Categories and
General Algebraic Structures
with Applications

In press.

On one-sided $\mathcal{U}$-ideals in monoidal categories

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Abstract. In this paper, we investigate one-sided (thick) ideals in monoidal categories and explore related concepts, including generating sets, idempotency, radicality and primeness. We establish several structural properties of one-sided ideals, drawing analogies with the ring-theoretic setting. Further results are obtained by specializing to pivotal categories. In particular, one-sided negligible objects are shown to provide examples of one-sided ideals, thanks to the well-established theory of categorical traces in this framework. Moreover, we introduce and study a generalization, called one-sided $\mathcal{U}$-ideals, where we show their nontrivial nature and establish various $\mathcal{U}$-analogues of the preceding results, examining the validity of several fundamental properties.

1 Introduction

Roughly speaking, there are typically two kinds of ideals in monoidal categories: tensor ideals and thick ideals. The former are defined on the class of morphisms of the category and are known to generalize the standard notion of ideals in ring theory [6, 10]. A more natural framework for study-

Keywords: Thick ideal, pivotal category, strong monoidal functor, negligible morphism.
Mathematics Subject Classification [2010]:15A99, 18A05, 18E05.
Received: 1 December 2025, Accepted: 20 February 2026.
ISSN: Print 2345-5853, Online 2345-5861.

ing tensor ideals is within additive categories, as convincingly confirmed by Mitchell [9], who showed that theorems in ring theory usually extend to additive categories. The interest of the current paper is towards thick ideals which are defined on the class of objects of the monoidal category and will be usually referred to as ideals in the sequel.

Monoidal categories with compatible duality structures are known to admit consistent theory of traces and dimensions which apply to many fields of both mathematics and physics, especially to low-dimensional topology. In particular, they are used to construct invariants of knots, links, 3-manifolds and other topological objects [11]. However, in many interesting cases, they yield only trivial invariants. To address this limitation, a theory of modified traces and dimensions has been developed, where these traces are defined on thick ideals of pivotal categories and other suitable monoidal categories [3–5]. The existence of such trace functions on thick ideals is, in general, difficult to establish. One notable example of thick ideals admitting modified trace functions is provided by the class of projective objects which coincides, in a pivotal category, with that of the injective ones [5]. Ideals generated by classes of objects were introduced in [5], and played an important role in establishing the existence and uniqueness of modified trace and dimension functions on ideals in pivotal categories.

The first aim of this paper is to provide a more detailed exposition of the structure of (one-sided thick) ideals in a monoidal category. A subclass of objects of a monoidal category is called a *left* (respectively, *right*) *ideal* if it is closed under retractions and under tensor products on the left (respectively, right). Examples of (one-sided) ideals, in a pivotal category, are given by (one-sided) negligible objects (Propositions 3.3 and 3.5). In analogy with the ring-theoretic setting, we derive and study several notions such as radicality, idempotency and primeness, and then establish similar results to those in the usual case. In particular, some characterizations (Theorem 3.9) and structure results (Corollary 3.10 and Proposition 3.12) are given. As usual, the radical of a one-sided ideal is not always an ideal. However, under some sufficient conditions, we prove that this holds true (Propositions 3.24 and 3.25). The main properties of the radical of a one-sided ideal are summarized in Theorem 3.26.

The second aim of the paper is the introduction and study of a generalization which we term *one-sided* $\mathcal{U}$*-ideals*. We show that this generalization is non trivial (Example 4.3) and establish several fundamental properties of the corresponding $\mathcal{U}$-versions, analogous to those satisfied by (one-sided) ideals (see Section 4).

2 Preliminaries

We start this section by fixing the necessary notations, which will be used in the sequel without further mention.

2.1 Notation Throughout, $\mathcal{C}$ will denote a (strict) monoidal category with unit I and tensor product $\otimes$, except when otherwise mentioned. We will use capital letters such as X, Y, Z, U, V, A, $B\ldots$, to denote the objects of $\mathcal{C}$, and we use lowercase and Greek letters f, g, h, α, $\beta\ldots$, to denote the morphisms of $\mathcal{C}$. The classes of objects and morphisms of $\mathcal{C}$ will be denoted by $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively. However, we will simply write $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$ and $f \in \mathcal{C}$ will mean that f is a morphism of $\mathcal{C}$. The class of morphisms $X \to Y$ will be denoted by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, and $\mathrm{Hom}_{\mathcal{C}}(X,X)$ will be written $\mathrm{End}_{\mathcal{C}}(X)$. For brevity, we will usually write fg to denote the composition $f \circ g$, and 1_X to denote the identity endomorphism on $X \in \mathcal{C}$. The tensor product $X^{\otimes n} := X \otimes \cdots \otimes X$ of X, n times, will be denoted simply by X^n. For $n = 0$, we set $X^0 = I$. As usual, $\mathbb{N}$ will denote the set of natural integers $\{0, 1, ...\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

2.2 Monoidal categories and functors The reminder of this section follows [1, 2, 7, 8, 11].

A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, l, r)$ is the following given data.

(1) A category $\mathcal{C}$.

(2) A tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(3) A unit object I.

(4) Natural isomorphisms $\alpha : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $l : I \otimes U \to U$ and $r : U \otimes I \to U$ for all objects $U, V, W \in \mathcal{C}$, called associativity constraint, left and right unitality constraints respectively, such that the pentagon and triangle axioms hold [8].

When α, l and r are identities, $\mathcal{C}$ is said to be strict. We will assume throughout the sequel that $\mathcal{C}$ is strict, and we write $(\mathcal{C}, \otimes, I, \alpha, l, r)$ simply as $(\mathcal{C}, \otimes, I)$. This is possible due to Mac Lane's Coherence Theorem [8].

Example 2.1. The category of left modules over a commutative ring R, and R-linear morphisms is a fundamental example of a monoidal category. The monoidal structure is given by the usual tensor product of modules over R, the unit object is R, and the isomorphisms α, l, and r are the obvious ones. Note that this category is not strict.

Example 2.2. Consider a commutative ring R with unit, and a multiplicative abelian group G with neutral element e. Let $\mathcal{W}$ be the category whose objects are the elements of G, and the morphisms are defined as follows: $\mathrm{Hom}_{\mathcal{W}}(g, g) = R$ and $\mathrm{Hom}_{\mathcal{W}}(g, h) = \{0\}$ if $g \neq h$, for all $g, h \in G$. The identity morphism is given by the unit of R, the tensor product of objects is defined by the product in G, and the composition and tensor product of morphisms are defined by the product in R. Therefore, $\mathcal{W}$ is a strict monoidal category.

A *strong monoidal functor* $\mathcal{U} : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \otimes', I')$ between strict monoidal categories is a triplet $(\mathcal{U}, \mathcal{U}_0, \mathcal{U}_2)$, where $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ is a functor, and $\mathcal{U}_2 : \mathcal{U}(U) \otimes' \mathcal{U}(V) \to \mathcal{U}(U \otimes V)$ and $\mathcal{U}_0 : I' \to \mathcal{U}(I)$ are isomorphisms in $\mathcal{D}$, for all objects $U, V \in \mathcal{C}$, satisfying the associativity, left unitality and right unitality constraints [11].

2.3 Duality Let $(\mathcal{C}, \otimes, I)$ be a (strict) monoidal category. A *left* (respectively, *right*) *duality* for $\mathcal{C}$ consists of a left (respectively, right) duality for every object V in $\mathcal{C}$. That is, the existence of an object V^* in $\mathcal{C}$, called its *left* (respectively, *right*) *dual*, along with morphisms

$$d_V : V^* \otimes V \to I \quad \big(\text{respectively,} \quad d'_V : V \otimes V^* \to I\big),$$

called the *evaluation*, and

$$b_V : I \to V \otimes V^* \quad \big(\text{respectively,} \quad b'_V : I \to V^* \otimes V\big),$$

called the *coevaluation*, such that the following triangular identities hold

$$(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V \quad \big(\text{respectively,} \quad (d'_V \otimes 1_V)(1_V \otimes b'_V) = 1_V\big), \tag{2.1}$$

$$(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*} \quad \big(\text{respectively,} \quad (1_{V^*} \otimes d'_V)(b'_V \otimes 1_{V^*}) = 1_{V^*}\big). \tag{2.2}$$

In this case, $\mathcal{C}$ is called *left* (respectively, *right*) *rigid*, and $\mathcal{C}$ will be called *rigid* if it is both left and right rigid.

For any morphism $f : U \to V$ of $\mathcal{C}$ between left dualizable objects, its *left dual* $f^* : V^* \to U^*$ is given by

$$f^* = (d_V \otimes 1_{U^*})(1_{V^*} \otimes f \otimes 1_{U^*})(1_{V^*} \otimes b_U).$$

Dually, its *right dual* can be defined.

For any object $V \in \mathcal{C}$ and composable morphisms f and g of $\mathcal{C}$, we immediately have that $(f \circ g)^* = g^* \circ f^*$ and $1_{V^*} = 1_V^*$.

2.4 Pivotal categories

Definition 2.3. [11, Sect. 1.7.1, page 26]. A *pivotal category* is a rigid category $\mathcal{C}$ with distinguished left duality (V^*, d_V, b_V) and right duality (V^*, d'_V, b'_V) structures for every object $V \in \mathcal{C}$, such that the induced left and right dual functors coincide. In other words, this means that we have

(1) $d'_V = d_V : I \to I^*$.

(2) For every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$, the following induced duals coincide

$$(d_V \otimes 1_{U^*})(1_{V^*} \otimes f \otimes 1_{U^*})(1_{V^*} \otimes b_U) : V^* \to U^*.$$
$$(1_{U^*} \otimes d'_V)(1_{U^*} \otimes f \otimes 1_{V^*})(b'_U \otimes 1_{V^*}) : V^* \to U^*.$$

(3) The following induced isomorphisms coincide

$$(d_V \otimes 1_{(U\otimes V)^*})(1_{V^*} \otimes d_U \otimes 1_{V\otimes(U\otimes V)^*})(1_{V^*\otimes U^*} \otimes b_{U\otimes V}) : V^* \otimes U^* \to (U \otimes V)^*.$$
$$(1_{(U\otimes V)^*} \otimes d'_V)(1_{(U\otimes V)^*\otimes U} \otimes d'_U \otimes 1_{U^*})(b'_{U\otimes V} \otimes 1_{V^*\otimes U^*}) : V^* \otimes U^* \to (U \otimes V)^*.$$

Notice that the triangular identities (2.1) and (2.2) corresponding to the duality structures (V^*, d_V, b_V) and (V^*, d'_V, b'_V) imply that the identity $b'_V = b_V : I^* \to I$ holds as well. Moreover, the family

$$\Phi = \{\ \Phi_V = (d'_V \otimes 1_{V^{**}})(1_V \otimes b_{V^*}) : V \to V^{**}\ \}_{V \in \mathcal{C}},$$

is a (monoidal) natural isomorphism, and it is referred to as the *pivotal structure*. Note here that for every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(V)$, we have $f^{**} = \Phi_V f \Phi_V^{-1}$.

