Categories and
General Algebraic Structures
with Applications

WWW.CGASA.ir

In press.

Generalization of Continuity

Omid Molaei, Ali Akbar Estaji*, and Mehdi Vatandoost

Abstract. This paper aims to redefine the concept of "continuity of a function at a point" from a set-theoretic perspective, providing a sufficiently flexible definition that encompasses the various forms of continuity found in the mathematical literature. Let $\mathscr{A}$ and $\mathscr{B}$ denote families of subsets of non-empty sets X and Y, respectively. We define an $\mathscr{A}$-$\mathscr{B}$-continuous map and examine some algebraic properties of structures related to the set $C_{(\mathscr{A},\mathscr{B})}(X,Y)$, which consists of all $\mathscr{A}$-$\mathscr{B}$-continuous maps from X to Y. Additionally, we show that $C_{(\mathscr{A},\mathcal{O}Y)}(X,Y) \cong C(X_z,Y)$, where Y is an f-ring and $\mathscr{A}$ is closed under finite intersections, and X_z is a topological space induced by $(X,\mathscr{A})$.

1 Introduction

Continuity is a central concept in topology and its interaction with algebraic structures, particularly rings of continuous functions, has been a rich source of research. Classical continuity is defined in terms of the open-set structure of a topological space. It has been extended in many directions to

* Corresponding author

Keywords: $\mathscr{A}$-open, $\mathscr{A}$-closed, $\mathscr{A}$-regular-open, $\mathscr{A}$-regular-closed, $\mathscr{A}$-$\mathscr{B}$-continuous map, f-ring.
Mathematics Subject Classification [2010]: 54C05, 54C30; Secondary 06F25, 26A46.
Received: 1 January 2026, Accepted: 14 April 2026.
ISSN: Print 2345-5853, Online 2345-5861.

capture finer structural and algebraic properties of function spaces. These generalized notions of continuity often lead to new classes of rings and to representation theorems connecting algebraic and topological structures.

Several variants of continuity have been introduced and studied in the literature, including super-continuity [20], strongly θ-continuity [21], z-supercontinuity [16], F-supercontinuity [17], clopen continuity [22], e-continuity [4], and R_{cl}-supercontinuity [12]. These notions have played a significant role in the study of algebraic and topological structures of rings of continuous functions, particularly in representation theorems for zero-dimensional spaces.

From a set-theoretic viewpoint, many of these notions can be unified by replacing the topology with suitable families of subsets that serve as generalized open sets. This approach allows one to study continuity in a purely algebraic and lattice-theoretic framework, leading to a unified treatment of topological and algebraic properties of function spaces.

The main purpose of this paper is to introduce and study the notion of $\mathscr{A}$-$\mathscr{B}$-continuity, which generalizes several existing concepts of continuity. We investigate the associated algebraic structures, separation axioms, and representation results for rings of $\mathscr{A}$-$\mathscr{B}$-continuous functions, and we establish conditions under which these rings are isomorphic to rings of continuous functions on suitable topological spaces.

The paper is organized as follows. In Section 3, we develop generalized interior and closure operators and study the lattice structure of $\mathscr{A}$-regular open and closed sets. In Section 4, we introduce $\mathscr{A}$-$\mathscr{B}$-continuous mappings and investigate the algebraic properties of the ring $C_{(\mathscr{A},\mathscr{B})}(X,Y)$, as well as generalized separation axioms.

2 Preliminaries

We denote by $\mathbb{R}$ the set of all real numbers endowed with the Euclidean topology. When X is a topological space, $C(X)$ refers to the ring of all real-valued continuous functions, and the set of all open subsets of X is denoted by $\mathcal{O}(X)$.

We recall from [6] that an *f-ring* is a lattice-ordered ring A, which satisfies $|ab| = |a||b|$ for every $a, b \in A$. Also, a lattice-ordered ring A with unit is called

- *strong* if every $a \geq 1$ is invertible in A.
- *bounded* if, for each $a \in A$, there exists an element n in $\mathbb{N}$ such that $|a| \leq n$.
- *Archimedean* if $0 \leq a, b$ and $na \leq b$ for all natural n implies $a = 0$.

Let R be a lattice-ordered ring. We set

$$U(R) := \{r\colon\ r \text{ is a unit element of } R\},$$

and

$$U^+(R) := \{r \in U(R)\colon\ r \text{ is a positive element of } R\}.$$

According to Császár [7], a collection $\mu \subseteq \mathcal{P}(X)$ is a *generalized topology* if $\emptyset \in \mu$ and μ is closed under arbitrary unions. Also, if μ is a generalized topology on X, then the pair (X, μ) is called a generalized topological space (also see [1, 2]). Let (X, μ) and (Y, μ') be two generalized topological spaces. We recall from [7] that a map $f\colon X \to Y$ is said to be (μ, μ')-*continuous* if

$$f^{-1}(V) \in \mu \quad \text{for every } V \in \mu'.$$

3 $\mathscr{G}$-regular open sets and $\mathscr{G}$-regular closed sets

In this section, $\mathscr{G}$-open and $\mathscr{G}$-closed subsets of a non-empty set X are defined via a non-empty collection $\mathscr{G} \subseteq \mathcal{P}(X)$. Also, $\mathscr{G}$-interior and $\mathscr{G}$-closure operators are introduced to characterize $\mathscr{G}$-regular-closed and $\mathscr{G}$-regular-open subsets, denoted as $RC_{(X,\mathscr{G})}(X)$ and $RO(X, \mathscr{G})$. Then, we show that $\big(RO(X, \mathscr{G}), \subseteq\ \big)$ and $\big(RC_{(X,\mathscr{G})}(X), \subseteq\ \big)$ are complete lattices. When $\mathscr{G}$ is closed under finite intersections, these become a complete distributive lattice. In addition, when $\mathscr{G}$ is a cover of X and has an empty intersection, these two lattices become a complete Boolean algebra.

We adopt the convention that the intersection over an empty index family of subsets of X equals X, and the corresponding union equals $\emptyset$.

Throughout this paper, we denote the topology generated by $\mathscr{G} \subseteq \mathcal{P}(X)$ on X with $\mathcal{T}(X, \mathscr{G})$ and denote the family of complements of elements of $\mathscr{D} \subseteq \mathcal{P}(X)$ with $\mathscr{D}^c := \big\{D^c\colon D \in \mathscr{D}\big\}$. Also, we define

$$\mathcal{T}_0(X, \mathscr{G}) := \left\{\bigcup \mathscr{D}\colon \mathscr{D} \subseteq \mathscr{G}\right\}.$$

Therefore, $\mathcal{T}_0(X, \mathscr{G})$ is a generalized topology on X. Clearly, $\emptyset \in \mathcal{T}_0(X, \mathscr{G})$, but X need not belong to $\mathcal{T}_0(X, \mathscr{G})$.

Definition 3.1. For every subset V of X, we put

- $\mathscr{I}_V := \{A \in \mathscr{G} \colon A \subseteq V\}$ and $\mathrm{int}_{(X,\mathscr{G})} V := \bigcup \mathscr{I}_V$.
- $\mathscr{C}_V := \{F \subseteq X \colon X \setminus F \in \mathscr{G}$ and $V \subseteq F\}$ and $\mathrm{cl}_{(X,\mathscr{G})} V := \bigcap \mathscr{C}_V$.

A subset V of X is called $\mathscr{G}$-*open* ($\mathscr{G}$-*closed*) in $(X, \mathscr{G})$ if

$$V = \mathrm{int}_{(X,\mathscr{G})} V \ (V = \mathrm{cl}_{(X,\mathscr{G})} V).$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$, with $\mathscr{G} = \{\{0, 1\}, \{1, 3\}, \{2, 3\}\}$, be given. If $V = \{0, 1, 2, 3\}$, then

$$\begin{aligned} \mathscr{I}_V = \{\{0, 1\}, \{1, 3\}, \{2, 3\}\} &\Rightarrow \bigcup \mathscr{I}_V = \{0, 1, 2, 3\} = V \\ &\Rightarrow \mathrm{int}_{(X,\mathscr{G})} V = V \end{aligned}$$

Therefore, there exists an $\mathscr{G}$-open subset V of X such that $V \notin \mathscr{G}$. A similar argument shows that there exists a pair $(X, \mathscr{G})$ together with an $\mathscr{G}$-closed subset $V \subseteq X$ whose complement $X \setminus V$ fails to belong to $\mathscr{G}$.

Remark 3.3. Let $\mathscr{G} \subseteq \mathcal{P}(X)$ and $X_0 := \bigcup \mathscr{G}$ be given. Clearly, X_0 is the largest $\mathscr{G}$-open set in the generalized topological space $(X, \mathcal{T}_0(\mathscr{G}))$, and $X \setminus X_0$ is the smallest $\mathscr{G}$-closed set in $(X, \mathcal{T}_0(\mathscr{G}))$. Now, if we take the ambient space to be X_0, then $\mathscr{G} \subseteq \mathcal{P}(X_0)$, and for any $A \subseteq X$, we have

- $\mathrm{int}_{(X,\mathscr{G})} A = \mathrm{int}_{(X_0,\mathscr{G})}(A \cap X_0)$,
- $\mathrm{cl}_{(X,\mathscr{G})} A = (X \setminus X_0) \cup \mathrm{cl}_{(X_0,\mathscr{G})}(A \cap X_0)$.

