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Frames in which double pseudocomplements of cozero elements are cozero elements

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Abstract. A Tychonoff space X is said to be a CAP-space if the closure of the interior of every zero-set of X is itself a zero-set of X. These spaces were introduced by Golrizkhatami and Taherifar [19]. With a view to supplementing the results in the cited paper, we extend this notion to the setting of point-free topology. We thus define a completely regular frame to be capped in case the double pseudocomplement of every cozero element of the frame is also a cozero element. This makes the comparison of this concept with other disconnectivity notions (such as basic disconnectedness) defined by imposing conditions on pseudocomplements very transparent.

1 Introduction

Golrizkhatami and Taherifar [19] introduced a large class of spaces called C-almost P-space (abbreviated CAP-space). The space consisting of those spaces in which the closure of the interior of every zero-set is a zero-set. The

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idea is to extend the results in [19] to the setting of pointfree topology. By doing so, we adapted the definition of a CAP-space to frames by replacing subspaces with sublocales. We shall call frames of these type “CAP-frames” or “capped frames”. Our aim is to characterize frames of these types and to analyse characterizations of capped frames across a homomorphisms, in either direction. We then realized that a frame is capped precisely when the double pseudocomplements of cozero elements are cozero elements. The authors in [1] call these types of frames “casi Oz-frames”. There are several properties of frames which are defined by imposing conditions that pseudocomplements of certain types of elements be cozero elements. The authors in [6] studied similar frames and they call them Oz-frames. Oz-frames are the natural pointfree counterpart of Oz-spaces, that is, those topological spaces in which every open set is z-embedded. We will then compare capped frames to other frames which are defined (or characterized) by imposing certain requirements on pseudocomplements. The aim is also to determine conditions under which the coproduct of two capped frames is capped and to investigate the ring-theoretic characterization of capped frames.

The paper is laid out in the following order. In Section 2, we give preliminaries. The main purposes are to fix notation and recall key terms relevant for this discussion.

In Section 3, we give frame-theoretic characterizations of capped frames. We show that a frame L is capped if for every $c \in \operatorname{Coz} L$, c^{**} is a cozero element in L, and conversely (Proposition 3.2). One of the characterizations is that a Tychonoff space is capped if and only if the frame of its open subsets is capped (Proposition 3.4). A frame L is *weak* O_z if $c^* \in \operatorname{Coz} L$ for every $c \in \operatorname{Coz} L$. In Proposition 3.5, we prove that a frame is a weak Oz-frame if and only if it is capped and cozero-complemented. We say a frame homomorphism $h\colon L \to M$ is $\mathrm{coz}_{\mathrm{reg}}$-*onto* if for every regular $d \in \operatorname{Coz} M$ there is a regular $c \in \operatorname{Coz} L$ such that $h(c) = d$, and *solid* if for every regular $d \in \operatorname{Coz} M$, $h_*(d)$ is a regular cozero element in L. For dense quotient maps, $\mathrm{coz}_{\mathrm{reg}}$-ontoness and solidity are equivalent (Theorem 3.8). Let $h\colon L \to M$ be a dense coz-onto frame homomorphism. In Theorem 3.9, we prove that if L is capped, then so is M, and if h is solid and M is capped, then L is capped. One of our main results is to show that a frame L is capped if and only if βL is capped (Theorem 3.11). A quotient map $h\colon L \to M$ is called *$z^{\#}$-quotient map* if for each $v \in \operatorname{Coz} M$ there is a $u \in \operatorname{Coz} L$ such that $h(u^*) = v^*$. If a homomorphism $h\colon L \to M$ commutes with pseudocomplements, in the sense that $h(x^*) = h(x)^*$ for every $x \in L$, then it is called *nearly open*. Let $h\colon L \to M$ be a $z^{\#}$-quotient map. In Proposition 3.13, we show that if h is nearly open and L is capped, then so is M, and if h is dense and M is capped, then so is L. In Proposition 3.14, we show that for any capped frame L, L is basically disconnected if and only if L is an F-frame if and only if L is an F' frame. We end this section by showing that if the coproduct of two Lindelöf frames is a capped frame, then each summand is also a capped frame (Theorem 3.16). Arising from Theorem 3.16, we deduce that if X and Y are Lindelöf spaces with one of them locally compact, then both X and Y are capped spaces if $X \times Y$ is a capped space (Corollary 3.17).

In Section 4, we then present the characterization of capped frames in terms of algebraic ring-theoretic properties of the ring of continuous real-valued functions on a frame. We show that a frame L is capped if and only if every basic d-ideal of $\mathcal{R}L$ is a basic z-ideal (Theorem 4.2). In Proposition 4.3, we show that every basic d-ideal of $\mathcal{R}L$ is a basic z-ideal if and only if every basic d-ideal of $\mathcal{R}^*L$ is a basic z-ideal.

2 Preliminaries

We assume familiarity with frames and locales. Our references are [20] and [25], and our notation will be of these reference, by and large. As in these references, we will not work strictly within either **Frm** or **Loc**, but rather we will avail ourselves tools from both categories even within the same proof in some instances. Throughout this section, and, in fact, throughout the paper, L denotes a frame.

2.1 Frames and their homomorphisms The asterisk will be used as a subscript to denote the right adjoint of a frame homomorphism, and as a superscript to denote the pseudocomplement of an element. All frames are assumed to be completely regular. We shall often say "homomorphism" when referring to a frame homomorphism. By a *quotient map* we mean a surjective homomorphism. A homomorphism $h\colon L \to M$ is called *dense* if, for any $a \in L$, the equality $h(a) = 0$ implies $a = 0$. This is true precisely when $h_*(0) = 0$. If h is dense quotient map, then:

(a) $h(a^*) = h(a)^*$ for every $a \in L$ [10, Lemma 2.1].

(b) $h_*(b^*) = h_*(b)^*$ for every $b \in M$ [4, Proof of Lemma 1].

In general, if a homomorphism $h\colon L \to M$ commutes with pseudocomplements, in the sense that $h(x^*) = h(x)^*$ for every $x \in L$, then it is called *nearly open* [10].