2.5 Trace and partial trace Let $\mathcal{C}$ be a pivotal category. For any object $V \in \mathcal{C}$ and endomorphism $f \in \mathrm{End}_{\mathcal{C}}(V)$, the *left trace* of f is the element of $\mathtt{K}_{\mathcal{C}}$ given by

$$Tr_l(f) = d_V(1_{V^*} \otimes f)b'_V.$$

Likewise, the *right trace* of f is the following element of $\mathtt{K}_{\mathcal{C}}$

$$Tr_r(f) = d'_V(f \otimes 1_{V^*})b_V.$$

The main properties of left and right traces can be found in [11, Sect. 2.6].

The *right partial trace* and *left partial trace* of an endomorphism $f \in \mathrm{End}_{\mathcal{C}}(U \otimes V)$ are, respectively, given by

$$pTr_r(f) = (1_U \otimes d'_V)(f \otimes 1_{V^*})(1_U \otimes b_V) \in \mathrm{End}_{\mathcal{C}}(U).$$

$$pTr_l(f) = (d_U \otimes 1_V)(1_{U^*} \otimes f)(b'_U \otimes 1_V) \in \mathrm{End}_{\mathcal{C}}(V).$$

The terminology "partial trace" is justified by the following equalities, for every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(U \otimes V)$:

$$Tr_l\big(pTr_l(f)\big) = Tr_l(f) \quad \text{and} \quad Tr_r\big(pTr_r(f)\big) = Tr_r(f).$$

For further details on partial traces, we refer to [5].

2.6 Negligibility Let $\mathcal{C}$ be a pivotal category and X, Y two objects of $\mathcal{C}$. A morphism $f \in \mathrm{Hom}_C(X, Y)$ is called *left negligible* if for every morphism $g \in \mathrm{Hom}_C(Y, X)$, one has $Tr_l(fg) = 0$. *Right negligibility* is defined similarly by replacing the left trace by the right trace in the previous identity. A morphism is *negligible* when it is both left and right negligible.

The previous notions of left and right negligible morphisms are generally distinguished in pivotal categories. However, they coincide in a spherical category, and in particular, in a ribbon category.

3 One-sided ideals: properties and related concepts

3.1 Ideals and related structures Let $\mathcal{C}$ be a monoidal category and X an object of $\mathcal{C}$. An object $Y \in \mathcal{C}$ is a *retract* of X provided that there exist morphisms $\alpha : Y \to X$ and $\beta : X \to Y$ such that $\beta\alpha = 1_Y$. This situation will be denoted by $(Y \lneq X, \alpha, \beta)$ or simply $Y \lneq X$.

A subclass $\mathcal{J}$ of $\mathcal{C}_0$ is said to be closed under left (respectively, right) tensor products if for every $X \in \mathcal{J}$ and $Y \in \mathcal{C}$, we have $Y \otimes X \in \mathcal{J}$ (respectively, $X \otimes Y \in \mathcal{J}$). $\mathcal{J}$ is said to be closed under retractions if for every $X \in \mathcal{J}$ and $Y \in \mathcal{C}$ such that $Y \lneq X$, we have $Y \in \mathcal{J}$.

Definition 3.1. Let $\mathcal{C}$ be a monoidal category. A subclass $\mathcal{J} \subseteq \mathcal{C}_0$ of objects is called a *left ideal* of $\mathcal{C}$ provide that

(1) $\mathcal{J}$ closed under tensor products on the left.
(2) $\mathcal{J}$ is closed under retractions.

In other words, $\mathcal{J}$ is a left ideal if for all objects $X \in \mathcal{J}$ and $Y, Z \in \mathcal{C}$ such that $Y \lneq Z \otimes X$, it follows that $Y \in \mathcal{J}$.
A *right ideal* is defined similarly by replacing "on the left", in the first condition above, by "on the right".

A *two-sided ideal*, or simply an *ideal*, is a subclass which is both a left and a right ideal.

Recall that a category is called *braided* provided that it is equipped with a *braiding* which is a family of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$, for all objects $X, Y \in \mathcal{C}$, satisfying the hexagon identities [11].

Remark 3.2. Let $\mathcal{C}$ be a monoidal category.

(1) A left (or right) ideal $\mathcal{J}$ of $\mathcal{C}$ contains the unit object I if and only if $\mathcal{J} = \mathcal{C}_0$. Indeed, for every object $Y \in \mathcal{C}$, we have $Y = Y \otimes I \in \mathcal{J}$. The converse is clear.
(2) If moreover $\mathcal{C}$ is a braided category, then left ideals and right ideals coincide with ideals. Indeed, the braiding plays a weak version of commutativity.

Proposition 3.3. *Let $\mathcal{C}$ be a pivotal category. Consider the following classes.*

(1) $\mathcal{N}^l(\mathcal{C}) = \{Y \in \mathcal{C},\ \text{every endomorphism } f \in \mathrm{End}_{\mathcal{C}}(Y) \text{ is left negligible}\}$.
(2) $\mathcal{N}^r(\mathcal{C}) = \{Y \in \mathcal{C},\ \text{every endomorphism } f \in \mathrm{End}_{\mathcal{C}}(Y) \text{ is right negligible}\}$.

(3) $\mathcal{N}(\mathcal{C}) = \{Y \in \mathcal{C}$, every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(Y)$ is negligible$\}$.

Then, $\mathcal{N}^l(\mathcal{C})$, $\mathcal{N}^r(\mathcal{C})$ and $\mathcal{N}(\mathcal{C})$ are, respectively, a left ideal, a right ideal and an ideal of $\mathcal{C}$.

Proof. We prove that $\mathcal{N}^l(\mathcal{C})$ is a left ideal. Let $Y \in \mathcal{N}^l(\mathcal{C})$, $Z \in \mathcal{C}$ and let's show that every endomorphism $g \in \mathrm{End}_{\mathcal{C}}(Z \otimes Y)$ is left negligible. For every $h \in \mathrm{End}_{\mathcal{C}}(Z \otimes Y)$, we consider the left partial trace $pTr_l(gh) \in \mathrm{End}_{\mathcal{C}}(Y)$ of gh. Since $Y \in \mathcal{N}^l(\mathcal{C})$, $pTr_l(gh)$ is left negligible. In particular, $Tr_l(pTr_l(gh)) = 0$. But, $Tr_l(gh) = Tr_l(pTr_l(gh))$, which implies $Tr_l(gh) = 0$. Thus, g is left negligible and so $Z \otimes Y \in \mathcal{N}^l(\mathcal{C})$. On the other hand, let $Y \in \mathcal{N}^l_{\mathcal{C}}$ and $(Z \lneqq Y, \alpha, \beta)$ any retract of Y, then for every $f, h \in \mathrm{End}_{\mathcal{C}}(Z)$, we have $Tr_l(fh) = Tr_l(f\alpha\beta h) = Tr_l(\beta h f \alpha) = 0$ by hypothesis, since the trace is cyclic and $\beta h f \alpha \in \mathrm{End}_{\mathcal{C}}(Y)$. Hence, f is left negligible and so $Z \in \mathcal{N}^l(\mathcal{C})$. This completes the proof. Similarly one can show that $\mathcal{N}^r(\mathcal{C})$ and $\mathcal{N}(\mathcal{C})$ are a right ideal and an ideal of $\mathcal{C}$, respectively. □

Recall that an object X of a pivotal category $\mathcal{C}$ is called *left negligible* if every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(X)$ is left negligible. Likewise, a *right negligible* object can be defined using right negligibility on endomorphisms of X. *Negligible* objects are those that are simultaneously left and right negligible. Hence, the classes $\mathcal{N}^l(\mathcal{C})$, $\mathcal{N}^r(\mathcal{C})$ and $\mathcal{N}(\mathcal{C})$ defined in the previous Proposition 3.3 are exactly the classes of left negligible, right negligible and negligible objects of $\mathcal{C}$, respectively.

Corollary 3.4. *The class of non left (respectively, right) negligible objects of a pivotal category $\mathcal{C}$ form a left (respectively, right) ideal of $\mathcal{C}$.*

Proof. Straightforward. □

Proposition 3.5. *Let $\mathcal{C}$ be a pivotal category and $X \in \mathcal{C}$ an object. Consider the following classes*

(1) $\mathcal{N}^l(X) = \{Y \in \mathcal{C},$ for every $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), f$ is left negligible$\}$.

(2) $\mathcal{N}^r(X) = \{Y \in \mathcal{C},$ for every $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), f$ is right negligible$\}$.

(3) $\mathcal{N}(X) = \{Y \in \mathcal{C},$ for every $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), f$ is negligible$\}$.

Then, $\mathcal{N}^l(X)$, $\mathcal{N}^r(X)$ and $\mathcal{N}(X)$ are a left ideal, a right ideal and an ideal of $\mathcal{C}$, respectively. Moreover, if X is left negligible (respectively, right negligible, negligible), then $\mathcal{N}^l(X) = \mathcal{C}_0$ (respectively, $\mathcal{N}^r(X) = \mathcal{C}_0$, $\mathcal{N}(X) = \mathcal{C}_0$).

Proof. We proceed as in Proposition 3.3. Let $Y \in \mathcal{N}^l(X)$, $A \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A \otimes Y, X)$. Then, for every $g \in \mathrm{Hom}_{\mathcal{C}}(X, A \otimes Y)$, we have $Tr_l(fg) = Tr_l\big(pTr_l(gf)\big) = 0$. Hence $A \otimes Y \in \mathcal{N}^l(X)$. Now let $Y \in \mathcal{N}^l(X)$ and $A \in \mathcal{C}$ such that $Y \lneq X$, then for every $f \in \mathrm{Hom}_{\mathcal{C}}(A \otimes Y, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X, A \otimes Y)$, we have $Tr_l(fg) = Tr_l(\alpha f g \beta) = 0$. Likewise, one shows the other statements about $\mathcal{N}^r(X)$ and $\mathcal{N}(X)$.