Proposition 3.4. *The following statements are true for a subset V of X:*

(1) $\mathscr{C}_V^c = \mathscr{I}_{X \setminus V}$ *and* $\mathscr{I}_V^c = \mathscr{C}_{X \setminus V}$.
(2) $X \setminus \mathrm{cl}_{(X,\mathscr{G})} V = \mathrm{int}_{(X,\mathscr{G})}(X \setminus V)$.
(3) $X \setminus \mathrm{int}_{(X,\mathscr{G})} V = \mathrm{cl}_{(X,\mathscr{G})}(X \setminus V)$.
(4) $V = \mathrm{int}_{(X,\mathscr{G})} \mathrm{cl}_{(X,\mathscr{G})} V$ *if and only if* $V^c = \mathrm{cl}_{(X,\mathscr{G})} \mathrm{int}_{(X,\mathscr{G})} V^c$.

Proof. It is evident that

$$\mathscr{C}_V^c := \{A\colon A \in \mathscr{G} \text{ and } V \subseteq X \setminus A\} = \mathscr{I}_{X\setminus V} \text{ and}$$
$$\mathscr{I}_V^c := \{X \setminus A\colon A \in \mathscr{G} \text{ and } A \subseteq V\} = \mathscr{C}_{X\setminus V}.$$

Then

$$\begin{aligned} X \setminus \mathrm{cl}_{(X,\mathscr{G})}V &= X \setminus \bigcap \mathscr{C}_V = \bigcup \mathscr{C}_V^c = \bigcup \mathscr{I}_{X\setminus V} \\ &= \mathrm{int}_{(X,\mathscr{G})}(X \setminus V) \end{aligned}$$

and

$$\begin{aligned} X \setminus \mathrm{int}_{(X,\mathscr{G})}V &= X \setminus \bigcup \mathscr{I}_V = \bigcap \mathscr{I}_V^c = \bigcap \mathscr{C}_{X\setminus V} \\ &= \mathrm{cl}_{(X,\mathscr{G})}(X \setminus V). \end{aligned}$$

Also, we get $V = \mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})}V$ if and only if

$$\begin{aligned} \mathrm{cl}_{(X,\mathscr{G})}\mathrm{int}_{(X,\mathscr{G})}V^c &= \mathrm{cl}_{(X,\mathscr{G})}(\mathrm{cl}_{(X,\mathscr{G})}V)^c \\ &= (\mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})}V)^c \\ &= V^c. \end{aligned}$$

□

Let L be a poset. We recall from [13] that

- a *projection operator* (shortly *projection*) is an idempotent, monotone self-map $p\colon L \to L$.
- a *closure operator* is a projection c on L with $1_L \leq c$.
- a *kernel operator* is a projection k on L with $k \leq 1_L$.

Proposition 3.5. *The following statements are true.*

(1) *The map* $\mathrm{cl}_{(X,\mathscr{G})}\big(V \mapsto \mathrm{cl}_{(X,\mathscr{G})}V\big)\colon \mathcal{P}(X) \to \mathcal{P}(X)$ *is a closure operator.*

(2) *The map* $\mathrm{int}_{(X,\mathscr{G})}\big(V \mapsto \mathrm{int}_{(X,\mathscr{G})}V\big)\colon \mathcal{P}(X) \to \mathcal{P}(X)$ *is a a kernel operator.*

(3) *If* $A \cap B \in \mathscr{G}$ *for every* $A, B \in \mathscr{G}$*, then for subsets* V *and* W *of* X*,*

  (a) $\mathrm{cl}_{(X,\mathscr{G})}(V \cup W) = \mathrm{cl}_{(X,\mathscr{G})}V \cup \mathrm{cl}_{(X,\mathscr{G})}W$.

  (b) $\mathrm{int}_{(X,\mathscr{G})}(V \cap W) = \mathrm{int}_{(X,\mathscr{G})}V \cap \mathrm{int}_{(X,\mathscr{G})}W$.

Proof. It is straightforward; we only prove (3)(a). Since, by our hypothesis, $X \setminus (F \cup K) \in \mathscr{G}$ for every $(F, K) \in \mathscr{C}_V \times \mathscr{C}_W$, we conclude that

$$\big\{F \cup K \colon F \in \mathscr{C}_V \text{ and } K \in \mathscr{C}_W\big\} \subseteq \mathscr{C}_{V \cup W},$$

which implies that

$$\begin{aligned}\mathrm{cl}_{(X,\mathscr{G})}(V \cup W) &\subseteq \bigcap \{F \cup K \colon F \in \mathscr{C}_V \text{ and } K \in \mathscr{C}_W\} \\ &= \mathrm{cl}_{(X,\mathscr{G})} V \cup \mathrm{cl}_{(X,\mathscr{G})} W.\end{aligned}$$

Also, by part (1), we have $\mathrm{cl}_{(X,\mathscr{G})} V \cup \mathrm{cl}_{(X,\mathscr{G})} W \subseteq \mathrm{cl}_{(X,\mathscr{G})}(V \cup W)$. Hence,

$$\mathrm{cl}_{(X,\mathscr{G})}(V \cup W) = \mathrm{cl}_{(X,\mathscr{G})} V \cup \mathrm{cl}_{(X,\mathscr{G})} W.$$

□

Example 3.6. Let $X = \{1, 2, 3\}$ with $\mathscr{G} = \big\{\{1\}, \{3\}, \{2, 3\}\big\}$ be given. Then

- $\mathrm{cl}_{(X,\mathscr{G})}(V \cup W) = \mathrm{cl}_{(X,\mathscr{G})} V \cup \mathrm{cl}_{(X,\mathscr{G})} W$ for every $V, W \in \mathcal{P}(X)$.
- $\mathrm{int}_{(X,\mathscr{G})}(V \cap W) = \mathrm{int}_{(X,\mathscr{G})} V \cap \mathrm{int}_{(X,\mathscr{G})} W$ for every $V, W \in \mathcal{P}(X)$.
- $\mathscr{G}$ is not closed under finite intersections.

Therefore, the converse of Proposition 3.5(3) is not true.

Proposition 3.7. *The following statements are equivalent:*

(1) $\mathcal{T}_0(X, \mathscr{G})$ *is closed under finite intersections.*

(2) $G_1 \cap G_2 \in \mathcal{T}_0(X, \mathscr{G})$ *for all* $G_1, G_2 \in \mathscr{G}$.

(3) $\mathscr{G}$ *is a base for a topology on* $\bigcup \mathscr{G}$.

(4) $\mathcal{T}_0(X, \mathscr{G}) = \mathcal{T}(X_0, \mathscr{G})$, *where* $X_0 = \bigcup \mathscr{G}$.

(5) $\mathrm{int}_{(X,\mathscr{G})}(A \cap B) = \mathrm{int}_{(X,\mathscr{G})} A \cap \mathrm{int}_{(X,\mathscr{G})} B$ *for all* $A, B \subseteq X$.

(6) $\mathrm{cl}_{(X,\mathscr{G})}(A \cup B) = \mathrm{cl}_{(X,\mathscr{G})} A \cup \mathrm{cl}_{(X,\mathscr{G})} B$ *for all* $A, B \subseteq X$.

Proof. It is straightforward; we only prove (2) $\Leftrightarrow$ (5). Let $A, B \in \mathcal{P}(X)$ be given. Since

$$\begin{aligned} \operatorname{int}_{(X,\mathscr{G})} A \cap \operatorname{int}_{(X,\mathscr{G})} B &= \bigcup \mathscr{I}_A \cap \bigcup \mathscr{I}_B \\ &= \bigcup \{V \cap W \colon (V, W) \in \mathscr{I}_A \times \mathscr{I}_B\} \\ &\subseteq \bigcup \mathscr{I}_{A \cap B}, \qquad\qquad \text{by assumption (2)} \\ &= \operatorname{int}_{(X,\mathscr{G})}(A \cap B), \end{aligned}$$

we conclude from Proposition 3.5(2) that

$$\operatorname{int}_{(X,\mathscr{G})}(A \cap B) = \operatorname{int}_{(X,\mathscr{G})} A \cap \operatorname{int}_{(X,\mathscr{G})} B.$$

Therefore, (2) implies (5).
Conversely, suppose $G_1, G_2 \in \mathscr{G}$. By assumption (5),

$$\begin{aligned} G_1 \cap G_2 &= \operatorname{int}_{(X,\mathscr{G})} G_1 \cap \operatorname{int}_{(X,\mathscr{G})} G_2 \\ &= \operatorname{int}_{(X,\mathscr{G})}(G_1 \cap G_2) \in \mathcal{T}_0(X, \mathscr{G}). \end{aligned}$$

Therefore, (5) implies (2). □

Proposition 3.8. *The following statements are true for subsets V and W of X.*

(1) $\operatorname{int}_{(X,\mathscr{G})}\operatorname{cl}_{(X,\mathscr{G})}\operatorname{int}_{(X,\mathscr{G})}\operatorname{cl}_{(X,\mathscr{G})}V = \operatorname{int}_{(X,\mathscr{G})}\operatorname{cl}_{(X,\mathscr{G})}V$.

(2) $\operatorname{cl}_{(X,\mathscr{G})}\operatorname{int}_{(X,\mathscr{G})}\operatorname{cl}_{(X,\mathscr{G})}\operatorname{int}_{(X,\mathscr{G})}V = \operatorname{cl}_{(X,\mathscr{G})}\operatorname{int}_{(X,\mathscr{G})}V$.

