, for each pair of elements a and b of an LG-space, aT_1b if and only if bT_1a. Also, aT_1b implies aT_0b and bT_0a. Thus, each T_1 lattice generalized topological space is T_0.

Proposition 5.2. *Suppose that F_τ is an LG-space. The following statements are equivalent.*

(a) *F_τ is T_1.*

(b) *For each pair distinct nonzero elements a and b of F we have $aT_{01}b$ and $aT_{02}b$.*

(c) *For each pair distinct nonzero elements a and b of F we have $aT_{01}b$.*

Proof. (a)⇒(b) and (b)⇒(c). These implications are clear.

(c)⇒(a). Suppose that a and b are two distinct nonzero elements of F. Since $aT_{01}b$, there are nonzero elements $x_1 \leqslant a$ and $y_1 \leqslant b$ in F and s in τ such that $x_1 \leqslant s$ and $s \wedge y_1 = 0$. Since $y_1T_{01}x_1$, there are nonzero elements $x \leqslant x_1$ and $y \leqslant y_1$ and t in τ such that $y \leqslant t$ and $x \wedge t = 0$. Then, clearly, $x \leqslant s$, $y \leqslant t$, $x \wedge t = 0$ and $s \wedge y = 0$. This shows that F_τ is T_1. $\square$

The following theorem presents some conditions equivalent to the T_1 separation axiom.

Theorem 5.3. *Suppose that F_τ is an LG-space. Then the following are equivalent.*

(a) *F_τ is T_1.*

(b) *For each pair nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge \overline{y} = 0$ and $y \wedge \overline{x} = 0$.*

(c) *For each pair nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $\overline{x} || \overline{y}$.*

(d) *For each pair nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ and s and t in τ such that $x \wedge s = 0$, $y \wedge s \neq 0$, $x \wedge t \neq 0$ and $y \wedge t = 0$.*

(e) *For each pair nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ and s and t in τ such that $x \wedge t \neq 0$, $b \wedge t^* \neq 0$, $y \wedge s \neq 0$ and $a \wedge s^* \neq 0$.*

Proof. (a)⇒(b). Suppose that a and b are two distinct elements of F. By the assumption, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ in F and s and t in τ such that

$$\left.\begin{cases} x \leqslant t & \\ y \wedge t = 0 \quad \Rightarrow \quad y \leqslant t^* & \end{cases}\right. \quad \Rightarrow \quad x \wedge \overline{y} \leqslant t \wedge t^* \quad \Rightarrow \quad x \wedge \overline{y} = 0$$

$$\left.\begin{cases} y \leqslant s & \\ x \wedge s = 0 \quad \Rightarrow \quad x \leqslant s^* & \end{cases}\right. \quad \Rightarrow \quad y \wedge \overline{x} \leqslant s \wedge s^* \quad \Rightarrow \quad y \wedge \overline{x} = 0$$

(b)⇒(c). By the assumption there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge \overline{y} = \overline{x} \wedge y = 0$. On contrary suppose that $\overline{x} \not\parallel \overline{y}$. Without loss of generality, we can assume that $\overline{x} \leqslant \overline{y}$. Then

$$0 = x \wedge \overline{y} = (x \wedge \overline{x}) \wedge \overline{y} = x \wedge (\overline{x} \wedge \overline{y}) = x \wedge \overline{x} = x$$

which is a contradiction.

(c)⇒(d). By the assumption, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $\overline{x} \parallel \overline{y}$. Thus, $\overline{y} \nleqslant \overline{x}$ and $\overline{x} \nleqslant \overline{y}$, so there exist s and t in τ such that $x \leqslant s^*$, $\overline{y} \nleqslant s^*$, $y \leqslant t^*$ and $\overline{x} \nleqslant t^*$. Hence, $y \wedge s \neq 0$, $x \wedge s = 0$, $x \wedge t \neq 0$ and $y \wedge t = 0$.

(d)⇒(e). By the assumption, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ and s and t in τ such that $x \wedge s = 0$, $y \wedge s \neq 0$, $y \wedge t = 0$ and $x \wedge t \neq 0$. Then

$$\begin{cases} x \leqslant s^* & \Rightarrow \quad 0 \neq x = x \wedge s^* \leqslant a \wedge s^* \quad \Rightarrow \quad a \wedge s^* \neq 0 \\ y \leqslant t^* & \Rightarrow \quad 0 \neq y = y \wedge t^* \leqslant b \wedge t^* \quad \Rightarrow \quad b \wedge t^* \neq 0 \end{cases}$$

(e)⇒(a). Suppose that a and b are two nonzero distinct elements of F. By Proposition 5.2, it suffices to show that $aT_{01}b$. By our assumption, there are nonzero elements $z \leqslant a$ and $t \in \tau$ such that $z \wedge t \neq 0$, $b \wedge t^* \neq 0$. Taking $x = z \wedge t$ and $y = b \wedge t^*$, it follows that $x \leqslant a$ and $y \leqslant b$ are nonzero elements, $x \leqslant t$ and $y \wedge t = 0$. Therefore, $aT_{01}b$. □

In the above theorem, if F has the ps-property, we can replace "distinct" by "disjoint".

Clearly, the T_1 separation axiom introduced in the context of LG-topology is an extension of the same axiom in general topology. Moreover, in the following remark we provide some known conditions through general topology context that are equivalent to the T_1 separation axiom.

Remark 5.4. Suppose that (X, τ) is a topological space and $F = P(X)$. We know that the following are equivalent.

- X is a T_1 topological space.