2.2 The coreflections βL, λL, υL and πL As usual,we denote by βL the Stone-vCech compactification of L. We view it as the frame of completely regular ideals of $\mathrm{Coz}\,L$; that is, the ideals $J \subseteq \mathrm{Coz}\,L$ such that for every $u \in J$ there is a $v \in J$ with $u \prec\!\!\prec v$. We write $j_L\colon \beta L \to L$ for the dense onto frame homomorphism $J \mapsto \vee J$. The right adjoint of j_L will be denoted by r_L. Recall that, for any $a \in L$,

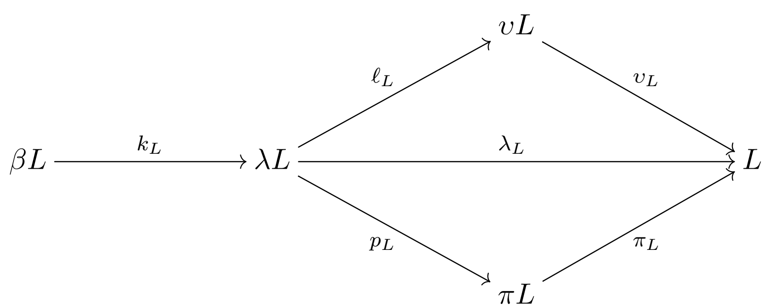
$$r_L(a) = \{c \in \mathrm{Coz}\,L \mid x \prec\!\!\prec a\}.$$

Working in the category **Loc**, Madden and Vermeer [21] showed that regular Lindelöf locales form a reflective subcategory of the category of locales. They showed that the reflection λL of any completely regular locale L is the intersection of all cozero-sublocales of βL that contain L. A frame-theoretic construction is as follows: An ideal I of $\mathrm{Coz}\,L$ is called a *σ-ideal* if it is closed under countable joins. The frame λL is the frame of all σ-ideals of $\mathrm{Coz}\,L$. It is a completely regular Lindelöf frame, and the map $\lambda L\colon \lambda L \to L$, given by join, is a dense onto frame homomorphism, and is, in fact, the coreflection to L from completely regular Lindelöf frames.

A frame L is *realcompact* if every maximal ideal of $\mathrm{Coz}\,L$ that has join equal to the top has a countable subset whose join is the top. Equivalently (this was observed by Marcus [22]), L is realcompact precisely when every point of βL with join equal to the top contains a countable subset whose join is the top. Realcompact completely regular frames form a coreflective subcategory of **CRFrm**. The construction of the coreflection is in [8]. As in spaces, the coreflection of L from realcompact frames is denoted by υL.

One way of describing the paracompact coreflection of L, denoted πL, is given in [8, Proposition 6]. For our purposes it suffices to recall that there is a nucleus $p\colon \lambda L \to \lambda L$ such that $\pi L = \mathrm{Fix}(p)$. As in the other cases, we write $p_L \colon \lambda L \to \pi L$ for the associated quotient map. The join map $\pi_L \colon \pi L \to L$ is a dense quotient map, and it is the coreflection map to L from paracompact frames. We describe its right adjoint in the next paragraph.

Summarizing what is mentioned above, let $k_L \colon \beta L \to \lambda L$ be the map given by $J \mapsto \langle J \rangle_\sigma$, where $\langle J \rangle_\sigma$ designates the σ-ideal of $\operatorname{Coz} L$ generated by J. Then k_L is a dense quotient map. We therefore have the commutative diagram



(‡)

where each morphism is a dense quotient map. Since the composite of any two dense homomorphisms is dense, the composite of any composable maps in Diagram (‡) is a dense quotient map. From the commutativity of the lower triangle in this diagram, we deduce that

$$(\lambda_L)_* = (p_L)_* \circ (\pi_L)_*.$$

Since p_L arises from a nucleus, its right adjoint is the identical embedding $\pi L \rightarrowtail \lambda L$. A consequence of this is that

$$(\pi_L)_* = (\lambda_L)_* = (\upsilon_L)_*.$$

2.3 Sublocales Throughout, we use the terminology and notation of [25] regarding sublocales. We thus denote by $\mathcal{S}(L)$ the co-frame of sublocales of L. Whenever we speak of a join of sublocales, the join will be meant in this lattice. Let us highlight just a few matters that are of relevance for our purposes in this paper. The open (resp. closed, resp. Boolean) sublocale of L associated to an element a of L is given by

$$\mathfrak{o}_L(a) = \{a \to x \mid x \in L\} = \{x \mid x = a \to x\},$$

$$\mathfrak{c}_L(a) = {\uparrow}a = \{x \in L \mid x \geq a\},$$

and

$$\mathfrak{b}_L(a) = \{a \to x \mid x \in L\};$$

with the subscript dropped if the context is clear. Let us recall that $\mathfrak{b}(a)$ is the smallest sublocale of L containing a.

As usual, by a point of L we mean an element $p \in L$ such that $p < 1$ and, for all $x, y \in L$,

$$x \wedge y \leq p \quad \Longrightarrow x \leq p \quad \text{or} \quad y \leq p.$$

Equivalently, the inequalities above can be replaced with equalities. Points are also called *prime elements*. The set of all points of L will be denoted by $\mathrm{Pt}(L)$. A *one-point sublocale* of L is a sublocale of the form $\mathfrak{b}(p) = \{p, 1\}$, for $p \in \mathrm{Pt}(L)$. A *maximal element* of L is an element which is maximal in the poset $L \smallsetminus \{1\}$. Maximal elements are points in any frame. In regular frames, maximal elements are exactly the points of the frame. A consequence of this is that, if L is a regular frame, then for any $p \in \mathrm{Pt}(L)$, $\mathfrak{p}(b = \mathfrak{c}(p))$.