(3) *If $A \cap B \in \mathscr{G}$ for every $A, B \in \mathscr{G}$, and V is $\mathscr{G}$-open, then*

- $\operatorname{cl}_{(X,\mathscr{G})}(V \cap W) = \operatorname{cl}_{(X,\mathscr{G})}\big(V \cap \operatorname{cl}_{(X,\mathscr{G})}W\big)$.
- $\operatorname{int}_{(X,\mathscr{G})}\operatorname{cl}_{(X,\mathscr{G})}(V \cap W) = \operatorname{int}_{(X,\mathscr{G})}\operatorname{cl}_{(X,\mathscr{G})}V \cap \operatorname{int}_{(X,\mathscr{G})}\operatorname{cl}_{(X,\mathscr{G})}W$.

(4) *If $A \cap B \in \mathscr{G}$ for every $A, B \in \mathscr{G}$, and V is $\mathscr{G}$-closed, then*

- $\operatorname{int}_{(X,\mathscr{G})}(V \cup W) = \operatorname{int}_{(X,\mathscr{G})}\big(V \cup \operatorname{int}_{(X,\mathscr{G})}W\big)$.
- $\operatorname{cl}_{(X,\mathscr{G})}\operatorname{int}_{(X,\mathscr{G})}(V \cup W) = \operatorname{cl}_{(X,\mathscr{G})}\operatorname{int}_{(X,\mathscr{G})}V \cup \operatorname{cl}_{(X,\mathscr{G})}\operatorname{int}_{(X,\mathscr{G})}W$.

Proof. It is straightforward; we only prove (3). By Proposition 3.5, it follows that

$$\mathrm{cl}_{(X,\mathscr{G})}(V\cap W)\subseteq \mathrm{cl}_{(X,\mathscr{G})}(V\cap \mathrm{cl}_{(X,\mathscr{G})}W).$$

By our hypothesis, $V=\bigcup \mathscr{I}_V$, then for every $F\in\mathscr{C}_{V\cap W}$, we have

$$\begin{aligned}
V\cap W\subseteq F &\Rightarrow \forall O\in\mathscr{I}_V,\ W\subseteq F\cup(X\setminus V)\subseteq F\cup(X\setminus O)\in\mathscr{G}^c\\
&\Rightarrow \forall O\in\mathscr{I}_V,\ \mathrm{cl}_{(X,\mathscr{G})}W\subseteq F\cup(X\setminus O)\\
&\Rightarrow \mathrm{cl}_{(X,\mathscr{G})}W\subseteq F\cup\bigcap_{O\in\mathscr{I}_V}(X\setminus O)=F\cup(X\setminus V)\\
&\Rightarrow V\cap \mathrm{cl}_{(X,\mathscr{G})}W\subseteq F\\
&\Rightarrow F\in\mathscr{C}_{V\cap \mathrm{cl}_{(X,\mathscr{G})}W},
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{cl}_{(X,\mathscr{G})}(V\cap \mathrm{cl}_{(X,\mathscr{G})}W)&=\bigcap\mathscr{C}_{V\cap \mathrm{cl}_{(X,\mathscr{G})}W}\\
&\subseteq\bigcap\mathscr{C}_{V\cap W}\\
&=\mathrm{cl}_{(X,\mathscr{G})}(V\cap W).
\end{aligned}$$

Therefore, $\mathrm{cl}_{(X,\mathscr{G})}(V\cap W)=\mathrm{cl}_{(X,\mathscr{G})}(V\cap \mathrm{cl}_{(X,\mathscr{G})}W)$.

For convenience, we use the notation. $\overline{A}$ for $\mathrm{cl}_{(X,\mathscr{G})}A$ and A° for $\mathrm{int}_{(X,\mathscr{G})}A$ for every subset $A\subseteq X$. As a consequence Proposition 3.5, it follows that $(\overline{V\cap W})^\circ\subseteq(\overline{V})^\circ\cap(\overline{W})^\circ$ and

$$\begin{aligned}
(\overline{V\cap W})^\circ&=(\overline{(\overline{V\cap W})^\circ})^\circ=\left(\overline{(\overline{(V\cap\overline{W})})^\circ}\right)^\circ\\
&\supseteq\left(\overline{(V\cap\overline{W})^\circ}\right)^\circ=\left(\overline{(V\cap(\overline{W})^\circ)}\right)^\circ\\
&=\left(\overline{(\overline{V}\cap(\overline{W})^\circ)}\right)^\circ\supseteq\left(\overline{V}\cap(\overline{W})^\circ\right)^\circ\\
&=(\overline{V})^\circ\cap(\overline{W})^\circ.
\end{aligned}$$

Therefore, $(\overline{V\cap W})^\circ=(\overline{V})^\circ\cap(\overline{W})^\circ$. □

Definition 3.9. A subset V of X is called an *$\mathscr{G}$-regular-closed* subset of X if

$$V=\mathrm{cl}_{(X,\mathscr{G})}\mathrm{int}_{(X,\mathscr{G})}V.$$

Also, a subset V of X is called *$\mathscr{G}$-regular-open* if

$$V = \mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})}V.$$

The set of all $\mathscr{G}$-regular-closed subsets of X is denoted by $RC(X, \mathscr{G})$, and the set of all $\mathscr{G}$-regular-open subsets of X is denoted by $RO(X, \mathscr{G})$.

Proposition 3.10. *The pair $\big(RO(X, \mathscr{G}), \subseteq\big)$ is a complete lattice.*

Proof. Let $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq RO(X, \mathscr{G})$ be given. Let $B \in RO(X, \mathscr{G})$ with $A_\lambda \subseteq B$ for every $\lambda \in \Lambda$ be given. Then $\bigcup_{\lambda\in\Lambda} A_\lambda \subseteq B$ implies

$$\mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})} \bigcup_{\lambda\in\Lambda} A_\lambda \subseteq \mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})}B = B,$$

and since $A_\lambda \subseteq \mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})} \bigcup_{\lambda\in\Lambda} A_\lambda$ for every $\lambda \in \Lambda$, we conclude that

$$\mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})} \bigcup_{\lambda\in\Lambda} A_\lambda = \bigvee_{\lambda\in\Lambda} A_\lambda \in RO(X, \mathscr{G}).$$

A similar argument shows that

$$\mathrm{int}_{(X,\mathscr{G})}\mathrm{cl}_{(X,\mathscr{G})} \bigcap_{\lambda\in\Lambda} A_\lambda = \bigwedge_{\lambda\in\Lambda} A_\lambda \in RO(X, \mathscr{G}).$$

Hence, the pair $\big(RO(X, \mathscr{G}), \subseteq\big)$ is a complete lattice. □

Proposition 3.11. *Let $\mathscr{G}$ be closed under finite intersections. Then, the following statements are true:*

(1) *The pair $\big(RO(X, \mathscr{G}), \subseteq\big)$ is a complete distributive lattice.*

(2) *If $\bigvee \mathscr{G} = X$ and $\bigwedge \mathscr{G} = \emptyset$, then $\big(RO(X, \mathscr{G}), \subseteq\big)$ is a complete Boolean algebra and the complement of A is equal to $\mathrm{int}_{(X,\mathscr{G})}A^c$ for every $A \in RO(X, \mathscr{G})$.*

Proof. To facilitate writing, we put $\mathrm{cl}_{(X,\mathscr{G})}A = \overline{A}$ and $\mathrm{int}_{(X,\mathscr{G})}A = A^\circ$ for every $A \subseteq X$.

(1) Let $\{A_1, A_2, A_3\} \subseteq RO(X, \mathscr{G})$ be given. Then, by Proposition 3.8, we get

$$\begin{aligned}
(A_1 \wedge A_2) \vee (A_1 \wedge A_3) &= \overline{(\overline{A_1 \cap A_2}^{\circ}) \cup (\overline{A_1 \cap A_3}^{\circ})}^{\circ} \\
&= \overline{(\overline{A_1}^{\circ} \cap \overline{A_2}^{\circ}) \cup (\overline{A_1}^{\circ} \cap \overline{A_3}^{\circ})}^{\circ} \\
&= \overline{(A_1 \cap A_2) \cup (A_1 \cap A_3)}^{\circ} \\
&= \overline{A_1 \cap (A_2 \cup A_3)}^{\circ} \\
&= \overline{A_1 \cap \overline{(A_2 \cup A_3}^{\circ})}^{\circ} \\
&= A_1 \wedge (A_2 \vee A_3)
\end{aligned}$$

Therefore, by Proposition 3.10, $\big(RO(X, \mathscr{G}), \subseteq\big)$ is a complete distributive lattice.

(2) Let $A \in RO(X, \mathscr{G})$ be given. Then, by Proposition 3.4,

$$\Big(\overline{(A^c)^{\circ}}\Big)^{\circ} = \big(((\overline{A})^{\circ})^c\big)^{\circ} = (A^c)^{\circ},$$

which implies that $(A^c)^{\circ} \in RO(X, \mathscr{G})$. Since $A \wedge (A^c)^{\circ} = \emptyset$ and

$$\begin{aligned}
A \vee (A^c)^{\circ} &= \overline{(A \cup (A^c)^{\circ})}^{\circ} = (\overline{A} \cup \overline{(A^c)^{\circ}})^{\circ} \\
&= \big(\overline{A} \cup \big((\overline{A})^{\circ}\big)^c\big)^{\circ} = (\overline{A} \cup A^c)^{\circ} \\
&= X,
\end{aligned}$$

we infer that the complement of A is equal to $\mathrm{int}_{(X,\mathscr{G})} A^c$. Hence, by part (1), $\big(RO(X, \mathscr{G}), \subseteq\big)$ is a complete Boolean algebra. □

Example 3.12. Let $X := \{1, 2, 3, 4, 5, 6\}$, and consider

$$\mathscr{G} := \{\emptyset, A, B, C, A^c, B^c, C^c, X\},$$

where $A := \{1, 2, 4\}$, $B := \{3, 4, 5\}$, and $C := \{1, 3, 6\}$. Then

- $\mathcal{RO}(X, \mathscr{G}) = \{\emptyset, A, B, C, A^c, B^c, C^c, X\}$, and
- in the lattice $\big(RO(X, \mathscr{G}), \subseteq\big)$, we observe that

$$A \wedge (B \vee C) = A \neq \emptyset = (A \wedge B) \vee (A \wedge C).$$

Hence, the pair $\big(RO(X,\mathscr{G}),\subseteq\ \big)$ is not a complete distributive lattice. Therefore, the condition "$\mathscr{G}$ is closed under finite intersections" is necessary in the Proposition 3.11.

