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When Freudenthal coincides with the smallest compactification with a categorical slant

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Abstract. In this note, for a certain class of regular continuous frames, we establish conditions that are equivalent to saying that the Freudenthal compactification and the smallest compactification are indistinguishable; in turn, this expands the list of conditions under which the smallest compactification is perfect, which is available in the literature. We define a new class of morphisms between frames, called F-maps, and provide a proof demonstrating that the category of compact regular frames and F-maps forms a coreflective full subcategory of the category of rim-compact frames and F-maps. This coreflection is evidenced by the join map associated with the Freudenthal compactification. Accordingly, this provides an affirmative answer to the question by Herrlich, which inquired whether the Freudenthal compactification can be regarded as a reflection with "sensible" maps.

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1 Introduction

A compactification of a frame is perfect if its associated right adjoint preserves disjoint binary joins. The Stone-Čech compactification of a completely regular frame (in the sense of [7]) and the Freudenthal compactification of a rim-compact frame (à la [1]) are salient examples of perfect compactifications. The Stone-Čech compactification of a frame has been extensively studied by several authors in different ways for distinct reasons in the literature, including Banaschewski and Mulvey [7–9], Johnstone [24, 25], Dube [14, 15], and Curi [11, 12], to mention a few. However, not a lot is known about the Freudenthal compactification of frames, except for the work by Baboolal [1] and that done by the first author [27]. It is the main aim of this note to contribute in this direction.

It is a result of Banaschewski ([6, Proposition 4]) that the smallest compactification exists precisely for frames that are regular and continuous. Since every regular continuous frame is rim-compact (see Proposition 3.7), every such frame has the Freudenthal compactification. In Theorem 3.9, we establish, for non-compact connected, locally connected, and regular continuous frames, a few conditions whereby the Freudenthal compactification coincides with the smallest compactification. The interest here is that if the latter holds, then the smallest compactification is perfect. Incidentally, this extends the list of conditions under which the smallest compactification is perfect, provided in [2].

It is part of the folklore that the category of compact regular frames forms a coreflective full subcategory of the category of completely regular frames, where all the maps are frame homomorphisms. In the last section of the paper (§4), we provide a frame-theoretic treatment of the question asked by Herrlich [23]: *Can the Freudenthal compactification of a space be regarded as a reflection in a "sensible" way?* The emphasis in this question is on the type of maps that witness the reflection. We provided an affirmative answer to this question in the category of frames, where the maps are what we call F-maps (Of course, "F" is for "Freudenthal"). Indeed, as mentioned in the abstract, the category of compact regular frames with F-maps is coreflective in the category of rim-compact frames and F-maps, the Freudenthal compactification being the desired coreflection (Theorem 4.12). This extends the scope of the main result in the short note by Hunsaker and Naimpally [22] where they deal with this question in the context of topological spaces.

2 Frames and compactifications

2.1 Frames A *frame* L is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

for all $x \in L$ and $S \subseteq L$. The top element and the bottom element of L will be denoted by 1 and 0, respectively. A *frame homomorphism* is a map $M \xrightarrow{h} L$, between the frames L and M, that preserves finite meets (including 1) and arbitrary joins (including 0). Therefore, we have the category **Frm** of frames and their homomorphisms. Thus, one has **Loc**:= $\mathbf{Frm}^{op}$, the category whose objects are frames (but, accordingly, called *locales*) and whose morphisms are called *localic maps* in [30]. Localic maps are just the right adjoints of frame homomorphisms. Indeed, for each frame homomorphism $M \xrightarrow{h} L$, the *right adjoint* (which is a morphism in the category **Loc**) of h is the map $L \xrightarrow{h_*} M$ satisfying $x \leq h_*(a) \Longleftrightarrow h(x) \leq a$ for all $a \in L$ and $x \in M$.

For any $a, b \in L$, we say that a is *rather below* b, written as $a \prec b$, if $a \wedge c = 0$ and $c \vee b = 1$, for some $c \in L$. Note that $a \prec b$ if and only if $a^* \vee b = 1$, where a^* is the *pseudocomplement* of a, that is, $a^* = \bigvee \{x \in L : x \wedge a = 0\}$. A frame L is called *regular* if $a = \bigvee \{x \in L : x \prec a\}$, for all $a \in L$. We say that a is *completely below* b in L, and write $a \prec\prec b$, if there are $a_r \in L$ where $r \in \mathbb{Q} \cap [0, 1]$ such that $a_0 = a, a_1 = b$ and $a_r \prec a_s$ for $r < s$. A frame L is said to be *completely regular* if $a = \bigvee \{x : x \prec\prec a\}$ for every $a \in L$. A frame L is *compact* if $S \subseteq L$ and $1 = \bigvee S$ implies that $1 = \bigvee S_0$ for some finite $S_0 \subseteq S$. A complete lattice L is called *continuous* if for $a \in L$, $a = \bigvee \{x \in L : x \ll a\}$, where $x \ll a$ means that for any $S \subseteq L$ with $a \leq \bigvee S$, there exists a finite $F \subseteq S$ such that $x \leq \bigvee F$. If $x \ll a$, we say that x is *well below* a. The well below relation interpolates if L is continuous, that is, if $x \ll y$ in L then $x \ll z \ll y$ for some $z \in L$. An element $c \in L$ is *connected* if whenever $c = a \vee b$ and $a \wedge b = 0$, then either $a = 0$ or $b = 0$. This is equivalent to saying that c different from 0 has the property that, whenever $c \leq a \vee b$ for some $a \wedge b = 0$ then either $c \leq a$ or $c \leq b$. A frame L is *connected* if its top element is connected, and L is *locally connected* if every element is expressible as a join of connected elements in L. For any frame L and $a \in L$, a *component* of a is a maximal connected element $c \leq a$. For more on the general theory of frames and locales, we refer the reader to [24], [30], and [31].