(C0-1) F_τ is a T_1 lattice generalized topological space.

(C1-1) For each pair of nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge \overline{y} = 0$ and $y \wedge \overline{x} = 0$.

(C1-2) For each pair of nonzero disjoint elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge \overline{y} = 0$ and $y \wedge \overline{x} = 0$.

(C2-1) For each pair of nonzero distinct elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $\overline{x} \parallel \overline{y}$.

(C2-2) For each pair of nonzero disjoint elements a and b of F, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $\overline{x} \parallel \overline{y}$.

(C3-1) For every nonzero element a in F there is a nonzero element $x \leqslant a$ such that $\overline{x} \leqslant a$.

(C3-2) For every nonzero element a in F there is a nonzero closed element $x \leqslant a$.

(C4-1) For every $1 \neq a \in F$, there is an element $1 \neq x \geqslant a$ such that $x^\circ \geqslant a$.

(C4-2) For every $1 \neq a \in F$, there is an open element $1 \neq x \geqslant a$.

(C5-1) For each pair of nonzero elements $a < b$ of F, there is a closed element $x \leqslant b$ such that $x \nleqslant a$.

(C5-2) $a = \bigvee\{t^* : t \in \tau\ ,\ t^* \leqslant a\}$, for all $a \in F$.

(C6-1) For each pair of nonzero elements $a < b$ of F, there is an open element $a \leqslant t$ such that $b \nleqslant t$.

(C6-2) $a = \bigwedge\{t \in \tau : a \leqslant t\}$, for all $a \in F$.

In the above theorem, we see that (C0-1), (C1-1), (C1-2), (C2-1), and (C2-2), are equivalent and it is easy to show that (Ci-1) and (Ci-2) are equivalent, for all $3 \leqslant i \leqslant 6$, for each LG-space with the ps-property. The following proposition states how the above conditions are related to each other in this case.

Proposition 5.5. *Suppose that F_τ is an LG-space and F has the ps-property. Then the following statements hold:*

(a) (C6) $\Rightarrow$ (C4).

(b) (C5) $\Rightarrow$ (C3).

(c) (C3) $\Rightarrow$ (C1).

(d) *If F is Boolean, then* (C3), (C4), (C5), *and* (C6) *are equivalent.*

Proof. (a). Suppose that $1 \neq a \in F$. Then, by the assumption, $1 \neq a = \bigwedge\{t \in \tau : a \leqslant t\}$ and therefore there is an open element $1 \neq t \geqslant a$.

(b). Suppose a is a nonzero element of F. Then, by the hypothesis, $0 \neq a = \bigvee\{t^* : t \in \tau, t^* \leqslant a\}$ and therefore a closed element t^* exists such that $0 \neq t^* \leqslant a$.

(c). Suppose that a and b are two nonzero disjoint elements of F. By the assumption, there are closed elements $x \leqslant a$ and $y \leqslant b$, then $\overline{x} \wedge y = x \wedge y \leqslant a \wedge b = 0$. Similarly, $x \wedge \overline{y} = 0$.

(d). By parts (a), (b), and (c), it is sufficient to show (C3) $\Leftrightarrow$ (C4), (C3) $\Rightarrow$ (C5), and (C5) $\Leftrightarrow$ (C6). Note that in this part, we have $\tau^* = \tau^c = \{t^c :\ t \in \tau\}$.

(C3) $\Rightarrow$ (C4). Suppose that $1 \neq a \in F$. Thus $a^c \neq 0$ and thus, by the assumption, $t \in \tau$ exists such that $0 \neq t^c \in \tau^*$ and $t^c \leqslant a^c$. Hence $1 \neq t \in \tau$ and $a \leqslant t$.

(C4) $\Rightarrow$ (C3). Suppose that $0 \neq a \in F$. Thus $a^c \neq 1$ and so, by the assumption, $1 \neq t \in \tau$ exists such that $a^c \leqslant t$. Hence $0 \neq t^c \in \tau^*$ and $t^c \leqslant a$.

(C3) $\Rightarrow$ (C5). Let $0 \neq a \in F$. On contrary, suppose that $\bigvee\{t^c : \ t \in \tau, \ \ t^c \leqslant a\} = b < a$. Set $x = b^c \wedge a$. Clearly, $x \neq 0$. Thus, by the assumption, $t \in \tau$ exists such that $0 \neq t^c \leqslant x = b^c \wedge a$. Thus, $t^c \leqslant a$ and so $t^c \leqslant b$. Hence, $t^c \leqslant b \wedge b^c = 0$, which is a contradiction.

(C5) $\Leftrightarrow$ (C6). Clearly, we can write

$$\begin{aligned} & \forall a \in F \setminus \{0\}, \quad a = \bigvee\{t^c : t \in \tau, \ t^c \leqslant a\} \\ \Leftrightarrow \quad & \forall a \in F \setminus \{0\}, \quad a^c = \bigwedge\{t : \ t \in \tau, \ a^c \leqslant t\} \\ \Leftrightarrow \quad & \forall b \in F \setminus \{1\}, \quad b = \bigwedge\{t : \ t \in \tau, \ b \leqslant t\}. \qquad \square \end{aligned}$$

We know that in general topology a space is T_1 if and only if each element of $P(X)$ is an intersection of open sets. In the following proposition, we show that one implication of the above fact holds for topoframes.