Now, if X is left negligible, then for every $Y \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, we have $Tr_l(fg) = Tr_l(gf) = 0$. Similarly, one obtains that $\mathcal{N}^r(X) = \mathcal{C}_0$ and $\mathcal{N}(X) = \mathcal{C}_0$ when X is right negligible and negligible, respectively. □

Let $\mathcal{O} \subseteq \mathcal{C}_0$ be a class of objects of a monoidal category $\mathcal{C}$. Recall [5] the following ideals

(1) The ideal generated, on the left, by $\mathcal{O}$ is the left ideal

$$\langle \mathcal{O} \rangle^l := \big\{ Y \in \mathcal{C},\ Y \lneq Z \otimes X,\ \text{with } Z \in \mathcal{C} \text{ and } X \in \mathcal{O} \big\}.$$

(2) The ideal generated, on the right, by $\mathcal{O}$ is the right ideal

$$\langle \mathcal{O} \rangle^r := \big\{ Y \in \mathcal{C},\ Y \lneq X \otimes Z,\ \text{with } Z \in \mathcal{C} \text{ and } X \in \mathcal{O} \big\}.$$

(3) The ideal generated by $\mathcal{O}$ is the ideal

$$\langle \mathcal{O} \rangle := \big\{ Y \in \mathcal{C},\ Y \lneq U \otimes X \otimes Z,\ \text{with } U, Z \in \mathcal{C} \text{ and } X \in \mathcal{O} \big\}.$$

Remark 3.6. Let $\mathcal{J} \subseteq \mathcal{C}_0$ be a subclass of objects of a monoidal category $\mathcal{C}$.

(1) If $\mathcal{J}$ is a left ideal, then $\langle \mathcal{J} \rangle^l = \mathcal{J}$.
(2) If $\mathcal{J}$ is a right ideal, then $\langle \mathcal{J} \rangle^r = \mathcal{J}$.
(3) If $\mathcal{J}$ is an ideal, then $\langle \mathcal{J} \rangle = \mathcal{J}$.
(4) $\langle I \rangle = \langle \mathcal{C}_0 \rangle = \mathcal{C}_0$.

Proposition 3.7. *Let $\mathcal{O} \subseteq \mathcal{C}_0$ be a subclass of objects of a pivotal category $\mathcal{C}$ and set $\mathcal{O}^{\vee} = \big\{ Y \in \mathcal{C},\ Y \lneq X^* \text{ and } X \in \mathcal{O} \big\}$. The following assertions hold.*

(1) $\langle \mathcal{O}^\vee \rangle^l = \langle \mathcal{O} \rangle^{r^\vee}$.

(2) $\langle \mathcal{O}^\vee \rangle^r = \langle \mathcal{O} \rangle^{l^\vee}$.

(3) $\langle \mathcal{O}^\vee \rangle = \langle \mathcal{O} \rangle^\vee$.

Proof. Let's prove the third assertion (3) and similar procedure can be conducted to prove (1) and (2) (see also [5, Lemma 8(c)]). We have

$$\begin{aligned}\langle \mathcal{O}^\vee \rangle &= \{U \in \mathcal{C},\ U \lneq Z_1 \otimes Y \otimes Z_2,\ \text{with } Z_1, Z_2 \in \mathcal{C} \text{ and } Y \in \mathcal{O}^\vee\} \\ &= \{U \in \mathcal{C},\ U \lneq Z_1 \otimes X^* \otimes Z_2,\ \text{with } Z_1, Z_2 \in \mathcal{C},\ \text{and } X \in \mathcal{O}\} \\ &= \{U \in \mathcal{C},\ U \lneq \left(Z_2^* \otimes X \otimes Z_1^*\right)^*,\ \text{with } Z_1, Z_2 \in \mathcal{C} \text{ and } X \in \mathcal{O}\}.\end{aligned}$$

But, $Z_2^* \otimes X \otimes Z_1^* \in \langle \mathcal{O} \rangle$. Thus, $\langle \mathcal{O}^\vee \rangle = \langle \mathcal{O} \rangle^\vee$. □

The next lemma will be useful to prove the subsequent Theorem 3.9.

Lemma 3.8. *Let $\mathcal{J}$ be a left ideal (respectively, right ideal, ideal) of a monoidal category $\mathcal{C}$ and X, Y two objects of $\mathcal{C}$.*

(1) *For every object $A \in \mathcal{J}$, we have $\langle A \rangle^l \subseteq \mathcal{J}$ (respectively, $\langle A \rangle^r \subseteq \mathcal{J}$, $\langle A \rangle \subseteq \mathcal{J}$).*

(2) *If $X \lneq Y$, then $\langle X \rangle^l \subseteq \langle Y \rangle^l$, $\langle X \rangle^r \subseteq \langle Y \rangle^r$ and $\langle X \rangle \subseteq \langle Y \rangle$.*

Proof. We prove the "left" versions, the other ones follow by analogous arguments.

(1) For every $Y \in \langle A \rangle^l$, there exists some object $Z \in \mathcal{C}$ such that $Y \lneq Z \otimes A$. Thus, $Y \in \mathcal{J}$ since $\mathcal{J}$ is a left ideal.

(2) Let $U \in \langle X \rangle^l$. Then, there exists some object $Z \in \mathcal{C}$ such that $U \lneq Z \otimes X$. Hence, $U \lneq Z \otimes Y$ by hypothesis. Thus, $U \in \langle Y \rangle^l$. □

The following theorem characterizes when a subclass of objects is a left (or right) ideal of $\mathcal{C}$.

Theorem 3.9. *Let $\mathcal{J} \subseteq \mathcal{C}_0$ be a subclass of a monoidal category $\mathcal{C}$. The following conditions are equivalent.*

(1) *$\mathcal{J}$ is a left (respectively, right) ideal of $\mathcal{C}$.*

(2) *For every $X \in \mathcal{J}$, we have $\langle X \rangle^l \subseteq \mathcal{J}$ (respectively, $\langle X \rangle^r \subseteq \mathcal{J}$).*

(3) *For every (non empty) class of objects $\mathcal{O} \subseteq \mathcal{C}_0$, we have $\mathcal{O} \cdot \mathcal{J} \subseteq \mathcal{J}$ (respectively, $\mathcal{J} \cdot \mathcal{O} \subseteq \mathcal{J}$).*

Proof. We will only prove the "left" versions since the "right" ones follow analogously.

(1) $\Longleftrightarrow$ (2) Let $X \in \mathcal{J}$. Then, for every $U \in \langle X \rangle^l$, there exists, by definition, an object $Z \in \mathcal{C}$ such that $U \lneq Z \otimes X$. But, $Z \otimes X \in \mathcal{J}$ since $\mathcal{J}$ is a left ideal. Thus, $U \in \mathcal{J}$.

Conversely, let $Y \lneq X \in \mathcal{J}$. Then, $\langle Y \rangle^l \subseteq \langle X \rangle^l \subseteq \mathcal{J}$ by hypothesis and Lemma 3.8, and hence $Y \in \mathcal{J}$. On the other hand, for every $Y \in \mathcal{C}$ and $X \in \mathcal{J}$ such that $Y \lneq X$, we have $Y \otimes X \in \mathcal{J}$ since $X \in \mathcal{J}$ implies by hypothesis that $\langle X \rangle^l \subseteq \mathcal{J}$. Thus, $Y \otimes X \in \mathcal{J}$.

(1) $\Longleftrightarrow$ (3) Let $\mathcal{O} \subseteq \mathcal{C}_0$. Then, the inclusion $\mathcal{O} \cdot \mathcal{J} \subseteq \mathcal{J}$ is clear since $\mathcal{J}$ is a left ideal of $\mathcal{C}$.

Conversely, let $Y \lneq X \in \mathcal{J}$. Then, $Y \otimes X \in \{Y\} \cdot \mathcal{J} \subseteq \mathcal{J}$, and hence $Y \otimes X \in \mathcal{J}$. On the other hand, for every $Y \in \mathcal{C}$ and $X \in \mathcal{J}$ such that $Y \lneq X$, we have $Y \in \{I\} \cdot \mathcal{J} \subseteq \mathcal{J}$. Thus, $Y \in \mathcal{J}$. □

Corollary 3.10. *Let $\mathcal{J}$ be an ideal of a monoidal category $\mathcal{C}$. Then*

$$\mathcal{J} = \bigcap_{X \in \mathcal{J}} \langle X \rangle.$$

A similar formula can be obtained in case $\mathcal{J}$ is a left (respectively, right) ideal (using $\langle X \rangle^l$ and $\langle X \rangle^r$, respectively).

Proof. This follows immediately from Theorem 3.9. □

Definition 3.11. Let $\mathcal{C}$ be a monoidal category and $\mathcal{J}, \mathcal{J}' \subseteq \mathcal{C}_0$ two subclasses. The *product* of $\mathcal{J}$ and $\mathcal{J}'$ is defined as follows

$$\mathcal{J} \cdot \mathcal{J}' := \big\{ Y \in \mathcal{C},\ Y \lneq X \otimes X',\ \text{with } X \in \mathcal{J} \text{ and } X' \in \mathcal{J}' \big\}.$$

Let $\mathcal{C}$ be a monoidal category and $\mathcal{O}, \mathcal{S} \subseteq \mathcal{C}_0$ subclasses. We denote by $\langle \mathcal{O} \rangle^l_{\mathcal{S}}$ and call, the *set generated on the left by $\mathcal{O}$ with respect to $\mathcal{S}$*, the class

$$\langle \mathcal{O} \rangle^l_{\mathcal{S}} := \big\{ Y \in \mathcal{C},\ Y \lneq Z \otimes X,\ \text{with } Z \in \mathcal{S},\ X \in \mathcal{O} \big\}.$$

Similarly is defined $\langle \mathcal{O} \rangle^r_{\mathcal{S}}$, *the set generated on the right by $\mathcal{O}$ with respect to $\mathcal{S}$* (by tensoring $X \in \mathcal{O}$ on the right this time by objects of $\mathcal{S}$) and $\langle \mathcal{O} \rangle_{\mathcal{S}}$, the *set generated by* $\mathcal{O}$ *with respect to* $\mathcal{S}$ (by tensoring $X \in \mathcal{O}$ on the left and right simultaneously by objects of $\mathcal{S}$).