We write S° for the *interior* of a sublocale S, and $\overline{S}$ for the closure of S. We recall that, for any $a \in L$,

$$\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*) \quad \text{and } \mathfrak{c}(a)^\circ = \mathfrak{o}(a^*).$$

A frame is called *spatial* if it is isomorphic to $\Omega(X)$, the frame of open subsets of X, for some topological space X. An internal characterization is that each element is a meet of points. To say a sublocale is spatial means that it is spatial as a frame. Because of the way joins are calculated in $\mathcal{S}(L)$; namely,

$$\bigvee_{i \in I} S_i = \Big\{ \bigwedge M \mid M \subseteq \bigcup_{i \in I} S_i \Big\},$$

and because the points of a sublocale are exactly the points of the containing frame that belong to the sublocale, that is, for any $S \in \mathcal{S}(L)$, $\mathrm{Pt}(S) = S \cap \mathrm{Pt}(L)$, it is easy to see that the join of spatial sublocales is a spatial sublocale.

2.4 The ring $\mathcal{R}L$ and the cozero map The ring $\mathcal{R}L$ has as its elements frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, where $\mathfrak{L}(\mathbb{R})$ denotes the frame of reals. We refer to [5] concerning the ring $\mathcal{R}L$, the cozero map $\mathrm{coz}\colon \mathcal{R}L \to L$, and the properties of the cozero map. As in this reference, we will denote its elements with lower case Greek letters. These rings are also discussed in Chapter XIV of [25]. We recall, in particular, that a frame homomorphism $h\colon M \to L$ induces a ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ is given by $(\mathcal{R}h)(\alpha) = h \circ \alpha$ and for which $\mathrm{coz}\big((\mathcal{R}h)(\alpha)\big) = h(\mathrm{coz}\,\alpha)$.

3 Capped frames

We adapt the definition of a CAP-space to frames by replacing subspaces with sublocales. We shall then characterize the frames so defined in terms of elements, without any mention of sublocales.

Definition 3.1. We say a completely regular frame L is *capped* if the closure of the interior of every zero-sublocale of L is a zero-sublocale of L.

Thus, L is capped if and only if for every $c \in \mathrm{Coz}\,L$, the sublocale $\overline{\mathfrak{c}(c)^{\circ}}$ is a zero-sublocale of L. By the description of closures of open sublocales and interiors of closed sublocales, this says $\mathfrak{c}(c^{**})$ is a zero-sublocale of L. We consequently have the following characterization of capped frames.

Proposition 3.2. *A completely regular frame L is capped if and only if for every $c \in \mathrm{Coz}\,L$, c^{**} is a cozero element in L.*

Proof. Suppose that L is capped, and let $c \in \mathrm{Coz}\,L$. Since $\mathfrak{c}(c)$ is a zero-sublocale of L, the closure of its interior is a zero-sublocale of L. So, there is a $d \in \mathrm{Coz}\,L$ such that $\overline{\mathfrak{c}(c)^{\circ}} = \mathfrak{c}(d)$. By the description of closures of open sublocales and interiors of closed sublocales, this says $\mathfrak{c}(c^{**}) = \mathfrak{c}(d)$; whence $c^{**} = d$, showing that $c^{**} \in \mathrm{Coz}\,L$.

Conversely, suppose that the double pseudocomplement of each cozero element of L is a cozero element. Let Z be a zero-sublocale of L. Pick $c \in \mathrm{Coz}\,L$ such that $Z = \mathfrak{c}(c)$. Then

$$\overline{Z^{\circ}} = \overline{\mathfrak{c}(c)^{\circ}} = \mathfrak{c}(c^{**}),$$

which implies that $\overline{Z^{\circ}}$ is a zero-sublocale of L since $c^{**} \in \mathrm{Coz}\,L$, by the current hypothesis. Therefore L is capped. □

Remark 3.3. The referee has pointed out that capped frames are precisely what the authors in [1] call *casi Oz-frames*, and that there is no mention of the paper of Golrizkhatami and Taherifar [19] in [1], which is a significant oversight. In this paper, we will use the term "capped frames" since they are precisely what the authors in [1] call "casi Oz-frames".

Let us use this characterization to show that the capped property in frames is a conservative extension of the CAP-property in spaces. Recall that, for any Tychonoff space X,

$$\mathrm{Coz}(\mathfrak{O}X) = \{U \subseteq X \mid U \text{ is a cozero-set of } X\}.$$

Also, if U is an open subset of X, so that $U \in \mathfrak{O}X$, then the pseudocomplement of U as an element of $\mathfrak{O}X$ is known to be

$$U^* = X \smallsetminus \overline{U} = (X \smallsetminus U)^\circ,$$

as a consequence of which

$$U^{**} = X \smallsetminus \overline{U^*} = X \smallsetminus \overline{(X \smallsetminus U)^\circ}.$$

Proposition 3.4. *A Tychonoff space is a CAP-space if and only if the frame of its open subsets is capped.*

Proof. Suppose that X is a CAP-space. Let $U \in \mathrm{Coz}(\mathfrak{O}X)$. Then U is a cozero-set of X, so that $X \smallsetminus U$ is a zero-set of X. Since X is a CAP-space, $\overline{(X \smallsetminus U)^\circ}$ is a zero-set of X, hence $X \smallsetminus \overline{(X \smallsetminus U)^\circ}$ is a cozero-set of X. In frame language, this says U^{**} is a cozero element of the frame $\mathfrak{O}X$. Therefore $\mathfrak{O}X$ is a capped frame.

The converse is virtually the same argument; keeping in mind that the zero-sets of X are exactly the set-theoretic complements of its cozero-sets. □

We now wish to compare capped frames to other frames which are defined (or characterized) by imposing certain requirements on pseudocomplements. Let us recall the pertinent definitions. A frame L is said to be:

(a) an *Oz-frame* [7] if $a^* \in \mathrm{Coz}\, L$ for every $a \in L$.

(b) a *weak Oz-frame* [6] if $c^* \in \operatorname{Coz} L$ for every $c \in \operatorname{Coz} L$.

(c) *basically disconnected* [3] if $c^* \vee c^{**} = 1$ for every $c \in \operatorname{Coz} L$.

(d) an *almost P-frame* [3] if $c = c^{**}$ for every $c \in \operatorname{Coz} L$.