Remark 3.13. Using a similar argument, we obtain

- the pair $\big(RC(X,\mathscr{G}),\subseteq\big)$ is a complete lattice.
- if $\mathscr{G}$ is closed under finite intersections, then the pair $\big(RC(X,\mathscr{G}),\subseteq\big)$ is a complete distributive lattice.
- if $\mathscr{G}$ is closed under finite intersections, $\bigvee\mathscr{G} = X$ and $\bigwedge\mathscr{G} = \emptyset$, then $\big(RC(X,\mathscr{G}),\subseteq\big)$ is a complete Boolean algebra, and the complement of A is equal to $\mathrm{cl}_{(X,\mathscr{G})}A^c$ for every $A \in RC(X,\mathscr{G})$.

4 $\mathscr{A}$-$\mathscr{B}$-continuous maps

In this section, we first introduce $\mathscr{A}$-$\mathscr{B}$-continuous mappings, and then we show that if $\mathscr{A}$ is closed under finite intersections, then there exists a topological space X_z such that

$$C_{\mathscr{A}}(X,Y) \cong C(X_z,Y).$$

In this section, X and Y denote non-empty sets, and $\mathscr{A}$ and $\mathscr{B}$ denote non-empty families of subsets of X and Y, respectively.

Definition 4.1. A function f from X into Y is called *$\mathscr{A}$-$\mathscr{B}$-continuous map* in $x \in X$ if for every $V \in \mathscr{B}$ with $f(x) \in V$, there exists an element U in $\mathscr{A}$ containing x such that $f(U) \subseteq V$. We say f is a $\mathscr{A}$-$\mathscr{B}$-continuous from X into Y if and only if f is $\mathscr{A}$-$\mathscr{B}$-continuous at each $x \in X$. We denote the set of all $\mathscr{A}$-$\mathscr{B}$-continuous maps from X into Y by $C_{(\mathscr{A},\mathscr{B})}(X,Y)$.

Example 4.2. Suppose that G is an abelian group. If $\mathscr{A}$ denotes the set of cyclic subgroups of G and $\mathscr{B}$ denotes the set of subgroups of G, then each group homomorphism f from G to G is an $\mathscr{A}$-$\mathscr{B}$-continuous map. The endomorphism ring of G is a subset of $C_{(\mathscr{A},\mathscr{B})}(G,G)$.

Example 4.3. The set

$$\mathcal{M}(X,\mathcal{A}) := \big\{f \in \mathbb{R}^X : \forall G \in \mathcal{O}(\mathbb{R}), f^{-1}(G) \in \mathcal{A}\big\}$$

is a subset of $C_{(\mathscr{A},\mathcal{O}(\mathbb{R}))}(X,\mathbb{R})$ (see [5]).

Proposition 4.4. *The following conditions are equivalent for a function f from X into Y:*

1. $f \in C_{(\mathscr{A},\mathscr{B})}(X,Y)$.
2. *There exists a subcollection $\mathscr{C}$ of $\mathscr{A}$ such that $f^{-1}(V) = \bigcup \mathscr{C}$ for every $V \in \mathscr{B}$.*
3. $f \in C_{(\mathcal{T}_0(\mathscr{A}),\mathscr{B})}(X,Y)$.
4. $f \in C_{(\mathcal{T}_0(\mathscr{A}),\mathcal{T}_0(\mathscr{B}))}(X,Y)$.
5. $f(\mathrm{cl}_{(X,\mathscr{A})}V) \subseteq \mathrm{cl}_{(Y,\mathscr{B})}f(V)$ *for subset V of X.*
6. $\mathrm{cl}_{(X,\mathscr{A})}f^{-1}(U) \subseteq f^{-1}(\mathrm{cl}_{(Y,\mathscr{B})}U)$ *for subset U of Y.*
7. $f^{-1}\left(\mathrm{int}_{(Y,\mathscr{B})}U\right) \subseteq \mathrm{int}_{(X,\mathscr{A})}f^{-1}(U)$ *for subset U of Y.*

Proof. The equivalence of statements (1), (2), (3), and (4) follows directly. Let us verify the remaining implications.

(4) $\Rightarrow$ (5). Let $G \in \mathscr{B}$ with $f(V) \subseteq Y \setminus G$ be given. By part (4), there exists a subcollection $\mathscr{C}$ of $\mathscr{A}$ such that $f^{-1}(G) = \bigcup \mathscr{C}$, which implies that

$$V \subseteq f^{-1}(Y \setminus G) = X \setminus f^{-1}(G) = \bigcap_{A \in \mathscr{C}} (X \setminus A),$$

this entails that $\mathrm{cl}_{(X,\mathscr{A})}V \subseteq f^{-1}(Y \setminus G)$, and so, $f(\mathrm{cl}_{(X,\mathscr{A})}V) \subseteq Y \setminus G$. Therefore,

$$f(\mathrm{cl}_{(X,\mathscr{A})}V) \subseteq \mathrm{cl}_{(Y,\mathscr{B})}f(V).$$

(5) $\Rightarrow$ (6). Let U be a subset of Y. Then, by part (5),

$$f(\mathrm{cl}_{(X,\mathscr{A})}f^{-1}(U)) \subseteq \mathrm{cl}_{(Y,\mathscr{B})}f(f^{-1}(U)) \subseteq \mathrm{cl}_{(Y,\mathscr{B})}U,$$

which gives $\mathrm{cl}_{(X,\mathscr{A})}f^{-1}(U) \subseteq f^{-1}(\mathrm{cl}_{(Y,\mathscr{B})}U)$.

(6) $\Rightarrow$ (7). Let U be a subset of Y. Then, by Part (6),

$$\begin{aligned} f^{-1}\left(\mathrm{int}_{(Y,\mathscr{B})}U\right) &= f^{-1}\left(Y \setminus \left(Y \setminus \mathrm{int}_{(Y,\mathscr{B})}U\right)\right) \\ &= X \setminus f^{-1}\left(Y \setminus \mathrm{int}_{(Y,\mathscr{B})}U\right) \\ &= X \setminus f^{-1}\left(\mathrm{cl}_{(Y,\mathscr{B})}(Y \setminus U)\right) \\ &\subseteq X \setminus \mathrm{cl}_{(X,\mathscr{A})}f^{-1}\left(Y \setminus U\right) \\ &= X \setminus \mathrm{cl}_{(X,\mathscr{A})}\left(X \setminus f^{-1}(U)\right) \\ &= \mathrm{int}_{(X,\mathscr{A})}f^{-1}(U). \end{aligned}$$

(7) $\Rightarrow$ (1). Let $p \in X$ with $f(p) \in U \in \mathscr{B}$ be given. Then, by part (7),

$$\begin{aligned} f^{-1}(U) &= f^{-1}\left(\mathrm{int}_{(Y,\mathscr{B})}U\right) \subseteq \mathrm{int}_{(X,\mathscr{A})} f^{-1}(U) \\ &\subseteq f^{-1}(U), \end{aligned}$$

which gives

$$\begin{aligned} f^{-1}(U) &= \mathrm{int}_{(X,\mathscr{A})} f^{-1}(U) \\ &= \bigcup \{A \in \mathscr{A} \colon A \subseteq f^{-1}(U)\}, \end{aligned}$$

and so, there exists an element A in $\mathscr{A}$ such that $p \in A$ and $f(A) \subseteq U$. Hence, $f \in C_{(\mathscr{A},\mathscr{B})}(X,Y)$. □

In this paper, we denote $C_{(\mathscr{A},\mathcal{O}(\mathbb{R}))}(X,\mathbb{R})$ by $C_{\mathscr{A}}(X)$.

Remark 4.5. It is evident that

- if $\mathscr{B} \subseteq \mathscr{F} \subseteq \mathcal{P}(Y)$, then $C_{(\mathscr{A},\mathscr{F})}(X,Y) \subseteq C_{(\mathscr{A},\mathscr{B})}(X,Y)$.
- if $\mathscr{A} \subseteq \mathscr{E} \subseteq \mathcal{P}(X)$, then $C_{(\mathscr{A},\mathscr{B})}(X,Y) \subseteq C_{(\mathscr{E},\mathscr{B})}(X,Y)$.

Suppose that $\mathscr{A}$ is closed under finite intersections. If X is endowed with the topology $\mathcal{T}(X,\mathscr{A})$, then the following statements are true:

- $C_{\mathscr{A}}(X) = C(X)$.
- The set $Z_{\mathscr{A}}(X) := \{\mathrm{Z}(f) \colon f \in C_{\mathscr{A}}(X)\}$ is closed under countable intersection.

In what follows, the term *f-ring* refers to a commutative f-ring with identity, and Y is assumed to be an Archimedean, strong, and bounded f-ring.