Thanks to the above notations, the product $\mathcal{J} \cdot \mathcal{J}'$ can be rewritten in terms of generating sets, as shown in the next proposition.

Proposition 3.12. *The product $\mathcal{J} \cdot \mathcal{J}'$ introduced in Definition 3.11 is then*

$$\mathcal{J} \cdot \mathcal{J}' = \langle \mathcal{J} \rangle^l_{\mathcal{J}'} = \langle \mathcal{J}' \rangle^r_{\mathcal{J}}.$$

Proof. Straightforward. □

The next lemma determines when the product $\mathcal{J} \cdot \mathcal{J}'$ is a left (or right) ideal.

Lemma 3.13. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{J}, \mathcal{J}' \subseteq \mathcal{C}_0$ two subclasses.*

(1) *If $\mathcal{J}$ is a left (respectively, right) ideal, then so is $\mathcal{J} \cdot \mathcal{J}'$.*
(2) *If $\mathcal{J}'$ is a right (respectively, left) ideal, then so is $\mathcal{J} \cdot \mathcal{J}'$.*

Consequently, if $\mathcal{J}$ is a left (respectively, right) ideal and $\mathcal{J}'$ is a right (respectively, left) ideal, then $\mathcal{J} \cdot \mathcal{J}'$ is an ideal.

Proof. Straightforward. □

Lemma 3.14. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{J}, \mathcal{J}' \subseteq \mathcal{C}_0$ two subclasses.*

(1) *If $\mathcal{J}'$ is a left ideal, then $\mathcal{J} \cdot \mathcal{J}' \subseteq \mathcal{J}$.*
(2) *If $\mathcal{J}$ is a right ideal, then $\mathcal{J} \cdot \mathcal{J}' \subseteq \mathcal{J}'$.*
(3) *If $\mathcal{J}'$ is a left ideal and $\mathcal{J}$ is a right ideal, such that $\mathcal{J} \cap \mathcal{J}' = \emptyset$, then also $\mathcal{J} \cdot \mathcal{J}' = \emptyset$.*

Proof. The statements (1) and (2) are clear and using them, we get $\mathcal{J} \cdot \mathcal{J}' \subseteq \mathcal{J} \cap \mathcal{J}'$, which proves (3). □

Let $\mathcal{C}$ be a monoidal category, $\mathcal{J}$ an ideal of $\mathcal{C}$, and $\mathcal{S} \subseteq \mathcal{C}_0$ a subclass. We set

$$\mathcal{J} : \mathcal{S} := \big\{X \in \mathcal{C},\ X \otimes Y \in \mathcal{J},\ \text{for every } Y \in \mathcal{S}\big\}.$$

Proposition 3.15. *Let $\mathcal{C}$ be a monoidal category, $\mathcal{J}$ an ideal of $\mathcal{C}$, and $\mathcal{S} \subseteq \mathcal{C}_0$ a subclass. If $\mathcal{S}$ is a left ideal of $\mathcal{C}$, then $\mathcal{J} : \mathcal{S}$ is an ideal of $\mathcal{C}$.*

Proof. Let $Z \in \mathcal{C}$ and $X \in \mathcal{J} : \mathcal{S}$. Then, $Z \otimes X \otimes Y \in \mathcal{J}$ for every $Y \in \mathcal{S}$ since $X \otimes Y \in \mathcal{J}$ and $\mathcal{J}$ is a left ideal. On the other hand, let $Z \lesssim X$ be any retract of X. Then, $Z \otimes Y \lesssim X \otimes Y$. Since $X \otimes Y \in \mathcal{J}$, we obtain $Z \otimes Y \in \mathcal{J}$, and hence $Z \in \mathcal{J} : \mathcal{S}$. Thus, $\mathcal{J} : \mathcal{S}$ is a left ideal. To prove that it is also a right ideal, we only need to prove that it is closed under right tensor products. For so, let $Z \in \mathcal{C}$ and $X \in \mathcal{J} : \mathcal{S}$. Then, $X \otimes Z \otimes Y \in \mathcal{J}$ for every $Y \in \mathcal{S}$, since $Z \otimes Y \in \mathcal{S}$ as $\mathcal{S}$ is a left ideal. □

We next obtain analogues of the classical results.

Proposition 3.16. *Let $\mathcal{J}$, $\mathcal{J}'$ and $\mathcal{J}''$ be three ideals of a monoidal category $\mathcal{C}$. The following assertions hold.*

(1) $\mathcal{J} \subseteq \mathcal{J} : \mathcal{J}'$.

(2) $(\mathcal{J} : \mathcal{J}') : \mathcal{J}'' = \mathcal{J} : \mathcal{J}'' \cdot \mathcal{J}'$.

(3) $(\mathcal{J} : \mathcal{J}') : \mathcal{J}' \subseteq \mathcal{J}$.

Proof. (1) Clear.

(2) Let $X \in (\mathcal{J} : \mathcal{J}') : \mathcal{J}''$ and $Y \in \mathcal{J}'' \cdot \mathcal{J}'$. Then, $Y \lesssim Y'' \otimes Y'$ for some $Y'' \in \mathcal{J}''$, $Y' \in \mathcal{J}'$, and we have $X \otimes Y \lesssim X \otimes Y'' \otimes Y' \in \mathcal{J}$ by assumption. Therefore, $X \otimes Y \in \mathcal{J}$.

Conversely, let $X \in \mathcal{J} : \mathcal{J}'' \cdot \mathcal{J}'$. Then, for every $Y'' \in \mathcal{J}''$ and $Y' \in \mathcal{J}'$, we clearly have $X \otimes Y'' \otimes Y' \in \mathcal{J}$ as $Y'' \otimes Y' \lesssim Y'' \otimes Y' \in \mathcal{J}'' \cdot \mathcal{J}'$.

(3) Let $X \in (\mathcal{J} : \mathcal{J}') : \mathcal{J}'$. Then, $X \lesssim Y \otimes Z$ for some $Y \in \mathcal{J} : \mathcal{J}'$ and $Z \in \mathcal{J}'$. Thus, $X \in \mathcal{J}$. □

Let $\mathcal{C}$ be a monoidal category. A subclass $\mathcal{J} \subseteq \mathcal{C}_0$ is called a *right* (respectively, *left*) *direct product* provided that there exists a subclass $\mathcal{J}' \subseteq \mathcal{C}_0$ such that $\mathcal{J} \cap \mathcal{J}' = \emptyset$ and $\mathcal{C}_0 = \mathcal{J} \cdot \mathcal{J}'$ (respectively, $\mathcal{C}_0 = \mathcal{J}' \cdot \mathcal{J}$). In this case, we write $\mathcal{C}_0 = \mathcal{J} \bigotimes \mathcal{J}'$ (respectively, $\mathcal{C}_0 = \mathcal{J}' \bigotimes \mathcal{J}$).

A subclass $\mathcal{J}$ is called *trivial* if $\mathcal{J} = \emptyset$ or $\mathcal{J} = \mathcal{C}_0$.

Proposition 3.17. *There are no non trivial right* (*or left*) *direct product ideals of $\mathcal{C}$.*

Proof. Assume that $\emptyset \neq \mathcal{J} \subseteq \mathcal{C}_0$ is a right (respectively, left) direct product ideal. Then, there exists $\mathcal{J}' \subseteq \mathcal{C}_0$ such that $\mathcal{C}_0 = \mathcal{J} \bigotimes \mathcal{J}'$ (respectively, $\mathcal{C}_0 = \mathcal{J}' \bigotimes \mathcal{J}$). In particular, we get $I = X \otimes X'$ for some objects $X \in \mathcal{J}$ and $X' \in \mathcal{J}'$ (respectively, $X \in \mathcal{J}'$ and $X' \in \mathcal{J}$). It follows that $I \in \mathcal{J}$. Thus, $\mathcal{J} = \mathcal{C}_0$. □

3.2 Idempotent and radical ideals

Definition 3.18. A left (or right) ideal $\mathcal{J}$ of a monoidal category $\mathcal{C}$ is called *idempotent* when $\mathcal{J}^2 = \mathcal{J}$, where $\mathcal{J}^2 := \mathcal{J} \cdot \mathcal{J}$.

Remark 3.19. The product $\cdot$ is *associative*. That is, $\big(\mathcal{J}_1 \cdot \mathcal{J}_2\big) \cdot \mathcal{J}_3 = \mathcal{J}_1 \cdot \big(\mathcal{J}_2 \cdot \mathcal{J}_3\big)$ for any subclasses $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \subseteq \mathcal{C}_0$. Indeed, this follows directly from the associativity of the tensor product of $\mathcal{C}$.

Lemma 3.20. *If $\mathcal{J}$ is a left (or right) ideal of a monoidal category $\mathcal{C}$, then $\mathcal{J}^n \subseteq \mathcal{J}$ for every $n \in \mathbb{N}^*$.*

Proof. Immediate. □

Definition 3.21. Let $\mathcal{J}$ be a left (or right) ideal of a monoidal category $\mathcal{C}$. The *radical* of $\mathcal{J}$, denoted by $\sqrt{\mathcal{J}}$, is the following subclass

$$\sqrt{\mathcal{J}} := \big\{X \in \mathcal{C}, X^n \in \mathcal{J} \text{for some } n \in \mathbb{N}\big\}.$$

Lemma 3.22. *Let $\mathcal{J}$ be a left (or right) ideal of $\mathcal{C}$ and $n \in \mathbb{N}$. Then*

$$\mathcal{J}^n \subseteq \mathcal{J}^{n-1} \subseteq \cdots \subseteq \mathcal{J} \subseteq \sqrt{\mathcal{J}}.$$

Proof. Straightforward. □

Definition 3.23. A left (or right) ideal $\mathcal{J}$ of $\mathcal{C}$ is called *radical* if $\sqrt{\mathcal{J}} = \mathcal{J}$.

Let $\mathcal{C}$ be a monoidal category. We call *object-center* of $\mathcal{C}$, and denote it by $Z_0(\mathcal{C})$, the subclass

$$Z_0(\mathcal{C}) := \big\{X \in \mathcal{C},\ X \otimes Y \simeq Y \otimes X,\ \text{for every } Y \in \mathcal{C}\big\}.$$

The elements of $Z_0(\mathcal{C})$ will be called *central*. Clearly, the unit object I is central and if $\mathcal{C}$ is braided, then $Z_0(\mathcal{C}) = \mathcal{C}_0$.

