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On Condition $(G - PCP)$ of acts over monoids

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Abstract. By a left PCP (or left $P(P)$) monoid we mean a monoid such that all its principal left ideals satisfy Condition (P) (see [4, 17]). In this paper, we generalize formalization of left PCP (left $P(P)$) monoids to right acts and will give characterizations of monoids by this property of their right acts.

1 Introduction and Preliminaries

Throughout this paper we used S to denote a monoid. We refer the reader to [8, 10] for basic results, definitions and terminologies related to semigroups and acts over monoids, and to [1, 12, 13] for definitions and results on flatness properties which are used in this paper. S is called right (left) reversible if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'(su = s'v)$.

A right ideal K of S is called left stabilizing (or satisfies Condition (LU)) if for every $k \in K$, there exists $l \in K$ such that $lk = k$.

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An element s of S is called right e-cancellable, for an idempotent $e \in S$, if $s = es$ and $ker\rho_s \leq ker\rho_e$, i.e. $ts = t's$, $t, t' \in S$, implies $te = t'e$. S is called left PP if every element $s \in S$ is right e-cancellable, for some idempotent $e \in S$. It is easy to see that S is left PP if and only if for every $s \in S$ there exsits $e \in E(S)$, such that $ker\rho_s = ker\rho_e$. This is equivalent to saying that every principal left ideal of S is projective. Similarly a right PP is defined. An element $s \in S$ is called right semi-cancellable if $ts = t's$, $t, t' \in S$, implies there exists $r \in S$ such that $s = rs$ and $tr = t'r$. S is called left PSF if all principal left ideals of S are strongly flat. It is easy to see that S is left PSF if and only if every element $s \in S$ is right semi-cancellable.

An element $s \in S$ is called regular, if $sxs = s$, for some $x \in S$. S is called regular if all its elements are regular. An element s of S is called left almost regular if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$ such that

$$\begin{gathered} s_1c_1 = sr_1 \\ s_2c_2 = s_1r_2 \\ \cdots \\ s_mc_m = s_{m-1}r_m. \\ s = s_mrs. \end{gathered}$$

If all elements of S are left almost regular, then S is called left almost regular. We can see that every left almost regular monoid is left PP (see [10, IV, Proposition 1.3]).

A right S-act is a non-empty set A, usually denoted by A_S, on which S acts unitarian from the right, that is, $(as)t = a(st)$ and $a1 = a$, for every $a \in A$, $s, t \in S$, where 1 is the identity of S. A right S-act A_S satisfies Condition (P) if for all $a, a' \in A_S, s, s' \in S$, $as = a's'$, implies that there exist $b \in A_S, u, v \in S$ such that $a = bu, a' = bv$ and $us = vs'$.

Monoids with all principal left ideals satisfying Condition (P) first introduce in [4], titled "left PCP" and three years later this concept introduce in [17], titled "left $P(P)$". In this paper we use this concept by left PCP. It can be readily checked that a monoid S is left PCP if and only if $as = bs$ for $a, b, s \in S$, implies the existence of $u, v \in S$ such that $au = bv$ and $us = vs = s$. Also a monoid S is called weakly left PCP (or weakly left $P(P)$) if the equalities $as = bs$, $xb = yb$ in S imply the existence of $r \in S$, such that $xar = yar$, $rs = s$. For any monoid S we have the following implications: left $PP \Rightarrow$ left $PSF \Rightarrow$ left $PCP \Rightarrow$ weakly left PCP.

Recall, from [6] that a right S-act A_S satisfies Condition (P') if for all $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply the existence $b \in A$ and $u, v \in S$ such that $a = bu$, $a' = bv$ and $us = vt$. A_S is said to satisfy Condition (E) if whenever $as = as'$ with $a \in A_S, s, s' \in S$, there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = us'$. Recall, from [2, 3, 12] that a right S-act A_S satisfies Condition (E') if $as = as'$ and $sz = s'z$, for $a \in A_S$ and $s, s', z \in S$, imply the existence $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = us'$. A right S-act A_S satisfies Condition (EP) if $as = at$, for $a \in A_S$, $s, t \in S$, implies the existence $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. Also, we say that A_S satisfies Condition $(E'P)$ if $as = at$ and $sz = tz$, for $a \in A_S$ and $s, t, z \in S$, imply the existence $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(P) \Rightarrow (EP) \Rightarrow (E'P)$, $(E) \Rightarrow (E') \Rightarrow (E'P)$, $(P) \Rightarrow (P') \Rightarrow (E'P)$ and $(E) \Rightarrow (EP)$.

We recall from [1, 12, 13] that:

A right S-act A_S is weakly pullback flat (WPF), if the corresponding ϕ is bijective for every pullback diagram $P(S, S, f, g, S)$.

A right S-act A_S is weakly kernel flat (WKF), if the corresponding ϕ is bijective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S.

A right S-act A_S is principally weakly kernel flat $(PWKF)$, if the corresponding ϕ is bijective for every pullback diagram $P(Ss, Ss, f, f, S)$, where $s \in S$.

A right S-act A_S is translation kernel flat (TKF), if the corresponding ϕ is bijective for every pullback diagram $P(S, S, f, f, S)$.

A right S-act A_S is weakly homoflat (Condition (WP)), if for all elements $s, t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$, all $a, a' \in A_S$, if $af(s) = a'f(t)$ then there exist $a'' \in A_S$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A \otimes_S (Ss \cup St)$ and $f(us') = f(vt')$.

A right S-act A_S is principally weakly homoflat (Condition (PWP)), if $as = a's$, for $a, a' \in A_S$, $s \in S$, implies the existence of $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$.

A right S-act A_S is called torsion free if for any $a, a' \in A_S$ and for any right cancellable element $c \in S$ the equality $ac = a'c$ implies $a = a'$.

A right S-act A_S is also called $\mathfrak{R}$-torsion free, if for any $a, b \in A$ and any right cancellable element $c \in S$, $ac = bc$ and $a\mathfrak{R}b$ ($\mathfrak{R}$ denotes Green's equivalence, as described in [18], imply that $a = b$.

Recall, from [5] that a right S-act A_S is called (P)-regular if all cyclic subacts of A satisfy Condition (P).

Recall, from [4] that a right S-act A_S is called strongly (P)-cyclic if for every $a \in A$ there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P).

Recall from [7] that the right S-act A_S is called GP-flat if $as = a's$ for $a, a' \in A_S$, $s \in S$ implies that there exists $n \in \mathbb{N}$, such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes Ss^n$. A right S-act A_S is called GPW-flat if for every $s \in S$, there exists $n = n_{(s,A_S)} \in \mathbb{N}$, such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $a \otimes s^n = a' \otimes s^n$ in $A \otimes_S (Ss^n)$ (see [14]).

2 General properties

Definition 2.1. We say that A_S satisfies Condition $(G - PCP)$ if $as = a's$ for $a, a' \in A_S$, $s \in S$, implies the existence of $u, v \in S$ such that $au = a'v$, $us = s = vs$.