2.2 Compactifications of frames A frame homomorphism h is *dense* if whenever $h(x) = 0$, then $x = 0$. A *compactification* of a frame L is a compact regular frame M together with a dense onto frame homomorphism $M \xrightarrow{h} L$. The set $\mathfrak{K}(L)$ of all compactifications is partially ordered by the order relation $\leq$ which is defined as follows: given compactifications $M \xrightarrow{h} L$ and $N \xrightarrow{f} L$ of a frame L, we say that $M \xrightarrow{h} L$ is *smaller* than $N \xrightarrow{f} L$, written as $M \xrightarrow{h} L \leq N \xrightarrow{f} L$, if there exists a frame homomorphism $M \xrightarrow{g} N$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[dr, "h"'] & N \arrow[d, "f"] \\
& L
\end{tikzcd}$$

Compactifications $M \xrightarrow{h} L$ and $N \xrightarrow{f} L$ of L are *isomorphic* or *equivalent*, written as $M \xrightarrow{h} L \cong N \xrightarrow{f} L$, if $M \xrightarrow{g} N$ is an isomorphism. Intimately related to compactifications of frames is the notion of a strong inclusion:

Definition 2.1. [6] A *strong inclusion* on a frame L is a binary relation $\lhd$ on L satisfying the following conditions for all $a, b \in L$:

1. If $x \leq a \lhd b \leq y$, then $x \lhd y$.
2. $\lhd$ is a sublattice of $L \times L$.
3. If $a \lhd b$, then $a \prec b$.
4. If $a \lhd b$, then $a \lhd c \lhd b$ for some $c \in L$ (the *interpolation property*).
5. If $a \lhd b$, then $b^* \lhd a^*$.
6. $a = \bigvee\{x \in L : x \lhd a\}$.

If $\lhd_1$ and $\lhd_2$ are strong inclusions on L, then $\lhd_1 \subseteq \lhd_2$ if and only if $a \lhd_1 b \Longrightarrow a \lhd_2 b$ is a partial order on the set, $\mathfrak{S}(L)$, of all strong inclusions on L.

Remark 2.2. Banaschewki [6] showed that there is a one-to-one correspondence between the elements of $\mathfrak{S}(L)$ and the elements of $\mathfrak{K}(L)$. Since we need the details of this correspondence, let us briefly recall its ingredients: On the one hand, for any compactification (M, h) of L, one has the corresponding strong inclusion $\lhd_M$ on L defined by

$$x \lhd_M y \quad \text{if and only if} \quad h_*(x) \prec h_*(y).$$

On the other hand, if $\lhd$ is a strong inclusion on L, then we call an ideal $J \subseteq L$ *strongly regular* with respect to $\lhd$, if for each $x \in J$, there exists $y \in J$ such that $x \lhd y$. Let $\mathfrak{I}_{\lhd}L$ be the set of all strongly regular ideals of L with respect to $\lhd$. Then, $\mathfrak{I}_{\lhd}L$ is a compact frame (being a subframe of the compact frame consisting of all ideals on L). We then have the compactification given by the join map $\mathfrak{I}_{\lhd}L \xrightarrow{\rho_L} L$ which has the right adjoint $L \xrightarrow{k_{\lhd}} \mathfrak{I}_{\lhd}L$ defined by $k_{\lhd}(a) = \{x \in L : x \lhd a\}$. So, for any $J \in \mathfrak{I}_{\lhd}L$ and any $a \in L$, we have $\rho_L(J) \leq a \iff J \subseteq k_{\lhd}(a)$. For the proofs of the results mentioned in this remark, see [6, Lemma 1, Lemma 2, Proposition 2].

Because of nature of the one-to-one correspondence between elements of $\mathfrak{S}(L)$ and that of $\mathfrak{K}(L)$, we have that: $M \xrightarrow{h} L \leq N \xrightarrow{f} L$ if and only if $\lhd_M \subseteq \lhd_N$, where $\lhd_M$ and $\lhd_N$ are the strong inclusions corresponding to compactifications $M \xrightarrow{h} L$ and $N \xrightarrow{f} L$, respectively. So, $M \xrightarrow{h} L \cong N \xrightarrow{f} L$ if and only if $\lhd_M = \lhd_N$. Equivalently, we have $M \xrightarrow{h} L \cong N \xrightarrow{f} L$ if and only if $M \xrightarrow{h} L \leq N \xrightarrow{f} L$ and $N \xrightarrow{f} L \leq M \xrightarrow{h} L$.

3 The Freudenthal and the smallest compactification of a frame

A compactification $M \xrightarrow{h} L$ is called *perfect* ([1]) if $h_*(u \vee u^*) = h_*(u) \vee h_*(u^*)$ for all $u \in L$. In this case h_* also preserves any disjoint binary joins, i.e., $h_*(u \vee v) = h_*(u) \vee h_*(v)$ whenever $u \wedge v = 0$.

