



The spectrum of σ -frames in the adjunction between σ -frames and σ -spaces

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Abstract. In this paper, we define an adjunction between two categories: σ -frames and σ -spaces, denoted by the pair (Σ^σ, Λ) . The functor Σ^σ is constructed using the concept of σ -points. We prove that σ -points are equivalent to σ -completely prime filters, but unlike in pointfree topology, they do not correspond to prime elements. While every prime element determines a corresponding σ -point, the converse fails. For σ -frames, we define the σ -spatiality condition, which is equivalent to having enough σ -points. Dually, for σ -spaces, the σ -soberness condition is shown to be equivalent to the conjunction of the σ_0 separation axiom and σ -pointedness properties. These conditions naturally emerge from careful analysis of the adjunction morphisms.

1 Introduction

Pointfree topology centers on the algebraic structure of open sets rather than spatial points, treating points as derivative constructs. This approach

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operates with abstract lattices called *frames*, which formally model "lattices of open sets". A frame is defined as a complete lattice L satisfying the distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$ for all $a \in L$ and subsets $S \subseteq L$. Frame homomorphisms are structure-preserving maps $h : L \rightarrow M$ that preserve finite meets (including the top element) and arbitrary joins (including the bottom element). The category **Frm** comprises these frames and their homomorphisms as central objects of study.

As noted in [19] (Chapter 2), the standard spectrum functor $\Sigma : \mathbf{Frm}^{\text{op}} \rightarrow \mathbf{Top}$ is typically defined as $\Sigma L = \text{Sp}(L)$, $\Sigma h(\mathcal{F}) = h^{-1}[\mathcal{F}]$, forming an adjunction with the open-set functor $\Omega : \mathbf{Top} \rightarrow \mathbf{Frm}^{\text{op}}$.

The fundamental adjunction between the category **Top** of topological spaces and the opposite category \mathbf{Frm}^{op} of frames was first established in Isbell's classical work [17] and is thoroughly documented in Johnstone's comprehensive exposition [18].

This foundational work by Isbell reflects the core philosophy of point-free topology: that disregarding and eliminating points from spaces not only preserves their topological structure, but points can actually be reconstructed from neighborhood systems. From a categorical perspective, this reconstruction is well-behaved and indeed forms an adjunction.

Similar adjunctions between spaces and their point-free structures appear in the works:

1. The adjunction between the category **Alex** of Alexandroff spaces (see Section 2 for the definition) and the opposite category $\mathbf{Reg}\sigma\mathbf{Frm}^{\text{op}}$ of regular σ -frames, as discussed in [14, 15].
2. As shown in [3], there exists an adjunction between the category **Bor** of Borel spaces with Borel maps and the opposite category $\mathbf{Bool}\sigma\mathbf{Frm}^{\text{op}}$ of Boolean σ -frames with σ -frame homomorphisms.

In this paper, we introduce a larger adjunction between broader categories that generalizes all three adjunctions mentioned above. Specifically, we establish an adjunction between:

- The category $\sigma\mathbf{Sp}$ of σ -spaces (Definition 3.1) and σ -continuous maps (Definition 3.2).
- The opposite category $\sigma\mathbf{Frm}^{\text{op}}$ of σ -frames.

Note that:

- All three categories **Top** (topological spaces), **Alex** (Alexandroff spaces), and **Bor** (Borel spaces) are subcategories of $\sigma\mathbf{Sp}$.
- On the algebraic side, the categories **Frm** (frames), **Reg σ Frm** (regular σ -frames), and **Bool σ Frm** (Boolean σ -frames) are all subcategories of $\sigma\mathbf{Frm}$

Therefore, this unified adjunction encompasses all three previous cases as distinct special instances.

Proposition 5.1 (using Propositions 3.14 and 3.15) shows that $\Sigma^\sigma : \sigma\mathbf{Frm}^{\text{op}} \rightarrow \sigma\mathbf{Sp}$ and $\Lambda : \sigma\mathbf{Sp} \rightarrow \sigma\mathbf{Frm}^{\text{op}}$ are functors that form an adjunction. Specifically, the functor Σ^σ is a left adjoint to Λ (Theorem 5.3)

After reviewing prerequisites on frames, σ -frames, Borel spaces (measurable spaces) and Alexandroff spaces in Section 2, we introduce and study the spectrum of a σ -frame in Section 3. We define σ -points of a σ -frame (Definition 3.3). Proposition 3.4 and Corollary 3.5 establish a one-to-one correspondence between σ -points and σ -completely prime filters (Corollary 3.6). While frames exhibit correspondence between prime elements and points, this fails for σ -frames (Proposition 3.7 and Remark 3.8). For each σ -frame, we define a collection (Definition 3.11) and prove that it forms a σ -space for σ -frames (Proposition 3.12 and Corollary 3.13).

In Section 4, σ -spatiality for σ -frames and σ -soberness for σ -spaces are defined (Definitions 4.1 and 4.6). σ -spatiality is equivalent to having enough σ -points (Theorem 4.2), while a σ -space is σ -sober if and only if it is a σ_0 -space (Proposition 4.7).

Finally, in Section 5, we state and prove the adjunction between the functors $\Sigma^\sigma : \sigma\mathbf{Frm}^{\text{op}} \rightarrow \sigma\mathbf{Sp}$ and $\Lambda : \sigma\mathbf{Sp} \rightarrow \sigma\mathbf{Frm}^{\text{op}}$.

2 Preliminaries

Here, we recall some definitions and results from the literature on frames and σ -frames. For further information see [19] on frame-theoretic concepts.

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by \top and \perp respectively.

A *frame homomorphism (frame map)* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

A σ -*frame* is a countably complete lattice L in which the countable distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and countable $S \subseteq L$.