Theorem 2.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) *S is left PSF.*

(2) *S is left PCP and for any $s \in S$, $[1]_{ker\rho_s}$ is a right collapsible submonoid of S.*

Proof. (1) $\Rightarrow$ (2) If S is left PSF, then S is obviously left PCP. Let $s \in S$ and $l_1, l_2 \in [1]_{ker\rho_s}$. Then $(1, l_1), (1, l_2) \in ker\rho_s$ and so $l_1 s = s$, $l_2 s = s$. Hence $l_1 l_2 s = l_1 s = s$ that is $l_1 l_2 \in ker\rho_s$. Thus $[1]_{ker\rho_s}$ is a submonoid of S. If on the other hand $l_1, l_2 \in [1]_{ker\rho_s}$, then $l_1 s = l_2 s = s$. Since S is left PSF the equality $l_1 s = l_2 s$ implies existence $r \in S$ such that $rs = s$ and $l_1 r = l_2 r$. The equality $rs = s$ implies $r \in [1]_{ker\rho_s}$ and so submonoid $[1]_{ker\rho_s}$ is right collapsible.

(2) $\Rightarrow$ (1) Suppose that $l_1, l_2, s \in S$ such that $l_1 s = l_2 s$. Since S is left PCP there exist $u, v \in S$ such that $l_1 u = l_2 v$ and $us = vs = s$. The equality $us = s$ implies $u \in [1]_{ker\rho_s}$. Similarly $vs = s$ implies $v \in [1]_{ker\rho_s}$. Since by assumption $[1]_{ker\rho_s}$ is right collapsible, there exists $t \in [1]_{ker\rho_s}$ such that $ut = vt$. Let $r = ut = vt$. Then $l_1 r = l_2 r$. Since $ts = s$, we have $rs = uts = vs = s$. Thus S is left PSF. □

Corollary 2.3. *Suppose that S is a left PCP monoid. S is not left PSF if and only if there exists $s \in S$ such that $[1]_{ker\rho_s}$ is not right collapsible.*

Theorem 2.4. *Let S be an aperiodic monoid. Then S is left PSF if and only if S is left PCP.*

Proof. Necessity. If S is left PSF then S is obviously left PCP.

Sufficiency. Suppose that S is left PCP. Using Theorem 2.2 it is sufficient to show that for every $s \in S$, $[1]_{ker\rho_s}$ is right collapsible. Let $l_1, l_2 \in [1]_{ker\rho_s}$. Thus $l_1 s = l_2 s = s$. Since S is left PCP, there exist $u, v \in S$ such that $l_1 u = l_2 v$ and $us = vs = s$. Equalities $us = vs = s$ imply that $u, v \in [1]_{ker\rho_s}$. Thus we show that $[1]_{ker\rho_s}$ is left reversible. Since S is aperiodic, every left reversible submonoid of S is right collapsible (by [15, Lemma 2.4]). Thus for every $s \in S$, $[1]_{ker\rho_s}$ is right collapsible. □

Theorem 2.5. *An act A_S satisfies Condition $(G-PCP)$ if and only if for any homomorphism $f :_S S \to_S S$ and for any $a, a' \in A_S$ and $s \in S$, $af(s) = a'f(s)$ implies the existence of elements $u, v \in S$ such that $uf(1) = vf(1) = f(1)$ and $a \otimes su = a' \otimes sv$ in $A_S \otimes_S S$.*

Proof. Necessity. Suppose that A_S satisfies Condition $(G-PCP)$. Let $f :_S S \to_S S$ be homomorphism such that $af(s) = a'f(s)$, for $a, a' \in A_S$ and $s \in S$. Then $asf(1) = a'sf(1)$. Since A_S satisfies Condition $(G-PCP)$ there exist $u, v \in S$ such that $asu = a'sv$ and $uf(1) = vf(1) = f(1)$. Thus $a \otimes su = a' \otimes sv$ in $A_S \otimes_S S$ (by [10, II, Proposition 5.13]).

Sufficiency. Suppose $as = a's$, $a, a' \in A_S$, $s \in S$. Define a mapping $f :_S S \to_S S$ by

$$f(x) = xs$$

for every $x \in S$. Clearly f is a homomorphism. The equalities $as = a's$ and $f(1) = s$ imply $af(1) = a'f(1)$. Thus by assumption there exist $u, v \in S$ such that $uf(1) = vf(1) = f(1)$ and $a \otimes u = a' \otimes v$ in $A_S \otimes_S S$. Since $f(1) = s$ we have $us = vs = s$. The equality $a \otimes u = a' \otimes v$ in $A_S \otimes_S S$ implies $au = a'v$ (by [10, II, Proposition 5.13]). Thus $au = a'v$ and $us = vs = s$. Hence A_S satisfies Condition $(G-PCP)$. □

Theorem 2.6. *The following statements hold:*

(1) Θ_S *satisfies Condition $(G-PCP)$.*

(2) *If A_S satisfies Condition $(G-PCP)$, then every subact of A_S satisfies it.*

(3) *If A_S satisfies Condition $(G-PCP)$, then every retract of A_S satisfies it.*

(4) *If $\prod_{i\in I} A_i$, where A_i, $i \in I$, are right S-acts, satisfies Condition $(G-PCP)$, then A_i satisfies it, for every $i \in I$.*

(5) *$\coprod_{i\in I} A_i$, where A_i, $i \in I$, are right S-acts, satisfies Condition $(G-PCP)$ if and only if A_i satisfies Condition $(G-PCP)$, for every $i \in I$.*

(6) *If $\{B_i \mid i \in I\}$ is a chain of subacts of A_S and every B_i, $i \in I$, satisfies Condition $(G-PCP)$, then $\bigcup_{i\in I} B_i$ satisfies it.*

(7) *S_S satisfies Condition $(G-PCP)$ if and only if S is left PCP (that is, for every $s \in S$, Ss satisfies Condition (P)).*

Proof. The proofs are straightforward. □

Theorem 2.7. *The following statements hold:*

(1) *If A_S satisfies Condition $(G-PCP)$, then A_S is principally weakly flat.*

(2) *For a left PSF monoid S, A_S is principally weakly flat if and only if A_S satisfies Condition $(G-PCP)$.*

Proof. (1). Let $as = a's$, for $a \in A$ and $s, s' \in S$. By assumption, there exist $u, v \in S$ such that $au = a'v$ and $us = vs = s$. Hence

$$a \otimes s = a \otimes us = au \otimes s = a'v \otimes s = a' \otimes vs = a' \otimes s$$

in $A \otimes Ss$. Thus A_S is principally weakly flat, as required.