Proposition 5.6. *Suppose that F_τ is a topoframe and for each $a \in F$, there is a family $\{t_\lambda\}_{\lambda\in\Lambda}$ of elements of τ such that $a = \bigwedge_{\lambda\in\Lambda} t_\lambda$, then F_τ is T_1.*

Proof. Let a and b be two nonzero distinct elements of F. Thus either $a \nleqslant b$ or $b \nleqslant a$. Suppose that $b \nleqslant a$. According to the assumption, $a = \bigwedge_{\lambda\in\Lambda} t_\lambda$, for some family $\{t_\lambda\}_{\lambda\in\Lambda}$ of elements of τ. Therefore, there is a $\lambda \in \Lambda$ such that $b \nleqslant t_\lambda$. Set $y = b \wedge t_\lambda^c \neq 0$ and $x = a$, then $x \leqslant a$, $y \leqslant b$, $x \leqslant t_\lambda$, $b \wedge t_\lambda = 0$, and thus $aT_{01}b$. If $a \nleqslant b$, then, similarly, we can show that $aT_{02}b$. This shows that F_τ is T_0 and therefore F has the ps-property. Thus, we can assume that a and b are nonzero disjoint elements, hence $a \nleqslant b$ and $b \nleqslant a$, and therefore, analogously, aT_0b and bT_0a. This shows that F_τ is a T_1 space. $\square$

Considering the properties of LG-topology, we introduce and study a new type of separation axioms, and subsequently show that this condition is equivalent to the T_0 separation axiom for general topology context.

Proposition 5.7. *Let F_τ be a Boolean LG-space. The following statements are equivalent*

(a) *τ is a meet dense subset of F.*

(b) *τ^* is a join dense subset of F.*

(c) *$\bigwedge h_\tau(x) \neq 1$, for every $1 \neq x \in F$.*

(d) *$\bigvee c_{\tau^*}(x) \neq 0$, for every $0 \neq x \in F$.*

Proof. (a)⇔(b). We proved this in Proposition 5.5(d).

(a)⇒(c). Suppose, on the contrary, that $h_\tau(x) = \{1\}$, for some $1 \neq x \in F$. Thus, $\bigwedge h_\tau(x) = 1 \neq x$, which is a contradiction.

(c)⇒(d). Suppose that $0 \neq x \in F$. Hence, $x^c \neq 1$ and by the assumption we have $h_\tau(x^c) \neq \{1\}$ and so $c_{\tau^*}(x) \neq \{0\}$.

(d)⇒(a). Suppose that $y = \bigwedge h_\tau(x)$. On the contrary, assume that $x < y$. Thus, $x^c \wedge y \neq 0$ and so, by the hypothesis, there exists $t \in \tau$ such that $0 \neq t^c \leqslant x^c \wedge y$. Therefore, $x \leqslant t$ and $y^c \leqslant t$. Since $x \leqslant t$, it follows that $y \leqslant t$. Hence, $1 = y \vee y^c \leqslant t$ and so $t = 1$. Consequently, $t^c = 0$ and this is a contradiction. □

Lemma 5.8. *Let F be a Boolean algebra, $x \in F$ and $A \subseteq h_F(x)$. Then $x = \bigwedge A$ if and only if for every nonzero element b disjoint from x there are nonzero elements $y \leqslant b$ in F and a_0 in A such that $y \wedge a_0 = 0$.*

Proof. (⇒). Suppose that $0 \neq b \in x^\perp$, so $b \leqslant x^c = \bigvee_{a \in A} a^c$ and thus $b = \bigvee_{a \in A} b \wedge a^c$. Hence $a_0 \in A$ exists such that $y = b \wedge a_0^c \neq 0$. Clearly, $y \leqslant b$ is a nonzero element of F, $a_0 \in A$, and $y \wedge a_0 = 0$.

(⇐). Suppose, on the contrary, that $x < \bigwedge A$. Thus, we can easily see that $b = x^c \wedge (\bigwedge A) \neq 0$. Since $b \wedge x = 0$, the assumption implies that a nonzero element $y \leqslant b$ in F and a_0 in A exist such that $y \wedge a_0 = 0$. Hence, $y \wedge (\bigwedge A) = 0$ and this contradicts $0 \neq y \leqslant b = x^c \wedge (\bigwedge A)$. □

Proposition 5.9. *If τ (respectively, τ^*) is a meet dense (respectively, join dense) subset of a Boolean LG-space F_τ, then F_τ is T_1.*

Proof. Suppose a and b are disjoint nonzero elements of F. According to the assumption $a = \bigwedge h_\tau(a)$ and $b = \bigwedge h_\tau(b)$. Then, by Lemma 5.8, there are $0 \neq y \leqslant b$, $s \in h_\tau(a)$, $0 \neq x \leqslant a$ and $t \in h_\tau(b)$ such that $s \wedge y = 0$ and $t \wedge x = 0$. Therefore, $0 \neq x \leqslant a$, $0 \neq y \leqslant b$, $x \leqslant s$, $y \leqslant t$, $s \wedge y = 0$, and $t \wedge x = 0$. This shows that F_τ is a T_1 space. The proof of other case is similar. $\square$

6 Generalizations of the separation axiom T_2 in LG-spaces

In this section, we introduce the last separation axiom in this paper, namely T_2, in LG-topology, and present the conditions equivalent to it.

Definition 6.1. Suppose that F_τ is an LG-space and a and b are two distinct elements of F. We write aT_2b if there are nonzero elements $x \leqslant a$ and $y \leqslant b$ and $s \in h_\tau(x)$ and $t \in h_\tau(y)$ such that $s \wedge t = 0$. F_τ is called T_2 or Hausdorff if for each nonzero distinct elements a and b we have aT_2b.