For each $(n,a) \in \mathbb{N} \times Y$, define

$$V_n(a) := \{x \in Y : |x - a| < 1/n\}.$$

We recall from [6] that the collection $\mathcal{B}_0 = \{V_n(a)\}_{(n,a)\in\mathbb{N}\times Y}$ forms a base for a *uniform topology* on Y. Henceforth, Y will denote an f-ring endowed with this uniform topology.

Remark 4.6. I would like to introduce this base in a more effective way and further analyze the resulting topology. For each $u \in U^+(Y)$ and each $a \in Y$, define

$$N_u(a) = \{x \in Y : |x - a| < u\}, \qquad \mathcal{B} = \{N_u(a) : u \in U^+(Y)\}.$$

It is clear that $\mathcal{B}_0 \subseteq \mathcal{B}$. We claim that

$$\mathcal{T}(\mathcal{B}_0) = \mathcal{T}(\mathcal{B}).$$

To prove this, it suffices to show that $\mathcal{B} \subseteq \mathcal{T}(\mathcal{B}_0)$. Let $u \in U^+(Y)$ be fixed, and suppose $b \in N_u(a)$. Then $u - |a - b| > 0$. Since Y is Archimedean, there exists $n \in \mathbb{N}$ such that $n(u - |a - b|) > 1$. Because Y is strong, we have $n \in U(Y)$, and therefore

$$\frac{1}{n} < u - |a - b|.$$

To complete the proof of the claim, we show that $N_{1/n}(b) \subseteq N_u(a)$. Indeed, if $x \in N_{1/n}(b)$, then

$$|x - a| \leq |x - b| + |a - b| < \frac{1}{n} + |a - b| < u - |a - b| + |a - b| = u,$$

which implies $x \in N_u(a)$. Thus, $\mathcal{T}(\mathcal{B}_0) = \mathcal{T}(\mathcal{B})$.

Proposition 4.7. *The following statements hold:*

(1) $U^+(Y)$ *is closed under addition.*

(2) $a + N_u(b) = N_u(a + b)$ *for all* $a, b \in Y$ *and all* $u \in U^+(Y)$.

(3) $N_u(a) + N_v(b) = N_{u+v}(a + b)$ *for all* $a, b \in Y$ *and all* $u, v \in U^+(Y)$.

(4) *Addition is continuous; i.e., as a map* $Y \times Y \to Y$, *it is continuous.*

(5) *Multiplication is continuous.*

(6) Y *is a topological ring.*

Proof. (1) Let $u, v \in U^+(Y)$. Then $1 + u^{-1}v \geq 1$, so $1 + u^{-1}v$ is invertible. Hence $u + v = u(1 + u^{-1}v)$ is invertible, and thus $u + v \in U^+(Y)$.

(2) Since

$$\begin{aligned} y \in a + N_u(b) &\Leftrightarrow \exists c \in N_u(b),\ y = a + c \\ &\Leftrightarrow |y - (a+b)| < u \\ &\Leftrightarrow y \in N_u(a+b), \end{aligned}$$

we conclude that $a + N_u(b) = N_u(a+b)$.

(3) First, we show $N_u(0) + N_v(0) = N_{u+v}(0)$. Let $a \in N_u(0)$ and $b \in N_v(0)$. Then clearly $|a+b| \le |a| + |b| < u + v$, i.e., $a + b \in N_{u+v}(0)$. Thus $N_u(0) + N_v(0) \subseteq N_{u+v}(0)$. Conversely, let $c \in N_{u+v}(0)$. Clearly,

$$\left|\frac{uc}{u+v}\right| < u, \qquad \left|\frac{vc}{u+v}\right| < v.$$

Hence $x = \frac{uc}{u+v} \in N_u(0)$ and $y = \frac{vc}{u+v} \in N_v(0)$, so

$$c = x + y \in N_u(0) + N_v(0).$$

Thus $N_{u+v}(0) \subseteq N_u(0) + N_v(0)$. Consequently, using part (2), for $a, b \in Y$ and $u, v \in U^+(Y)$,

$$N_u(a) + N_v(b) = a + N_u(0) + b + N_v(0) = a + b + N_{u+v}(0) = N_{u+v}(a+b).$$

(4) Let $\varphi : Y \times Y \to Y$ denote addition, $(a,b) \in Y \times Y$, and let $N_\varepsilon(\varphi(a,b)) = N_\varepsilon(a+b)$ be a basic open set containing $a+b$. Setting $r = \varepsilon/2$, it is easy to see that

$$\varphi(N_r(a) \times N_r(b)) = N_r(a) + N_r(b) \subseteq N_\varepsilon(a+b).$$

(5) Let $\phi : Y \times Y \to Y$ denote multiplication, $(a,b) \in Y \times Y$, and let $N_\varepsilon(\phi(a,b)) = N_\varepsilon(ab)$ be a basic open set containing ab. Let $r = \min(\varepsilon/3, 1)$. Then there exist $u, v \in U^+(Y)$ such that for all $(x,y) \in N_u(a) \times N_v(b)$,

$$|b - y|,\ |a - x|,\ |a||b - y|,\ |b||a - x| < r.$$

Hence for all $(x,y) \in N_u(a) \times N_v(b)$,

$$\begin{aligned} |xy - ab| &= |(x-a)(y-b) + a(y-b) + b(x-a)| \\ &\le |x-a||y-b| + |a||y-b| + |b||x-a| \\ &< 3r = \varepsilon. \end{aligned}$$

Thus $\phi(N_u(a) \times N_v(b)) = N_u(a)N_v(b) \subseteq N_\varepsilon(ab)$.

(6) This follows immediately from (4) and (5). □

The collection Y^X of all functions from X into Y, equipped with pointwise addition and multiplication,

$$(f+g)(x) := f(x)+g(x), \qquad (fg)(x) := f(x)g(x),$$

forms a commutative ring with identity.

For each $f \in Y^X$, define the zero set and cozero set by

$$Z(f) := \{x \in X : f(x) = 0\}, \qquad \mathrm{coz}(f) := X \setminus Z(f).$$

Proposition 4.8. *The following statements are true for $f, g \in Y^X$:*

1. $\mathrm{Z}(f) \subseteq \mathrm{Z}(f^n)$ *for every* $n \in \mathbb{N}$.
2. *If Y is a semi-prime ring (i.e., $y^2 = 0$ implies $y = 0$ for every $y \in Y$), then* $\mathrm{Z}(f) = \mathrm{Z}(f^n)$ *for every* $n \in \mathbb{N}$.
3. $\mathrm{Z}(f) \cup \mathrm{Z}(g) \subseteq \mathrm{Z}(fg)$.
4. *If Y is an integral domain, then* $\mathrm{Z}(f) \cup \mathrm{Z}(g) = \mathrm{Z}(fg)$.
5. $\mathrm{Z}(f) \cap \mathrm{Z}(g) \subseteq \mathrm{Z}(f+g)$.
6. *If $f(x) \geq 0$ and $g(x) \geq 0$ for every $x \in X$, then* $\mathrm{Z}(f) \cap \mathrm{Z}(g) = \mathrm{Z}(f+g)$.

Proof. The proof is quite straightforward. □

Proposition 4.9. *If $\mathscr{A}$ is closed under finite intersections, then $C_{(\mathscr{A},\mathcal{O}(Y))}(X,Y)$ is a subring Y^X with unity element that contains all constant functions from X into Y.*

Proof. It is known that the intersection of the empty family of subsets of X equals X. Therefore, $X \in \mathscr{A}$ and for every $y_0 \in Y$, the constant function

$$f(x \mapsto y_0) \colon X \to Y$$

belongs to $C_{(\mathscr{A},\mathcal{O}(Y))}(X,Y)$. Hence, $C_{(\mathscr{A},\mathcal{O}(Y))}(X,Y)$ contains all constant functions from X into Y.

Suppose that $f, g \in C_{(\mathscr{A},\mathcal{O}(Y))}(X,Y)$. Let $a \in X$ and $n \in \mathbb{N}$ be given. Then, there are $A_{2n}, B_{2n} \in \mathscr{A}$ such that $a \in A_{2n} \cap B_{2n}$, $f(A_{2n}) \subseteq V_{2n}(f(a))$ and $g(B_{2n}) \subseteq V_{2n}(g(a))$. Since, by our hypothesis, $A_{2n} \cap B_{2n} \in \mathscr{A}$ and

$$\begin{aligned} |(f+g)(x)-(f+g)(a)| &\leq |f(x)-f(a)|+|g(x)-g(a)| \\ &< \frac{1}{2n}+\frac{1}{2n} = \frac{1}{n} \end{aligned}$$

for every $x \in A_{2n} \cap B_{2n}$, we conclude that $f + g \in C_{(\mathscr{A},\mathcal{O}(Y))}(X, Y)$.