(2). Necessity. Let $as = a's$, for $a \in A$ and $s, s' \in S$. By assumption, there exist $n \in \mathbb{N}$ and elements $a_1, ..., a_n \in A_S$, $s_1, t_1, ..., s_n, t_n \in S$ such that

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 s &= t_1 s \\ a_2 t_2 &= a_3 s_3 & s_2 s &= t_2 s \\ &\cdots & &\cdots \\ a_n t_n &= a' & s_n s &= t_n s. \end{aligned}$$

Since S is left PSF, $s_1s = t_1s$ implies the existence of $v_1 \in S$ such that $v_1s = s$ and $s_1v_1 = t_1v_1$. Then $s_2v_1s = t_2v_1s$ implies the existence of $v_2 \in S$ such that $v_2s = s$ and $s_2v_1v_2 = t_2v_1v_2$. If $v = v_1v_2$, then

$$vs = v_1v_2s = s, \; s_1v = s_1v_1v_2 = t_1v_1v_2 = t_1v, \; s_2v = t_2v.$$

Continuing this procedure, there exists $u \in S$ such that $us = s$ and $s_iu = t_iu$, for $1 \leq i \leq n$. Thus we have

$$au = (a_1s_1)u = a_1(s_1u) = a_1(t_1u) = (a_1t_1)u = \ldots = (a_nt_n)u = a'u.$$

So A_S satisfies Condition $(G - PCP)$, as required.

Sufficiency. This is true, by (1). □

Theorem 2.8. *Let S be a monoid. Then the following statements are equivalent:*

(1) *S is left PSF.*

(2) *S is left PCP and $D(S)$ satisfies Condition $(G - PCP)$.*

(3) *S is weakly left PCP and $D(S)$ satisfies Condition $(G - PCP)$.*

Proof. (1) ⇒ (2). S is left PCP and $D(S)$ is principally weakly flat (by [17, Thorem 2.5]). Since S is left PSF by part (2) of Theorem 2.7, $D(S)$ satisfies Condition $(G - PCP)$.

(2) ⇒ (3). Since by [17, Proposition 2.2] left PCP implies weakly left PCP, the result is obtained.

(3) ⇒ (1). $D(S)$ is principally weakly flat, by part (1) of Theorem 2.7. Thus by [17, Thorem 2.5], S is left PSF. □

Now we give an equivalence for a cyclic S-act to satisfy Condition $(G - PCP)$.

Lemma 2.9. *Let ρ be a right congruence on S. Then the S-act S/ρ satisfies Condition $(G - PCP)$ if and only if*

$$(\forall x, y, s \in S)\Big((xs)\rho(ys) \Rightarrow (\exists u, v \in S)\Big((xu)\rho(yv) \wedge us = s = vs\Big)\Big).$$

Proof. This is a straightforward result of the definition.

□

Corollary 2.10. *Let $z \in S$. Then the principal right ideal zS satisfies Condition $(G - PCP)$ if and only if $zxs = zys$, $x, y, z \in S$, implies the existence of $u, v \in S$ such that $us = s = vs$ and $zxu = zyv$.*

Proof. This is an immediate result of the definition. □

Theorem 2.11. *Let $w \in S$ and $\rho = \rho(w, 1)$. The cyclic S-act S/ρ satisfies Condition $(G - PCP)$ if and only if for every $x, y, s \in S$ and $m, n \in \mathbb{N}_0$*

$$w^m xs = w^n ys \Rightarrow (\exists p, q \in \mathbb{N}_0)(\exists u, v \in S)(w^p xu = w^q yv \wedge us = vs = s).$$

Proof. Necessity. Let $w^m xs = w^n ys$, for $x, y, s \in S$ and $m, n \in \mathbb{N}_0$. Then we have $(xs)\rho(ys)$ (by [10, III, Corollary 8.7]). By Lemma 2.9, there exist $u, v \in S$ such that $us = vs = s$ and $(xu)\rho(yv)$. Thus, by [10, III, Corollary 8.7], there exist $p, q \in \mathbb{N}_0$ such that $w^p xu = w^q yv$.

Sufficiency. Let $(xs)\rho(ys)$, for $x, y, s \in S$. Then, by [10, III, Corollary 8.7], there exist $m, n \in \mathbb{N}_0$ such that $w^m xs = w^n ys$. By assumption, there exist $u, v \in S$ and $p, q \in \mathbb{N}_0$ such that $us = vs = s$ and $w^p xu = w^q yv$. Hence $(xu)\rho(yv)$, by [10, III, Corollary 8.7], and so S/ρ satisfies Condition $(G - PCP)$, by Lemma 2.9. □

Notice that in the above theorem, if $w = 1$, then $S/\rho = S/\rho(w, 1) = S/\rho(1, 1) = S/\Delta_S \cong S_S$, and so the part (7) of Theorem 2.6 is obtained.

Corollary 2.12. *Let S be a monoid, $e \in E(S)$ and let $\rho = \rho(e, 1)$. Then the cyclic right S-act S/ρ satisfies Condition $(G - PCP)$ if and only if for every $x, y, s \in S$*

$$exs = eys \Rightarrow (\exists u, v \in S)(exu = eyv \wedge us = vs = s).$$

Proof. Since $e^2 = e$, it follows by Theorem 2.11. □

Corollary 2.13. *Let S be a monoid, $e \in E(S)$ be a left zero element of S and let $\rho = \rho(e, 1)$. Then the cyclic right S-act S/ρ satisfies Condition $(G - PCP)$.*

Proof. In Corollary 2.12, it suffices to take $u = v = 1$. □

Corollary 2.14. *Let S be a monoid and $e \in E(S)$. The following statements are equivalent:*

(1) *The monocyclic right S-act $S/\rho(e,1)$ satisfies Condition $(G-PCP)$.*

(2) *eS satisfies Condition $(G-PCP)$.*

(3) $(\forall x,y,s\in S)(exs=eys\Rightarrow(\exists u,v\in S)(exu=eyv\wedge us=vs=s))$.

Proof. It follows by Corollaries 2.10 and 2.12. □

Theorem 2.15. *Let K_S be a right ideal of S. Then the right Rees factor act S/K_S satisfies Condition $(G-PCP)$ if and only if K_S satisfies Conditions*

(1) $(\forall x,s\in S)(xs\in K_S\Rightarrow(\exists u\in S)(xu\in K_S\wedge us=s))$.

(2) $(\forall x,y,s\in S)(xs=ys\in S\setminus K_S)\Rightarrow(\exists u,v\in S)(xu=yv\in S\setminus K_S\wedge us=vs=s)$.

Proof. Necessity. (1). Suppose $xs\in K$, for $x,s\in S$. Then $(xs)\rho_K(ks)$ for $k\in K$. By Lemma 2.9, there exist $u,v\in S$ such that $(xu)\rho_K(kv)$ and $us=vs=s$. Thus $xu,kv\in K$ or $xu=kv\in K$. Thus $xu\in K$ and $us=s$ and so we are done.

(2). Suppose $xs=ys\in S\setminus K_S$, for $x,y,s\in S$. Then $(xs)\rho_K(ys)$. By Lemma 2.9, there exist $u,v\in S$ such that $(xu)\rho_K(yv)$ and $us=vs=s$. $(xu)\rho_K(yv)$ implies $xu,yv\in K$ or $xu=yv\in S\setminus K_S$. If $xu\in K$ then $xs=x(us)=(xu)s\in K$ which is a contradiction. Thus $xu=yv\in S\setminus K_S$ and so we are done.