A σ -*frame homomorphism (σ -frame map)* is a map between σ -frames which preserves finite meets, including the top element, and countable joins, including the bottom element.

A σ -frame in which every element is complemented is called a **Boolean σ -frame**.

A *Borel space* is a pair $(X, \mathcal{B}X)$, where $\mathcal{B}X \subseteq \mathcal{P}(X)$ is a collection of subsets satisfying:

1. $\emptyset, X \in \mathcal{B}X$;
2. Closed under countable unions: For any countable family $\{U_n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}X$, we have $\bigcup_{n \in \mathbb{N}} U_n \in \mathcal{B}X$;
3. Closed under complements: For any $U \in \mathcal{B}X$, we have $U^c = X \setminus U \in \mathcal{B}X$.

For two Borel spaces $(X, \mathcal{B}X)$ and $(Y, \mathcal{B}Y)$, a function $f : X \rightarrow Y$ is called a *Borel map* if it satisfies:

$$f^{-1}(U) \in \mathcal{B}X \quad \text{for all } U \in \mathcal{B}Y.$$

The collection of all Borel spaces with Borel maps forms a category, which we denote by **Bor**.

Remarkably, the Borel structure $\mathcal{B}X$ of any Borel space X naturally carries the algebraic structure of a Boolean σ -frame.

For a comprehensive treatment of Borel spaces and their measure-theoretic foundations, we refer to standard references in analysis and measure theory [11, 16]. Key aspects of Borel structures and their applications are developed in detail in these works. In the literature, Borel spaces and Borel maps

are sometimes called measurable spaces and measurable functions respectively, with Rudin's classical work [20] being the most prominent example of this terminology.

Another important concept that needs to be introduced here is that of Alexandroff spaces, which play a significant role in our analysis.

An *Alexandroff space* is a pair (X, \mathcal{Z}) satisfying the following conditions:

- (a) \mathcal{Z} is closed under finite intersections and countable unions;
- (b) For any $A, B \in \mathcal{Z}$ with $A \cap B = \emptyset$, there exist $C, D \in \mathcal{Z}$ such that $A \cap C = \emptyset = B \cap D$ and $C \cup D = X$;
- (c) For any $A \in \mathcal{Z}$, there exists a sequence $\{A_n\}_{n \geq 1}$ in \mathcal{Z} such that $X \setminus A = \bigcup_{n \geq 1} A_n$;
- (d) For any distinct points $x, y \in X$, there exists $A \in \mathcal{Z}$ containing exactly one of them.

For two Alexandroff spaces (X, \mathcal{Z}) and (Y, \mathcal{Z}') , a *coz-map* is a function $f : X \rightarrow Y$ satisfying $f^{-1}(Z') \in \mathcal{Z}$ for all $Z' \in \mathcal{Z}'$. The category of Alexandroff spaces with coz-maps as morphisms is denoted by **Alex**.

For a comprehensive treatment of Alexandroff spaces and their properties, particularly their connections to regular σ -frames, we refer to the fundamental works of Gilmour [14, 15].

For an introduction and review of the categorical concepts and tools needed in this article, we refer to [1].

3 σ -Spaces and spectrum of σ -frames

Analogous to the correspondence between frames and topological spaces, in this section we establish a relationship between σ -frames and certain structures called σ -spaces, following the definition introduced by Clarke [10]. We recall the definition:

Definition 3.1. Let X be a set. A pair $(X, \Lambda(X))$ is called a σ -space if $\Lambda(X) \subseteq \mathcal{P}(X)$ is a collection of subsets satisfying:

- (i) $\emptyset, X \in \Lambda(X)$;
- (ii) If $A, B \in \Lambda(X)$ then $A \cap B \in \Lambda(X)$;
- (iii) If $\{A_n\}_{n \in \mathbb{N}} \subseteq \Lambda(X)$, then $\bigcup_{n \in \mathbb{N}} A_n \in \Lambda(X)$.

As we can observe, since topological spaces are, in particular, closed under countable unions, the definition of σ -spaces generalizes that of topological spaces. Moreover, Borel spaces and Alexandroff spaces also satisfy the axioms (1)-(3), and thus, they constitute two other subclasses of σ -spaces.

Analogous to continuous functions between topological spaces, Borel maps between Borel spaces, and coz-maps between Alexandroff spaces, we now define σ -continuous functions between σ -spaces, which generalizes all three cases simultaneously.

Definition 3.2. Let $(X, \Lambda(X))$ and $(Y, \Lambda(Y))$ be two σ -spaces. A function $f : X \rightarrow Y$ is said to be σ -continuous if $f^{-1}(B) \in \Lambda(X)$ for all $B \in \Lambda(Y)$.

We denote the category of all σ -spaces with σ -continuous functions by $\sigma\mathbf{Sp}$. Note that all three categories **Top**, **Bor**, and **Alex** are subcategories of $\sigma\mathbf{Sp}$.

As established in [19], there exists a one-one correspondence between three objects: frame homomorphisms from a frame L to 2 , completely prime filters, and prime elements of L . Based on this fact, any of these three classes may be conventionally taken as the points of the frame, depending on preference.

In this section, we demonstrate that for σ -frames, prime elements become distinct from the other two notions. Here, we employ the terms " σ -point" for σ -frame homomorphisms to 2 , and " σ -completely prime filter". We prove that these two notions remain in bijective correspondence (Corollary 3.6).

However, regarding prime elements, while every prime element yields a σ -point (Proposition 3.7), the converse fails: there exist σ -points that cannot be associated with any corresponding prime element (Remark 3.8).

Definition 3.3. Let H be a σ -frame. We define:

- (i) A σ -point of H is a σ -frame map $\pi : H \rightarrow 2$;
- (ii) A σ -completely prime filter is a filter \mathcal{F} of H satisfying the condition that for every countable subset $S \subseteq H$, $\bigvee S \in \mathcal{F}$ if and only if $S \cap \mathcal{F} \neq \emptyset$.