Now, we show that $fg \in C_{(\mathscr{A},\mathcal{O}(Y))}(X, Y)$. Let $a \in X$ and $n \in \mathbb{N}$ be given. Since Y is a bounded f-ring, there exists an element $m \in \mathbb{N}$ such that

$$||g(a)| + 1 + |f(a)|| < m.$$

Then, there are $A_{mn}, B_{mn}, D_n \in \mathscr{A}$ such that $a \in A_{mn} \cap B_{mn} \cap D_n$,

$$f(A_{mn}) \subseteq V_{mn}\left(f(a)\right),\ g(B_{mn}) \subseteq V_{mn}\left(g(a)\right)$$

and $g(D_n) \subseteq V_n\left(g(a)\right)$. Since, by our hypothesis, $A_{mn} \cap B_{mn} \cap D_n \in \mathscr{A}$ and

$$\begin{aligned} |fg(x) - fg(a)| &= |f(x)g(x) - f(a)g(x) + \\ &\qquad f(a)g(x) - f(a)g(a)| \\ &\leq |g(x)||f(x) - f(a)| + \\ &\qquad |f(a)||g(x) - g(a)| \\ &< (|g(a)| + \frac{1}{n})\frac{1}{mn} + |f(a)|\frac{1}{mn} \\ &\leq \big((|g(a)| + 1) + |f(a)|\big)\frac{1}{mn} \\ &\leq ||g(a)| + 1 + |f(a)||\frac{1}{mn} \\ &\leq \frac{1}{n} \end{aligned}$$

for every $x \in A_{mn} \cap B_{mn} \cap D_n$, we conclude that $fg \in C_{(\mathscr{A},\mathcal{O}(Y))}(X, Y)$. □

Proposition 4.9 is also true when Y is a metric ring (ring with absolute value).

Throughout this paper, we denote $C_{(\mathscr{A},\mathcal{O}(Y))}(X, Y)$ by $C_{\mathscr{A}}(X, Y)$.

Remark 4.10. Suppose that $\mathscr{A}$ is closed under finite intersections. If X is endowed with the topology $\mathcal{T}(X, \mathscr{A})$, then $C(X, Y)$ is a subring Y^X with unity element.

Below, we present examples of the application of the previous proposition, as employed by researchers in different ways.

Example 4.11. Let (X, τ) be a topological space. If $\mathscr{A}$ is one of the following sets, then $C_{\mathscr{A}}(X, Y)$ is a subring Y^X with a unity element.

(1) $\{\mathrm{int}_X \mathrm{cl}_X G \colon G \subseteq X\}$ (see [20]).

(2) $\{G \in \tau \colon \mathrm{cl}_X G \in \tau\}$ (see [4]).

(3) $\{G \subseteq X \colon G$ is a clopen subset of $X\}$ (see [22]).

(4) $\{G \subseteq X \colon G$ is a r_{cl}-open set of $\}$ (see [12]).

(5) $\Big\{H \subseteq X \colon \exists \{G_n\}_{n\in\mathbb{N}} \subseteq \tau, H = \bigcap_{n\in\mathbb{N}} \mathrm{int}_X(X \setminus G_n)\Big\}$ (see [17, 19]).

(6) $\{U \in \tau \colon \forall x \in U \exists f \in C(X)\big(x \in \mathrm{Z}(f) \subseteq U\big)\}$ (see [17, 19]).

(7) $\{U \in \tau \colon \forall x \in U \exists F\big(X \setminus F \in \tau,\ x \in F \subseteq U\big)\}$ (see [18]).

Example 4.12. If $\mathscr{A}$ is a filter on set $(P(X), \subseteq)$, then $C_{\mathscr{A}}(X, Y)$ is a subring Y^X with a unity element.

Example 4.13. If $(X, \mathscr{A})$ is a measurable space, then $C_{\mathscr{A}}(X, Y)$ is a subring Y^X with unity element (see [9–11, 15, 23]).

Proposition 4.14. *The following statements are true.*

(1) *The function $f : Y \to Y$ given by $f(x) = |x|$ is continuous.*

(2) *For every $a \in Y$, $\uparrow a$ is closed.*

(3) *$a \in U(Y)$ if and only if $|a| \in U(Y)$.*

Proof. (1) Since $||x| - |a|| \leq |x - a|$ for every $a \in R$, this is immediate.

(2) Since addition is continuous, it suffices to show that $\uparrow 0$ is closed. Suppose $\{x_n\}$ is a sequence in $\uparrow 0$ converging to $a \in Y$. Because the absolute value is continuous and the limit in this space is unique,

$$\begin{aligned} x_n \to a \ \Rightarrow \ a &= \lim_{n\to\infty} x_n = \lim_{n\to\infty} |x_n| = |a| \\ \Rightarrow \ a &= |a| \in \uparrow 0. \end{aligned}$$

(3) Since Y is an f-ring, it is a sublattice of the direct product of completely ordered rings. Therefore, each element $a \in Y$ can be regarded as $a = (a_i)$, where $a_i \in R_i$ and R_i is a totally ordered ring for every $i \in I$. Hence $|a|$ is invertible if and only if $|a_i|$ is invertible in every factor; equivalently, a is invertible. $\square$

Lemma 4.15. *The map $\phi\colon Y \to Y$ is defined by*

$$\phi(x) = \begin{cases} x^{-1} & \text{if } |x| \geq 1 \\ x & \text{if } |x| \leq 1 \end{cases}$$

is continuous.

Proof. We set

- $C_1 := \{x \in Y : |x| \leq 1\} = {\downarrow}1 \cap {\uparrow} -1$, and
- $C_2 := \{x \in Y : |x| \geq 1\} = {\uparrow}1 \cup {\downarrow} -1$.

Then, C_1 and C_2 are closed. First, consider the restriction of ϕ to C_1. For all $x \in C_1$, we have $\phi(x) = x$. This is the identity mapping, which is trivially continuous with respect to the topology on Y.

Second, consider the restriction of ϕ to C_2. Let $a \in C_2$ and fix $n \in \mathbb{N}$. If $x \in V_n(a) \cap C_2$, then $|x| \geq 1$ and $|a| \geq 1$. Hence $|x^{-1}| \leq 1$ and $|a^{-1}| \leq 1$. Moreover,

$$\begin{aligned} |\phi(x) - \phi(a)| &= |x^{-1} - a^{-1}| \\ &= |x^{-1}|\ |a - x|\ |a^{-1}| \\ &\leq |x - a| \\ &< \frac{1}{n}. \end{aligned}$$

Therefore, ϕ is continuous at a. By the Gluing Lemma, it follows that ϕ is continuous. □

We recall from [8] that a commutative ring is called a Gelfand ring if for every $m \in R$, there exists $a, b \in R$ such that $(1 - am)(1 - bm_0) = 0$, where $m_0 = 1 - m$.

Proposition 4.16. *If $C_{\mathscr{A}}(X, Y)$ is a subring with unit of Y^X, then it is a Gelfand ring.*

Proof. Let $f \in C_{\mathscr{A}}(X, Y)$. We must find $g, h \in C_{\mathscr{A}}(X, Y)$ such that $(1 - gf)(1 - hf_0) = 0$, where $f_0 = 1 - f$. Define $\phi(x) = \frac{1}{x}$ for $|x| > 1$, and $\phi(x) = x$ otherwise. Now, we put $g\colon = \phi \circ f$ and $h\colon = \phi \circ f_0$. Then, by Lemma 4.15, $g, h \in C_{\mathscr{A}}(X, Y)$. By direct calculation, we can demonstrate that $(1 - gf)(1 - hf_0) = 0$, which implies that $C_{\mathscr{A}}(X, Y)$ is a Gelfand ring. □

Definition 4.17. A set X together with $\mathscr{A}$ is called

- a $T_0^{\mathscr{A}}$*-set* if and only if whenever $x, y \in X$ with $x \neq y$, there exist an element G in $\mathscr{A}$ such that ($x \in G$ and $y \notin G$) or ($y \in G$ and $x \notin G$).
- a $T_1^{\mathscr{A}}$*-set* if and only if whenever $x, y \in X$ with $x \neq y$, there exist $G, H \in \mathscr{A}$ such that $x \in G \setminus H$ and $y \in H \setminus G$.
- a $T_2^{\mathscr{A}}$*-set* if and only if whenever $x, y \in X$ with $x \neq y$, there exist $G, H \in \mathscr{A}$ such that $x \in G$, and $y \in H$ and $G \cap H = \emptyset$.

Throughout this paper, we define

$$Q_x := \bigcap \{A \in \mathscr{A} : x \in A\}$$

for every $x \in X$ (see [3]).

Proposition 4.18. *The following statements are true for* $(X, \mathscr{A})$.

(1) *For every* $a, b \in X$, $Q_a \subseteq Q_b$ *if and only if* $a \in Q_b$.

(2) X *is a* T_1^{A}*-set if and only if* $Q_x = \{x\}$ *for every* $x \in X$.

(3) *If* X *is a* $T_0^{\mathscr{A}}$*-set and* $x \in Q_y$ *with* $x \neq y$, *then* $y \notin Q_x$.

(4) *For every* $(x, f) \in X \times C_{(\mathscr{A},\mathscr{B})}(X, Y)$, $f(Q_x) \subseteq Q_{f(x)}$.

(5) *If* $\mathscr{A}$ *is closed under complement, then* $\{Q_x : x \in X\}$ *is a partition of* X.

Proof. The proofs of (1), (2), and (3) are straightforward.

(4) Let $(x, f) \in X \times C_{(\mathscr{A},\mathscr{B})}(X, Y)\}$ be given. If $f(x) \in Y \setminus \bigcup \mathscr{B}$, then it immediately implies $f(Q_x) \subseteq Q_{f(x)} = Y$. Now, suppose that there exists $B \in \mathscr{B}$ with $f(x) \in B$. Since $f \in C_{(\mathscr{A},\mathscr{B})}(X, Y)\}$, there exists an element A in $\mathscr{A}$ such that $x \in A$ and $f(A) \subseteq B$, which implies from $Q_x \subseteq A$ that $f(Q_x) \subseteq Q_{f(x)}$.