Sufficiency. Suppose $(xs)\rho_K(ys)$, for $x,y,s\in S$. Thus $xs,ys\in K$ or $xs=ys\in S\setminus K$. If $xs,ys\in K$ then by condition (1), there exist $u,v\in S$ such that $xu,yv\in K$ and $us=vs=s$. Thus $(xu)\rho_K(yv)$ and so by Lemma 2.9, we are done. If $xs=ys\in S\setminus K$ then by condition (2), there exist $u,v\in S$ such that $xu=yv$ and $us=vs=s$. $xu=yv$ implies $(xu)\rho_K(yv)$ and so by Lemma 2.9, we are done. □

3 Classification by Condition $(G-PCP)$ of Acts

In this section we give a classification of monoids for which all (cyclic, monocyclic) right acts satisfy Condition $(G-PCP)$ and give a classification of monoids for which all right acts satisfying some other flatness properties have Condition $(G-PCP)$.

Theorem 3.1. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition* $(G-PCP)$.

(2) *All finitely generated right S-acts satisfy Condition* $(G-PCP)$.

(3) *All cyclic right S-acts satisfy Condition* $(G-PCP)$.

(4) *All monocyclic right S-acts satisfy Condition* $(G-PCP)$.

(5) *All monocyclic right S-acts of the form $S/\rho(s,s^2)$ $(s \in S)$ satisfy Condition $G-PCP$.*

(6) $(\forall x,y,s \in S)\ (\exists u,v \in S)(us = vs = s \wedge xu\boldsymbol{\rho(xs,ys)}yv)$.

(7) $(\forall x,y,s \in S)\ (\exists u \in S)(us = s \wedge xu\boldsymbol{\rho(xs,ys)}yu)$.

(8) *All right Rees factor S-acts satisfy Condition* $(G-PCP)$.

(9) *All right Rees factor S-acts of the form S/sS $(s \in S)$ satisfy Condition* $(G-PCP)$.

(10) *All divisible right S-acts satisfy Condition* $(G-PCP)$.

(11) *All principally weakly injective right S-acts satisfy Condition* $(G-PCP)$.

(12) *All finitely generated weakly injective S-acts satisfy Condition* $(G-PCP)$.

(13) *All weakly injective right S-acts satisfy Condition* $(G-PCP)$.

(14) *All injective right S-acts satisfy Condition* $(G-PCP)$.

(15) *All cofree right S-acts satisfy Condition* $(G-PCP)$.

(16) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), (1) $\Rightarrow$ (8) $\Rightarrow$ (9), (1) $\Rightarrow$ (10) and (7) $\Rightarrow$ (6) are obvious.

Since cofreeness $\Rightarrow$ injectivity $\Rightarrow$ weak injectivity $\Rightarrow$ fg-weak injectivity $\Rightarrow$ principal weak injectivity $\Rightarrow$ divisibility, implications (10) $\Rightarrow$ (11) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14) $\Rightarrow$ (15) are obvious.

(5) $\Rightarrow$ (16). By part (1) of Theorem 2.7, all monocyclic right S-acts of the form $S/\rho(s,s^2)$ $(s \in S)$ are principally weakly flat. It follows by [10, IV, Theorem 6.6] that S is regular.

(9) $\Rightarrow$ (16). By part (1) of Theorem 2.7, all right Rees factor S-acts of the form S/sS $(s \in S)$ are principally weakly flat. It follows by [10, IV, Theorem 6.6] that S is regular.

(6) $\Rightarrow$ (4). Let $x,y\in S$. Suppose that $as\rho(x,y)a's$, for $a,a',s\in S$. By assumption there exist $u,v\in S$ such that $us=vs=s$ and $au\rho(as,a's)a'v$. Since $\rho(as,a's)\subseteq\rho(x,y)$ then $au\boldsymbol{\rho(x,y)}a'v$ and so $S/\rho(x,y)$ satisfies $(G-PCP)$, by Lemma 2.9.

(15) $\Rightarrow$ (16). Since every right S-act can be embedded in a cofree right S-act, by the assumption, every right S-act is a subact of $(G-PCP)$ right S-act. By part (2) of Theorem 2.6, all right S-acts satisfy Condition $(G-PCP)$. It follows, by (1) of Theorem 2.7, that all right S-acts are principally weakly flat. Thus by [10, IV, Theorem 6.6], S is regular.

(16) $\Rightarrow$ (1). Suppose that A_S is a right S-act and $as=a's$ for $s\in S$ and $a,a'\in A_S$. Since s is regular, there exists $x\in S$ such that $s=sxs$. Let $e=sx$. Then $ae=asx=a'sx=a'e$ and $es=sxs=s$. Thus A_S satisfies Condition $(G-PCP)$.

(16) $\Rightarrow$ (7). Let $x,y,s\in S$. Since S is regular there exists $s'\in S$ such that $s=ss's$. Let $u=ss'$. Thus $us=s$ and $xu\boldsymbol{\rho(xs,ys)}yu$. □

Theorem 3.2. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition $(G-PCP)$.*

(2) *All right S-acts satisfying Condition $(E'P)$ satisfy Condition $(G-PCP)$.*

(3) *All right S-acts satisfying Condition (EP) satisfy Condition $(G-PCP)$.*

(4) *All right S-acts satisfying Condition (E') satisfy Condition $(G-PCP)$.*

(5) *All right S-acts satisfying Condition (E) satisfy Condition $(G-PCP)$.*

(6) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious because $(E)\Rightarrow(EP)\Rightarrow(E'P)$ and $(E)\Rightarrow(E')\Rightarrow(E'P)$.

(5) $\Rightarrow$ (6). Let $s\in S$. If $sS=S$, then there exists $x\in S$ such that $sx=1$. Thus $sxs=s$, that is, s is regular. Let $sS\neq S$. Put

$$A=S\coprod^{sS}S=\big\{(l,x)|\quad l\in S\setminus sS\big\}\,\dot\cup\, sS\,\dot\cup\,\{(t,y)|\ t\in S\setminus sS\}.$$

Indeed,

$$B=\{(l,x)|\ l\in S\setminus sS\big\}\,\dot\cup\, sS\cong S_S\cong\{(t,y)|\ t\in S\setminus sS\}\,\dot\cup\, sS=C,$$

B and C are subacts of A which are generated by $(1, x)$ and $(1, y)$, respectively. A is generated by $(1, x)$ and $(1, y)$ because $A = B \cup C$. By the above isomorphisms, B and C satisfy Condition (E) and so A satisfies Condition (E). Thus, by assumption, A satisfy Condition $(G - PCP)$. Then $(1, x)s = (1, y)s$ implies that there exist $u, v \in S$ such that $us = vs = s$ and $(1, x)u = (1, y)v$. Therefore $u \in sS$ and so there exists $x \in S$ such that $u = sx$. Hence $s = us = sxs$, that is, s is regular. Thus S is regular, as required.

(4) $\Rightarrow$ (1). This is true, by Theorem 3.1. □

By the proof of Theorem 3.2, we conclude that the above theorem is true for finitely generated and generated by at most (exactly) two elements right S-acts.