Proposition 3.4. Let H be a σ -frame and \mathcal{F} a filter on H . The following conditions are equivalent:

- (i) The filter \mathcal{F} is σ -completely prime;
- (ii) The characteristic function $\chi_{\mathcal{F}} : H \rightarrow 2$ constitutes a σ -frame map (that is, a σ -point).

Proof. We prove both implications separately.

(i) \Rightarrow (ii): Since \mathcal{F} is a filter, it follows immediately that $\chi_{\mathcal{F}}$ preserves finite meets, i.e., $\chi_{\mathcal{F}}(x \wedge y) = \chi_{\mathcal{F}}(x) \wedge \chi_{\mathcal{F}}(y)$ for all $x, y \in H$. The boundary conditions $\chi_{\mathcal{F}}(\perp) = \perp$ and $\chi_{\mathcal{F}}(\top) = \top$ hold since $\perp_H \notin \mathcal{F}$ and $\top_H \in \mathcal{F}$.

For countable joins, let $S \subseteq H$ be countable. The σ -complete primality of \mathcal{F} yields:

$$\begin{aligned} \chi_{\mathcal{F}}(\bigvee S) = \top &\Leftrightarrow \bigvee S \in \mathcal{F} \\ &\Leftrightarrow S \cap \mathcal{F} \neq \emptyset \\ &\Leftrightarrow \exists s \in S, \chi_{\mathcal{F}}(s) = \top \\ &\Leftrightarrow \bigvee_{s \in S} \chi_{\mathcal{F}}(s) = \top. \end{aligned}$$

Thus, $\chi_{\mathcal{F}}$ preserves countable joins.

(ii) \Rightarrow (i): Assume $\chi_{\mathcal{F}}$ is a σ -frame map. The boundary conditions imply $\perp \notin \mathcal{F}$ and $\top \in \mathcal{F}$. Preservation of finite meets ensures \mathcal{F} is a filter.

For countable joins, given $S \subseteq H$ countable:

$$\begin{aligned} \bigvee S \in \mathcal{F} &\Leftrightarrow \chi_{\mathcal{F}}(\bigvee S) = \top \\ &\Leftrightarrow \bigvee_{s \in S} \chi_{\mathcal{F}}(s) = \top \\ &\Leftrightarrow S \cap \mathcal{F} \neq \emptyset, \end{aligned}$$

establishing σ -complete primality.

This completes the proof. \square

Corollary 3.5. *For any σ -frame H , the following statements are equivalent:*

- (i) *The map $\pi : H \rightarrow 2$ defines a σ -point;*
- (ii) *The preimage $\pi^{-1}(\top)$ forms a σ -completely prime filter.*

Proof. The result follows immediately from the identity $\chi_{\pi^{-1}(\{\top\})} = \pi$ combined with Proposition 3.4. \square

Corollary 3.6. *There exists a bijective correspondence between:*

- (i) *The σ -points (that is, σ -frame maps) of H ;*
- (ii) *The σ -completely prime filters of H .*

But what is the relation between prime elements and σ -points? The next proposition shows that every prime element induces a σ -point, though the converse does not hold in general (see Remark 3.8).

Proposition 3.7. *Let H be a σ -frame and $p \in H$ a prime element. The mapping $\varphi_p : H \rightarrow 2$ defined by*

$$\varphi_p(x) = \perp \text{ if and only if } x \leq p$$

constitutes a σ -point of H .

Proof. First, observe that φ_p satisfies $\varphi_p(\perp) = \perp$ and $\varphi_p(\top) = \top$ by definition. For any elements $x, y \in H$, we establish the preservation of meets through the following equivalences:

$$\begin{aligned} \varphi_p(x \wedge y) = \perp &\Leftrightarrow x \wedge y \leq p \\ &\Leftrightarrow x \leq p \text{ or } y \leq p \\ &\Leftrightarrow \varphi_p(x) \wedge \varphi_p(y) = \perp. \end{aligned}$$

For countable joins, consider any countable subset $S \subseteq H$:

$$\begin{aligned} \varphi_p(\bigvee S) = \perp &\Leftrightarrow \bigvee S \leq p \\ &\Leftrightarrow \forall s \in S, s \leq p \\ &\Leftrightarrow \forall s \in S, \varphi_p(s) = \perp \\ &\Leftrightarrow \bigvee_{s \in S} \varphi_p(s) = \perp. \end{aligned}$$

These computations demonstrate that φ_p preserves both finite meets and countable joins, thereby completing the proof that it is a σ -frame map. \square

Remark 3.8. It is well-known that points of frames correspond bijectively to prime elements [19]. However, this equivalence fails for σ -frames. While Proposition 3.7 establishes that prime elements induce σ -points, the converse does not hold. Specifically, there exist σ -points that cannot be expressed as φ_p for any prime $p \in H$. The following concrete examples illustrate this phenomenon.

Example 3.9 exhibits a σ -point on the σ -frame of countable and co-countable subsets of an uncountable set, which does not correspond to any prime element.

Example 3.10 presents a simpler σ -frame, consisting of all countable subsets together with the whole set, and a σ -point that is only true for the top element, again not arising from a prime.

These examples collectively underscore the breakdown of the classic frame-theoretic correspondence in the σ -frame setting.

Example 3.9. Consider the following counterexample. Let X be an uncountable set (for instance, $X = \mathbb{R}$) and define

$$H = \{A \subseteq X \mid A \text{ is countable or } A \text{ is co-countable}\},$$

where “co-countable” means $X \setminus A$ is countable. Then (H, \subseteq) is a Boolean σ -frame. Note that finite meets are intersections, and countable joins are unions, because the union of a countable family of countable sets is countable, while the union of any family containing a co-countable set is co-countable.