Example 3.1. The relation $\prec\!\prec$ is a strong inclusion on a completely regular frame. In [7], Banaschewski and Mulvey defined the *Stone-Čech compactification* of a completely regular frame L to be the compactification

$\Im_{\prec\prec}L \xrightarrow{\rho_L} L$. So, $\Im_{\prec\prec}L$ is the collection of all ideals of L which are strongly regular with respect to $\prec\prec$. An example of an ideal that is strongly regular with respect to $\prec\prec$ is $J_a = \{x \in L : x \prec\prec a\}$, for any $a \in L$. In fact, the right adjoint of the join map $\Im_{\prec\prec}L \xrightarrow{\rho_L} L$ is given by $k_{\prec\prec}(a) = J_a$, for each $a \in L$. Finally, for any $a, b \in L$ with $a \wedge b = 0$, we always have $k_{\prec\prec}(a \vee b) = k_{\prec\prec}(a) \vee k_{\prec\prec}(b)$, by the corollary of [4, Lemma 1.9]. So, $\Im_{\prec\prec}L \xrightarrow{\rho_L} L$ is perfect.

Recall from [30] that in a frame L, one has the binary operation $\rightarrow$ (called the *Heyting operation*) such that for any $a, b, c \in L$, the following holds:

$$c \leq a \rightarrow b \quad \text{if and only if} \quad c \wedge a \leq b.$$

Definition 3.2. [30] A subset $S \subseteq L$ is called a *sublocale* of a frame L if S is closed under arbitrary meets, and for each $x \in L$ and each $s \in S$, $x \rightarrow s \in S$.

Example 3.3. (1) We have the *trivial* sublocale $\mathrm{O} := \{1\}$. Note that for any sublocale S of L, $\mathrm{O} \subseteq S$, and therefore $1 \in S$. However, we do not necessarily have $0 \in S$. In fact, if $0 \in S$, we say that S is *dense* in L.

(2) A typical example of a sublocale that is not trivial (unless if $a = 1$) is the so-called *closed sublocale* associated with $a \in L$, which is given by

$$\mathfrak{c}_L(a) = {\uparrow}a = \{x \in L : a \leq x\}.$$

Let us introduce some nomenclature:

Definition 3.4. A *π-element* of a frame L is an element $a \in L$ such that $\mathfrak{c}_L(a \vee a^*)$ is compact. A *rim-compact* frame is a regular frame that has a basis of π-elements.

Definition 3.5. [1] A *π-compact* basis on a frame L, is a basis B satisfying the following properties:

(b1) $a, b \in B$ implies that $a \wedge b, a \vee b \in B$.

(b2) $a^* \in B$ for all $a \in B$.

(b3) a is a π-element in L for all $a \in B$.

Notice that a regular frame that has a π-compact basis is rim-compact by (b3). Let L be a regular frame that has a π-compact basis B. The relation $\lhd_B$ defined by

$$a \lhd_B b \Longleftrightarrow a \prec c \prec b \quad \text{for some} \quad b \in B$$

is a strong inclusion on L (see [1, Proposition 4.6]). The compactification $\mathfrak{J}_{\lhd_B} L \xrightarrow{\rho_L} L$ corresponding to $\lhd_B$ is called a *π-compactification* of a frame L.

Example 3.6. Let L be a regular frame. Let

$$B_L = \{a \in L : a \text{ is a } \pi\text{-element in } L\}.$$

Then B_L is a π-compact basis on L ([1, Remark 4.4]). The *Freudenthal compactification* is the compactification $\mathfrak{J}_{\lhd_{B_L}} L \xrightarrow{\rho_L} L$ corresponding to the strong inclusion $\lhd_{B_L}$. To simplify notation, We write γL for $\mathfrak{J}_{\lhd_{B_L}} L$. The Freudenthal compactification is perfect, by [1, Proposition 4.10]. In particular, it is the minimal perfect compactification, by [27, Proposition 5.5].

Proposition 3.7. *Any regular continuous frame is rim-compact.*

Proof. Suppose that L is a regular continuous frame and let $a \in L$. Then $a = \bigvee\{x \in L : x \ll a\}$. But, $x \ll a$ and $a \leq 1$ implies that $x \ll 1$. So $\mathfrak{c}_L(x^*)$ is compact, by [2, Proposition 3.3]. Hence $\mathfrak{c}_L(x \vee x^*)$ is compact. Thus, x is a π-element. It follows that $a = \bigvee\{x \in L : x \text{ is a } \pi\text{-element}\}$. That is, L is rim-compact. $\square$

So, regular continuous frames have the Freudenthal compactification. Since, up to isomorphism, the smallest compactification exists if and only if the frame at hand is regular continuous, a natural question arises: *under what conditions is the Freudenthal compactification isomorphic to the smallest one?* Before we address this question, let us recall some important concepts and results from the literature that will make our result intelligible:

Remark 3.8. (1) In the lattice $\mathcal{S}(L)$ of all sublocales of a frame L, the meets are precisely the intersections, O is the bottom element, and L is the top element of $\mathcal{S}(L)$. The joins are defined by the formula:

$$\bigvee_{i \in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i \in I} S_i\}$$

for any $\{S_i\}_{i\in I} \subseteq \mathcal{S}(L)$. The *supplement* of a sublocale S of a frame L, denoted by $L \smallsetminus S$, is the sublocale

$$L \smallsetminus S = \bigvee\{T \in \mathcal{S}(L) : T \cap S = \mathsf{O}\}.$$

For more on the lattice $\mathcal{S}(L)$, see [30].