It is clear that aT_2b if and only if bT_2a. Moreover, aT_2b implies aT_1b; for each distinct elements a and b of an LG-space. Hence, each Hausdorff LG-space is T_1.

Lemma 6.2. *Let F_τ be an LG-space and a and b are two nonzero elements of F. Then $a \nleqslant \overline{b}$ if and only if there are nonzero elements $x \leqslant a$ and $t \in h_\tau(x)$ such that $t \wedge b = 0$.*

Proof. Clearly, we can write

$$\begin{aligned} a \nleqslant \overline{b} \quad &\Leftrightarrow \quad \exists t \in \tau; \quad b \leqslant t^* \quad \text{and} \quad a \nleqslant t^* \\ &\Leftrightarrow \quad \exists t \in \tau \quad \exists x \in F; \quad b \leqslant t^* \quad \text{and} \quad x = a \wedge t \neq 0 \\ &\Leftrightarrow \quad \exists t \in \tau \quad \exists x \neq 0; \quad b \wedge t = 0 \quad , \quad x \leqslant a \quad \text{and} \quad x \leqslant t \\ &\Leftrightarrow \quad \exists 0 \neq x \leqslant a \quad \exists t \in h_\tau(x); \quad b \wedge t = 0. \end{aligned}$$

$\square$

Theorem 6.3. *For every LG-space F_τ, the following statements are equivalent.*

- (a) *F_τ is Hausdorff.*
- (b) *For every nonzero distinct elements a and b of F there are two disjoint elements s and t such that $a \wedge s \neq 0$ and $b \wedge t \neq 0$.*
- (c) *For every nonzero distinct elements a and b in F, there are nonzero elements $x \leqslant a$, $y \leqslant b$, $s \in h_\tau(x)$ and $t \in h_\tau(y)$ such that $\overline{s} \wedge y = 0$ and $\overline{t} \wedge x = 0$.*

(d) *For every nonzero distinct elements a and b in F, there are nonzero elements $x \leqslant a$, $y \leqslant b$ and $s \in h_\tau(x)$ such that $\overline{s} \wedge y = 0$.*

Proof. (a)⇒(b). Suppose that a and b are two distinct elements of F. By the hypothesis, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ and $s \in h_\tau(x)$ and $t \in h_\tau(y)$ such that $s \wedge t = 0$. Then $0 \neq x \leqslant a \wedge s$ and $0 \neq y \leqslant b \wedge t$.

(b)⇒(c). Suppose that a and b are two distinct elements of F. By the hypothesis, there are disjoint elements s and t of τ such that $x = a \wedge s \neq 0$ and $y = b \wedge t \neq 0$. Then $s \in h_\tau(x)$ and $t \in h_\tau(y)$ and

$$s \wedge t = 0 \ \Rightarrow \ t \leqslant s^* \ \Rightarrow \ \bar{t} \leqslant s^* \ \Rightarrow \ \bar{t} \wedge x \leqslant \bar{t} \wedge s = 0 \ \Rightarrow \ \bar{t} \wedge x = 0$$

Similarly, we can show that $\overline{s} \wedge y = 0$.

(c)⇒(d). It is evident.

(d)⇒(a). Suppose that a and b are two nonzero distinct elements of F. According to the assumption, there are nonzero elements $x \leqslant a$, $y \leqslant b$ and $s \in h_\tau(x)$ such that $\overline{s} \wedge y = 0$ and thus $y \nleqslant \overline{s}$. By Lemma 6.2, there is a nonzero element $z \leqslant y$ and $t \in h_\tau(z)$ such that $s \wedge t = 0$. Then $x \leqslant a$, $z \leqslant b$ are nonzero elements of F, $s \in h_\tau(x)$, $t \in h_\tau(z)$, $s \wedge t = 0$, and therefore aT_2b. This shows that F_τ is Hausdorff. □

In the following corollary, we show that the Hausdorff separation axiom introduced here is an extension of the Hausdorff separation axiom in general topology.

Corollary 6.4. *(X, τ) is a Hausdorff topological space if and only if $(P(X), \tau)$ is a Hausdorff LG-space.*

Proof. ⇒. Suppose $A, B \in P(X)$ are two distinct nonzero elements. Then there are two distinct elements $a \in A$ and $b \in B$. By the assumption, there are $U, V \in \tau$ such that $a \in U$ and $b \in V$ such that $U \cap V = \emptyset$. It is enough to set $A' = A \cap U$ and $B' = B \cap V$. Then $A' \subseteq A$, $B' \subseteq B$ are subsets of X, $U \in h_\tau(A')$, $V \in h_\tau(B')$ and $U \cap V = \emptyset$. This shows that $(P(X), \tau)$ is a Hausdorff LG-space.

⇐. Suppose that a and b are two distinct elements of X. By the assumption, for two distinct elements $\{a\}$ and $\{b\}$ of $P(X)$, there are nonempty subsets $A \subseteq \{a\}$, $B \subseteq \{b\}$, and disjoint open sets U and V such that $\{a\} = A \subseteq U$ and $\{b\} = B \subseteq V$. Then $a \in U$, $b \in V$ and $U \cap V = \emptyset$. This shows that X is Hausdorff. □

7 Related topics to the separation axioms

First, in this section, we show that separation axioms and pre-separation axioms are hereditary, and they transfer from products to components and from components to products.

Proposition 7.1. *Suppose that* (F,τ) *is an LG-space,* $a \in F$, $F_a = h_F(a)$, *and* $\tau_a = \{t \wedge a : t \in \tau\}$.