Define $\pi: H \rightarrow \mathbf{2}$ by

$$\pi(A) = \begin{cases} \top & \text{if } A \text{ is co-countable,} \\ \perp & \text{if } A \text{ is countable.} \end{cases}$$

We now verify that π is a σ -point. Clearly, $\pi(\emptyset) = \perp$ (since \emptyset is countable) and $\pi(X) = \top$ (since X is co-countable).

Preservation of finite meets: Let $A, B \in H$.

- If both A and B are co-countable, then $A \cap B$ is co-countable (its complement $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ is a countable union of countable sets). Hence $\pi(A \cap B) = \top = \pi(A) \wedge \pi(B)$.
- If at least one of A or B is countable, then $A \cap B$ is countable (as a subset of a countable set). Thus $\pi(A \cap B) = \perp = \pi(A) \wedge \pi(B)$.

Therefore $\pi(A \cap B) = \pi(A) \wedge \pi(B)$ for all $A, B \in H$.

Preservation of countable joins: Let $\{A_n\}_{n \in \mathbb{N}} \subseteq H$.

- If every A_n is countable, then $\bigcup_n A_n$ is countable, so $\pi(\bigcup_n A_n) = \perp = \bigvee_n \pi(A_n)$.
- If some A_k is co-countable, then $\bigcup_n A_n$ is co-countable (it contains the co-countable set A_k). Hence $\pi(\bigcup_n A_n) = \top$. Moreover $\pi(A_k) = \top$, so $\bigvee_n \pi(A_n) = \top$.

Thus $\pi(\bigcup_n A_n) = \bigvee_n \pi(A_n)$ in all cases.

Consequently, π is a σ -frame homomorphism, i.e., a σ -point of H .

Our second claim is that there is no prime element $p \in H$ such that $\pi = \varphi_p$. Recall that for a prime element p , the associated σ -point is defined by $\varphi_p(A) = \perp \iff A \subseteq p$.

Assume, for contradiction, that such a p exists. For every $x \in X$, we have $\pi(\{x\}) = \perp$; by the hypothesis $\pi = \varphi_p$, this forces $\{x\} \subseteq p$. Hence p must contain X , which is impossible since a prime element cannot be the top element. This contradiction shows that no prime $p \in H$ satisfies $\pi = \varphi_p$.

Example 3.10. Let X be an uncountable set (e.g., $X = \mathbb{R}$) and consider

$$H = \{A \subseteq X \mid A \text{ is countable}\} \cup \{X\}.$$

Then H is a σ -frame under inclusion: finite meets are intersections, countable joins are unions (a countable union of countable sets is countable, and any union containing X equals X).

Define $\pi : H \rightarrow \mathbf{2}$ by $\pi(A) = \top$ iff $A = X$, and $\pi(A) = \perp$ otherwise. Clearly $\pi(\emptyset) = \perp$ and $\pi(X) = \top$. For $A, B \in H$, if both are X then $A \cap B = X$ and $\pi(A \cap B) = \top = \pi(A) \wedge \pi(B)$; otherwise $A \cap B$ is countable, so $\pi(A \cap B) = \perp = \pi(A) \wedge \pi(B)$. For a countable family $\{A_n\} \subseteq H$, if every A_n is countable then $\bigcup A_n$ is countable and $\pi(\bigcup A_n) = \perp = \bigvee \pi(A_n)$; if some $A_k = X$ then $\bigcup A_n = X$ and $\pi(\bigcup A_n) = \top = \bigvee \pi(A_n)$ (since $\pi(A_k) = \top$). Thus π is a σ -point.

Now suppose there exists a prime element $p \in H$ such that $\pi = \varphi_p$, i.e., $\varphi_p(A) = \perp \iff A \subseteq p$. For each $x \in X$, the singleton $\{x\}$ is countable, so $\{x\} \in H$ and $\pi(\{x\}) = \perp$, hence $\{x\} \subseteq p$. Therefore p contains every singleton, forcing $p = X$. But $\varphi_X(X) = \perp$ while $\pi(X) = \top$, a contradiction. Hence no prime element induces π .

Definition 3.11. For a given σ -frame H , we define:

- (i) The collection of all σ -points of H as $\Sigma^\sigma H$;
- (ii) For each element $a \in H$, the subset $\Sigma_a^\sigma = \{\pi \in \Sigma^\sigma H \mid \pi(a) = \top\}$.

Proposition 3.12. For any σ -frame H , the following properties hold:

- (i) $\Sigma_\perp^\sigma = \emptyset$ and $\Sigma_\top^\sigma = \Sigma^\sigma H$;
- (ii) For all $a, b \in H$, we have $\Sigma_{a \wedge b}^\sigma = \Sigma_a^\sigma \cap \Sigma_b^\sigma$;
- (iii) For any countable subset $\{a_n\}_{n \in \mathbb{N}} \subseteq H$, the equality $\Sigma_{\bigvee_{n \in \mathbb{N}} a_n}^\sigma = \bigcup_{n \in \mathbb{N}} \Sigma_{a_n}^\sigma$ holds;
- (iv) For complemented elements $a \in H$, the complement satisfies $(\Sigma_a^\sigma)^c = \Sigma_{a^c}^\sigma$.