Let us denote by **OzFrm**, **WOzFrm**, **BdFrm**, and **AlPFrm** the subcategories of **CRFrm** consisting of the frames described in (a) to (d) above, respectively. Let us write **CpdFrm** for the subcategory of **CRFrm** whose objects are the capped frames. We then have the following containments

$$\mathbf{BdFrm} \subseteq \mathbf{WOzFrm} \subseteq \mathbf{CpdFrm}, \quad \mathbf{OzFrm} \subseteq \mathbf{WOzFrm},$$

$$\mathbf{AlPFrm} \subseteq \mathbf{CpdFrm};$$

the first one holding because in a basically disconnected frame, c^* is complemented for every $c \in \operatorname{Coz} L$, and is therefore a cozero element. The second containment follows by applying the weak Oz property to $c \in \operatorname{Coz} L$ to obtain that $c^* \in \operatorname{Coz} L$, and then applying it to c^* to obtain that $c^{**} \in \operatorname{Coz} L$. The other two containments are immediate.

Recall from [16] that a frame L is said to be *cozero complemented* if for every $c \in \operatorname{Coz} L$, there is a $d \in \operatorname{Coz} L$ such that $c \wedge d = 0$ and $c \vee d$ is dense. Weak Oz-frames are cozero complemented because $c \wedge c^* = 0$ and $c \vee c^*$ is dense. So, a weak Oz-frame is both cozero complemented and capped. The converse also holds, as the following result shows.

Proposition 3.5. *A frame is a weak Oz-frame if and only if it is capped and cozero-complemented.*

Proof. The forward implication follows from the preceeding paragraph.

To show the converse, assume that a frame L is capped and cozero-complemented, and let $a \in \operatorname{Coz} L$. We claim that $a^* \in \operatorname{Coz} L$. Since L is cozero complemented, there exists $d \in \operatorname{Coz} L$ such that $a \wedge d = 0$ and $a \vee d$ is dense. Observe that $a \wedge d = 0$ implies that $a \leq d^*$ and the density of $a \vee d$ implies that

$$(a \vee d)^* = a^* \wedge d^* = a \wedge d = 0. \qquad (\dagger)$$

It is clear from (†) that $d^* = a$. Now we have $d^{**} = a^*$ and since L is also capped, it follows from Proposition 3.2 that $d^{**} = a^*$ is a cozero element of L. This completes the proof. □

The foregoing proof actually agrees with [6, Proposition 5.2] that says that a frame L is a weak Oz-frame if and only if every regular element of L is c^* for some $c \in \operatorname{Coz} L$. Next, we investigate how the property of being capped is transported across a homomorphisms, in either direction. As in general algebraic structures, by a *quotient* of a frame L, we mean a homomorphic image of L. That is, M is a quotient of L if there is a quotient map $L \to M$. Recall that a frame homomorphism $h\colon L \to M$ is *coz-onto* if for every $d \in \operatorname{Coz} M$ there exists a $c \in \operatorname{Coz} L$ such that $h(c) = d$. See [18] for a detailed study of these homomorphisms. Since our frames are completely regular (so that they are $\bigvee$-generated by their cozero parts), coz-onto homomorphisms are surjective.

In one part of the upcoming theorem we are going to impose a certain condition on homomorphisms. Let us first show that it does hold for some familiar homomorphisms. Let c be a regular cozero element of L. Since, for any $x \in L$, $[x] = (\lambda_L)_*(x)$, and since right adjoints of dense onto homomorphisms commute with pseudocomplements, it follows that $[c]$ is a regular cozero element of λL. Hence, for any regular cozero element d of L, $(\lambda_L)_*(d)$ is a regular cozero element of λL. Let us name homomorphisms with this feature.

Definition 3.6. We say a frame homomorphism $h\colon L \to M$ is *solid* if for every regular $d \in \operatorname{Coz} M$, $h_*(d)$ is a regular cozero element in L.

We have just seen that $\lambda_L\colon \lambda L \to L$ is solid. So are the maps $\upsilon_L\colon \upsilon L \to L$ and $\pi_L\colon \pi L \to L$. From the discussion in the preceding paragraph, we see that $\lambda_L\colon \lambda L \to L$ also has the property that for every regular cozero element $c \in L$, $[c]$ is a regular cozero element of λL such that $\lambda_L([c]) = c$. Let us give this property a name.

Definition 3.7. We say a frame homomorphism $h\colon L \to M$ is $\mathrm{coz}_{\mathrm{reg}}$-*onto* if for every regular $d \in \operatorname{Coz} M$ there is a regular $c \in \operatorname{Coz} L$ such that $h(c) = d$.

A comment on the choice of the name "$\mathrm{coz}_{\mathrm{reg}}$-onto" is in order. The coz-onto homomorphisms are precisely the $h\colon L \to M$ such that the restriction $h_{|\operatorname{Coz} L}\colon \operatorname{Coz} L \to \operatorname{Coz} M$ is onto. Let us write $\mathrm{Coz}_{\mathrm{r}} L$ for the set of regular cozero elements of L. An arbitrary homomorphism does not necessarily send regular elements to regular elements. The ones called *nearly open* in [10], meaning those (like the dense onto ones) which commute with pseudocomplements, do send regular elements to regular elements, and hence regular cozero elements to regular cozero elements. So, a nearly open homomorphism $h\colon L \to M$ is $\mathrm{coz}_{\mathrm{reg}}$-onto if and only if the restriction $h_{|\,\mathrm{Coz_r}\, L}\colon \mathrm{Coz_r}\, L \to \mathrm{Coz_r}\, M$ is onto.

Consider the following properties that a homomorphism $h\colon L \to M$ can have

$$({}_*\mathrm{CC}) \qquad h_*[\mathrm{Coz}\, M] \subseteq \mathrm{Coz}\, L$$

$$({}_*\mathrm{C_rC}) \qquad h_*[\mathrm{Coz_r}\, M] \subseteq \mathrm{Coz}\, L$$

$$({}_*\mathrm{C_rC_r}) \qquad h_*[\mathrm{Coz_r}\, M] \subseteq \mathrm{Coz_r}\, L.$$

For dense quotient maps, $\mathrm{coz}_{\mathrm{reg}}$-ontoness and solidity are equivalent. To prove this, let us first do some ground clearing.