The following two propositions establish sufficient conditions under which the radical of a one-sided ideal carries the structure of an ideal.

Proposition 3.24. *Let $\mathcal{J}$ be a left (or right) ideal of a monoidal category $\mathcal{C}$. Then*

(1) *$\sqrt{\mathcal{J}}$ is closed under retractions.*

(2) *If $\sqrt{\mathcal{J}} \subseteq Z_0(\mathcal{C})$, then $\sqrt{\mathcal{J}}$ is an ideal of $\mathcal{C}$.*

Proof. (1) Let $Y \lneqq X \in \sqrt{\mathcal{J}}$. Then, $X^n \in \mathcal{J}$ for some $n \in \mathbb{N}$. It follows that $Y^n \lneqq X^n$. Since $\mathcal{J}$ is closed under retractions, we get $Y^n \in \mathcal{J}$, and hence $Y \in \sqrt{\mathcal{J}}$.

(2) Let $Y \in \mathcal{C}$ and $X \in \sqrt{\mathcal{J}}$. Then, $X^n \in \mathcal{J}$ for some $n \in \mathbb{N}$, and hence $(Y \otimes X)^n \simeq Y^n \otimes X^n$ (respectively, $(X \otimes Y)^n \simeq X^n \otimes Y^n$) by the assumption $\sqrt{\mathcal{J}} \subseteq Z_0(\mathcal{C})$. It follows that $Y \in \sqrt{\mathcal{J}}$. Using Item (1), we conclude that $\sqrt{\mathcal{J}}$ is an ideal of $\mathcal{C}$. □

Let $\mathcal{J}$ be a left (respectively, right) ideal of a monoidal category $\mathcal{C}$. Obviously, if $\sqrt{\mathcal{J}} \subseteq \mathcal{J}$, then $\sqrt{\mathcal{J}}$ is a left (respectively, right) ideal of $\mathcal{C}$. More broadly, we have the following result.

Proposition 3.25. *Let $\mathcal{J}$ be a left (respectively, right) ideal of a pivotal category $\mathcal{C}$ and $X \in \mathcal{J}$, an object satisfying $d_X b'_X = 1_I$ (respectively, $d'_X b_X = 1_I$). Then*

(1) *$\mathcal{J}$ is radical.*

(2) *$\sqrt{\mathcal{J}} = \mathcal{C}_0$.*

In particular, $\sqrt{\mathcal{J}}$ is an ideal of $\mathcal{C}$.

Proof. (1) Let $X \in \mathcal{J}$ be as assumed. Then, $X^* \otimes X \in \mathcal{J}$ (respectively, $X \otimes X^* \in \mathcal{J}$). By hypothesis, I is a retract of $X^* \otimes X$ (respectively, $X \otimes X^*$). Hence, $I \in \mathcal{J}$, and thus $\mathcal{J} = \mathcal{C}_0$ by Remark 3.2(1).

(2) Since $\mathcal{J} \subseteq \sqrt{\mathcal{J}}$, we immediately get $\sqrt{\mathcal{J}} = \mathcal{C}_0$ by Item (1).
The particular statement is clear. □

The main properties of the radical of a one-sided ideal are summarized in the following theorem and the subsequent corollaries.

Theorem 3.26. *Let $\mathcal{J}$ be a right ideal and $\mathcal{J}'$ a left ideal of a pivotal category $\mathcal{C}$.*

(1) *If $\mathcal{J} \subseteq \mathcal{J}'$, then $\sqrt{\mathcal{J}} \subseteq \sqrt{\mathcal{J}'}$.*

(2) $\sqrt{\mathcal{J} \cdot \mathcal{J}'} = \sqrt{\mathcal{J} \cap \mathcal{J}'} = \sqrt{\mathcal{J}} \cap \sqrt{\mathcal{J}'}$.

(3) *For every $n \in \mathbb{N}$, we have $\sqrt{\mathcal{J}^n} = \sqrt{\mathcal{J}}$.*

(4) *$\sqrt{\mathcal{J}} = \mathcal{C}_0$ if and only if $\mathcal{J} = \mathcal{C}_0$.*

Proof. (1) Clear.

(2) For the first equality, since $\mathcal{J} \cdot \mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J} \cdot \mathcal{J}' \subseteq \mathcal{J}'$ by Lemma 3.14(1) and (2), we get $\mathcal{J} \cdot \mathcal{J}' \subseteq \mathcal{J} \cap \mathcal{J}'$. Using Proposition 3.26(1) we obtain $\sqrt{\mathcal{J} \cdot \mathcal{J}'} \subseteq \sqrt{\mathcal{J} \cap \mathcal{J}'}$. For the reciprocal inclusion, let $X \in \sqrt{\mathcal{J} \cap \mathcal{J}'}$, then $X^n \in \mathcal{J} \cap \mathcal{J}'$ for some $n \in \mathbb{N}$, which implies that $X^n \otimes X^n \in \mathcal{J} \cdot \mathcal{J}'$. Since $\mathcal{J}$ is a right ideal, we get $X^n \otimes \left(X^n\right)^* \otimes X^n \in \mathcal{J} \cdot \mathcal{J}'$. But $\left(X^n \lneq X^n \otimes \left(X^n\right)^* \otimes X^n, b_X \otimes 1_X, d_X \otimes 1_X\right)$, then $X^n \in \mathcal{J} \cdot \mathcal{J}'$, which proves the second inclusion.

For the second equality, since $\mathcal{J} \cap \mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J} \cap \mathcal{J}' \subseteq \mathcal{J}'$, we get $\sqrt{\mathcal{J} \cap \mathcal{J}'} \subseteq \sqrt{\mathcal{J}}$ and $\sqrt{\mathcal{J} \cap \mathcal{J}'} \subseteq \sqrt{\mathcal{J}'}$, and hence $\sqrt{\mathcal{J} \cap \mathcal{J}'} \subseteq \sqrt{\mathcal{J}} \cap \sqrt{\mathcal{J}'}$. For the reciprocal inclusion, let $X \in \sqrt{\mathcal{J}} \cap \sqrt{\mathcal{J}'}$, then $X \in \sqrt{\mathcal{J}}$ and $X \in \sqrt{\mathcal{J}'}$, and so $X^n \in \mathcal{J}$ and $X^{n'} \in \mathcal{J}'$ for some $n \in \mathbb{N}$ and $n' \in \mathbb{N}$. Assume, without loss of generality, that $n > n'$ (the same procedure applies if $n' > n$), then $n = n' + p$ for some $p \in \mathbb{N}$. Since $\mathcal{J}'$ is a left ideal, $X^n = X^p \otimes X^{n'} \in \mathcal{J}'$. Hence, $X^n \in \mathcal{J} \cap \mathcal{J}'$. Therefore, $X \in \sqrt{\mathcal{J} \cap \mathcal{J}'}$.

(3) Let's proceed by induction. For $n = 2$, by Theorem 3.26(2.), we have $\sqrt{\mathcal{J}^2} = \sqrt{\mathcal{J}} \cap \sqrt{\mathcal{J}} = \sqrt{\mathcal{J}}$. Assume that the assertion holds true for $n - 1$, then we have

$$\sqrt{\mathcal{J}^n} = \sqrt{\mathcal{J}^{n-1} \cdot \mathcal{J}} = \sqrt{\mathcal{J}^{n-1}} \cap \sqrt{\mathcal{J}} = \sqrt{\mathcal{J}} \cap \sqrt{\mathcal{J}} = \sqrt{\mathcal{J}}.$$

(4) Assume that $\sqrt{\mathcal{J}} = \mathcal{C}_0$. Then, $I \in \sqrt{\mathcal{J}}$, and so $I^n = I \in \mathcal{J}$ for some $n \in \mathbb{N}$. Thus, $\mathcal{J} = \mathcal{C}_0$.

Conversely, if $\mathcal{J} = \mathcal{C}_0$, then $\sqrt{\mathcal{J}} = \mathcal{C}_0$ since $\mathcal{J} \subseteq \sqrt{\mathcal{J}}$. □

Corollary 3.27. *Let $\mathcal{J}$ be a right ideal and $\mathcal{J}'$ a left ideal of a pivotal category $\mathcal{C}$. If $\mathcal{J}' \subseteq \mathcal{J}$, then $\sqrt{\mathcal{J} \cdot \mathcal{J}'} = \sqrt{\mathcal{J}'}$.*

Proof. Clear from Theorem 3.26(1) and (2). □

Corollary 3.28. *Let $\mathcal{J}$ and $\mathcal{J}'$ be (left or right) ideals of a pivotal category $\mathcal{C}$. If $\mathcal{J}'$ is radical and $\mathcal{J}' \subseteq \mathcal{J}$, then $\mathcal{J}$ is also radical.*

Proof. This holds from Theorem 3.26(1) and (4). □

Corollary 3.29. *Let $\mathcal{J}$ and $\mathcal{J}'$ be two ideals of a pivotal category $\mathcal{C}$. The following conditions are equivalent.*

(1) *$\mathcal{J} \cdot \mathcal{J}'$ is a radical ideal of $\mathcal{C}$.*

(2) *$\mathcal{J}' \cap \mathcal{J}$ is a radical ideal of $\mathcal{C}$.*

Moreover, if one of the above conditions holds, then both of $\mathcal{J}$ and $\mathcal{J}'$ are radical ideals.

Proof. The equivalence holds evidently from Proposition 3.26(2). The second assertion holds by combining Lemmata 3.13 and 3.14, and Theorem 3.26(1). □

Corollary 3.30. *Let $\mathcal{J}$ and $\mathcal{J}'$ be (left or right) ideals of a pivotal category $\mathcal{C}$. If $\mathcal{J}$ is radical and $\mathcal{J} \cap \sqrt{\mathcal{J}'} = \emptyset$, then $\mathcal{J} \cdot \mathcal{J}' = \emptyset$.*

Proof. This follows clearly from Theorem 3.26(2). □

Corollary 3.31. *The intersection of a nonempty finite family of radical ideals of a pivotal category $\mathcal{C}$ is a radical ideal of $\mathcal{C}$.*

Proof. Let $\{\mathcal{J}_i\}_i$ be a nonempty finite family of radical ideals and $\mathcal{J} = \cap_i \mathcal{J}_i$. Then, we have $\sqrt{\mathcal{J}} = \cap_i \sqrt{\mathcal{J}_i} = \mathcal{J}$ by Theorem 3.26(2). □

3.3 Prime and primary ideals An ideal $\mathcal{J}$ of $\mathcal{C}$ is called *principal* provided that it is generated by a singleton. That is, $\mathcal{J} = \langle X \rangle$, for some $X \in \mathcal{C}$. Consequently, as proved is the next proposition, the product "$\cdot$" of principal ideals generated by central objects is again principal and generated by a central object.