Proof. The first statement follows immediately by definition. For the second property, consider any elements $a, b \in H$ and observe:

$$\begin{aligned} \pi \in \Sigma_{a \wedge b}^\sigma &\Leftrightarrow \pi(a \wedge b) = \top \\ &\Leftrightarrow \pi(a) \wedge \pi(b) = \top \\ &\Leftrightarrow \pi(a) = \top \text{ and } \pi(b) = \top \\ &\Leftrightarrow \pi \in \Sigma_a^\sigma \cap \Sigma_b^\sigma. \end{aligned}$$

To establish the third property, note that for any countable subset $\{a_n\}_{n \in \mathbb{N}}$:

$$\bigvee_{n \in \mathbb{N}} \pi(a_n) = \top \quad \text{if and only if} \quad \pi(a_n) = \top \text{ for some } n \in \mathbb{N}.$$

Finally, the complement property follows from the observation that for any σ -point $\pi \in \Sigma^\sigma H$:

$$\pi(a^c) = (\pi(a))^c,$$

and consequently $\pi(a^c) = \top$ precisely when $\pi(a) = \perp$. This completes the proof of all four properties. \square

Corollary 3.13. *Let H be a σ -frame. Then:*

- (i) *The pair $(\Sigma^\sigma H, \{\Sigma_a^\sigma : a \in H\})$ forms a σ -space;*
- (ii) *When H is a Boolean σ -frame, the structure $(\Sigma^\sigma H, \{\Sigma_a^\sigma : a \in H\})$ naturally becomes a Borel space.*
- (iii) *For any regular σ -frame H , the pair $(\Sigma^\sigma H, \{\Sigma_a^\sigma \mid a \in H\})$ defines an Alexandroff space.*

Proof. (i) and (ii) follow immediately from Proposition 3.12. For (iii), see [14], Proposition 3.1 and the paragraph preceding it. Note that in [14], σ -completely prime filters are used instead of σ -points, and the sets $\Psi_a = \{F \mid a \in F\}$ are used in place of Σ_a^σ . By Corollary 3.5, these two formulations are equivalent, so the proof in terms of Σ_a^σ follows accordingly. \square

Let G and H be σ -frames, and let $f : G \rightarrow H$ be a σ -frame map. We define the induced map

$$\Sigma^\sigma f : \Sigma^\sigma H \rightarrow \Sigma^\sigma G$$

by the composition

$$(\Sigma^\sigma f)(\pi) = \pi \circ f$$

for every σ -point $\pi : H \rightarrow 2$.

Proposition 3.14. *For any σ -frames G and H and a σ -frame map $f : G \rightarrow H$, the following hold:*

- (i) *The preimage satisfies $(\Sigma^\sigma f)^{-1}(\Sigma_a^\sigma) = \Sigma_{f(a)}^\sigma$ for every element $a \in G$;*
- (ii) *The induced map $\Sigma^\sigma f : \Sigma^\sigma H \rightarrow \Sigma^\sigma G$ is σ -continuous.*

Proof. We establish each part systematically.

(i) For any element $a \in G$, we verify the equality through the following chain of equivalences:

$$\begin{aligned} \pi \in (\Sigma^\sigma f)^{-1}(\Sigma_a^\sigma) &\Leftrightarrow \pi \circ f \in \Sigma_a^\sigma \\ &\Leftrightarrow \pi(f(a)) = \top \\ &\Leftrightarrow \pi \in \Sigma_{f(a)}^\sigma. \end{aligned}$$

(ii) This follows immediately from part (i) combined with Definition 3.2 of σ -continuous functions. \square

Proposition 3.15. *Let $(X, \Lambda(X))$ and $(Y, \Lambda(Y))$ be σ -spaces and $f : X \rightarrow Y$ a σ -continuous function. Then the induced map*

$$\Lambda f : \Lambda(Y) \rightarrow \Lambda(X)$$

is a σ -frame homomorphism between the corresponding σ -frames.

Proof. The result follows directly from the definition $\Lambda f(A) = f^{-1}(A) \in \Lambda(X)$, which holds for all $A \in \Lambda(Y)$. \square

4 σ -Spatial σ -frames and σ -sober σ -spaces

According to [19], for frames, a spatial frame is a frame L that is isomorphic to ΩX for some topological space X , which is equivalent to L being isomorphic to $\Omega \Sigma L$. Another equivalent condition for a frame L to be spatial is that " L has enough points". Here, we extend these notions to σ -frames by replacing:

- frames with σ -frames,
- topological spaces with σ -spaces,
- points with σ -points.

Definition 4.1. For a σ -frame H , we define:

(i) H is σ -spatial if it is isomorphic as a σ -frame to $\Lambda(X)$ for some σ -space $(X, \Lambda(X))$;

(ii) H has enough σ -points if for any elements $a \not\leq b$ in H , there exists a σ -point $\pi : H \rightarrow 2$ satisfying $\pi(a) = \top$ and $\pi(b) = \perp$.

Theorem 4.2. For any σ -frame H , the following conditions are equivalent:

- (i) H is σ -spatial;
- (ii) H has enough σ -points;
- (iii) For all elements $a \not\leq b$ in H , there exists a σ -completely prime filter \mathcal{F} containing a but not b ;
- (iv) The map $\eta_H : H \rightarrow \Lambda\Sigma^\sigma H$ given by $\eta_H(a) = \Sigma_a^\sigma$ is an isomorphism of σ -frames.

Proof. We prove the implications systematically.

(i) \Rightarrow (ii): Assume H is σ -spatial with isomorphism $H \cong \Lambda(X)$. To show $\Lambda(X)$ has enough σ -points, consider $A, B \in \Lambda(X)$ with $A \not\leq B$. Take $a \in A \setminus B$ and define the evaluation map:

$$\pi_a : \Lambda(X) \rightarrow 2, \quad \pi_a(C) = \top \text{ if and only if } a \in C.$$

This satisfies $\pi_a(A) = \top$ and $\pi_a(B) = \perp$. The map π_a preserves:

- Boundaries: $\pi_a(\emptyset) = \perp, \pi_a(X) = \top$
- Finite meets: $\pi_a(C \cap D) = \pi_a(C) \wedge \pi_a(D)$
- Countable joins: For $\{C_n\}_{n \in \mathbb{N}} \subseteq \Lambda(X)$,

$$\begin{aligned} \pi_a\left(\bigcup_n C_n\right) = \top &\Leftrightarrow a \in \bigcup_n C_n \\ &\Leftrightarrow \exists n, a \in C_n \\ &\Leftrightarrow \bigvee_n \pi_a(C_n) = \top. \end{aligned}$$

Thus π_a is the required σ -point in H .