Observe that a dense homomorphism (not necessarily onto) is one-one on regular elements. Indeed, let $h\colon L \to M$ be a dense homomorphism, and consider two regular elements $x, y \in L$ with $h(x) = h(y)$. Then

$$h(x \wedge y^*) = h(x) \wedge h(y^*) \leq h(x) \wedge h(y)^* = h(y) \wedge h(y)^* = 0,$$

which implies $x \wedge y^* = 0$ by the density of h, and so $x \leq y^{**} = y$, since y is regular. Similarly, $y \leq x$, and hence $x = y$. Next, recall that in a regular frame every element is a join of the regular elements below it because for any a in a regular frame, $x \prec a$ implies $x^{**} \leq a$, so that

$$\begin{aligned} a &= \bigvee \{x \mid x \prec a\} \\ &\leq \bigvee \{x^{**} \mid x \prec a\} \\ &\leq \bigvee \{y \mid y \text{ is regular and } y \leq x\} \\ &\leq a. \end{aligned}$$

Theorem 3.8. *A dense onto homomorphism $h\colon L \to M$ is solid if and only if it is* $\mathrm{coz}_{\mathrm{reg}}$*-onto.*

Proof. Suppose that h is solid. Let d be a regular cozero element in M. Then $h_*(d)$ is a regular cozero element in L. Since $h(h_*(d)) = d$, as h is onto, it follows that h is coz^{**}-onto.

Conversely, suppose that h is coz^{**}-onto. Let d be a regular cozero element in M. By hypothesis, there is a regular $c \in \mathrm{Coz}\, L$ such that $h(c) =$

d. We are going to argue that $h_*(c) = d$. We immediately have $c \leq h_*(d)$. To reverse the inequality, consider any regular $x \in L$ with $x \leq h_*(d)$. Then,

$$h(x) \leq h(h_*(d)) \leq d = h(c).$$

So, $h(x) = h(x) \wedge h(c) = h(x \wedge c)$. Since the meet of two regular elements in any frame is regular, $x \wedge c$ is regular, and so, since h is one-one on regular elements, $x = x \wedge c$, which implies $x \leq c$. Since $h_*(d)$ is the join of the regular elements below it, we infer that $h_*(d) \leq c$. Therefore $h_*(d) = c$, and it follows that h is solid. □

In the proof below we use the result in [18, Proposition 3.3] that says for any homomorphism $h\colon L \to M$, h is coz-onto if and only if for every $a, b \in \operatorname{Coz} M$ such that $a \wedge b = 0$, there exist $c, d \in \operatorname{Coz} L$ such that $c \wedge d = 0$, $h(c) = a$ and $h(d) = b$. Furthermore, in the proof below, and elsewhere, we shall use the fact that if $h\colon L \to M$ is dense quotient map, then $h(a^*) = h(a)^*$ for every $a \in L$. The authors in [9, Theorem 3.5] claim that $h(a^*) = h(a)^*$ if and only if $h(a^{**}) = h(a)^{**}$.

Theorem 3.9. *Let $h\colon L \to M$ be a dense coz-onto frame homomorphism.*

(a) *If L is capped, then so is M.*

(b) *If, furthermore, h is solid and M is capped, then L is capped.*

Proof. (a) Let $d \in \operatorname{Coz} M$. We claim that $d^{**} \in \operatorname{Coz} M$. Since h is coz-onto, there exists $c \in \operatorname{Coz} L$ such that $h(c) = d$. Since L is capped, then $c^{**} \in \operatorname{Coz} L$. We know that since our frames are completely regular, coz-onto homomorphisms are surjective. Then h is dense onto and by the result of Banaschewski and Pultr cited above, we have $h(c^{**}) = h(c)^{**} = d^{**}$. Therefore $d^{**} \in \operatorname{Coz} M$, and hence M is capped.

(b) Let $c \in \operatorname{Coz} L$. We must show that $c^{**} \in \operatorname{Coz} L$. The element $h(c)$ is a cozero element in M because dense coz-onto frame homomorphism preserves cozero elements, and so since M is capped, $h(c)^{**} \in \operatorname{Coz} M$ and so $h(c^{**}) \in \operatorname{Coz} M$ because dense onto maps commutes with pseudocomplements. Since our frames are completely regular, coz-onto homomorphisms are onto, and since h is solid by hypothesis, then $h_*(h(c^{**})) = c^{**}$. Hence c^{**} is a regular cozero element in L, and therefore L is capped. □

Since the homomorphisms $\lambda_L\colon \lambda L \to L$, $\upsilon_L\colon \upsilon L \to L$ and $\pi_L\colon \pi L \to L$ are dense, coz-onto, and solid, we have the following corollary.

Corollary 3.10. *The following are equivalent for L.*

(a) *L is capped.*

(b) *λL is capped.*

(c) *υL is capped.*

(d) *πL is capped.*

As the referee has pointed out, the results pertaining to the paracompact and realcompact coreflections follow simply from the fact that all the frames have the same cozeros (when considered as sublocales of λL). Recall that the mapping

$$j_L\colon \beta L \to L \quad \text{defined by} \quad j_L(I) = \bigvee I$$

is dense onto, and is the coreflection map to L from compact completely regular frames. Next, we show that a frame L is capped precisely when βL is capped.

Theorem 3.11. *A completely regular frame L is capped if and only if βL is capped.*

Proof. ($\Rightarrow$): Since the homomorphism $\beta L \to L$ is dense and coz-onto (the latter by [7, Corollary 5]), it follows from Theorem 3.9 that L is capped if βL is capped.

($\Leftarrow$): Assume that a frame L is capped. If $I \in \operatorname{Coz}\beta L$, then there is a sequence (c_n) in L such that $c_k \prec\!\!\prec c_{k+1}$ for each k and I is an ideal generated by the c_n. Put $c = \bigvee c_n$ and notice that $c \in \operatorname{Coz} L$ and $c = \bigvee I$. Now $I^{**} = r_L(c^{**})$ is a cozero element of βL by hypothesis. Let $x \in I^{**}$. Then $x \prec\!\!\prec c^{**}$. Pick $s \in \operatorname{Coz} L$ such that $x \wedge s = 0$ and $s \vee \bigvee c_n = 1$. Since L is a capped frame, then

$$(x \wedge s)^{**} = x \wedge s = 0 \quad \text{and} \quad (s \vee \bigvee c_n)^{**} = s \vee \bigvee c_n = 1$$

are cozero elements of L. In consequence, $x \prec\!\!\prec c$ implying that $x \in I$. Therefore $I^{**} = I$. □

We now recite the definition below sourced from [16, Definition 3.4] that we shall use in the upcoming proposition.