Proposition 3.32. *Let X and Y be two objects of a monoidal category $\mathcal{C}$.*

(1) *If X is central, then $\langle X \rangle^l \cdot \langle Y \rangle^l = \langle X \otimes Y \rangle^l$.*
(2) *If Y is central, then $\langle X \rangle^r \cdot \langle Y \rangle^r = \langle X \otimes Y \rangle^r$.*
(3) *If both X and Y are central, then $\langle X \rangle \cdot \langle Y \rangle = \langle X \otimes Y \rangle$.*

Proof. We prove only the first assertion, as the remaining ones follow by analogous arguments. Let $U \in \langle X \rangle^l \cdot \langle Y \rangle^l$. Then, $U \lneq A \otimes B$ for some $A \in \langle X \rangle^l$ and $B \in \langle Y \rangle^l$. Hence, there exist objects $Z_1, Z_2 \in \mathcal{C}$ such that $A \lneq Z_1 \otimes X$ and $B \lneq Z_2 \otimes Y$, and consequently $U \lneq Z_1 \otimes X \otimes Z_2 \otimes Y$. It follows that $U \lneq Z_1 \otimes Z_2 \otimes X \otimes Y$ by assumption, and thus $U \in \langle X \otimes Y \rangle^l$.

Conversely, let $U \in \langle X \otimes Y \rangle^l$. Then, there exists $Z \in \mathcal{C}$ such that $U \lneq Z \otimes X \otimes Y$. It follows that $U \lneq Z \otimes X \otimes I \otimes Y$, and thus $U \in \langle X \rangle^l \cdot \langle Y \rangle^l$. □

Definition 3.33. An ideal $\mathcal{J}$ of a monoidal category $\mathcal{C}$ is called *proper* if $\mathcal{J} \neq \mathcal{C}_0$.

It is clear, from Remark 3.2(1), that an ideal $\mathcal{J}$ of $\mathcal{C}$ is proper if and only if $I \notin \mathcal{J}$.

We now investigate primeness for two-sided ideals in analogy with the ring-theoretic setting.

Definition 3.34. A proper ideal $\mathcal{J}$ of a monoidal category $\mathcal{C}$ is *prime* if, for all objects $X, Y \in \mathcal{C}$ such that $X \otimes Y \in \mathcal{J}$, it follows that $X \in \mathcal{J}$ or $Y \in \mathcal{J}$.

Clearly, if $\mathcal{J}$ is prime, then $\mathcal{J}$ is radical.

Theorem 3.35. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{J}$ an ideal of $\mathcal{C}$. If $\mathcal{J}$ is prime, then for any ideals $\mathcal{J}'$ and $\mathcal{J}''$ of $\mathcal{C}$, the condition $\mathcal{J}' \cdot \mathcal{J}'' \subseteq \mathcal{J}$ implies that $\mathcal{J}' \subseteq \mathcal{J}$ or $\mathcal{J}'' \subseteq \mathcal{J}$. If $Z_0(\mathcal{C}) = \mathcal{C}_0$, then the converse holds as well.*

Proof. Assume that $\mathcal{J}$ is prime. Let $\mathcal{J}'$ and $\mathcal{J}''$ be two ideals of $\mathcal{C}$ such that $\mathcal{J}' \cdot \mathcal{J}'' \subseteq \mathcal{J}$. Suppose that $\mathcal{J}'' \nsubseteq \mathcal{J}$. Then, there exist two objects $Y'' \in \mathcal{J}''$ and $Y'' \notin \mathcal{J}$. For every $Y' \in \mathcal{J}'$, we have $Y' \otimes Y'' \in \mathcal{J}' \cdot \mathcal{J}'' \subseteq \mathcal{J}$. It follows that $Y' \in \mathcal{J}$ or $Y'' \in \mathcal{J}$ since $\mathcal{J}$ is prime. As $Y'' \notin \mathcal{J}$, we get $Y' \in \mathcal{J}$. Thus, $\mathcal{J}'' \subseteq \mathcal{J}$.

Conversely, let $X, Y \in \mathcal{C}$ such that $X \otimes Y \in \mathcal{J}$ and consider the ideals $\langle X \rangle$ and $\langle Y \rangle$. By Proposition 3.32 and Lemma 3.8, we have $\langle X \rangle \cdot \langle Y \rangle = \langle X \otimes Y \rangle \subseteq \mathcal{J}$ since X, Y are central idempotent objects. Then, by assumption, we have $\langle X \rangle \subseteq \mathcal{J}$ or $\langle Y \rangle \subseteq \mathcal{J}$. It follows that $X \in \mathcal{J}$ or $Y \in \mathcal{J}$. □

Corollary 3.36. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{P}$ a prime ideal of $\mathcal{C}$. If $\mathcal{J}$ and $\mathcal{J}'$ are two ideals of $\mathcal{C}$ such that $\mathcal{J} \cap \mathcal{J}' \subseteq \mathcal{P}$, then $\mathcal{J} \subseteq \mathcal{P}$ or $\mathcal{J}' \subseteq \mathcal{P}$.*

Proof. This follows from Theorem 3.35 since $\mathcal{J} \cdot \mathcal{J}' \subseteq \mathcal{J} \cap \mathcal{J}'$ by Lemma 3.14(1) and (2). □

Example 3.37. Let $\mathcal{C}$ be a monoidal category such that its ground ring $\mathtt{K}_{\mathcal{C}}$ is a field. Then, the class $\mathcal{N}(\mathcal{C})$ of negligible objects of $\mathcal{C}$ is a prime ideal.

Definition 3.38. A proper ideal $\mathcal{J}$ of a monoidal category $\mathcal{C}$ is called *primary* if, for all objects $X, Y \in \mathcal{C}$ such that $X \otimes Y \in \mathcal{J}$, it follows that $X \in \mathcal{J}$ or $Y \in \sqrt{\mathcal{J}}$.

Clearly, every prime ideal is primary. If moreover $\mathcal{C}$ is braided, then $\mathcal{J}$ is primary if and only if, for all objects $X, Y \in \mathcal{C}$ such that $X \otimes Y \in \mathcal{J}$, it follows that $Y \in \mathcal{J}$ or $X \in \sqrt{\mathcal{J}}$.

Although the intersection of prime ideals of $\mathcal{C}$ need not be prime, the intersection of primary ideals of $\mathcal{C}$ sharing the same radical is a primary ideal of $\mathcal{C}$. An ideal $\mathcal{J}$ will be called $\mathcal{K}$-primary if $\mathcal{J}$ is primary and $\sqrt{\mathcal{J}} = \mathcal{K}$.

Theorem 3.39. *Let $\{\mathcal{J}_i\}_{i\in\Omega}$ be a nonempty finite family of $\mathcal{K}$-primary ideals of $\mathcal{C}$, and set $\mathcal{J} = \cap_i \mathcal{J}_i$. Then, $\mathcal{J}$ is $\mathcal{K}$-primary.*

Proof. First, we have $\sqrt{\mathcal{J}} = \cap_i \sqrt{\mathcal{J}_i} = \cap_i \mathcal{K} = \mathcal{K}$ by Theorem 3.26(2). On the other hand, let $X, Y \in \mathcal{C}$ be with $X \otimes Y \in \mathcal{J}$ and $X \notin \mathcal{J}$. Then, $X \notin \mathcal{J}_k$ for some $k \in \Omega$. Since $X \otimes Y \in \mathcal{J}_k$ and $\mathcal{J}_k$ is $\mathcal{K}$-primary, we get $Y \in \sqrt{\mathcal{J}_k} = \mathcal{K} = \sqrt{\mathcal{J}}$. Thus, $\mathcal{J}$ is $\mathcal{K}$-primary. □

Remark 3.40. In ring theory, the elementwise definition of primeness, as we adopted here, turns out not to be a good idea. Indeed, many noncommutative rings R have no factor rings $\frac{R}{P}$ which are domains (which is equivalent, in the commutative case, to P being a prime ideal of R); for example, matrix ring over a field. Thus, a more relaxed definition of primeness is desirable in the noncommutative setting. The idea is to adopt the idealwise equivalent formulation of the commutative definition; that is, replacing products of elements with products of ideals, as first proposed by Krull in 1928. The same phenomenon occurs for primariness, where the radical of an ideal is defined as the intersection of all prime ideals conaining it. In our framework, the product of ideals is well-defined in view of Definition 3.11, which opens the door to further future developments.

4 One-sided $\mathcal{U}$-ideals: a generalization

In this section, we introduce one-sided $\mathcal{U}$-ideals, for $\mathcal{U}$ an endofunctor of a given monoidal category $\mathcal{C}$, as a generalization of one-sided ideals. Analogously to the ordinary case, we study various basic properties and establish some characterizations of $\mathcal{U}$-ideals.

Definition 4.1. Let $\mathcal{C}$ be a monoidal category, $\mathcal{J} \subseteq \mathcal{C}_0$ a subclass, and $\mathcal{U}$ an endofunctor of $\mathcal{C}$. We say that $\mathcal{J}$ is a left $\mathcal{U}$-ideal of $\mathcal{C}$ provided that

(1) For all objects $X \in \mathcal{J}$ and $Y \in \mathcal{C}$, we have $\mathcal{U}(Y) \otimes X \in \mathcal{J}$ (**closedness under left $\mathcal{U}$-tensor products**).

(2) For all objects $X \in \mathcal{J}$ and $Y \in \mathcal{C}$ such that $\mathcal{U}(Y) \lneq X$ is a retract of X, we have $\mathcal{U}(Y) \in \mathcal{J}$ (**closedness under $\mathcal{U}$-retractions**).