(a) *If* F *has the ps-property, then* F_a *also has the ps-property.*

(b) *If* (F,τ) *is* T_0 (*respectively,* T_1, T_2), *then* (F_a,τ_a) *is as well.*

Proof. The proofs are straightforward. □

Proposition 7.2. *Suppose that* (F_i,τ_i) *is an LG-space, for every* $i \in I$, $F = \prod_{i\in I} F_i$, τ_p *is the product topology on* F *and* τ_b *is the box topology on* F.

(a) F *has the ps-property* (*respectively, sps-property*) *if and only if* F_i *has the ps-property* (*respectively, sps-property*), *for every* $i \in I$.

1. (b) (F,τ_p) (*respectively,* (F,τ_b)) *is* T_0 *if and only if* (F_i,τ_i) *is* T_0, *for every* $i \in I$.

(c) (F,τ_p) (*respectively,* (F,τ_b)) *is* T_1 *if and only if* (F_i,τ_i) *is* T_1, *for every* $i \in I$.

(d) *If* (F,τ_p) *is* T_2, *then* (F_i,τ_i) *is* T_2, *for every* $i \in I$.

(e) (F,τ_b) *is* T_2 *if and only if* (F_i,τ_i) *is* T_2, *for every* $i \in I$.

(f) *If* I *is infinite, then* (F,τ_p) *is not* T_2 *at all. But, if* I *is finite, then* (F,τ_p) *is* T_2 *if and only if* (F_i,τ_i) *is* T_2 *for every* $i \in I$.

Proof. (a)$\Rightarrow$. Suppose that a_i and b_i are two distinct nonzero elements of F_i. For each $j \neq i$, set $a_j = b_j = 0$. Then $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$ are two distinct elements of F. Now, by the assumption, there are two nonzero elements $x \leqslant a$ and $y \leqslant b$ such that $x \wedge y = 0$. Then $x_i \leqslant a_i$ and $y_i \leqslant b_i$ are two nonzero elements of F_i and $x_i \wedge y_i = 0$. This shows that F_i has the ps-property.

(a)$\Leftarrow$. Suppose that a and b are two nonzero distinct elements of F. Thus, there is some $i \in I$ such that $a_i \neq b_i$. If $a_i, b_i > 0$, then, by hypothesis, there exist nonzero elements $c_i, d_i \in F_i$ such that $c_i \leqslant a_i$, $d_i \leqslant b_i$, and $c_i \wedge d_i = 0$. We define $x, y \in F$ as follows

$$x_r = \begin{cases} 0 & r \neq i \\ c_i & r = i \end{cases} \qquad\qquad y_r = \begin{cases} 0 & r \neq i \\ d_i & r = i \end{cases}$$

Clearly, $0 < x \leqslant a$, $0 < y \leqslant b$, and $x \wedge y = 0$.

Now suppose that either a_i or b_i is zero. Without loss of generality, we can suppose that $a_i = 0$ and $b_i > 0$. In this case, there exists $j \neq i$ such that $a_j > 0$. We define $x, y \in F$ as follows

$$x_r = \begin{cases} 0 & r \neq j \\ a_j & r = j \end{cases} \qquad\qquad y_r = \begin{cases} 0 & r \neq i \\ b_i & r = i \end{cases}$$

Clearly, $0 < x \leqslant a$, $0 < y \leqslant b$, and $x \wedge y = 0$. The proof for the sps-property case is similar.

(b)$\Rightarrow$. Suppose that a_j and b_j are two distinct nonzero elements of F_j. Set $a_i = b_i = 0$, for all $i \neq j$, then $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$ are two distinct elements of F. Thus, by the assumption, there are two distinct nonzero elements $x \leqslant a$ and $y \leqslant b$ and $t \in \tau_p$ (respectively, $t \in \tau_b$) such that either $x \leqslant a$ and $y \wedge t = 0$ or $y \leqslant b$ and $x \wedge t = 0$. Hence $x_j \leqslant a_j$ and $y_j \leqslant b_j$ are two nonzero distinct elements of F_j such that either $x_j \leqslant a_j$ and $y_j \wedge t_j = 0$ or $y_j \leqslant b_j$ and $x_j \wedge t_j = 0$. This implies that F_j is T_0.

(b)$\Leftarrow$. Suppose that a and b are two nonzero distinct elements of F, so j in I exists such that $a_j \neq b_j$.

If a_j and b_j are nonzero, then, by the assumption, there are nonzero elements $x_j \leqslant a_j$ and $y_j \leqslant b_j$ and $t_j \in \tau_j$ such that either $x_j \leqslant t_j$ and $y_j \wedge t_j = 0$ or $y_j \leqslant t_j$ and $x_j \wedge t_j = 0$. Set $x_i = y_i = 0$ and $t_i = 1$, for all $i \neq j$. Then $x \leqslant a$, $y \leqslant b$ and $t \in \tau_p$ and either $x \leqslant t$ and $y \wedge t = 0$ or $y \leqslant t$ and $x \wedge t = 0$.

Now suppose, without loss of generality, $a_j = 0$ and $b_j \neq 0$. Since a is not zero, k in I exists such that $a_k \neq 0$ and therefore $k \neq j$. Set

$$x_i = \begin{cases} a_k & i = k \\ 0 & i \neq k \end{cases}; \qquad y_i = \begin{cases} b_j & i = j \\ 0 & i \neq j \end{cases}; \quad \text{and;} \qquad t_i = \begin{cases} 0 & i = k \\ 1 & i \neq k \end{cases}.$$

Then $0 \neq x \leqslant a$, $0 \neq y \leqslant b$, $t \in \tau_p$, $x \wedge t = 0$ and $y \leqslant t$

This implies that (F,τ_p) and (F,τ_b) are T_0.