(ii) \Rightarrow (iii): For $a \not\leq b$ in H , there exists a σ -point π with $\pi(a) = \top$ and $\pi(b) = \perp$. The filter $\mathcal{F} = \pi^{-1}(\top)$ is σ -completely prime by Corollary 3.5, containing a but not b .

(iii) \Rightarrow (ii): The converse follows similarly from Corollary 3.5.

(ii) \Rightarrow (iv): The map η_H is:

- A σ -frame map (Proposition 3.12).
- Surjective by construction.
- Injective: If $\eta_H(a) = \eta_H(b)$ but $a \not\leq b$, existence of a σ -point with $\pi(a) = \top$, $\pi(b) = \perp$, by (ii), yields $\pi \in \Sigma_a^\sigma \setminus \Sigma_b^\sigma$, a contradiction.

Hence η_H is an isomorphism.

(iv) \Rightarrow (i): Immediate since $H \cong \Lambda(\Sigma^\sigma H)$. □

Definition 4.3. Given a σ -space $(X, \Lambda(X))$, we define for each point $x \in X$:

(i) The evaluation map $\pi_x : \Lambda(X) \rightarrow 2$ by

$$\pi_x(A) = \top \text{ if and only if } x \in A, \quad \text{for all } A \in \Lambda(X),$$

which clearly constitutes a σ -point of $\Lambda(X)$;

(ii) The map $\varepsilon_X : X \rightarrow \Sigma^\sigma \Lambda(X)$ via $\varepsilon_X(x) = \pi_x$.

A σ -space $(X, \Lambda(X))$ is called a σ_0 -space (in the sense of Clarke [10]) if for any distinct $x, y \in X$, there exists $S \in \Lambda(X)$ containing exactly one of them. This definition represents the σ -space counterpart of the T_0 separation axiom for topological spaces. It is also the natural analogue of the T -measurability axiom defined in measurable spaces [12].

Lemma 4.4. *The map $\varepsilon_X : X \rightarrow \Sigma^\sigma \Lambda(X)$ satisfies:*

- (i) *It is σ -continuous;*
- (ii) *It is injective if and only if $(X, \Lambda(X))$ is a σ_0 -space.*

Proof. We establish both claims systematically.

First, to prove ε_X is σ -continuous, consider any set $A \in \Lambda(X)$ and its corresponding $\Sigma_A^\sigma \in \Lambda(\Sigma^\sigma \Lambda(X))$. The preimage computation reveals:

$$\begin{aligned} \varepsilon_X^{-1}(\Sigma_A^\sigma) &= \{x \in X \mid \pi_x \in \Sigma_A^\sigma\} \\ &= \{x \in X \mid \pi_x(A) = \top\} \\ &= \{x \in X \mid x \in A\} \\ &= A \\ &\in \Lambda(X), \end{aligned}$$

confirming the σ -continuity.

For injectivity, observe that for distinct points $x, y \in X$:

$$\begin{aligned} \pi_x \neq \pi_y &\Leftrightarrow \exists A \in \Lambda(X), \pi_x(A) \neq \pi_y(A) \\ &\Leftrightarrow (x \in A \wedge y \notin A) \vee (x \notin A \wedge y \in A). \end{aligned}$$

This equivalence demonstrates that ε_X is injective precisely when $(X, \Lambda(X))$ satisfies the σ_0 separation axiom. \square

Remark 4.5. ε_X may not be surjective. Indeed, there exists a σ -space $(X, \Lambda(X))$ with a σ -point $\pi : \Lambda(X) \rightarrow 2$ such that $\pi \neq \pi_x$ for any $x \in X$.

Consider the example constructed in Remark 3.8: Let $X = [0, 1]$ with $\Lambda(X) = \{A \subseteq X : A \text{ is Lebesgue measurable}\}$. Define $\pi : \Lambda(X) \rightarrow 2$ by $\pi(A) = \perp$ if and only if A is countable.

If $\pi = \pi_x$ were to hold for some $x \in X$, we would have the contradiction:

$$\perp = \pi(\{x\}) = \pi_x(\{x\}) = \top.$$

Definition 4.6. For a σ -space $(X, \Lambda(X))$, we define:

- (i) $(X, \Lambda(X))$ is σ -pointed if the map $\varepsilon_X : X \rightarrow \Sigma^\sigma \Lambda(X)$ is surjective, meaning every σ -point $\pi \in \Sigma^\sigma \Lambda(X)$ equals π_x for some $x \in X$;
- (ii) $(X, \Lambda(X))$ is σ -sober if ε_X is a σ -space isomorphism.

Proposition 4.7. For any σ -space $(X, \Lambda(X))$, the following conditions are equivalent:

- (i) The σ -space $(X, \Lambda(X))$ is σ -sober;
- (ii) The σ -space $(X, \Lambda(X))$ is both σ -pointed and σ_0 ;
- (iii) There exists a σ -frame H for which $X \cong \Sigma^\sigma H$ as σ -spaces.