In other words, $\mathcal{J}$ is a left $\mathcal{U}$-ideal if, for all $X \in \mathcal{J}$ and $Y, Z \in \mathcal{C}$, the condition $\mathcal{U}(Y) \lneq \mathcal{U}(Z) \otimes X$ implies that $\mathcal{U}(Y) \in \mathcal{J}$.
A right $\mathcal{U}$-ideal is defined similarly by replacing "$\mathcal{U}(Y) \otimes X \in \mathcal{J}$", in the first condition (1), by "$X \otimes \mathcal{U}(Y) \in \mathcal{J}$".
A $\mathcal{U}$-ideal is a subclass of $\mathcal{C}_0$ which is both a left and right $\mathcal{U}$-ideal.

Remark 4.2. A subclass $\mathcal{J} \subseteq \mathcal{C}_0$ is a left (respectively, right) $\mathcal{U}$-ideal of $\mathcal{C}$ if and only if $\mathcal{J}$ is a left (respectively, right) ideal of $\mathcal{U}(\mathcal{J})$.

Left $\mathcal{U}$-ideals and right $\mathcal{U}$-ideals in a pivotal category are generally distinguished classes. However, if the category is moreover braided, then both notions coincide.

Clearly, any left (respectively, right) ideal can be seen as a left (respectively, right) $\mathcal{U}$-ideal for every endofunctor $\mathcal{U}$ of $\mathcal{C}$. However, the converse does not hold in general, as established by the next example.

Example 4.3. Let $\mathcal{U}$ be the identity endofunctor on $\mathcal{C}$ and $\mathcal{J} \subseteq \mathcal{C}_0$ any subclass containing the unit object, such that $\mathcal{J} \neq \mathcal{C}_0$. Then, $\mathcal{J}$ is a $\mathcal{U}$-ideal which is not an ideal of $\mathcal{C}$. Indeed, it is straightforward that $\mathcal{J}$ is not an ideal. To see that it is a $\mathcal{U}$-ideal, let $X \in \mathcal{J}$ and $Y \in \mathcal{C}$, then we have $\mathcal{U}(Y) \otimes X = I \otimes X = X \in \mathcal{J}$. On the other hand, the fact that $I \in \mathcal{J}$ ensures the second condition of the definition of a $\mathcal{U}$-ideal.

Proposition 4.4. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{U}$ an endofunctor of $\mathcal{C}$. If $\mathcal{U}$ is essentially surjective, then every left (respectively, right) $\mathcal{U}$-ideal of $\mathcal{C}$ is a left (respectively, right) ideal of $\mathcal{C}$.*

Proof. Let $\mathcal{J}$ be any left (respectively, right) $\mathcal{U}$-ideal. For every $X \in \mathcal{J}$ and $Y \in \mathcal{C}$, $Y \simeq \mathcal{U}(Y')$ for some $Y' \in \mathcal{C}$ by hypothesis. Since $\mathcal{J}$ is particularly replete and closed under left (respectively, right) $\mathcal{U}$-tensor products, we have $Y \in \mathcal{J}$. On the other hand, for every $Y \in \mathcal{C}$ such that $Y \lneq X$ and $X \in \mathcal{J}$, we have $Y \simeq \mathcal{U}(Y')$ for some $Y' \in \mathcal{C}$, which implies that $\mathcal{U}(Y') \lneq X$. Hence, $\mathcal{U}(Y') \in \mathcal{J}$. Thus, $Y \in \mathcal{J}$. $\square$

Proposition 4.5. *The intersection of left (respectively, right) $\mathcal{U}$-ideals of $\mathcal{C}$ is a left (respectively, right) $\mathcal{U}$-ideal of $\mathcal{C}$.*

Proof. Straightforward. □

Example 4.6. Let $\mathcal{J}$ be an ideal of $\mathcal{C}$, $A \in Z_0(\mathcal{C})$ and $\mathcal{U}_A$ the endofunctor of $\mathcal{C}$ sending any object $Y \in \mathcal{C}$ to A and any morphism of $\mathcal{C}$ to the identity 1_A. Then, although $\sqrt{\mathcal{J}}$ is not necessarily an ideal, it is a $\mathcal{U}$-ideal. Indeed, $\sqrt{\mathcal{J}}$ is closed under retractions by Proposition 3.24(1). On the other hand, for all objects $Y \in \mathcal{C}$ and $X \in \sqrt{\mathcal{J}}$, we have $X^n \in \mathcal{J}$ for some $n \in \mathbb{N}$, and hence $(\mathcal{U}(Y) \otimes X)^n \simeq A^n \otimes X^n \in \mathcal{J}$. Similarly, we get $(X \otimes \mathcal{U}(Y))^n \in \mathcal{J}$. Thus, $\sqrt{\mathcal{J}}$ is also closed under left and right $\mathcal{U}$-tensor products.

Proposition 4.7. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{U}$ a strong monoidal endofunctor of $\mathcal{C}$. If $\mathcal{J}$ is a left (respectively, right) ideal of $\mathcal{C}$, then $\mathcal{U}(\mathcal{J})$ is a left (respectively, right) $\mathcal{U}$-ideal of $\mathcal{C}$.*

Proof. Let $Y \lneq X$ be such that $\mathcal{U}(Y)$ is a retract of $X \in \mathcal{U}(\mathcal{J})$. Then, $\mathcal{U}(Y) \lneq \mathcal{U}(X')$ for some $X' \in \mathcal{J}$. But, $\mathcal{U}(X') \in \mathcal{U}(\mathcal{J})$, then $\mathcal{U}(Y) \in \mathcal{U}(\mathcal{J})$. On the other hand, for every $X \in \mathcal{U}(\mathcal{J})$ and $Y \in \mathcal{C}$, we have $\mathcal{U}(Y) \otimes X = \mathcal{U}(Y) \otimes \mathcal{U}(X') \simeq \mathcal{U}(Y \otimes X')$ for some $X' \in \mathcal{J}$, since $\mathcal{U}$ is a strong monoidal functor. Thus, $\mathcal{U}(Y) \otimes X \in \mathcal{U}(\mathcal{J})$. Likewise, one can show that $\mathcal{U}(\mathcal{J})$ is a right ideal in the other statement. □

Let $\mathcal{C}$ be a monoidal category and $\mathcal{J}$ a left (respectively, right) ideal of $\mathcal{C}$. Then, one may easily check that the pre-image $\mathcal{U}^{-1}(\mathcal{J})$ of $\mathcal{J}$ is again a left (respectively, right) ideal of $\mathcal{C}$. The following proposition provides a weaker version of this fact.

Proposition 4.8. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{U}$ a strong monoidal endofunctor of $\mathcal{C}$. If $\mathcal{J}$ is a left (respectively, right) $\mathcal{U}$-ideal of $\mathcal{C}$, then $\mathcal{U}^{-1}(\mathcal{J})$ is a left (respectively, right) ideal of $\mathcal{C}$.*

Proof. Let $Y \lneq X$ be a retract of X and let's show that $\mathcal{U}(Y) \in \mathcal{J}$. Since $Y \lneq X$, we get $\mathcal{U}(Y) \lneq \mathcal{U}(X)$. But, $\mathcal{U}(X) \in \mathcal{U}(\mathcal{J}) \subseteq \mathcal{J}$, then $\mathcal{U}(Y) \in \mathcal{J}$, i.e., $Y \in \mathcal{U}^{-1}(\mathcal{J})$. On the other hand, for every $X \in \mathcal{U}^{-1}(\mathcal{J})$ and $Y \in \mathcal{C}$, we have $\mathcal{U}(Y \otimes X) \simeq \mathcal{U}(Y) \otimes \mathcal{U}(X) \in \mathcal{J}$ since $\mathcal{J}$ is closed under left $\mathcal{U}$-tensor products, then $Y \otimes X \in \mathcal{U}^{-1}(\mathcal{J})$. Thus, $\mathcal{U}^{-1}(\mathcal{J})$ is a left ideal. Similarly, one can show that $\mathcal{U}^{-1}(\mathcal{J})$ is a right ideal in the other statement. □

The following result establishes the connection between one-sided ideals and one-sided $\mathcal{U}$-ideals.

Corollary 4.9. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{U}$ a strong monoidal endofunctor of $\mathcal{C}$. Then, $\mathcal{J}$ is a $\mathcal{U}$-ideal of $\mathcal{C}$ if and only if $\mathcal{U}^{-1}(\mathcal{J})$ is an ideal of $\mathcal{C}$.*

Proof. The direct implication being done by Proposition 4.8, the converse holds by Proposition 4.7. □

The following lemma is instrumental in proving Theorem 4.11, which characterizes one-sided $\mathcal{U}$-ideals.

Lemma 4.10. *Let $\mathcal{C}$ be a pivotal category, $\mathcal{U}$ a strong monoidal endofunctor, and $\mathcal{J} \subseteq \mathcal{C}_0$ a subclass such that $\mathcal{U}(\mathcal{J}) \subseteq \mathcal{J}$.*

(1) *If $\mathcal{J}$ is a left (or right) $\mathcal{U}$-ideal, then $\mathcal{J}$ is closed under $\mathcal{U}$-biduality in the sense that $\mathcal{U}(X)^{**} \in \mathcal{J}$ for every $X \in \mathcal{J}$.*

(2) *If $\mathcal{J}$ is a $\mathcal{U}$-ideal, then $\mathcal{J}$ is closed under $\mathcal{U}$-duality in the sense that $\mathcal{U}(X)^* \in \mathcal{J}$ for every $X \in \mathcal{J}$.*

Proof. (1) Let $X \in \mathcal{J}$. Then, $\mathcal{U}(X) \lneqq \mathcal{U}(X)$, and so $\mathcal{U}(X)^{**} \lneqq \mathcal{U}(X)$ since $\mathcal{U}(X)^{**} \lneqq \mathcal{U}(X)$ via the pivotal structure. But, $\mathcal{U}(X) \in \mathcal{U}(\mathcal{J}) \subseteq \mathcal{J}$ and $\mathcal{U}(X)^{**} \simeq \mathcal{U}(X^{**})$ by the fact that $\mathcal{U}$ being a strong monoidal functor, it preserves duals. Thus, $\mathcal{U}(X)^{**} \in \mathcal{J}$ since $\mathcal{J}$ is closed under $\mathcal{U}$-retractions.