Theorem 3.3. *The following statments are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right S-acts satisfy Condition $(G - PCP)$.*

(2) *All $\mathfrak{R}$-torsion free finitely generated right S-acts satisfy Condition $(G - PCP)$.*

(3) *All $\mathfrak{R}$-torsion free right S-acts generated by at most two elements satisfy Condition $(G - PCP)$.*

(4) *All $\mathfrak{R}$-torsion free right S-acts generated by exactly two elements satisfy Condition $(G - PCP)$.*

(5) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \{(l, x) | \ l \in S \setminus sS\} \,\dot{\cup}\, sS \,\dot{\cup}\, \{(t, y) | \ t \in S \setminus sS\}.$$

By the proof of part (5) $\Rightarrow$ (6) of Theorem 3.2, A is an S-act which is generated by exactly two different elements $(1, x)$ and $(1, y)$ and also satisfy Condition (E). Every S-act satisfying Condition (E) is $\mathfrak{R}$-torsion free, (by [18, Proposition 1.2]). Thus A is $\mathfrak{R}$-torsion free and so, by assumption, satisfies Condition $(G - PCP)$. Hence, by the proof of part (5) $\Rightarrow$ (6) of Theorem 3.2, s is regular. Therefore S is regular, as required.

(5) $\Rightarrow$ (1). Every $\mathfrak{R}$-torsion free S-act is principally weakly flat (by [18, Theorem 4.5]). Since every regular monoid is left PP and so is left PSF, principally weakly flat is equivalent to Condition $(G-PCP)$ of right acts, by part (2) of Theorem 2.7. Therefore every $\mathfrak{R}$-torsion free S-act satisfies Condition $(G-PCP)$. $\square$

Theorem 3.4. *Let (U) be a property on S-acts such that implies Condition (PWP) and S_S satisfies property (U). Then the following statments are equivalent:*

(1) *All right S-acts satisfying property (U) satisfy Condition $(G-PCP)$.*

(2) *All finitely generated right S-acts satisfying property (U) satisfy Condition $(G-PCP)$.*

(3) *All cyclic right S-acts satisfying property (U) satisfy Condition $(G-PCP)$.*

(4) *S_S satisfies Condition $(G-PCP)$.*

(5) *S is left PCP.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). S_S is cyclic right S-act satisfies property (U). Thus by assumption, S_S satisfies Condition $(G-PCP)$.

(4) $\Rightarrow$ (5). It follows by part (7) of Theorem 2.6.

(5) $\Rightarrow$ (1). Suppose that A_S satisfies property (U). We show that A_S satisfies Condition $(G-PCP)$. Suppose that $as = a's$ for some $s \in S$ and $a, a' \in A_S$. Since property (U) satisfies Condition (PWP), A_S satisfies Condition (PWP) and so $as = a's$ implies the existence of $a'' \in A_S$ and $w_1, w_2 \in S$ such that $a = a''w_1, a' = a''w_2$ and $w_1s = w_2s$. Since S is left PCP by assumption, the equality $w_1s = w_2s$ implies the existence of $u, v \in S$ such that $w_1u = w_2v$ and $us = vs = s$. Thus $au = a''w_1u = a''w_2v = a'v$ and $us = vs = s$ that is A_S satisfies Condition $(G-PCP)$. $\square$

Corollary 3.5. *Let (U) be one of the following properties for acts over monoids such as:*

free, projective generator, projective, strongly flat, weakly pullback flat, weakly kernel flat, principally weakly kernel flat, translation kernel flat, Condition (WP), Condition (P), Condition (P'), Condition (PWP).

Then the following statments are equivalent:

(1) *All right S-acts satisfying property (U) satisfy Condition $(G-PCP)$.*

(2) *All finitely generated right S-acts satisfying property (U) satisfy Condition $(G-PCP)$.*

(3) *All cyclic right S-acts satisfying property (U) satisfy Condition $(G-PCP)$.*

(4) *S_S satisfies Condition $(G-PCP)$.*

(5) *S is left PCP.*

Proof. Since all properties mentioned in the statment of the corollary imply Condition (PWP) and S_S satisfies these properties, by Theorem 3.4, the result follows. □

Theorem 3.6. *The following statments are equivalent:*

(1) *All torsion free right S-acts satisfy Condition $(G-PCP)$.*

(2) *All torsion free finitely generated right S-acts satisfy Condition $(G-PCP)$.*

(3) *All torsion free cyclic right S-acts satisfy Condition $(G-PCP)$.*

(4) *All torsion free right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(5) *S is left almost regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). It follows from part (1) of Theorem 2.7 and [10, IV, Theorem 6.5].

(5) ⇒ (1). Assume that A_S is torsion free right S-act. Let $as = a's$ for $a, a' \in A_S$, $s \in S$. Since s is left almost regular, there exist elements $r, r_1, ..., r_m, s_1, s_2, ..., s_m \in S$ and right cancellable elements $c_1, ..., c_m \in S$ such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\cdots \\ s_mc_m &= s_{m-1}r_m \\ s &= s_mrs \end{aligned}$$

we have:

$$s_1c_1 = sr_1 = s_mrsr_1 = s_mrs_1c_1 \Rightarrow s_1 = s_mrs_1$$

and so

$$s_2c_2 = s_1r_2 = s_mrs_1r_2 = s_mrs_2c_2 \Rightarrow s_2 = s_mrs_2$$

By continuing this process, $s_i = s_mrs_i$ for every $1 \leq i \leq m$. Thus $s_m = s_mrs_m$. Equality $as = a's$ implies $asr_1 = a'sr_1$ and so $as_1c_1 = a's_1c_1$. Since A_S is torsion free, $as_1 = a's_1$. Thus $as_1r_2 = a's_1r_2$ and so $as_2c_2 = a's_2c_2$. Again, since A_S is torsion free, $as_2 = a's_2$. By continuing this process, $as_m = a's_m$. Let $e = s_mr$. Thus $ae = as_mr = a's_mr = a'e$ and $es = s_mrs = s$. Hence A_S satisfies Condition $(G - PCP)$. □

Theorem 3.7. *The following statements are equivalent:*

(1) *All faithful right S-acts satisfy Condition $(G - PCP)$.*

(2) *All faithful finitely generated right S-acts satisfy Condition $(G-PCP)$.*

(3) *All faithful right S-acts generated by at most two elements satisfy Condition $(G - PCP)$.*

(4) *All faithful right S-acts generated by exactly two elements satisfy Condition $(G - PCP)$.*

(5) *S is regular.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Let $sS \neq S$. Put $A_S = S \coprod^{sS} S$. By the proof of part $(5) \Rightarrow (6)$ of Theorem 3.2, A_S is an S-act which is generated by exactly two different elements $(1, x)$ and $(1, y)$. Since $B_S \cong S_S \cong C_S$ and S_S is faithful, then B_S and C_S are faithful and so A_S is faithful, because $A_S = B_S \cup C_S$. Thus by assumption, A_S satisfies Condition $(G - PCP)$. Hence, by the proof of part $(5) \Rightarrow (6)$ of Theorem 3.2, s is regular. Therefore S is regular, as required.