Proof. We establish the equivalences systematically:

- The equivalence (i) \Rightarrow (ii) follows immediately from Lemma 4.4.
- The implication (i) \Rightarrow (iii) is a direct consequence of Definition 4.6.
- For (iii) \Rightarrow (i), we demonstrate that $(\Sigma^\sigma H, \Lambda(\Sigma^\sigma H))$ is σ -sober. Given any σ -point $f : \Lambda(\Sigma^\sigma H) \rightarrow 2$, define $\pi_f : H \rightarrow 2$ by:

$$\pi_f(a) = f(\Sigma_a^\sigma).$$

This construction yields:

- $\pi_f \in \Sigma^\sigma H$ since f is a σ -frame map;
- The equivalence $f(\Sigma_a^\sigma) = \top \Leftrightarrow \pi_f \in \Sigma_a^\sigma$ holds for all $a \in H$;
- Uniqueness follows since any π' satisfying $\pi'(a) = f(\Sigma_a^\sigma)$. must coincide with π_f .
- $\varepsilon_{\Sigma^\sigma H}(\pi_f) = f$

This means, $\varepsilon_{\Sigma^\sigma H} : \Sigma^\sigma H \rightarrow \Sigma^\sigma \Lambda(\Sigma^\sigma H)$ is an isomorphism. Therefore, $\Sigma^\sigma H$ is σ -sober. □

Lemma 4.8. *For a σ -space $(X, \Lambda(X))$, the following are equivalent:*

- (i) *The σ -space $(X, \Lambda(X))$ is σ -sober;*
- (ii) *Every σ -point $f : \Lambda(X) \rightarrow \mathbf{2}$ corresponds to a unique $x \in X$ such that, for all $A \in \Lambda(X)$, $f(A) = \top$ if and only if $x \in A$;*
- (iii) *Every σ -completely prime filter \mathcal{F} of $\Lambda(X)$ equals $\{A \in \Lambda(X) \mid x \in A\}$ for a unique $x \in X$.*

Proof. The equivalence (ii) \Leftrightarrow (iii) follows directly from Proposition 3.4 and Corollary 3.5. Furthermore, condition (ii) is equivalent to the bijectivity of the map ε_X . □

Corollary 4.9. *A σ -space $(X, \Lambda(X))$ is σ -sober if and only if:*

- (i) *$(X, \Lambda(X))$ satisfies the σ_0 separation axiom;*
- (ii) *The spectrum decomposes as $\Sigma^\sigma \Lambda(X) = \{\pi_x \mid x \in X\}$, where each $\pi_x : \Lambda(X) \rightarrow \mathbf{2}$ is the evaluation map defined by $\pi_x(A) = \chi_A(x) = \varepsilon_X(x)(A)$.*

Proof. We establish two key properties:

1. The equality $\Sigma^\sigma \Lambda(X) = \{\pi_x \mid x \in X\}$ precisely characterizes the surjectivity of ε_X , as each σ -point π_x corresponds to some $x \in X$;
2. The σ_0 separation property for $(X, \Lambda(X))$ is equivalent to injectivity of ε_X , as distinct points induce distinct σ -points.

□

5 The adjunction between $\sigma\mathbf{Frm}^{\text{op}}$ and $\sigma\mathbf{Sp}$

The category of all σ -frames with σ -frame maps is denoted by $\sigma\mathbf{Frm}$, while the category of all σ -spaces with σ -continuous functions is denoted by $\sigma\mathbf{Sp}$. In this section, we construct an adjunction between these categories through the following functors:

- The spectrum functor $\Sigma^\sigma: \sigma\mathbf{Frm}^{\text{op}} \rightarrow \sigma\mathbf{Sp}$;
- The measurable structure functor $\Lambda: \sigma\mathbf{Sp} \rightarrow \sigma\mathbf{Frm}^{\text{op}}$.

The functor Σ^σ is defined as follows:

- **On objects:** For a σ -frame H , we set $\Sigma^\sigma(H) := \Sigma^\sigma H$;
- **On morphisms:** For any σ -frame map $\varphi: H \rightarrow G$, the induced map $\Sigma^\sigma \varphi: \Sigma^\sigma G \rightarrow \Sigma^\sigma H$ is given by: $(\Sigma^\sigma \varphi)(\pi) = \pi \circ \varphi$ for all $\pi \in \Sigma^\sigma G$.

Proposition 3.14 verifies that Σ^σ is indeed a well-defined functor.

The functor Λ is defined by:

- **On objects:** For a σ -space $(X, \Lambda(X))$, we take $\Lambda(X, \Lambda(X)) := \Lambda(X)$;
- **On morphisms:** For any σ -continuous function $f: X \rightarrow Y$, the σ -frame map $\Lambda f: \Lambda(Y) \rightarrow \Lambda(X)$ is given by preimages: $\Lambda f(A) = f^{-1}(A)$ for all $A \in \Lambda(Y)$.

Proposition 3.15 establishes that Λ is properly defined.

Proposition 5.1. *The assignments*

$$\Sigma^\sigma: \sigma\mathbf{Frm}^{\text{op}} \rightarrow \sigma\mathbf{Sp} \quad \text{and} \quad \Lambda: \sigma\mathbf{Sp} \rightarrow \sigma\mathbf{Frm}^{\text{op}}$$

constitute well-defined functors between the specified categories.

Proof. We verify the functorial properties for both Σ^σ and Λ :

- **Preservation of identities:**

$$\begin{aligned} \Sigma^\sigma(1_H) &= 1_{\Sigma^\sigma H}, \\ \Lambda(1_X) &= 1_{\Lambda(X)}. \end{aligned}$$

- **Σ^σ preserves composition:** For σ -frame maps $\varphi: G \rightarrow H$ and $\psi: H \rightarrow K$, and any $\pi \in \Sigma^\sigma K$:

$$\Sigma^\sigma(\psi \circ \varphi)(\pi) = \pi \circ (\psi \circ \varphi) = (\pi \circ \psi) \circ \varphi = (\Sigma^\sigma \varphi \circ \Sigma^\sigma \psi)(\pi).$$

Thus Σ^σ is a functor.