(2) For a dense onto frame homomorphism $M \xrightarrow{h} L$, the right adjoint of h is an injective localic map $L \xrightarrow{h_*} M$ whose image is a dense sublocale of M. Let us underscore that $h_*[L]$ is isomorphic to L (as locales). Given this background, we often identify $h_*[L]$ with L and consider L as a dense sublocale of M. Remainders of frames in their compactifications have been explored by a few authors, including Baboolal [1, 2], Dube [16, 17], Ferreira et al. [18], and the first author [27–29]. We remind the reader about this notion below:

> Let $M \xrightarrow{h} L$ be a compactification of L with the right adjoint $L \xrightarrow{h_*} M$. The *remainder* of L in this compactification is the sublocale $M \smallsetminus h_*[L]$. We often write $M \smallsetminus L$ for $M \smallsetminus h_*[L]$.

It was shown in [18, Proposition 6.5] (see also [2, Section 2]) that for any compactification $M \xrightarrow{h} L$ of a regular continuous frame L,

$$M \smallsetminus L = \mathfrak{c}_M(a_L) = {\uparrow} a_L$$

where $a_L = \bigvee\{h_*(x) : x \ll 1\}$.

(3) Let $\blacktriangleleft$ be the strong inclusion corresponding to the smallest compactification of a regular continuous frame L. Recall from [6] that for $a, b \in L$:

$$a \blacktriangleleft b \text{ if and only if } a \prec b, \text{ and either } \mathfrak{c}_L(a^*) \text{ or } \mathfrak{c}_L(b) \text{ is compact.}$$

So, one has the smallest compactification $\mathfrak{J}_{\blacktriangleleft} L \xrightarrow{\rho_L} L$. One of the interesting properties of this compactification is that $\mathfrak{J}_{\blacktriangleleft} L \smallsetminus L$ is always connected when viewed as a frame (see [28, Proposition 3.3]). That is, ${\uparrow} J_L$ is a connected frame, where $J_L = \bigvee\{k_{\blacktriangleleft}(x) : x \ll 1\}$ and $k_{\blacktriangleleft}$ is the right adjoint of the join map $\mathfrak{J}_{\blacktriangleleft} L \xrightarrow{\rho_L} L$.

Now, to the main result of this section:

Theorem 3.9. *Let L be a non-compact connected, locally connected, and regular continuous frame. The following conditions are equivalent:*

1. $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$.
2. *$M \smallsetminus L$ is connected for any compactification $M \xrightarrow{h} L$.*
3. *If $k \in L$ and $\mathfrak{c}_L(k)$ is compact, then k has exactly one component c such that $\mathfrak{c}_L(c^*)$ is not compact.*

Proof. (1) $\implies$ (2). If $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$, then $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L$ is perfect. Hence, $M \smallsetminus L$ is connected for any compactification $M \xrightarrow{h} L$, by [2, Theorem 4.2].

(2) $\implies$ (3). Suppose that (2) is true. We shall derive a contradiction by assuming that the conclusion of condition (3) is not true for any closed compact sublocale. To this end, let $k \in L$ be such that $\mathfrak{c}_L(k)$ is compact. Since L is regular continuous, connected, and locally connected, then, by [3, Lemma 2.5], we can find $r \in L$ such that: $r \prec k$, $\mathfrak{c}_L(r)$ is compact, and r has only finitely many components. Let us denote these components by $u_1, u_2, \ldots, u_n$. Then, by local connectedness of L and [4, Corollary 1.4], $r = u_1 \vee u_2 \vee \cdots \vee u_n$. Since $\mathfrak{c}_L(r)$ is compact, then by the assumption, we have two cases to consider:

Case 1: There is no u_i such that $\mathfrak{c}_L(u_i^*)$ is not compact. This implies that $\mathfrak{c}_L(u_i^*)$ is compact for all $1 \leq i \leq n$. Therefore, $\bigvee\limits_{i=1}^{n} \mathfrak{c}_L(u_i^*)$ is compact. Notice that

$$\bigvee_{i=1}^{n} \mathfrak{c}_L(u_i^*) = \mathfrak{c}_L\left(\bigwedge_{i=1}^{n} u_i^*\right) = \mathfrak{c}_L\left((\bigvee_{i=1}^{n} u_i)^*\right) = \mathfrak{c}_L(r^*).$$

Therefore, $\mathfrak{c}_L(r^*)$ is compact. Since $\mathfrak{c}_L(r)$ is also compact, and $\mathfrak{c}_L(r^*) \vee \mathfrak{c}_L(r) = \mathfrak{c}_L(r^* \wedge r) = \mathfrak{c}_L(0) = L$, then L is compact. This is a contradiction.