Definition 3.12. A quotient map $h\colon L \to M$ is $z^{\#}$*-quotient map* if for each $v \in \operatorname{Coz} M$ there is a $u \in \operatorname{Coz} L$ such that $h(u^*) = v^*$.

Clearly, any coz-onto frame homomorphism is $z^{\#}$-quotient map. Nearly open homomorphisms include dense onto homomorphisms (see [10, Lemma 2.1]), and they are *skeletal*, which is to mean that h maps dense elements to dense elements. As observed in [24], that h is skeletal if and only if, for any $a, b \in M$,

$$a^* = b^* \implies h(a)^* = h(b)^*.$$

It is shown in [9, Theorem 3.5] that $h(a^*) = h(a)^*$ implies that $h(a^{**}) = h(a^*)^*$.

Proposition 3.13. *Let $h\colon L \to M$ be a $z^{\#}$-quotient map.*

(a) *If h is nearly open and L is capped , then so is M.*

(b) *If h is dense and M is capped, then so is L.*

Proof. (a) Assume that h is nearly open and L is capped, and let $d \in \operatorname{Coz} M$. Since h is a $z^{\#}$-quotient map, there exists $c \in \operatorname{Coz} L$ such that $h(c^*) = d^*$. We are going to argue that $h(c^{**}) = d^{**}$. We immediately have $c^{**} \leq h_*(d^{**})$ and since L is capped, then $c^{**} \in \operatorname{Coz} L$. To reverse the inequality, consider any cozero element x such that $x \leq h_*(d^{**})$. Then

$$h(x) \leq h(h_*(d^{**})) \leq d^{**} = h(c^*)^*.$$

Since h is nearly open, then $h(c^*)^* = h(c^{**})$. So $h(x) = h(x) \wedge h(c^{**}) = h(x \wedge c^{**})$. Since the meet of two cozero element in any frame is cozero element, $x \wedge c^{**}$ is a cozero element, and so, since h is one-one on cozero elements, $x = x \wedge c^{**}$, which implies $x \leq c^{**}$. Since $h_*(d^{**})$ is the join of the cozero elements below it, we infer that $h_*(d^{**}) \leq c^{**}$. Therefore $h_*(d^{**}) = c^{**}$ which implies $h(c^{**}) = d^{**}$ and it follows that M is capped.

(b) Assume that h is dense and M is capped. Recall that dense onto frame homomorphism preserves pseudocomplements, and let $y \in \operatorname{Coz} M$. Then $y^{**} \in \operatorname{Coz} M$. Since h is $z^{\#}$-quotient, there exists $x \in \operatorname{Coz} L$ such that $h(x^*) = y^*$. We claim that $x^{**} \in \operatorname{Coz} L$. We are going to argue that $h(x^{**}) = y^{**}$. We immediately have $x^{**} \leq h_*(y^{**})$. To reverse the inequality, consider any cozero element c such that $c \leq h_*(y^{**})$. Now

$$h(c) \leq h(h_*(y^{**})) \leq y^{**} = h(x^*)^* = h(x^{**}).$$

So $h(c) = h(c) \wedge h(x^{**}) = h(c \wedge x^{**})$. Since the meet of two cozero element in any frame is cozero element, $c \wedge x^{**}$ is a cozero element, and so, since h is one-one on cozero elements, $c = c \wedge x^{**}$, which implies $c \leq x^{**}$. Since $h_*(y^{**})$ is the join of the cozero elements below it, we infer that $h_*(y^{**}) \leq x^{**}$. Therefore $h_*(y^{**}) = x^{**}$ which implies that $h(x^{**}) = y^{**}$. Observe that $h_*(y^{**}) = x^{**}$ is a cozero element of L which implies that L is capped. □

We now focus on the following proposition. Before we state the proposition, we shall remind the reader of the following from [6]. A frame L is called

- an *F-frame* if all cozero quotients are C^*-quotient.
- an *F'-frame* if disjoint cozero elements have covering pseudocomplements.
- *basically disconnected* if $c^* \vee c^{**} = 1$ for each $c \in \operatorname{Coz} L$.

In [6, Proposition 5.4], the authors show that, within the class of Oz-frames, the F-frames, F'-frames, and basically disconnected frames coincide.

Proposition 3.14. *For any capped frame L,the following are equivalent*

(a) *L is basically disconnected.*

(b) *L is an F-frame.*

(c) *L is an $F^{'}$ frame.*

Proof. A reflection on the definition in [6] of each and [6, Proposition 5.4] makes it clear that (a), (b) and (c) are equivalent. □

Remark 3.15. The foregoing proposition agrees with [1, Remark 7.11 (4)] which says that basically disconnected frame implies the frame is capped; in fact it implies that the frame is weak Oz. The latter follows since for basically disconnected frames, $(\operatorname{Coz} L)^{**} = (\operatorname{Coz} L)^*$. The results in [1, Remark 7.11 (4)] and [6, Proposition 5.4] make the foregoing proposition obvious. In [6, Proposition 5.2], the authors show that a weak Oz-frame L is Oz if and only if every regular element of L is c^* for some $c \in \operatorname{Coz} L$. The most interesting results in [1, Corollary 7.14 (2)] is that if L is *coole*, that is the Booleanization map is coz-onto, then a frame L is capped iff it is Oz iff it is weak Oz. We refer the reader to [1] for some more interesting results concerning capped frames.