(c). The proof is similar to (b).

(d) and (e)$\Rightarrow$. The proofs are similar to that of (b)$\Rightarrow$.

(e)$\Leftarrow$. Suppose that a and b are two nonzero distinct elements of F, so j in I exists such that $a_j \neq b_j$.

If a_j and b_j are nonzero, then, by the assumption, there are nonzero elements $x_j \leqslant a_j$ and $y_j \leqslant b_j$ of F_j and s_j and t_j in τ_j such that $x_j \leqslant s_j$, $y_j \leqslant t_j$ and $s_j \wedge t_j = 0$. Set $x_i = y_i = s_i = t_i = 0$, for all $i \neq j$. Then $x = (x_i)_{i\in I} \leqslant a$, $y = (y_i)_{i\in I} \leqslant b$ are nonzero elements of F and $s = (s_i)_{i\in I}$ and $t = (t_i)_{i\in I}$ are two elements of τ_b such that $x \leqslant s$, $y \leqslant t$ and $s \wedge t = 0$.

Now, without loss of generality, we assume that $a_j \neq 0$ and $b_j = 0$. Since $b \neq 0$, there is some $j \neq k \in I$ such that $b_k \neq 0$ and therefore $k \neq j$. Set

$$x_i = \begin{cases} a_j & i=j \\ 0 & i\neq j \end{cases}; \quad y_i = \begin{cases} b_k & i=k \\ 0 & i\neq k \end{cases}; \quad s_i = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}; \quad \text{and;} \quad t_i = \begin{cases} 1 & i=k \\ 0 & i\neq k \end{cases}.$$

Then $x = (x_i)_{i\in I} \leqslant a$, $y = (y_i)_{i\in I} \leqslant b$ are nonzero elements of F and $s = (s_i)_{i\in I}$ and $t = (t_i)_{i\in I}$ are two elements of τ_b such that $x \leqslant s$, $y \leqslant t$ and $s \wedge t = 0$. These imply that (F,τ_b) is T_2.

(f). If s and t are two elements of τ_p, then there is a finite subset A of I such that for all $i \in I \setminus A$, $s_i = t_i = 1$ and thus $(s\wedge t)_i = s_i \wedge t_i = 1$. Since I is infinite, there is some $j \in I$ such that $(s \wedge t)_j = 1$. Hence $s \wedge t \neq 0$. This shows that (F,τ_p) is not T_2.

If I is finite, then $\tau_p = \tau_b$ and therefore, by part (e), (F,τ_p) is T_2 if and only if (F_i,τ_i) is T_2, for all i in I. $\square$

Proposition 7.3. *Suppose that (F_i,τ_i) is an LG-space, for every $i \in I$, $F = \prod_{i\in I} F_i$, α is an infinite cardinal and*

$$\tau_\alpha = \Big\{t \in \prod_{i\in I} \tau_i : \big|\{i \in I : t_i \neq 1\}\big| \leqslant \alpha\Big\} \cup \Big\{0\Big\}.$$

(a) *τ_α is a subframe of F and so (F,τ_α) is an LG-space.*

(b) *(F,τ_α) is T_0 (respectively, T_1) if and only if (F_i,τ_i) is T_0 (respectively, T_1) for every $i \in I$.*

(c) *(F,τ_α) is T_2 if and only if $|I| \leqslant \alpha$ (that is, $\tau_\alpha = \tau_b$).*

Proof. (a). This is straightforward.

(b). The proof is similar to the proof of Proposition 7.2(c).

(c)$\Rightarrow$. Assume that $\alpha < |I|$ and r and s are two nonzero elements of τ_α. Put $A = \{i \in I : r_i \neq 1\}$ and $B = \{i \in I : s_i \neq 1\}$. Therefore, there exists $j \in I \setminus A \cup B$. Thus, $(r \wedge s)_j = r_j \wedge s_j = 1$ and so $r \wedge s \neq 0$. This concludes that (F, τ_α) is not T_2.

(c)$\Leftarrow$. It follows from Proposition 7.2(e). □

It is easy to see that for every infinite cardinal number α, we have $\tau_p \subseteq \tau_\alpha \subseteq \tau_b$. Also, if $\alpha \leqslant \beta$, then $\tau_\alpha \subseteq \tau_\beta$.

At the end of this paper, we examine which of the properties mentioned in Corollary 5.4 are preserved under which mappings.

Proposition 7.4. *Assume that* (F_1, τ_1) *and* (F_2, τ_2) *are two LG-spaces,* $f : F_1 \to F_2$ *is an onto, open, and order preserving map and* $f^{-1}(0) = \{0\}$. *If* (F_1, τ_1) *is* T_0 (*respectively,* T_1), *then* (F_2, τ_2) *is too.*