Case 2: At least two of the components $u_1, u_2, \ldots, u_n$ satisfies the conclusion of condition (3). Without loss of generality, we may suppose u_1 and u_2 are two of such components. That is, $\mathfrak{c}_L(u_1^*)$ and $\mathfrak{c}_L(u_2^*)$ are not compact. Let $u = u_1$ and $v = u_2 \vee \cdots \vee u_n$. Since L is locally connected, then by [4, Corollary 1.3], we have that $u \wedge v = 0$. Observe that $\mathfrak{c}_L(u \vee v) = \mathfrak{c}_L(r)$, so $\mathfrak{c}_L(u \vee v)$ is compact. We show that $\mathfrak{c}_L(u)$ and $\mathfrak{c}_L(v)$ are not compact. First, note that $u_2 \leq v \implies \mathfrak{c}_L(u_2^*) \subseteq \mathfrak{c}_L(v^*)$. On the one hand, if $\mathfrak{c}_L(u)$ is compact, then $u \wedge v = 0 \Longrightarrow u \leq v^* \Longrightarrow \mathfrak{c}_L(v^*) \subseteq \mathfrak{c}_L(u) \Longrightarrow \mathfrak{c}_L(v^*)$ is compact. The latter implies that $\mathfrak{c}_L(u_2^*)$ is compact, and this is a contradiction. On the other hand, if $\mathfrak{c}_L(v)$ is compact, then $u \wedge v = 0 \Longrightarrow v \leq u^* = u_1^* \Longrightarrow \mathfrak{c}_L(u_1^*) \subseteq \mathfrak{c}_L(v) \Longrightarrow \mathfrak{c}_L(u_1^*)$ is compact. Again, we get a contradiction. Thus, in summary, we have that $u \wedge v = 0$, $\mathfrak{c}_L(u \vee v)$ is compact, and neither $\mathfrak{c}_L(u)$ nor $\mathfrak{c}_L(v)$ is compact. Hence, by [2, Theorem 3.7], there is a compactification $M \xrightarrow{f} L$ such that $N \smallsetminus L$ is disconnected. This contradicts the hypothesis. Therefore, for any $k \in L$ with $\mathfrak{c}_L(k)$ compact, we always have that k has exactly one component c such that $\mathfrak{c}_L(c^*)$ is not compact.

(3) $\Longrightarrow$ (1). To show that $\mathfrak{J}_{\blacktriangleleft} L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$, it is enough to show that $\blacktriangleleft = \lhd_{B_L}$. We always have that $\blacktriangleleft \subseteq \lhd_{B_L}$, since $\blacktriangleleft$ is the smallest strong inclusion on L, by [6, Proposition 4]. To show that $\lhd_{B_L} \subseteq \blacktriangleleft$, suppose $x \lhd_{B_L} y$ in L. Then there exists $u \in B_L$ such that $x \prec u \prec y$. Since $\mathfrak{c}_L(u \vee u^*)$ is compact, then by [3, Lemma 2.5], there exists $r \in L$ such that $r \prec u \vee u^*$, $\mathfrak{c}_L(r)$ is compact and r has finitely many components $b_1, b_2, \ldots, b_n$. By the latter and the fact that L is locally connected, we can write $r = \bigvee_{i=1}^n b_i$ (see [4, Corollary 1.4]). Since each b_i is a component of r, we have that $b_i \wedge b_j = 0$ for $i \neq j$ by [4, Corollary 1.3]. Therefore, $b_i \leq b_j^*$ for $i \neq j$. It is now lucid that

$$r \wedge b_1^* \wedge b_2^* \wedge \cdots \wedge b_{k-1}^* \wedge b_{k+1}^* \wedge \cdots \wedge b_n^* = b_k,$$

for any $1 \leq k \leq n$. By the hypothesis, we can assume without loss of generality that b_k is such that $\mathfrak{c}_L(b_k^*)$ is not compact. Therefore, the sublocales

$$\mathfrak{c}_L(b_1^*), \mathfrak{c}_L(b_2^*), \ldots, \mathfrak{c}(b_{k-1}^*), \mathfrak{c}(b_{k+1}^*), \ldots, \mathfrak{c}_L(b_n^*)$$

are all compact. Recall that $\mathfrak{c}_L(r)$ is compact, so the finite join

$$\mathfrak{c}_L(r) \vee \mathfrak{c}_L(b_1^*) \vee \mathfrak{c}_L(b_2^*) \vee \cdots \vee \mathfrak{c}(b_{k-1}^*) \vee \mathfrak{c}(b_{k-1}^*) \vee \cdots \vee \mathfrak{c}_L(b_n^*)$$

is also compact. That is, $\mathfrak{c}_L(r \wedge b_1^* \wedge b_2^* \wedge \cdots \wedge b_{k-1}^* \wedge b_{k+1}^* \wedge \cdots \wedge b_n^*) = \mathfrak{c}_L(b_k)$ is compact. Since $b_k \leq r \prec u \vee u^*$ then $b_k \leq u \vee u^*$. But b_k is connected, so $b_k \leq u$ or $b_k \leq u^*$, and so $\mathfrak{c}_L(u) \subseteq \mathfrak{c}_L(b_k)$ or $\mathfrak{c}_L(u^*) \subseteq \mathfrak{c}_L(b_k)$. Hence, $\mathfrak{c}_L(u)$ or $\mathfrak{c}_L(u^*)$ is compact. On the one hand, if $\mathfrak{c}_L(u)$ is compact, then $\mathfrak{c}_L(y)$ is compact since $\mathfrak{c}_L(y) \subseteq \mathfrak{c}_L(u)$. On the other hand, if $\mathfrak{c}_L(u^*)$ is compact, then $\mathfrak{c}_L(x^*)$ is compact since $u^* \leq x^*$. Therefore $x \blacktriangleleft y$. $\square$

We want to provide an example of a regular continuous frame L such that $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$. For this, recall from [29] that a frame L is called a *J-frame* if whenever $a \wedge b = 0$ in L and $\mathfrak{c}_L(a \vee b)$ is compact, then $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(b)$ is compact. The following result will be handy:

Proposition 3.10. *Let L be a regular continuous frame. Then $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$ if and only if L is a J-frame.*

Proof. By the equivalence of (1) and (3) of [2, Theorem 4.2], $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L$ is perfect if and only if L is a J-frame. Since the $\gamma L \xrightarrow{\rho_L} L$ is the smallest perfect compactification of L (by [27, Proposition 5.5]) and $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L$ is the smallest compactification, then $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L$ is perfect precisely when $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$. Thus, $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$ if and only if L is a J-frame. ☐

The frame of open sets of a locally compact space is a continuous frame (for example, see [30, Chapter VII.5.1.1]).