We shift our focus to the product of capped frames. We could not show that if the coproduct of two arbitrary frames is capped then each summand is capped, but we could prove a similar result for Lindelöf capped frames. Recall that if $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$, then $c \oplus d \in \operatorname{Coz}(L \oplus M)$ because $c \oplus d = i_L(c) \wedge i_M(d)$ for the coproduct injections $i_L : L \to L \oplus M$ and $i_M = M \to L \oplus M$. It is shown in [10] that, for any $x \in L$ and $y \in M$, $(x \oplus y)^{**} = x^{**} \oplus y^{**}$. Thus, if $x \oplus y$ is a cozero element, then both x and y are cozero elements. It is known that a space X is *locally compact* if each point in X has a neighbourhood base consisting of compact sets, and every compact Hausdorff space is locally compact.

Theorem 3.16. *If the coproduct of two Lindelöf frames is a capped frame, then each summand is also a capped frame.*

Proof. Suppose that L and M are frames such that $L \oplus M$ capped. Let $a \in \operatorname{Coz} L$ and $b \in \operatorname{Coz} M$, so that $a \oplus 1$ and $b \oplus 1$ are cozero elements of $L \oplus M$. Since $L \oplus M$ is capped by hypothesis, it follows that

$$(a \oplus 1)^{**} = a^{**} \oplus 1^{**} = a^{**} \oplus 1 \in \operatorname{Coz}(L \oplus M)$$

and

$$(b \oplus 1)^{**} = b^{**} \oplus 1^{**} = b^{**} \oplus 1 \in \operatorname{Coz}(L \oplus M).$$

We claim that there are sequences (u_n) and (v_n) in $\operatorname{Coz} L$ and $\operatorname{Coz} M$ respectively such that

$$U = \bigvee_{n=1}^{\infty} (u_n \oplus v_n).$$

To show this, we write U as a join of basic elements, say $U = (k_\alpha \oplus s_\alpha)$. By complete regularity, for every α there are cozero elements $\{u_i^{(\alpha)}\}$ in L and cozero elements $\{v_j^{(\alpha)}\}$ in M such that

$$k_\alpha = \bigvee_i u_i^{(\alpha)} \qquad \text{and} \qquad s_\alpha = \bigvee_j v_j^{(\alpha)}.$$

Consequently,

$$k_\alpha \oplus s_\alpha = \bigvee_i u_i^{(\alpha)} \oplus \bigvee_j v_j^{(\alpha)} = \bigvee_{i,j} (u_i^{(\alpha)} \oplus v_j^{(\alpha)}),$$

so that

$$U = \bigvee_{\alpha,i,j} (u_i^{(\alpha)} \oplus v_j^{(\alpha)}).$$

Since U is a cozero element in Lindelöf frame, it is a Lindelöf element, and hence we can find countably many $u_n \in \operatorname{Coz} L$ and countably many $v_n \in \operatorname{Coz} M$ such that $U = \bigvee_{n=1}^{\infty} (u_n \oplus v_n)$. Now

$$(a \oplus 1)^{**} = a^{**} \oplus 1 = \bigvee_{n=1}^{\infty} (u_n^{**} \oplus 1) = \bigvee_{n=1}^{\infty} u_n^{**} \oplus 1. \qquad (\#)$$

Thus, the equality in (#) implies that $a^{**} = \bigvee_{n=1}^{\infty} u_n^{**}$, and whence a^{**} is a cozero element of L. Therefore L is a capped frame. Similarly, M is a capped frame. □

Corollary 3.17. *Let X and Y be Lindelöf spaces with one of them locally compact. If $X \times Y$ is a capped space, then both X and Y are capped spaces.*

Proof. By [20, Chapter II, Proposition 2.13], $\mathfrak{O}(X \times Y)$ is isomorphic to $\mathfrak{O}X \oplus \mathfrak{O}Y$. Therefore $\mathfrak{O}X \oplus \mathfrak{O}Y$ is a capped frame, and so $\mathfrak{O}X$ and $\mathfrak{O}Y$ are capped frames which imply that X and Y are capped spaces. □

Remark 3.18. The fact that L and M are Lindelöf was used only to enable us to write the cozero element U as a join of countable many of "cozero rectangles" $u_n \oplus v_n$. The term "cozero rectangle" is borrowed from [11]. The result therefore is true for any pair (L, M) of frames for which every cozero element of $L \oplus M$ is a join of countably many cozero rectangles.

4 Ring-theoretic characterizations of capped frames

In this section, we give ring-theoretic characterization of capped frames . Before we state the result that we are aiming for, we need some preliminaries which we deliberately did not include in Section 2 because they are of relevance only in the upcoming theorem. For an element a of a ring A, we write $\mathfrak{M}(a)$ for the sets

$$\mathfrak{M}(a) = \{M \in \mathrm{Max}(A) \mid a \in M\},$$

and M_a shall denote the intersection of all maximal ideals containing a. As defined in [23], an ideal I of A is a *z-ideal* if, for any $a, \in A$,

$$\mathfrak{M}(a) = \mathfrak{M}(b) \text{ and } a \in I \quad \Longrightarrow \quad b \in I.$$

A useful characterization (which is known in the literature) is that I is a z-ideal if and only if $M_a \subseteq I$ for every $a \in I$. Observe, as well, that M_a is the smallest z-ideal containing a, and it is called the *basic z-ideal.* We denote by $\mathrm{Min}(A)$ the set of all minimal prime ideals of a ring A. For an element a of a ring A, we write $\mathfrak{V}(a)$ for the sets

$$\mathfrak{V}(a) = \{Q \in \mathrm{Min}(A) \mid a \in Q\}$$

and P_a shall denote the intersection of all minimal ideals containing a. As defined in [2], an ideal I of A is a *d-ideal* if, for any $a, b \in A$

$$\mathfrak{V}(a) = \mathfrak{V}(b) \text{ and } a \in I \quad \Longrightarrow \quad b \in I.$$

A useful characterization (which is known in the literature) is that Q is a d-ideal if and only if $P_a \subseteq Q$ for every $a \in Q$. Observe, as well, that P_a is the smallest d-ideal containing a, and it is called the *basic d-ideal.* We recall that the annihilator of a singleton $\{a\}$ of a ring A is the ideal

$$\mathrm{Ann}(a) = \{x \in A : ax = 0\},$$

and the *double annihilator* is the ideal

$$\mathrm{Ann}(\mathrm{Ann}(a)) = \{x, \in A, ; xy = 0 \text{ for all } y \in \mathrm{Ann}(a)\}.$$