Proof. Suppose that c and d are two nonzero distinct elements of F_2. Since f is onto, there are nonzero distinct elements a and b in F_1 such that $f(a) = c$ and $f(b) = d$. Since F_1 is T_0, there are nonzero elements $x \leqslant a$ and $y \leqslant b$ and t in τ such that either $x \leqslant t$ and $y \wedge t = 0$ or $x \wedge t = 0$ and $y \leqslant t$, since f is an open, order preserving map and $f^{-1}(0) = \{0\}$, it follows that $0 \neq f(x) \leqslant f(a)$, $0 \neq f(y) \leqslant f(b)$, $f(t) \in \tau_2$ and either $f(x) \leqslant f(t)$ and $f(y) \wedge f(t) = 0$ or $f(x) \wedge f(t) = 0$ and $f(y) \leqslant f(t)$. This shows that (F_2, τ_2) is T_0. Similarly, the T_1 state can be proven. □

Proposition 7.5. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two LG-spaces,* $f : F_1 \to F_2$ *is an onto, open, and order preserving map and* $f^{-1}(1) \subseteq \{1\}$. *If* (F_1, τ_1) *satisfies (C4), then* (F_2, τ_2) *satisfies* (C4) *too.*

Proof. Suppose that $1 \neq b \in F_2$. Since $f^{-1}(1) = \{1\}$ and f is onto, there is an element $1 \neq a \in F_1$ such that $f(a) = b$. Since F satisfies in (C4), there is $1 \neq t \in \tau_2$ such that $a \leqslant t$. Also, since $f^{-1}(1) = \{1\}$ and f is open, it follows that $f(t) \neq 1$ and $b = f(a) \leqslant f(t) \in \tau_2$. This shows that (F_2, τ_2) satisfies in (C4). □

Proposition 7.6. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two LG-spaces,* $f : F_1 \to F_2$ *is an o-continuous map,* $f^{-1}(1) \subseteq \{1\}$ *and* $f_*^{-1}(1) = \{1\}$. *If* (F_2, τ_2) *satisfies in* (C4), *then* (F_1, τ_1) *satisfies in* (C4) *too.*

Proof. Suppose that $1 \neq a \in F_1$, since $f^{-1}(1) \subseteq \{1\}$, it follows that $f(a) \neq 1$, since (F_2, τ_2) satisfies in (C4), there is $1 \neq t \in \tau_2$ such that $f(a) \leqslant t$, since f is o-continuous and $f_*^{-1}(1) = \{1\}$, it follows that $a \leqslant f_*f(a) \leqslant f_*(t) \in \tau_1$ and $f_*(t)$ is not the identity. This shows that (F_1, τ_1) satisfies in (C4). □

Proposition 7.7. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two LG-spaces and* $f : F_1 \to F_2$ *is an onto, closed, and left adjoint map. If* (F_1, τ_1) *satisfies in* (C5)*, then* (F_2, τ_2) *satisfies in* (C5) *too.*

Proof. Suppose that $c < d$ are two elements of F_2. Since f is onto, there are a and b in F_1 such that $f(a) = c$ and $f(b) = d$. Then $f(a) < f(b)$, so $f_*f(a) \leqslant f_*f(b)$. If $f_*f(a) = f_*f(b)$, then $c = f(a) = ff_*f(a) = ff_*f(b) = f(b) = d$, which is a contradiction. Hence $f_*f(a) < f_*f(b)$, since (F_1, τ_1) satisfies in (C5), it follows that there is a closed element $x \leqslant f_*f(b)$ such that $x \nleqslant f_*f(a)$, thus $f(x) \leqslant ff_*f(b) = f(b) = d$. If $f(x) \leqslant c = f(a)$, then $x \leqslant f_*f(x) \leqslant f_*f(a)$, which contradicts our assumption. Hence $f(x) \nleqslant c$. Since f is closed, $f(x)$ is closed and thus (F_2, τ_2) satisfies in (C5). □

Proposition 7.8. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two topoframes and* $f : F_1 \to F_2$ *is an onto and one-to-one o-continuous map. If* (F_2, τ_2) *satisfies in* (C5)*, then* (F_1, τ_1) *satisfies in* (C5) *too.*

Proof. Suppose that $a < b$ are two elements of F_1. Since f is one-to-one, it follows that $f(a) < f(b)$ and, by the hypothesis, there is a closed element $x \leqslant f(b)$ such that $x \nleqslant f(a)$. Since f is a one-to-one and onto o-continuous map, f is RL-adjoint, thus, by Proposition 2.3, $f_*(x)$ is a closed element of F_1 and, since f is one-to-one, it follows that $f_*(x) \leqslant f_*f(b) = b$ and $f_*(x) \nleqslant f_*f(a) = a$. This shows that (F_1, τ_1) satisfies in (C5). □

Proposition 7.9. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two LG-spaces and* $f : F_1 \to F_2$ *is an onto, one-to-one, open and order preserving map. If* (F_1, τ_1) *satisfies in* (C6)*, then* (F_2, τ_2) *satisfies in* (C6) *too.*

Proof. It is straightforward. Note that, by hypothesis $f : F_1 \to F_2$ is a frame isomorphism. □

Proposition 7.10. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two LG-spaces and* $f : F_1 \to F_2$ *is a one-to-one o-continuous map. If* (F_2, τ_2) *satisfies in* (C6)*, then* (F_1, τ_1) *satisfies in* (C6) *too.*

Proof. Assume that $a < b$ are two elements of F_1, since f is one-to-one and order preserving, it follows $f(a) < f(b)$. Since (F_2, τ_2) satisfies in (C6), there is $t \in \tau_2$ such that $f(a) \leqslant t$ and $f(b) \nleqslant t$. Then $a \leqslant f_*f(a) \leqslant f_*(t)$, and, since f is o-continuous, $f_*(t)$ is open. If $b \leqslant f_*(t)$, then $f(b) \leqslant ff_*(t) \leqslant t$, which is a contradiction. Thus $b \nleqslant f_*(t)$ and therefore (F_1, τ_1) satisfies in (C6) too. ☐