(5) Let $t \in Q_x \cap Q_y$ be given. Then, by part (1), $Q_t \subseteq Q_x$. Suppose that $Q_t \subset Q_x$, then there is an element A in $\mathscr{A}$ such that $t \in A$ and $x \notin A$, which implies from our hypothesis that $t \notin X \setminus A \in \mathscr{A}$ and $x \in X \setminus A \in \mathscr{A}$, and this is a contradiction to the fact that $t \in Q_x$. Hence, $Q_x = Q_t = Q_y$. □

Proposition 4.19. *Let* $(Y, \mathscr{B})$ *be a* $T_1^{\mathscr{B}}$*-set. Then, the following statements are true.*

(1) *f is constant on Q_x for every $(x, f) \in X \times C_{(\mathscr{A},\mathscr{B})}(X, Y)$.*

(2) *If $X \neq \bigcup \mathscr{A}$, then $C_{(\mathscr{A},\mathscr{B})}(X,Y) = \emptyset$.*

(3) *If there exists an $f \in C_{(\mathscr{A},\mathscr{B})}(X, Y)$ such that f is one-to-one, then Q_x is singleton for every $x \in X$.*

(4) *If $\mathscr{A}$ is closed under complement, then*

$$Q_x = \{y \in X : f(x) = f(y) \text{ for every } f \in C_{(\mathscr{A},\mathscr{B})}(X, Y)\}$$

for every $x \in X$.

Proof. (1) The proof of this case is obvious in view of Proposition 4.18(4).

(2) If $|Y| = 1$, it is evident. So we can assume $|Y| \geq 2$, which implies that $Y = \bigcup \mathscr{B}$. If $f \in C_{(\mathscr{A},\mathscr{B})}(X,Y) \neq \emptyset$ and $x \in X \setminus \bigcup \mathscr{A}$ and $B \in \mathscr{B}$ with $f(x) \in B$, then there exists no set $A \in \mathscr{A}$ containing x such that $f(A) \subseteq B$, which is a contradiction.

(3) Suppose $x_1, x_2 \in Q_x$. Since f is constant on Q_x, we have $f(x_1) = f(x_2)$. The injectivity of f implies that $x_1 = x_2$. Hence Q_x is a singleton.

(4) If $y \in Q_x$, then, by part (1), $f(x) = f(y)$ for each $f \in C_{(\mathscr{A},\mathscr{B})}(X, Y)$. Conversely, suppose that $y \notin Q_x$. There exists an element U in $\mathscr{A}$ such that $x \in U$ and $y \notin U$. Define f as follows: $f(t) = x$ for $t \in U$, and $f(t) = y$ for $t \notin U$. Since, by our hypothesis, $X \setminus U \in \mathscr{A}$, we conclude that $f \in C_{(\mathscr{A},\mathscr{B})}(X, Y)$ and $f(x) \neq f(y)$, completing the proof. □

Corollary 4.20. *For every $f \in C_{\mathscr{A}}(X, Y)$,*

$$\mathrm{Z}(f) = \bigcup_{x \in \mathrm{Z}(f)} Q_x.$$

Proof. If $y \in \mathrm{Z}(f)$, then $y \in \bigcup_{x\in \mathrm{Z}(f)} Q_x$, and conversely, if $y \in \bigcup_{x\in \mathrm{Z}(f)} Q_x$, there exists an element x in X such that $y \in Q_x$ and $x \in \mathrm{Z}(f)$. By part (1) of Proposition 4.19, $f(y) = f(x) = 0$. Consequently, $y \in \mathrm{Z}(f)$. □

Proposition 4.21. *The following statements are equivalent.*

(1) *$C_{(\mathcal{A},\mathcal{B})}(X, Y) \neq \emptyset$.*

(2) *$Y \setminus \bigcup \mathcal{B} \neq \emptyset$ or $X = \bigcup \mathcal{A}$.*

(3) *$C_{(\mathcal{A},\mathcal{B})}(X, Y)$ contains all constant functions.*

Proof. (1) $\Rightarrow$ (2). Let $f \in C_{(\mathcal{A},\mathcal{B})}(X,Y)$ be given. We argue by contradiction. Let us assume that $x \in X \setminus \bigcup \mathcal{A}$ and $f(x) \in V \in \mathcal{B}$, then there exists no set $U \in \mathscr{A}$ containing x such that $f(U) \subseteq V$, which is a contradiction.

(2) $\Leftrightarrow$ (3). Suppose $f : (X,\mathcal{A}) \to (Y,\mathcal{B})$ a constant function. Then, assuming that $f(X) = \{y_0\}$, we have $f \in C_{(\mathcal{A},\mathcal{B})}(X,Y)$ if and only if $y_0 \in Y \setminus \bigcup \mathcal{B}$ or $X = \bigcup \mathcal{A}$.

(3) $\Rightarrow$ (1). It is evident. □

Inspired by the construction presented in Section 3 of [3], we consider the equivalence relation $\sim$ on X given by $x \sim y$ if and only if $Q_x = Q_y$, and use the corresponding equivalence classes to define a topological space (X_z, τ_z).

For each $a \in X$, the equivalence class of a is given by

$$[a] := \{x \in X : Q_x = Q_a\}.$$

The associated quotient set is denoted by $X_z := X/\sim$. Finally, the collection that we claim to be the family of all open sets of X_z is defined as follows:

$$\tau_z := \left\{V \subseteq X_z \colon \bigcup V = \bigcup \mathscr{C} \ for some \ \mathscr{C} \subseteq \mathscr{A}\right\}.$$

Now, if $P_z(x \mapsto [x]) \colon X \to X_z$ is the natural map, then

$$\tau_z = \{V \subseteq X_z : \ P_z^{-1}(V) \in \mathcal{T}_0(\mathscr{A})\}.$$

In fact, one can say that this is the largest generalized topology on X_z under which P_z is $(\mathcal{T}_0(\mathscr{A}), \tau_z)$-continuous.

Proposition 4.22. *Suppose $\mathscr{A}$ is closed under finite intersections. Then, the following statements are true.*

(1) The set τ_z is a topology on X_z.

(2) For every $V \subseteq X_z$ and every $\mathscr{C} \subseteq \mathscr{A}$, $\bigcup_{[x]\in V}[x] = \bigcup \mathscr{C}$ if and only if $V = \left\{[x] \colon x \in \bigcup \mathscr{C}\right\}$. Especially, $V \in \tau_z$ if and only if there exists a subcollection $\mathscr{C}$ of $\mathscr{A}$ such that $V = \left\{[x] \colon x \in \bigcup \mathscr{C}\right\}$, and as a result $\left\{\{[a] \colon a \in A\}\right\}_{A \in \mathscr{A}}$ is a base for τ_z.

(3) $P_z \in C_{(\mathcal{A},\tau_z)}(X, X_z)$.

(4) If $\mathscr{A}$ is closed under complement, then for every $A \in \mathscr{A}$, the set $\{[a]\colon a \in A\}$ is clopen in X_z, and especially, X_z is a zero-dimensional space.

(5) The topological space (X_z, τ_z) is a T_0 space.

Proof. (1) The union over an empty indexed family of subsets of X equals $\emptyset$, and therefore $\emptyset \in \tau_Z$. Since $\bigcup X_z = X = \bigcap \emptyset \in \mathscr{A}$, we conclude that $X_z \in \tau_z$. Every union of members of τ_z belongs to τ_z. Given $H, K \in \tau_z$, then there exist $\mathscr{C} \subseteq \mathscr{A}$ and $\mathscr{D} \subseteq \mathscr{A}$ such that $\bigcup H = \bigcup_{C \in \mathscr{C}} C$ and $\bigcup K = \bigcup_{D \in \mathscr{D}} D$. Since X_z is a partition of X, we conclude that

$$\big\{[x] \cap [y]\colon ([x],[y]) \in H \times K\big\} \setminus \big\{\emptyset\big\} \subseteq X_z.$$

Hence,

$$\begin{aligned} \bigcup_{[x] \in H \cap K} [x] &= \bigcup_{([x],[y]) \in H \times K} ([x] \cap [y]) = \bigcup H \cap \bigcup K \\ &= \bigcup_{(C,D) \in \mathscr{C} \times \mathscr{D}} (C \cap D) \end{aligned}$$

and this entails that $H \cap K \in \tau_z$. Therefore, τ_z is a topology on X_z.

(2) By definition of $[a]$ we can infer that $[a] \subseteq Q_a \subseteq A$ for very $a \in A \in \mathscr{A}$. Hence, $A = \bigcup_{a \in A} [a]$, which implies that $A_z := \big\{[a]\colon a \in A\big\} \in \tau_z$, and this entails that $\bigcup \mathscr{C} = \bigcup_{x \in \bigcup \mathscr{C}} [x]$ for every $\mathscr{C} \subseteq \mathscr{A}$. Given $V \subseteq X_z$ and $\mathscr{C} \subseteq \mathscr{A}$. Since X_z is a partition of X, we conclude that

$$\begin{aligned} \bigcup_{[x] \in V} [x] = \bigcup \mathscr{C} \Leftrightarrow \bigcup_{[x] \in V} [x] &= \bigcup_{x \in \bigcup \mathscr{C}} [x] \\ \Leftrightarrow V &= \big\{[x]\colon x \in \bigcup \mathscr{C}\big\}. \end{aligned}$$

The proofs of (3) and (4) are straightforward.