Furthermore, $\mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(a))) = \mathrm{Ann}(a)$ and it is shown in [23] that $P_a = \mathrm{Ann}(\mathrm{Ann}(a))$. For any sublocale S of βL the ideals, the following ideals of the ring $\mathcal{R}L$ are given as

$$\boldsymbol{M}^S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{c}_{\beta L}(r_L(\mathrm{coz}\,\alpha))\}$$

and

$$\begin{aligned}\boldsymbol{O}^S &= \{\alpha \in \mathcal{R}L \mid S \subseteq \mathrm{int}_{\beta L}\,\mathfrak{c}_{\beta L}(r_L(\mathrm{coz}\,\alpha))\} \\ &= \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{o}_{\beta L}(r_L(\mathrm{coz}\,\alpha)^*)\}.\end{aligned}$$

Definition 4.1. For any sublocale S of L, we define the ideals $\boldsymbol{O}_S$ and $\boldsymbol{M}_S$ of $\mathcal{R}L$ to be the ideals $\boldsymbol{O}_S = \boldsymbol{O}^{r_L[S]}$and $\boldsymbol{M}_S = \boldsymbol{M}^{r_L[S]}$.

A pleasant observation from [13] is that, for S a sublocale of L, the ideals $\boldsymbol{O}_S$ and $\boldsymbol{M}_S$ can be described solely in terms of L without invoking β, as follows:

$$\boldsymbol{O}_S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{o}_L(\operatorname{coz}\alpha)^*\}$$

and

$$\boldsymbol{M}_S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{c}_L(\operatorname{coz}\alpha)\}.$$

In [12, Lemma 3.1], it is shown that for any set $S \subseteq \mathcal{R}L$, the annihilator of S is, in our present notation,

$$\operatorname{Ann}(S) = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(a^*))},$$

where $a = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in S\}$. In the same lemma it is shown that, in fact, the set of annihilator ideals of $\mathcal{R}L$ is the collection

$$\{\boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(b^*))} \mid b \in L\}.$$

In [17, Section 4], the authors proved that these ideals are precisely the ideals $\boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(b))}$. For completeness, in [15, Lemma 3.8], it is shown that for any $\gamma \in \mathcal{R}L$, the double annihilator of γ is

$$\operatorname{Ann}(\operatorname{Ann}(\gamma)) = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L((\operatorname{coz}\gamma)^{**}))},$$

and in the same lemma it is shown that, in fact, the double annihilator ideals of $\mathcal{R}L$ is the collection

$$\{\boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L((\operatorname{coz}\gamma)^{**}))} \mid \gamma \in \mathcal{R}L\}.$$

Clearly $P_b = \operatorname{Ann}(\operatorname{Ann}(b)) = \boldsymbol{O}^{\mathfrak{o}_{\beta L}(r_L(b))}$ for any $b \in \operatorname{Coz} L$. It is shown in [14, Proof of Lemma 3.3] that $\boldsymbol{M}_{\operatorname{coz}\tau} = M_\tau$. In the proof below, we use the fact that for a sublocale A of a frame L, $\boldsymbol{M}_A = \boldsymbol{O}_A$ if and only if A is contained in the interior of every zero-sublocale of L containing it [13, Proposition 2.2].

Theorem 4.2. *A frame L is capped if and only if every basic d-ideal of $\mathcal{R}L$ is a basic z-ideal.*

Proof. (⇒): Assume that a frame L is capped and let P_α be a basic d-ideal of the ring $\mathcal{R}L$. By current hypothesis, there exists $d \in \operatorname{Coz} L$ such that $\overline{(\operatorname{coz}\alpha)^\circ} = \operatorname{coz}\gamma$, where $d = \operatorname{coz}\gamma$. Thus, we have

$$P_\alpha = \boldsymbol{O}_{(\operatorname{coz}\alpha)^\circ} = \boldsymbol{M}_{(\operatorname{coz}\alpha)^\circ} = \boldsymbol{M}_{\overline{(\operatorname{coz}\alpha)^\circ}} = \boldsymbol{M}_{\operatorname{coz}\gamma} = M_\gamma.$$

Thus every basic d-ideal of $\mathcal{R}L$ is a basic z-ideal.

(⇐): Assume that every basic d-ideal of $\mathcal{R}L$ is a basic z-ideal and let $c \in \operatorname{Coz} L$. Then there is $\tau \in \mathcal{R}L$ such that $c = \operatorname{coz}\tau$ and there exists $\gamma \in \mathcal{R}L$ such that $P_\tau = M_\gamma$. Since $P_\tau = \boldsymbol{O}_{(\operatorname{coz}\tau)^\circ}$ and $M_\gamma = \boldsymbol{M}_{\operatorname{coz}\gamma}$, the equality of $P_\tau = M_\gamma$ implies that $\overline{(\operatorname{coz}\tau)^\circ} = \operatorname{coz}\gamma$. Hence L is a capped frame. □

We remind the reader that the *bounded part* of A, denoted A^*, is the sub-f-ring

$$A^* = \{a \in A\colon |a| \le n \text{ for some } n \in \mathbb{N}\}.$$

The bounded part of $\mathcal{R}L$ (as in the case of $C^*(X)$) is written as $\mathcal{R}^*L$. We end this section by showing the following.

Proposition 4.3. *Every basic d-ideal of $\mathcal{R}L$ is a basic z-ideal if and only if every basic d-ideal of $\mathcal{R}^*L$ is a basic z-ideal.*

Proof. Recall that for any frame L, $\mathcal{R}^*L$ and $\mathcal{R}(\beta L)$ are isomorphic as f-rings. A frame L is capped if and only if every basic d-ideal of $\mathcal{R}L$ is a basic z-ideal (Theorem 4.2) if and only if βL is capped (Theorem 3.11) if and only if every basic d-ideal of $\mathcal{R}(\beta L)$ is a basic z-ideal if and only if every basic d-ideal of $\mathcal{R}^*L$ is a basic z-ideal. □

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