$(5) \Rightarrow (1)$. All right S-acts satisfy Condition $(G - PCP)$, by Theorem 3.1. Thus all faithful right S-acts satisfy Condition $(G - PCP)$. □

Notation: C_l (C_r) is the set of all left (right) cancellable elements of S.

Lemma 3.8. *The following statements are equivalent:*

(1) *There exists at least one strongly faithful right S-act.*

(2) sS *as a right S-act is strongly faithful, for every* $s \in S$.

(3) S *as a right S-act is strongly faithful.*

(4) $sS \subseteq C_l$, *for some* $s \in S$.

(5) S *is left cancellative.*

Proof. Implications (2) $\Rightarrow$ (1), (5) $\Rightarrow$ (4) and (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (5). Let A be a strongly faithful right S-act and $sl = st$, for $s, l, t \in S$. Then $asl = ast$, for $a \in A$. Since A is strongly faithful and $as \in A$, $l = t$. Therefore S is left cancellative.

(5) $\Rightarrow$ (3). Let S be left cancellative and $sl = st$, for $l, t, s \in S$. By assumption, $l = t$ and so S_S is strongly faithful, as an S-act.

(4) $\Rightarrow$ (5). Let $sS \subseteq C_l$, for some $s \in S$ and $rt = rl$, for $r, t, l \in S$. Then $(sr)t = (sr)l$. By assumption, $t = l$ and so S is left cancellative.

(5) $\Rightarrow$ (2). Let $skt = skl$, for $sk \in sS$ and $t, l \in S$. By assumption, $t = l$ and so sS is strongly faithful as a right S-act. □

By the above lemma, for a monoid S there exists no strongly faithful right S-act if and only if S is not left cancellative.

Theorem 3.9. *The following statements are equivalent:*

(1) *All strongly faithful right S-acts satisfy Condition* $(G - PCP)$.

(2) *All strongly faithful finitely generated right S-acts satisfy Condition* $(G - PCP)$.

(3) *All strongly faithful right S-acts generated by at most two elements satisfy Condition* $(G - PCP)$.

(4) *All strongly faithful right S-acts generated by exactly two elements satisfy Condition* $(G - PCP)$.

(5) *Either S is not left cancellative or it is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). If S is not left cancellative, then (5) is satisfied. Let S be left cancellative and $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Now let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \{(l,x)|\ l \in S \setminus sS\} \dot\cup\ sS \dot\cup\ \{(t,y)|\ t \in S \setminus sS\}.$$

We have

$$B = \{(l,x)|\ l \in S \setminus sS\} \dot\cup\ sS \cong S_S \cong \{(t,y)|\ t \in S \setminus sS\} \dot\cup\ sS = C$$

and

$$A =< (1,x),(1,y) >= B \cup C.$$

Since S is left cancellative, S_S is strongly faithful, by Lemma 3.8. By the above isomorphisms, B and C are strongly faithful as subacts of A. Thus A is strongly faithful. Since A is generated by two different elements $(1,x)$ and $(1,y)$, by assumption, A satisfies Condition $(G-PCP)$. By the proof of part (5) $\Rightarrow$ (6) of Theorem 3.2, s is regular and so S is regular. Thus for every $s \in S$ there exists $x \in S$ such that $sxs = s$. Since S is left cancellative, $xs = 1$. Thus every element in S has left inverse and so S is a group.

(5) $\Rightarrow$ (1). If S is left cancellative, then there exists at least one strongly faithful right S-act, by Lemma 3.8. Since S is a group, it is regular and so (1) is satisfied, by Theorem 3.1. □

Lemma 3.10. *Let ρ be a right congruence on S. Then the following statements are equivalent:*

(1) *S/ρ is a strongly faithful cyclic right S-act.*

(2) *$\rho = \Delta_S$ and S is left cancellative.*

Proof. (1) $\Rightarrow$ (2). Since S/ρ is strongly faithful as a right S-act, there exists at least one strongly faithful S-act. Hence S is left cancellative, by Lemma 3.8. Now let $(s,t) \in \rho$, for $s,t \in S$. Then $[1]_\rho \cdot s = [s]_\rho = [t]_\rho = [1]_\rho \cdot t$. Thus $s = t$, since S/ρ is strongly faithful, and so $\rho = \Delta_S$.

(2) $\Rightarrow$ (1). $S/\rho = S/\Delta_S \cong S_S$. Since S is left cancellative, $S_S \cong S/\rho$ is strongly faithful, by Lemma 3.8. □

Theorem 3.11. *The following statements are equivalent:*

(1) *Every strongly faithful cyclic right S-act satisfies Condition $(G-PCP)$.*

(2) *Either S is not left cancellative or it is left PCP.*

Proof. (1) $\Rightarrow$ (2). If S is not left cancellative, then (2) is satisfied. Let S be left cancellative. Then S, as a cyclic right S-act, is strongly faithful, by Lemma 3.8. Thus, by assumption, S_S satisfies Condition $(G-PCP)$ and so S is left PCP, by part (7) of Theorem 2.6.

(2) $\Rightarrow$ (1). If S is not left cancellative, then there exists no strongly faithful right S-act, by Lemma 3.8. Thus (1) is satisfied. If S is left cancellative, then there exists at least one strongly faithful cyclic right S-act, by Lemma 3.8. If S/ρ is a strongly faithful cyclic right S-act, then $\rho = \Delta_S$, by Lemma 3.10, and so $S/\rho \cong S_S$. By assumption, S is left PCP and so $S/\rho \cong S_S$ satisfies Condition $(G-PCP)$, by part (7) of Theorem 2.6. □

Theorem 3.12. *The following statements are equivalent:*

1. *There exists at least one strongly faithful cyclic right S-act satisfying Condition $(G-PCP)$.*
2. *S is left cancellative and every strongly faithful cyclic right S-act satisfies Condition $(G-PCP)$.*
3. *S is left cancellative and it is left PCP.*

Proof. (1) $\Rightarrow$ (2). Since there exists at least one strongly faithful cyclic right S-act, S is left cancellative, by Lemma 3.8. If S/ρ is a strongly faithful cyclic right S-act satisfying Condition $(G-PCP)$, then $\rho = \Delta_S$, by Lemma 3.10. Thus $S/\rho \cong S_S$ satisfies Condition $(G-PCP)$ and so S is left PCP, by part (7) of Theorem 2.6. Therefore every strongly faithful cyclic right S-act satisfies Condition $(G-PCP)$, by Theorem 3.11.

(2) $\Rightarrow$ (3). This is true, by Theorem 3.11.