Example 3.11. Let us now concretize:

(a) It is well-known that the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the standard topology is a locally compact Hausdorff (and therefore, a regular) space. Therefore, the frame, $\mathfrak{O}\mathbb{R}^n$, of open sets of $\mathbb{R}^n$ is regular and continuous as a frame. Moreover, by [29, Example 4.8 (2)], $\mathfrak{O}\mathbb{R}^n$ is a J-frame whenever $n > 1$. Thus, for $n > 1$, the frame $L = \mathfrak{O}\mathbb{R}^n$ has the property that $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \cong \gamma L \xrightarrow{\rho_L} L$, by Proposition 3.10.

(b) Consider the real line $\mathbb{R}$ endowed with the standard topology. The frame $L = \mathfrak{O}\mathbb{R}$ is regular and continuous. But L is not a J-frame, by [29, Example 4.8 (1)]. Thus, by Proposition 3.10, $\mathfrak{J}_{\blacktriangleleft}L \xrightarrow{\rho_L} L \ncong \gamma L \xrightarrow{\rho_L} L$.

It is worth noting that for any positive integer n, $\mathbb{R}^n$ (and therefore $\mathbb{R}$) is a non-compact, locally connected, and connected topological space. Therefore, in (a) and (b) above, L is a regular continuous frame which is non-compact, locally connected, and connected.

Corollary 3.12. *If L is a non-compact connected, locally connected, and regular continuous frame that satisfies condition (2) or (3) of the theorem above, then the smallest compactification of L is perfect.*

4 Freudenthal as a sensible coreflection: a categorical slant

Herrlich [23] posed the question: *Can the Freudenthal compactification of a space be regarded as a reflection in a "sensible" way?* Hunsaker and Naimpally [22] provided an affirmative answer to this question in the context of topological spaces. The main aim of this section is to show that a frame-theoretic treatment of this question also yields a positive answer, where the "sensible" maps are what we call F-maps. Our category of discourse is the category of frames with frame homomorphisms, so we shall speak of a coreflection rather than a reflection. For basic categorical concepts not defined here, the reader is referred to the book by Mac Lane [26].

4.1 Proximal maps

Let us first recall some nomenclature:

Definition 4.1. [20, 21] A frame L is called *proximal* if there is a strong inclusion on it. If $\lhd$ is a strong inclusion on frame L, we shall refer to the pair $(L, \lhd)$ as a *proximal frame.* Let $(L, \lhd_1)$ and $(M, \lhd_2)$ be proximal frames. A function $(L, \lhd_1) \xrightarrow{f} (M, \lhd_2)$ is called a *proximal map* if $L \xrightarrow{f} M$ a frame homomorphism and $x \lhd_1 y \Longrightarrow f(x) \lhd_2 f(y)$. Let **ProxFrm** denote the category of proximal frames and proximal maps.

For more details about the category **ProxFrm**, see [20] and [21]. The content of the following lemma is part of the folklore (for example, see [5] and [30]), so we omit the proofs:

Lemma 4.2. (1) *Let $h : M \longrightarrow L$ be a frame homomorphism. Then $a \prec b \Longrightarrow h(a) \prec h(b)$, and $a \leq h_*h(a)$, for all $a \in M$.*

(2) *If $\lhd$ is a strong inclusion on a compact regular frame L, then $\lhd = \prec$.*

(3) *For any frames L, M, suppose that $h : M \longrightarrow L$ is dense onto frame homomorphism. If $a \prec b$ in M, then $h_*h(a) \prec b$ in L.*

Let us now show that any strong inclusion gives rise to a proximal map:

Proposition 4.3. *Let $(L, \lhd)$ be a proximal frame and $\Im_\lhd L \xrightarrow{\rho_L} L$ be the compactification associated with the strong inclusion $\lhd$. Then the join map $(\Im_\lhd L, \prec) \xrightarrow{\rho_L} (L, \lhd)$ is a proximal map.*

Proof. Suppose that $J \prec I$ in $\Im_\lhd L$. We need to show that $\rho_L(J) \lhd \rho_L(I)$. That is, we need to show that $k_\lhd(\rho_L(J)) \prec k_\lhd(\rho_L(I))$, where $L \xrightarrow{k_\lhd} \Im_\lhd L$ is the right adjoint of $\Im_{\lhd}L \xrightarrow{\rho_L} L$. On the one hand, $J \prec I \Longrightarrow k_{\lhd}(\rho_L(J)) \prec I$, by Lemma 4.2 (3). On the other hand, $I \subseteq k_{\lhd}(\rho_L(I))$, by Lemma 4.2 (1). Therefore $k_{\lhd}(\rho_L(J)) \prec k_{\lhd}(\rho_L(I))$. ☐

Mutatis mutandis, the following result appears as Lemma 6 in [10], so we omit the proof.