- **Λ preserves composition:** For σ -continuous $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, and any $A \in \Lambda(Z)$:

$$\Lambda(g \circ f)(A) = (g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) = (\Lambda f \circ \Lambda g)(A).$$

Hence Λ is a functor. □

Now, consider two functors $\Sigma^\sigma: \sigma\mathbf{Frm}^{\text{op}} \rightarrow \sigma\mathbf{Sp}$ and $\Lambda: \sigma\mathbf{Sp} \rightarrow \sigma\mathbf{Frm}^{\text{op}}$. We will prove that Σ is left adjoint to the functor Λ .

Lemma 5.2. *For any σ -frame H , the assignment $\eta_H: H \rightarrow \Lambda(\Sigma^\sigma H)$ defined by:*

$$\eta_H(a) = \Sigma_a^\sigma \quad \text{for all } a \in H$$

yields a σ -frame homomorphism. Furthermore, η_H represents the universal morphism from H to the functor $\Lambda: \sigma\mathbf{Sp} \rightarrow \sigma\mathbf{Frm}^{\text{op}}$.

Proof. We establish the universal property in two steps:

1. By Proposition 3.12, η_H is indeed a σ -frame homomorphism.
2. For universality, given any σ -space $(X, \Lambda(X))$ and σ -frame map $\varphi: H \rightarrow \Lambda(X)$, define $\bar{\varphi}: X \rightarrow \Sigma^\sigma H$, where, for each $x \in X$, $\bar{\varphi}(x): H \rightarrow 2$ is the σ -point in H given by $\bar{\varphi}(x)(a) = \chi_{\varphi(a)}(x)$, for all $a \in H$. This construction satisfies:

- σ -continuity: $\bar{\varphi}^{-1}(\Sigma_a^\sigma) = \varphi(a) \in \Lambda(X)$ for each $a \in H$;
- Uniqueness: $\bar{\varphi}$ is uniquely determined by the commutativity condition $\Lambda \bar{\varphi} \circ \eta_H = \varphi$,

$$\begin{array}{ccc} H & \xrightarrow{\eta_H} & \Lambda \Sigma^\sigma H \\ & \searrow \varphi & \downarrow \Lambda \bar{\varphi} \\ & & \Lambda(X) \end{array}$$

Thus η_H satisfies the universal property. \square

Theorem 5.3. *The spectrum functor $\Sigma^\sigma: \sigma\mathbf{Frm}^{\text{op}} \rightarrow \sigma\mathbf{Sp}$ serves as a left adjoint to the functor $\Lambda: \sigma\mathbf{Sp} \rightarrow \sigma\mathbf{Frm}^{\text{op}}$, establishing an adjunction:*

$$\Sigma^\sigma \dashv \Lambda$$

between the respective categories.

Proof. We establish that the family of σ -frame maps $\{\eta_H\}_{H \in \sigma\mathbf{Frm}}$ constitutes the unit of the adjunction. By Lemma 5.2, it suffices to verify the naturality condition for $\eta: \mathbf{1} \rightarrow \Lambda\Sigma^\sigma$.

For any σ -frame homomorphism $f: H \rightarrow G$ and element $a \in H$, we compute:

$$\begin{aligned} (\Lambda\Sigma^\sigma f)(\eta_H(a)) &= (\Lambda\Sigma^\sigma f)(\Sigma_a^\sigma) \\ &= \Sigma_{f(a)}^\sigma \\ &= \eta_G(f(a)). \end{aligned}$$

This computation demonstrates the commutativity of the diagram:

$$\begin{array}{ccc} H & \xrightarrow{\eta_H} & \Lambda\Sigma^\sigma H \\ f \downarrow & & \downarrow \Lambda\Sigma^\sigma f \\ G & \xrightarrow{\eta_G} & \Lambda\Sigma^\sigma G \end{array}$$

confirming that η is indeed a natural transformation. \square

Remark 5.4. Since every adjunction induces an equivalence when restricted to suitable subcategories (see [1] for categorical foundations), consider the adjunction constructed in Theorem 5.3. This adjunction yields a dual equivalence between the following two categories:

1. The full subcategory $\mathbf{Spat}\sigma\mathbf{Frm}$ of $\sigma\mathbf{Frm}$, of all σ -spatial σ -frames.
2. The full subcategory $\mathbf{Pntd}\sigma_0\mathbf{Sp}$ of $\sigma\mathbf{Sp}$, of all σ -pointed σ_0 -spaces.
Note that σ -pointed σ_0 -spaces are also known as σ -sober σ -spaces.

The details follow from these facts:

- By Theorem 4.2, $\eta_H: H \rightarrow \Lambda\Sigma^\sigma H$ is an isomorphism if and only if H is σ -spatial.

- By Proposition 4.7, $\varepsilon_X : X \rightarrow \Sigma^\sigma \Lambda(X)$ is an isomorphism if and only if X is σ -pointed and σ_0 .

To clarify the connection between this paper and [3], observe that the adjunction established in their Theorem 2.2:

$$\mathbf{Bor} \underset{\Gamma}{\overset{\mathcal{B}}{\rightleftarrows}} \mathbf{Bool}\sigma\mathbf{Frm}^{\text{op}}$$

obtained by restricting the adjunction from our Theorem 5.3, which is:

$$\sigma\mathbf{Sp} \underset{\Sigma^\sigma}{\overset{\Lambda}{\rightleftarrows}} \sigma\mathbf{Frm}$$

Furthermore, the resulting equivalence in [3, Theorem 2.7]:

$$\mathbf{BorSob} \underset{\Gamma}{\overset{\mathcal{B}}{\rightleftarrows}} \mathbf{SpatBool}\sigma\mathbf{Frm}^{\text{op}}$$

is precisely the restriction of our equivalence

$$\mathbf{Pntd}\sigma_0\mathbf{Sp} \underset{\Sigma^\sigma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Spat}\sigma\mathbf{Frm}^{\text{op}}$$

to the corresponding subcategories.

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