(3) $\Rightarrow$ (1). Since S is left cancellative, there exists at least one strongly faithful cyclic right S-act, by Lemma 3.8. If S/ρ is a strongly faithful cyclic right S-act, then $\rho = \Delta_S$, by Lemma 3.10. Thus $S/\rho = S/\Delta_S \cong S_S$. Since S is left PCP, $S/\rho \cong S_S$ satisfies Condition $(G-PCP)$, by part (7) of Theorem 2.6. □

Theorem 3.13. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition $(G-PCP)$ are (strongly) faithful.*
2. *All finitely generated right S-acts satisfying Condition $(G-PCP)$ are (strongly) faithful.*

(3) *All cyclic right S-acts satisfying Condition $(G-PCP)$ are (strongly) faithful.*

(4) *All right Rees factor acts of S satisfying Condition $(G-PCP)$ are (strongly) faithful.*

(5) $S=\{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By part (1) of Theorem 2.6, $\Theta_S \cong S/S_S$ satisfies Condition $(G-PCP)$ and so by assumption $\Theta_S \cong S/S_S$ is (strongly) faithful. Thus for every $s,t\in S$, equality $\theta s=\theta t$ implies $s=t$ and so $S=\{1\}$.

(5) $\Rightarrow$ (1). It is obvious. □

Theorem 3.14. *The following statements are equivalent:*

(1) *All right Rees factor acts of S satisfying Condition (P) satisfy Condition $(G-PCP)$.*

(2) *All weakly pullback flat right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(3) *All strongly flat right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(4) *All projective right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(5) *All projective generator right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(6) *All free right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(7) *Either S does not contain a left zero or S is left PCP.*

Proof. By [10, III, Theorem 16.7, Proposition 17.5] and [13, Theorem 21], we have projective $\Rightarrow$ strongly flat $\Leftrightarrow$ pullback flat $\Rightarrow$ weakly pullback flat $\Rightarrow$ Condition (P). Also we have free $\Rightarrow$ projective generator $\Rightarrow$ projective. Thus the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (7). Suppose that S contains a left zero element as z. Thus $K_S=zS=\{z\}$ is a right ideal of S and so $S/K_S\cong S_S$ is free. By assumption S_S satisfies Condition $(G-PCP)$ and so S is left PCP by part (7) of Theorem 2.6.

(7) $\Rightarrow$ (1). Suppose K_S is a right ideal of S and S/K_S satisfies Condition (P). We have the following cases:

Case 1. If $K_S = S$ then $S/K_S = S/S_S \cong \Theta_S$. Thus $S/K_S \cong \Theta_S$ satisfies Condition $(G - PCP)$ by part (1) of Theorem 2.6.

Case 2. If $K_S \neq S$ then $|K_S| = 1$, by [10, III, Proposition 13.9]. Suppose $z \in K_S$. Thus $K_S = zS = \{z\}$, that is S has a left zero element and so S is left PCP monoid, by assumption. Thus $S/K_S \cong S_S$ satisfies Condition $(G - PCP)$ by part (7) of Theorem 2.6. □

Theorem 3.15. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition $(G-PCP)$ are (projective-) generator.*

(2) *All finitely generated right S-acts satisfying Condition $(G-PCP)$ are (projective-) generator.*

(3) *All cyclic right S-acts satisfying Condition $(G-PCP)$ are (projective-) generator.*

(4) *All right Rees factor acts of S satisfying Condition $(G - PCP)$ are (projective-) generator.*

(5) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). The right S-act $S/S_S \cong \Theta_S$ satisfies Condition $(G - PCP)$ by part (1) of Theorem 2.6. Thus $S/S_S \cong \Theta_S$ is generator, by assumption and so there exists an epimorphism $\pi : \Theta_S \to S_S$ by [10, II, Thorem 3.16], that is $S = \{1\}$.

(5) ⇒ (1). It is obvious. □

Corollary 3.16. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition $(G - PCP)$ are free.*

(2) *All finitely generated right S-acts satisfying Condition $(G-PCP)$ are free.*

(3) *All cyclic right S-acts satisfying Condition $(G - PCP)$ are free.*

(4) *All right Rees factor acts of S satisfying Condition $(G - PCP)$ are free.*

(5) $S = \{1\}$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. All right Rees factor S-acts satisfying Condition $(G-PCP)$ are generator. Thus $S = \{1\}$ by Theorem 3.15.

$(5) \Rightarrow (1)$. It is obvious. □

Theorem 3.17. *The following statements are equivalent:*

(1) *All indecomposable right S-acts satisfy Condition $(G-PCP)$.*

(2) *All indecomposable finitely generated right S-acts satisfy Condition $(G-PCP)$.*

(3) *All indecomposable finitely generated right S-acts by at most two elements satisfy Condition $(G-PCP)$.*

(4) *All indecomposable right S-acts generated by exactly two elements satisfy Condition $(G-PCP)$.*

(5) *S is regular.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Let $sS \neq S$. Put

$$A_S = S \coprod^{sS} S = \{(l,x) |\ l \in S \setminus sS\} \,\dot\cup\, sS \,\dot\cup\, \{(t,y) |\ t \in S \setminus sS\}.$$

By the proof of part $(5) \Rightarrow (6)$ of Theorem 3.2, A_S is an S-act which is generated by exactly two different elements $(1,x)$ and $(1,y)$ and also is indecomposable. Thus by assumption, A_S satisfy Condition $(G-PCP)$. Hence, by the proof of part $(5) \Rightarrow (6)$ of Theorem 3.2, s is regular. Therefore S is regular, as required.

$(5) \Rightarrow (1)$. By Theorem 3.1, the result follows. □

Theorem 3.18. *The following statements are equivalent:*

(1) *All regular right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(2) *If S is right PP and contains a left zero, then S is left PCP.*

Proof. $(1) \Rightarrow (2)$. Let $z \in S$ be left zero element. Thus $K_S = zS = \{z\}$ and so $S/K_S \cong S_S$. Since S is right PP, $S/K_S \cong S_S$ is regular. By assumption, $S/K_S \cong S_S$ is $(G-PCP)$ and so by part (7) of Theorem 2.6, is right PCP.

(2) $\Rightarrow$ (1). Let K_S be a right ideal of S such that S/K_S is regular. we have two cases:

case 1. If $K_S = S$, then $S/K_S = S/S_S \cong \Theta_S$ satisfies Condition $(G-PCP)$, by part (1) of Theorem 2.6.

case 2. If $K_S \neq S$ then $|K_S| = 1$ and S is right PP, by [10, III, Proposition 19.6]. Suppose $z \in K_S$. Thus $K_S = zS = \{z\}$, that is S has a left zero element and so S is left PCP monoid, by assumption. Thus $S/K_S \cong S_S$ satisfies Condition $(G-PCP)$ by part (7) of Theorem 2.6. □

Theorem 3.19. *The following statements are equivalent:*

(1) *All (P)-regular right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(2) *All strongly (P)-cyclic right Rees factor acts of S satisfy Condition $(G-PCP)$.*

(3) *If S is right PCP and contains a left zero, then S is left PCP.*

Proof. (1) $\Leftrightarrow$ (3). By [5, Thorem 3.1], the proof is similar to that of Theorem 3.18.

(2) $\Leftrightarrow$ (3). By [4, Thorem 3.1], the proof is similar to that of Theorem 3.18. □

Theorem 3.20. *The following statements are equivalent:*

(1) *All torsionless right S-acts satisfy Condition $(G-PCP)$.*

(2) *For every torsionless right S-act A_S we have:*
$(\forall a, a' \in A_S)(\forall s \in S)(as = a's \Rightarrow (\exists u, v \in S)(au = a'v \wedge us = s = vs))$.