Lemma 4.4. *Let $M \xrightarrow{h} L$ be a dense frame homomorphism. If M and L are regular frames, then h is left cancellative. Moreover, if M and L are compact regular frames, then h is one-to-one.*

The following result prevails:

Proposition 4.5. *The category of compact regular frames and proximal maps is a coreflective full subcategory of the category of proximal frames and proximal maps.*

Proof. Note that any frame homomorphism $L_1 \xrightarrow{f} L_2$ between compact regular frames is a proximal map since $\prec$ is the only strong inclusion on these frames, and we have already seen that $a \prec b$ implies that $f(a) \prec f(b)$, from Lemma 4.2 (1). Now, let $(M, \lhd)$ be a proximal frame. Suppose that $(L, \prec) \xrightarrow{h} (M, \lhd)$ is a proximal map, where L is compact regular. We need a proximal map (simply a frame homomorphism) $(L, \prec) \xrightarrow{g} (\Im_{\lhd}M, \prec)$ that makes the following diagram commute:

$$\begin{tikzcd}
 & (M, \lhd) \\
(L, \prec) \arrow[ur, "h"] \arrow[r, dashed, "g"'] & (\Im_{\lhd}M, \prec) \arrow[u, "\rho_M"']
\end{tikzcd}$$

where ρ_M is the join map associated with the compactification $\Im_{\lhd}M$. From [10, Proposition 1], we do have the following commutative diagram:

$$\begin{tikzcd}
(L, \prec) \arrow[r, "h"] & (M, \lhd) \\
(\Im_{\prec}L, \prec) \arrow[u, "\rho_L"] \arrow[r, "\Im h"] & (\Im_{\lhd}M, \prec) \arrow[u, "\rho_M"']
\end{tikzcd}$$

where $\Im h(J)$ is an ideal in $\Im M$ generated by $h(J)$. Note that, by Lemma 4.4, $\Im_{\prec} L \xrightarrow{\rho_L} L$ is an isomorphism since it is dense onto homomorphism between compact regular frames. Now, $\rho_M(\Im h) = h\rho_L \Longrightarrow h = \rho_M\Bigg((\Im h)(\rho_L)^{-1}\Bigg)$. We now have $h = \rho_M g$, where $g = (\Im h)(\rho_L)^{-1}$. This factorization is unique: for if $h = \rho_M g'$, then $\rho_M g = \rho_M g'$, and therefore $g = g'$. The latter being the case since $\Im M \xrightarrow{\rho_M} M$ is a dense map between regular frames, whence, it is left cancellative by Lemma 4.4. $\square$

We now specialize our discussion of proximal maps to the setting where all the frames involved are rim-compact. We are particularly interested in the case where our frames are endowed with the strong inclusions associated with their Freudenthal compactifications. Recall that we write B_L for the Freudenthal basis associated with a rim-compact frame L.

Proposition 4.6. *A frame homomorphism* $(L, \lhd_{B_L}) \xrightarrow{f} (M, \lhd_{B_M})$ *is proximal if and only if whenever* $a \vee b = 1$ *where* $a, b \in B_L$*, there exists* $c, d \in B_M$ *such that* $c \vee d = 1$ *and* $c \leq f(a), d \leq f(b)$*.*

Proof. ($\Longrightarrow$) Suppose that $(L, \lhd_{B_L}) \xrightarrow{f} (M, \lhd_{B_M})$ is a proximal map. Take $a \vee b = 1$ with $a, b \in B_L$. Then, by [1, Lemma 4.5], there exists $s, t \in B_L$ such that $s \prec a, t \prec b$ and $s \vee t = 1$. Since $s \leq s^{**}$, we have $s^{**} \vee t = 1$. Thus, $s^* \prec t \prec b$. From this, it follows that $s^* \lhd_{B_L} b$. Therefore $f(s^*) \lhd_{B_M} f(b)$, since f is a proximal map. We use the interpolating property of the strong inclusion $\lhd_{B_M}$ to get $x, y \in B_M$ such that $f(s^*) \lhd_{B_M} x \lhd_{B_M} y \lhd_{B_M} f(b)$. Therefore, $x^* \vee y = 1$ and $x^* \leq f(s^*)^*$, and whence $x^* \wedge f(s^*) = 0$. Now, $s \prec a \Longrightarrow s^* \vee a = 1 \Longrightarrow f(s^*) \vee f(a) = 1 \Longrightarrow x^* \wedge \Bigg(f(s^*) \vee f(a)\Bigg) = x^* \wedge 1 \Longrightarrow \Bigg(x^* \wedge f(s^*)\Bigg) \vee \Bigg(x^* \wedge f(a)\Bigg) = x^* \Longrightarrow x^* \wedge f(a) = x^* \Longrightarrow x^* \leq f(a)$. Let us point it out that $x^* \in B_M$ (since B_M is a π-compact basis), so $\mathfrak{c}_L(x^* \vee x^{**})$ is compact. But $y \leq f(b)$ and $\mathfrak{c}_L(y \vee y^*)$ is compact, hence we conclude the proof by taking $c = x^*$ and $d = y$.

($\Longleftarrow$) Now, suppose that $a \lhd_{B_L} b$ in L. We need to show that $f(a) \lhd_{B_M} f(b)$. Take $x, y \in B_L$ such that $a \lhd_{B_L} x \lhd_{B_L} y \lhd_{B_L} b$. Observe that $x^* \vee y = 1$ and $x^*, y \in B_L$. Therefore, by the hypothesis, there exists $c, d \in B_M$ such that $c \leq f(x^*), d \leq f(y)$ and $c \vee d = 1$. Notice that $c \wedge f(x) \leq f(x^*) \wedge f(x) =$

$f(x^* \wedge x) = f(0) = 0$. Thus, $c \wedge f(x) = 0$. It follows from the latter and the equation $c \vee d = 1$ that $d \wedge f(x) = f(x)$. That is, $f(x) \leq d$. Now, note that $a \prec x \prec y \prec b$, so $f(a) \prec f(x) \leq d \leq f(y) \prec f(b)$. Hence, $f(a) \prec d \prec f(b)$ and $d \in B_M$; which implies that $f(a) \lhd_{B_M} f(b)$. $\square$

4.2 Freudenthal and F-maps In a general setting, a frame homomorphism satisfying the conditions in Proposition 4.6 is what we define to be an F-map:

Definition 4.7. Let L and M be frames. We call a frame homomorphism $L \xrightarrow{f} M$ an *F-map* if whenever $a \vee b = 1$ in L where a, b are π-elements in L, then there exists π-elements c, d in M such that $c \leq f(a), d \leq f(b)$ and $c \vee d = 1$.