(2) Let $X \in \mathcal{J}$. We will show that $X^* \in \mathcal{J}$. By the triangular identities (2.1) and (2.2) on X, X^* is a retract of $X^* \otimes X \otimes X^*$ via the morphisms $\alpha = (1_X \otimes b_X)$ and $\beta = (d_X \otimes 1_X)$. Then, $\mathcal{U}(X^*)$ is a retract of $\mathcal{U}(X^* \otimes X \otimes X^*) \simeq \mathcal{U}(X^*) \otimes \mathcal{U}(X) \otimes \mathcal{U}(X^*)$. But, $\mathcal{U}(X) \in \mathcal{U}(\mathcal{J}) \subseteq \mathcal{J}$ and $\mathcal{J}$ is a $\mathcal{U}$-ideal. Hence, $\mathcal{U}(X^*) \otimes \mathcal{U}(X) \otimes \mathcal{U}(X^*) \in \mathcal{J}$. Thus, $\mathcal{U}(X)^* \simeq \mathcal{U}(X^*) \in \mathcal{J}$. □

Theorem 4.11. *Let $\mathcal{U}$ be a strong monoidal endofunctor of a pivotal category $\mathcal{C}$ and $\mathcal{J} \subseteq \mathcal{C}_0$ a subclass such that $\mathcal{U}(\mathcal{J}) \subseteq \mathcal{J}$. Set $\mathcal{J}^{\vee} = \{Y \in \mathcal{C},\ Y \lneqq X^*,\ X \in \mathcal{J}\}$. Then*

(1) *$\mathcal{J}$ is a left (respectively, right) $\mathcal{U}$-ideal if and only if $\mathcal{J}^{\vee}$ is a right (respectively, left) $\mathcal{U}$-ideal.*

(2) *$\mathcal{J}$ is a $\mathcal{U}$-ideal if and only if $\mathcal{J}$ is a left (or right) $\mathcal{U}$-ideal and $\mathcal{J}^{\vee} = \mathcal{J}$.*

Proof. (1) Let $\mathcal{J}$ be a left $\mathcal{U}$-ideal, $Y \in \mathcal{J}^{\vee}$ and $Z \in \mathcal{C}$, such that $\mathcal{U}(Z) \lneq Y$. Then, $\mathcal{U}(Z) \lneq X^*$ for some $X \in \mathcal{J}$, and so $\mathcal{U}(Z) \in \mathcal{J}^{\vee}$. On the other hand, let $Z \in \mathcal{C}$ and $Y \in \mathcal{J}^{\vee}$. Since $\mathcal{U}(Z^*) \simeq \mathcal{U}(Z)^*$, we have

$$Y \otimes \mathcal{U}(Z) \lneq X^* \otimes \mathcal{U}(Z) \simeq (\mathcal{U}(Z)^* \otimes X)^*.$$

Hence, $Y \otimes \mathcal{U}(Z) \in \mathcal{J}^{\vee}$ since $\mathcal{U}(Z)^* \otimes X \in \mathcal{J}$. A similar proof can be obtained for the case when $\mathcal{J}$ is a right $\mathcal{U}$-ideal.

Conversely, if $\mathcal{J}^{\vee}$ is a right (respectively, left) $\mathcal{U}$-ideal, then by the previous direct implication, $\mathcal{J}^{\vee\vee}$ is a left (respectively, right) $\mathcal{U}$-ideal. But, clearly $\mathcal{J}^{\vee\vee} = \mathcal{J}$.

(2) The direct implication holds by Lemma 4.10(2).

Conversely, if $\mathcal{J}$ is a left (respectively, right) $\mathcal{U}$-ideal, then by Item (1) of this proposition, $\mathcal{J}$ is also a right (respectively, left) $\mathcal{U}$-ideal, and hence an ideal. □

Now, we introduce a $\mathcal{U}$-version of ideals generated by classes of objects. Such latter ideals have played an important role in establishing the existence and uniqueness of modified trace and dimension functions on ideals in pivotal categories [5].

Definition 4.12. Let $\mathcal{C}$ be a monoidal category, $\mathcal{O} \subseteq \mathcal{C}_0$ a subclass, and $\mathcal{U}$ an endofunctor of $\mathcal{C}$. We define the following classes

(1) $\langle \mathcal{O} \rangle_{\mathcal{U}}^{l} = \big\{Y \in \mathcal{C},\, \mathcal{U}(Y) \lneq \mathcal{U}(Z) \otimes X, \text{ with } Z \in \mathcal{C},\ X \in \mathcal{O}\big\}$.
(2) $\langle \mathcal{O} \rangle_{\mathcal{U}}^{r} = \big\{Y \in \mathcal{C},\, \mathcal{U}(Y) \lneq X \otimes \mathcal{U}(Z), \text{ with } Z \in \mathcal{C},\ X \in \mathcal{O}\big\}$.
(3) $\langle \mathcal{O} \rangle_{\mathcal{U}} = \big\{Y \in \mathcal{C},\, \mathcal{U}(Y) \lneq \mathcal{U}(Z) \otimes X \otimes \mathcal{U}(Z'), \text{ with } Z, Z' \in \mathcal{C},\ X \in \mathcal{O}\big\}$.

Next we establish the relationship between the above defined classes and the corresponding "dual" sets.

Corollary 4.13. *Let $\mathcal{O} \subseteq \mathcal{C}_0$ be a subclass and $\mathcal{U}$ a strong monoidal endofunctor of $\mathcal{C}$.*

(1) $\langle \mathcal{O}^{\vee} \rangle_{\mathcal{U}}^{l} = \langle \mathcal{O} \rangle_{\mathcal{U}}^{r\vee}$.
(2) $\langle \mathcal{O}^{\vee} \rangle_{\mathcal{U}}^{r} = \langle \mathcal{O} \rangle_{\mathcal{U}}^{l\vee}$.
(3) $\langle \mathcal{O}^{\vee} \rangle_{\mathcal{U}} = \langle \mathcal{O} \rangle_{\mathcal{U}}^{\vee}$.

Proof. This holds analogously to Proposition 3.7. □

Theorem 4.14. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{U}$ a strong monoidal endofunctor of $\mathcal{C}$. The classes $\langle\mathcal{O}\rangle^l_{\mathcal{U}}$, $\langle\mathcal{O}\rangle^r_{\mathcal{U}}$ and $\langle\mathcal{O}\rangle_{\mathcal{U}}$ are, respectively, a left $\mathcal{U}$-ideal, a right $\mathcal{U}$-ideal and a $\mathcal{U}$-ideal of $\mathcal{C}$. Moreover, these are, respectively, the smallest left $\mathcal{U}$-ideal, right $\mathcal{U}$-ideal and $\mathcal{U}$-ideal containing the class $\mathcal{O}$.*

Proof. We will only prove the proposition for $\langle\mathcal{O}\rangle^l_{\mathcal{U}}$ and a similar procedure can be adapted for the other classes. Let $A \in \mathcal{C}$ and $Y \in \mathcal{O}$ be two objects such that $\mathcal{U}(A) \lneq Y$. Then, we obtain $\mathcal{U}\big(\mathcal{U}(A)\big) \lneq \mathcal{U}(Y) \lneq \mathcal{U}(Z) \otimes X$ for some $Z \in \mathcal{C}$ and $X \in \mathcal{O}$. Thus, $\mathcal{U}(A) \in \langle\mathcal{O}\rangle^l_{\mathcal{U}}$. On the other hand, let $Y \in \langle\mathcal{O}\rangle^l_{\mathcal{U}}$ and $A \in \mathcal{C}$. By definition, there exist objects $Z \in \mathcal{C}$ and $X \in \mathcal{O}$ such that $\mathcal{U}(Y) \lneq \mathcal{U}(Z) \otimes X$. Let's show that $\mathcal{U}(A) \otimes Y \in \langle\mathcal{O}\rangle^l_{\mathcal{U}}$. We have $\mathcal{U}\big(\mathcal{U}(A) \otimes Y\big) \simeq \mathcal{U}\big(\mathcal{U}(A)\big) \otimes \mathcal{U}(Y)$ and we also have $\mathcal{U}\big(\mathcal{U}(A)\big) \otimes \mathcal{U}(Y) \lneq \mathcal{U}\big(\mathcal{U}(A)\big) \otimes \mathcal{U}(Z) \otimes X$, or again $\mathcal{U}\big(\mathcal{U}(A)\big) \otimes \mathcal{U}(Y) \lneq \mathcal{U}\big(\mathcal{U}(A) \otimes Z\big) \otimes X$. Thus, $\mathcal{U}(A) \otimes Y \in \langle\mathcal{O}\rangle^l_{\mathcal{U}}$. For the last statement, $\langle\mathcal{O}\rangle^l_{\mathcal{U}}$ is the smallest left $\mathcal{U}$-ideal containing $\mathcal{O}$. Indeed, let $\mathcal{J}$ be another left $\mathcal{U}$-ideal containing $\mathcal{O}$ and let's show that $\langle\mathcal{O}\rangle^l_{\mathcal{U}} \subseteq \mathcal{J}$. Let $Y \in \langle\mathcal{O}\rangle^l_{\mathcal{U}}$, then there exist objects $Z \in \mathcal{C}$ and $X \in \mathcal{O}$ such that $\mathcal{U}(Y) \lneq \mathcal{U}(Z) \otimes X$. But, $\mathcal{O} \subseteq \mathcal{J}$, and so $X \in \mathcal{J}$, which implies that $\mathcal{U}(Z) \otimes X \in \mathcal{J}$. Thus, we also get $\mathcal{U}(Y) \in \mathcal{J}$ since $\mathcal{J}$ is closed under $\mathcal{U}$-retractions. □

Corollary 4.15. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{U}$ a strong monoidal endofunctor of $\mathcal{C}$. Then, $\langle\mathcal{O}\rangle^l_{\mathcal{U}} = \bigcap_{\mathcal{J} \supseteq \mathcal{O}} \mathcal{J}$, $\langle\mathcal{O}\rangle^r_{\mathcal{U}} = \bigcap_{\mathcal{J}' \supseteq \mathcal{O}} \mathcal{J}'$ and $\langle\mathcal{O}\rangle_{\mathcal{U}} = \bigcap_{\mathcal{J}'' \supseteq \mathcal{O}} \mathcal{J}''$, where $\mathcal{J}$, $\mathcal{J}'$ and $\mathcal{J}''$ run over all left $\mathcal{U}$-ideals, right $\mathcal{U}$-ideals and $\mathcal{U}$-ideals of $\mathcal{C}$, respectively.*

Proof. Straightforward. □

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