Proposition 7.11. *Suppose that* (F_1, τ_1) *and* (F_2, τ_2) *are two LG-spaces and* $f : F_1 \to F_2$ *is a one-to-one o-continuous map. If* (F_2, τ_2) *is* T_0 *(respectively,* T_1 *or* T_2*), then* (F_1, τ_1) *is too.*

Proof. Suppose a and b are two distinct elements of F_1. Since f is one-to-one, $f(a)$ and $f(b)$ are distinct and since (F_2, τ_2) is T_0, there are nonzero elements $x \leqslant f(a)$ and $y \leqslant f(b)$ and t in τ_2 such that either $x \leqslant t$ and $y \wedge t = 0$ or $x \wedge t = 0$ and $y \leqslant t$. Then $f_*(x) \leqslant f_*f(a)$, $f_*(y) \leqslant f_*f(b)$ and either $f_*(x) \leqslant f_*(t)$ and $f_*(y) \wedge f_*(t) = f_*(0) = 0$ or $f_*(y) \leqslant f_*(t)$ and $f_*(x) \wedge f_*(t) = f_*(0) = 0$. Since f is o-continuous, $f_*(t) \in \tau_1$. It is sufficient to show that $f(x)$ and $f(y)$ are nonzero. Otherwise, since f is one-to-one, then $x = f_*f(x) = f_*(0) = 0$ and $y = f_*f(y) = f_*(0) = 0$, which is a contradiction.

Similarly, we can prove the T_1 and T_2 states. ☐

Acknowledgment

The author would like to thank the referee for the valuable suggestions and comments. The valuable guidance of the reviewer led the authors to conduct a more thorough review of the article and improved it by correcting its errors.

The first author grateful to the Research Council of Shahid Chamran University of Ahvaz for financial support (GN: SCU.MM1404.839).

References

[1] Aliabad, A.R. and Sheykhmiri, A., *LG-topology*, Bull. Iranian Math. Soc. 41(1) (2015), 239-258.

[2] Aliabad, A.R. and Zarepour, H., *Continuous functions on LG-spaces*, J. Algebr. Syst. 8(2) (2021), 181-200.

[3] Alexandroff, P., *Über die Metrisation der im Kleinen kompakten topologische Räume*, Math. Ann. 92 (1924), 294-301.

[4] Alexandroff, P. and Urysohn, P., *Sur les espaces topologiques compacts*, Bull. Intern. Acad. Pol. Sci. Sér. A (1923), 5-8.

[5] Alexandroff, P. and Urysohn, P., *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch. Amsterdam 14 (1929), 1–96.

[6] Badie, M., Kasiri, H., and Shahidikia, A., *On continuous functions on LG-topology*, J. Adv. Math. Model. 11(4) (2021), 666-678.

[7] Bruns, G., *Darstellungen und Erweiterungen geordneter Mengen II*, J. Reine Angew. Math. 210 (1962), 1-23.

[8] Engelking, R., "General Topology", Heldermann Verlag, 1989.

[9] Estaji, A.A., Karimi Feizabadi, A., and Zarghani, M., *The ring of realcontinuous functions on a topoframe*, Categ. Gen. Algebr. Struct. Appl. 4(1) (2016), 75-94.

[10] Estaji, A.A., Karimi Feizabadi, A., and Zarghani, M., *Zero elements and z-ideals in modified pointfree topology*, Bull. Iranian Math. Soc. 43(7) (2017), 2205-2226.

[11] Givant, S. and Halmos, P., "Introduction to Boolean Algebras", Springer+Business Media, LLC, 2007.

[12] Johnstone, P.T., "Stone Spaces", Cambridge University Press, 1982.

[13] Lindelöf, E., *Sur quelques points de la théorie des ensembles*, C.R. Acad. Paris 137 (1903), 697-700.

[14] Rudin, W., "Real and Complex Analysis", Third Edition, McGraw-Hill Book Company, 1987.

[15] Thron, W.J., *Lattice-equivalence of topological spaces*, Duke Math. J. 29 (1962), 671-679.

[16] Valov, V., *Function spaces*, Topology Appl. 81 (1997), 1-22.

[17] Wang, G.J., *Topological molecular lattices (I)*, Kexue Tongbao 28(18) (1983), 1089-1091.

[18] Wang, G.J., *Theory of topological molecular lattices*, Fuzzy Sets and Systems 47(3) (1992), 351-376.

[19] Willard, S., "General Topology", Addison-Wesley Publishing Co., 1970.

[20] Zarghani, M., "The Ring of Real-Continuous Functions on a Topoframe", Ph.D. Thesis, Hakim Sabzevari University, 2017.

[21] Zarghani, M., Estaji, A.A., and Karimi Feizabadi, A., *A new pointfree form of topology*, 47th Annual Iranian Math. Conf. (AIMC47), Kharazmi Univ., Karaj, Iran, 2016.

[22] Zarghani, M., Estaji, A.A., and Karimi Feizabadi, A., *Some separation axioms in topoframes*, Algebr. Struct. Appl. 9(2) (2022), 37-56.

Ali R. Alibabad *Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran*

Email: aliabady_r@scu.ac.ir

Mehdi Badie *Department of Basic Sciences, Jundi-Shapur University of Technology, Dezful, Iran*

Email: badie@jsu.ac.ir

Mohammad Hossein Hajian *Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran*

Email: hossein.mhh71@gmail.com