(3) *For every torsionless right S-act A_S we have:*
$(\forall a, a' \in A_S)(\forall s \in S)(as = a's \Rightarrow (\exists r \in S)(ar = a'r \wedge rs = s))$.

(4) *For every torsionless right S-act A_S we have:*
$(\forall a, a' \in A_S)(\forall s \in S)(as = a's \Rightarrow (\exists e \in E(S))(ae = a'e \wedge es = s))$.

(5) *For every non-empty set I, S_S^I satisfies Condition $(G-PCP)$.*

(6) *For every non-empty set I, S_S^I satisfies:*
$(\forall s, x_i, y_i \in S, i \in I)((x_i)_I s = (y_i)_I s \Rightarrow (\exists r \in S)((x_i)_I r = (y_i)_I r \wedge rs = s))$.

(7) *For every non-empty set I, S_S^I satisfies:* $(\forall s, x_i, y_i \in S, i \in I)((x_i)_I s = (y_i)_I s \Rightarrow (\exists e \in E(S))((x_i)_I e = (y_i)_I e \wedge es = s))$.

(8) $S_S^{S\times S}$ *satisfies Condition $(G-PCP)$.*

(9) $(\forall s, x_i, y_i \in S, i \in I = S \times S)((x_i)_I s = (y_i)_I s \Rightarrow (\exists r \in S)((x_i)_I r = (y_i)_I r \wedge rs = s))$.

(10) $(\forall s, x_i, y_i \in S, i \in I = S \times S)((x_i)_I s = (y_i)_I s \Rightarrow (\exists e \in E(S))((x_i)_I e = (y_i)_I e \wedge es = s))$.

(11) *S is left PP.*

Proof. Implications (6) $\Rightarrow$ (9), (5) $\Rightarrow$ (8), (7) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (8), (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (5) are obvious.

(1) $\Leftrightarrow$ (2). It follows by definition.

(2) $\Rightarrow$ (5). Let I be a non-empty set. S_S^I is torsionless, by [11, Proposition 2.6]. Thus S_S^I satisfies Condition $(G-PCP)$.

(7) $\Rightarrow$ (4). Obviously, if S_S^I, for any non-empty set I, satisfies (7), then all subacts of S_S^I satisfies (7). Thus we have (4), by [11, Proposition 2.6].

(8) $\Rightarrow$ (11). By part (1) of Theorem 2.7, $S_S^{S\times S}$ is principally weakly flat. Let $I = S \times S$ and define $f : S_S \to S_S^I$ by $f(s) = (s_i)_I$, for all $i \in I$; $s_i = s$. Clearly, f is S-monomorphism, and so $S_S \cong Imf \leq S^I$. Since $S_S^I = S_S^{S\times S}$ satisfies Condition $(G-PCP)$, thus by (2) of Theorem 2.6, Imf satisfies Condition $(G-PCP)$ and so S_S satisfies Condition $(G-PCP)$. Thus by (7) of Theorem 2.6, S is left PCP. Since $S_S^{S\times S}$ is principally weakly flat, by [16, Proposition 2.2], S_S^I, for every non-empty set I, is principally weakly flat. Thus S is left PP, by [17, Corollary 2.6].

(11) $\Rightarrow$ (7). Let I be a non-empty set. Suppose $(x_i)_I s = (y_i)_I s$ in S_S^I. Since S is left PP there exists $e \in E(S)$ such that $ker\rho_s = ker\rho_e$. we have:

$$(x_i)_I s = (y_i)_I s \Rightarrow \forall i \in I, x_i s = y_i s \Rightarrow \forall i \in I, (x_i, y_i) \in ker\rho_s = ker\rho_e \Rightarrow$$

$$\forall i \in I, x_i e = y_i e, es = s \Rightarrow (x_i)_I e = (y_i)_I e, es = s.$$

□

Lemma 3.21. *Let $S \neq C_r$. Then the following statements hold:*

(1) *$I = S \setminus C_r$ is a proper right ideal of S.*

(2) *If S is left PSF, then the right ideal $I = S \setminus C_r$ satisfies Condition (LU).*

Proof. (1). Since $S \neq C_r$ and $1 \in C_r$, $\emptyset \neq I \subset S$. Let $i \in I$ and $s \in S$. Then there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 i = l_2 i$. Thus $l_1 is = l_2 is$. If $is \in C_r$, then $l_1 = l_2$, which is a contradiction. Therefore $is \in I$ and so I is a proper right ideal of S, as required.

(2). Let $i \in I$. Then i is not right cancellable. Thus there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 i = l_2 i$. Since S is left PSF, there exists $r \in S$ sucht hat $l_1 r = l_2 r$ and $ri = i$. If $r \notin I$, then $l_1 = l_2$, which is a contradiction. Thus $r \in I$ and $ri = i$. Therefore I satisfies Condition (LU). □

Theorem 3.22. *Let (U) be a property on S-acts such that*

$$\text{Condition } (G - PCP) \Rightarrow \text{Property } (U) \Rightarrow \text{torsion free}$$

Then the following statments are equivalent:

(1) *S is left PSF and property (U) implies $PWKF$.*

(2) *S is left PSF and property (U) implies TKF.*

(3) *S is left PSF and property (U) implies Condition (PWP).*

(4) *S is left PSF and property (U) implies Codition (P').*

(5) *S is right cancellative.*

Proof. Implications (1) ⇒ (2) ⇒ (3) and (4) ⇒ (3) are obvious.

(3) ⇒ (5). Suppose that S is not right cancellative and take $I = S \setminus C_r$. By Lemma 3.21, I is a proper right ideal of S satisfies Condition (LU). By [10, III, Proposition 12.19], $A = S \coprod^I S$ is principally weakly flat. Since S is left PSF, by (2) of Theorem 2.7, $A = S \coprod^I S$ satisfies Condition $(G - PCP)$ and so satisfies property (U). By assumption, $A_S = S \coprod^I S$ satisfies Condition (PWP). Let $i \in I$. Thus equality $(1, x)i = (1, y)i$ implies there exist $u, v \in S$ and $\alpha \in A_S$ such $(1, x) = \alpha u$, $(1, y) = \alpha v$ and $us = vs$. The equality $(1, x) = \alpha u$ implies there exists $l \in S \setminus I$ such that $\alpha = (l, x)$ and so $(1, y) = \alpha v = (l, x)v$, which is a contradiction. Thus S is right cancellative.

(5) ⇒ (1). Since S is right cancellative, S is left PSF. By [9, Lemma 3.13], torsion freeness and $PWKF$ are equivalent. Thus property (U) implies $PWKF$.

$(5) \Rightarrow (4)$. Since S is right cancellative, S is left PSF. By [9, Lemma 3.13], torsion freeness and Condition (P') are equivalent. Thus property (U) implies Condition (P'). □

Notice that in the above theorem Property (U) can be replaced by GP-flat, GPW-flat, principal weakly flat and Condition $(G - PCP)$.

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