(5) Let $[x]$ and $[y]$ be two distinct elements of X_z. By definition $\sim$, we have:

$$\begin{aligned} \big[x\big] \neq \big[y\big] &\Leftrightarrow Q_x \neq Q_y \\ &\Leftrightarrow Q_x \nsubseteq Q_y \textit{ or } Q_y \nsubseteq Q_x \\ &\Leftrightarrow \exists z \in X\, (z \in Q_x \text{ and } z \notin Q_y) \textit{ or } \exists z \in X\, (z \in Q_y \text{ and } z \notin Q_x)\,. \end{aligned}$$

Suppose that $z \in Q_x$ and $z \notin Q_y$. Then

$$\begin{aligned}
&\forall V \in \mathscr{A}(x \in V \Rightarrow z \in V) \text{ and } \exists U \in \mathscr{A}(y \in U \text{and } z \notin U)\\
\Rightarrow &\forall V \in \mathscr{A}(z \notin V \Rightarrow x \notin V) \text{ and } \exists U \in \mathscr{A}(y \in U \text{and } z \notin U)\\
\Rightarrow &\exists U \in \mathscr{A}(y \in U \text{and } x \notin U)
\end{aligned}$$

A similar argument shows that if $z \in Q_y$ and $z \notin Q_x$, then there exists an element U in $\mathscr{A}$ such that $x \in U$ and $y \notin U$. Therefore, the topological space (X_z, τ_z) is a T_0 space. □

Proposition 4.23. *Suppose $\mathscr{A}$ is closed under finite intersections. If X is a $T_i^{\mathscr{A}}$-set, then X_z is a T_i space for $i = 1, 2$.*

Proof. By proof of Proposition 4.22(2), we get $A = \bigcup_{a\in A}[a]$ for every $A \in \mathscr{A}$. Accordingly, the proof is clear. □

Lemma 4.24. *Let $\mathscr{A}$ be closed under finite intersections. If $f \in C_{\mathscr{A}}(X, Y)$ and $f_z\big([x] \mapsto f(x)\big)\colon X_z \to Y$, then $f_z \in C(X_z, Y)$.*

Proof. Given $a, b \in X$ with $[a] = [b]$, then $Q_a = Q_b$, which implies from Proposition 4.19 that $f(a) = f(b)$ and this entails that f_z is a function. Let $x \in X$ and $n \in \mathbb{N}$ be given. Then there exists an element A in $\mathscr{A}$ such that $f(A) \subseteq V_n\big(f(x)\big)$. Since, by proof of Proposition 4.22(2), $A = \bigcup_{a\in A}[a]$, we infer that $[x] \in \big\{[a] \colon a \in A\big\} \in \tau_z$ and

$$f_z\left(\big\{[a] \colon a \in A\big\}\right) = f(A) \subseteq V_n\big(f(x)\big).$$

Hence, f is continuous at every point of X_z. □

Henceforth, we define f_z as in Lemma 4.24 for every $f \in C_{\mathscr{A}}(X, Y)$ and $A_z = \big\{[a] \colon a \in A\big\}$ for every $A \subseteq X$. If $\mathscr{D} \subseteq \mathcal{P}(X)$, then $\bigcup_{D\in\mathscr{D}} D_z = (\bigcup \mathscr{D})_z$ and $(\bigcap \mathscr{D})_z \subseteq \bigcap_{D\in\mathscr{D}} D_z$. Also, if $\mathscr{D} \subseteq \mathscr{A}$, then $\bigcap_{D\in\mathscr{D}} D_z = (\bigcap \mathscr{D})_z$ and for every $A \in \mathscr{A}$, $(X \setminus A)_z = X_z \setminus A_z$.

Proposition 4.25. *If $\mathscr{A}$ is closed under finite intersections, then*

$$C_{\mathscr{A}}(X, Y) \cong C(X_z, Y).$$

Proof. We claim that the function

$$\theta(f \mapsto f_z)\colon C_{\mathscr{A}}(X,Y) \to C(X_z,Y)$$

is an isomorphism. It is clear that $\theta(f+g) = \theta(f) + \theta(g)$ and $\theta(fg) = \theta(f)\theta(g)$ for every $f, g \in C_{\mathscr{A}}(X,Y)$. Let $f \in C_{\mathscr{A}}(X,Y)$ with $\theta(f) = 0$ be given. Then $f(x) = f_z([x]) = \theta(f)([x]) = 0$ for every $x \in X$, which implies that $f = 0$, and this entails that θ is one-one. To complete the claim, we must show that θ is onto. To this end, let $g \in C(X_z, Y)$. We consider the function $f(x \mapsto g([x]))\colon X \to Y$. Fix $x \in X$ and choose an element n in $\mathbb{N}$. Then there exists $G \in \tau_z$ such that $[x] \in G$ and $g(G) \subseteq V_n(g([x]))$. Since, by Proposition 4.22(2), there exists a subcollection $\mathscr{C}$ of $\mathscr{A}$ such that $x \in \bigcup_{[t]\in G}[t] = \bigcup \mathscr{C}$ and $G = \left\{[t]\colon t \in \bigcup \mathscr{C}\right\}$, we conclude that there exists an element A in $\mathscr{A}$ such that $x \in A$ and $f(A) = g\left(A_z\right) \subseteq g(G) \subseteq V_n(g([x]))$. Hence, $f \in C_{\mathscr{A}}(X,Y)$ and $\theta(f) = g$. □

Let $\mathscr{A}$ be closed under finite intersections. Then, by Proposition 4.25 and [14, Theorem 3.9], there exists a completely regular Hausdorff space Z such that $C_{\mathscr{A}}(X) \cong C(Z)$. Therefore, the algebraic properties of $C(X)$ may be studied by working with weaker set-theoretic structures rather than the topology.

Acknowledgements. The authors would like to thank the referees for their valuable comments and constructive suggestions, which significantly improved the quality of the paper. In particular, Proposition 4.7 and Remarks 3.3 and 4.6 were contributed by one of the referees.

References

[1] Aliabad, A.R. and Sheykhmiri, A., *LG-topology*, Bull. Iran. Math. Soc. 41 (2015), 239-258.

[2] Aliabad, A.R. and Zarepour, H., *Continuous functions on LG-spaces*, J. Algebr. Syst. 8 (2021), 181-200.

[3] Afrooz, S., Azarpanah, F., and Etebar, M., *On rings of real valued clopen continuous functions*, Appl. Gen. Topol. 19 (2018), 203–216.

[4] Afrooz, S., Azarpanah, F., and Hajee, N.H., *On e-spaces and rings of real valued e-continuous functions*, Appl. Gen. Topol. 24 (2023), 433-448.

[5] Amini, A., Amini, B., Momtahan, E., and Shirdareh Haghighi, M.H., *Generalized rings of measurable and continuous functions*, Bull. Iran. Math. Soc. 39 (2013), 49-64.

[6] Banaschewski, B., "The real numbers in pointfree topology", Textos de Matemática (Universidade de Coimbra): Série B, Universidad de Coimbra, 1997.

[7] Császár, Á., *Generalized topology, generalized continuity*, Acta Math. Hungar. 96 (2002), 351-357.

[8] De Marco, G. and Orsatti, A., *Commutative rings in which every prime ideal is contained in a unique maximal ideal*, Proc. Amer. Math. Soc. 30 (1971), 459-466.

[9] Estaji, A.A. and Mahmoudi Darghadam, A., *Some properties of algebras of real-valued measurable functions*, Arch. Math. (Brno) 59 (2023), 383-395.

[10] —, *Rings of real measurable functions vanishing at infinity on a measurable space*, J. Frame Matrix Theory 1 (2024), 1-21.

[11] Estaji, A.A., Mahmoudi Darghadam, A., and Yousefpour, H., *Maximal ideals in rings of real measurable functions*, Filomat 32 (2018), 5191-5203.

[12] Etebar, M. and Soltanpour, S., *On rings of real-valued R_{cl}-supercontinuous functions*, J. Math. Ext. 16 (2022), 13, Id/No 1.

[13] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., and Scott, D.S., "Continuous Lattices and Domains", Cambridge University Press, 2003.

[14] Gillman, L. and Jerison, M., "Rings of Continuous Functions", D. Van Nostrand Company, Inc., 1960.

[15] Hager, A.W., *Algebras of measurable functions*, Duke Math. J. 38 (1971), 21-27.

[16] Kohli, J.K. and Kumar, R., *Z-supercontinuous functions*, Indian J. Pure Appl. Math. 33 (2002), 1097-1108.

[17] Kohli, J.K., Singh, D., and Aggarwal, J., *F-supercontinuous functions*, Appl. Gen. Topol. 10 (2009), 69-83.

[18] —, *R-supercontinuous functions*, Demonstr. Math. 43 (2010), 703-723.

[19] Mack, J., *Countable paracompactness and weak normality properties*, Trans. Am. Math. Soc. 148 (1970), 265-272.

[20] Munshi, B.M. and Bassan, D.S., *Super-continuous mappings*, Indian J. Pure Appl. Math.13 (1982), 229-236.

[21] Noiri, T., *On delta-continuous functions*, J. Korean Math. Soc. 16 (1980), 161-166.

[22] Reilly, I.L. and Vamanamurthy, M.K., *On super-continuous mappings*, Indian J. Pure Appl. Math. 14 (1983), 767-772.

[23] Yousefpour, H., Estaji, A.A., Mahmoudi Darghadam, A., and Sadeghi, Gh., *m-topology on the ring of real-measurable functions*, J. Algebr. Syst. 9 (2021), 83-106.

Omid Molaei *Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Sabzevar, Iran*

Email: omid.molaei.ir@gmail.com

Ali Akbar Estajii *Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Sabzevar, Iran*

Email: aaestaji@hsu.ac.ir

Mehdi Vatandoost *Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Sabzevar, Iran*

Email: m.vatandoost@hsu.ac.ir