Remark 4.8. Note that, in Proposition 4.6, we have shown that a frame homomorphism $(L, \lhd_{B_L}) \xrightarrow{f} (M, \lhd_{B_M})$ is a proximal map if and only if it is an F-map. We will see below, however, that an F-map need not be a proximal map.

Example 4.9. We provide here some examples of F-maps:

(a) The identity map from a frame to itself is an F-map.

(b) Let $\mathbb{B}$ be any complete Boolean algebra. Then any frame homomorphism $f : \mathbb{B} \longrightarrow \mathbb{B}$ is an F-map. Indeed, any element $b \in \mathbb{B}$ is a π-element since $\mathfrak{c}_L(b \vee b^*) = \mathfrak{c}_L(1) = \mathrm{O}$. Moreover, if $a \vee b = 1$ in $\mathbb{B}$, then $c = f(a)$ and $d = f(b)$ are the required π-elements.

(c) Denote the three-element chain by $\mathbf{3}$. For any frame L, a frame homomorphism $f : L \longrightarrow \mathbf{3}$ is an F-map. Indeed, for any element π-element $a, b \in L$ with $a \vee b = 1$, we have the π-element $c = f(a)$ and $d = f(b)$ in $\mathbf{3}$ with the required properties. Needless to say, $\mathbf{3}$ can be chosen to be any finite frame. The emphasis on $\mathbf{3}$ as a codomain is that it is known that $\mathbf{3}$ has no strong inclusion on it (for example, see [20, Example 4.6 (iii)]). Hence, not all F-maps are proximal maps.

The following shows that in the setting where the codomain is compact, F-maps are just frame homomorphisms.

Proposition 4.10. *If L is any frame and M is a compact frame, then every frame homomorphism $L \xrightarrow{f} M$ is an F-map.*

Proof. Let $a \vee b = 1$ in L where a, b are π-elements in L, and $L \xrightarrow{f} M$ be a frame homomorphism. In the definition of an F-map, take $c = f(a)$ and $d = f(b)$. Since M is compact, then $\mathfrak{c}_M(c \vee c^*)$ and $\mathfrak{c}_M(d \vee d^*)$ are compact, by [24, Proposition III.1.2]. That is, c, d are π-elements in M. Moreover, $c \vee d = f(a) \vee f(b) = f(a \vee b) = f(1) = 1$. Therefore, $L \xrightarrow{f} M$ is a frame homomorphism if and only if $L \xrightarrow{f} M$ is an F-map. $\square$

Proposition 4.11. *The Freudenthal join map $\gamma L \xrightarrow{\rho_L} L$ is an F-map.*

Proof. Note that $(\gamma L, \prec) \xrightarrow{\rho_L} (L, \lhd_{B_L})$ is a proximal map, by Proposition 4.3. Thus, $\gamma L \xrightarrow{\rho_L} L$ is an F-map by Proposition 4.6. $\square$

We conclude our paper by proving the main result of this section.

Theorem 4.12. *The category of compact regular frames and F-maps is a coreflective full subcategory of the category of rim-compact frames and F-maps. The coreflection is evidenced by the Freudenthal compactification.*

Proof. Let L be a rim-compact frame and $\lhd_{B_L}$ be the strong inclusion associated with the Freudenthal compactification $\gamma L \xrightarrow{\rho_L} L$. The join map $\gamma L \xrightarrow{\rho_L} L$ is an F-map by Proposition 4.11. We also know that the join map $\rho_L : (\gamma L, \prec) \longrightarrow (L, \lhd_{B_L})$ is a proximal map, by Proposition 4.3. Now, suppose that $M \xrightarrow{f} L$ is an F-map where M is a compact regular frame. Since M is compact regular, then it has only one strong inclusion, namely $\prec$. Therefore, the strong inclusion $\lhd_{B_M}$ associated with the Freudenthal compactification of M is the same as $\prec$. Hence, $(M, \prec) \xrightarrow{f} (L, \lhd_{B_L})$ is proximal by Proposition 4.6. So, by the proof of Proposition 4.5, there exists a frame homomorphism $M \xrightarrow{g} \gamma L$ such $\rho_L g = f$. We, therefore, have the following commutative diagram:

$$\begin{tikzcd}
 & L \\
M \arrow[ur, "f"] \arrow[r, dashed, "g"'] & \gamma L \arrow[u, "\rho_L"']
\end{tikzcd}$$

Since γL is compact, then $g : M \longrightarrow \gamma L$ is an F-map by Proposition 4.10. Hence, the diagram above is commutative, and all the maps are F-maps. The uniqueness of g follows from the fact that L and γL are regular together with the fact that $\gamma L \xrightarrow{\rho_L} L$ is dense and onto, and therefore $\gamma L \xrightarrow{\rho_L} L$ is left cancellative, by Lemma 4.4. ☐

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