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Nucleus topology in equality algebras

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Abstract. In this paper, we define the concept of nucleus map on equality algebras and study related results. Then, using this concept and upsets, a topology on equality algebras is constructed and it is shown that the equality algebra with this topology becomes a topological space. In addition, some properties of topological space such as compactness and connectedness are investigated. Moreover, we study the continuity of all operations with respect to the topology on equality algebras. Finally, the relations between the two topologies in quotient equality algebras are revealed.

1 Introduction and Preliminaries

Algebra and topology are indeed fundamental areas of mathematics that complement each other in many ways. Algebra deals with the study of mathematical operations and structures, while topology focuses on properties that are preserved under continuous transformations, such as bending and stretching. Together, these two fields provide a powerful toolkit for understanding and solving mathematical problems. They are used together

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in the field of mathematics known as algebraic topology. Algebraic topology is a branch of mathematics that uses algebraic tools to study topological spaces. Topological spaces are mathematical structures that capture the notion of closeness or continuity, while algebraic structures provide a way to analyze these spaces using algebraic techniques. By combining algebraic and topological methods, algebraic topology allows mathematicians to study properties of spaces that are not easily accessible through purely topological or algebraic means.

Topology on logical algebraic structures is a fascinating field that explores the relationship between the algebraic properties of structures like lattices, Boolean algebras, etc., and their topological properties. It involves studying concepts like open and closed sets, convergence, continuity, and compactness within the context of these structures.

The motivation for introducing equality algebras came from EQ-algebras which are defined by Novák in [18]. In EQ-algebras, compared to equality algebras, there is an additional operation $\otimes$, called product, which is very loosely related to the other operations. Therefore, there might not exist deep algebraic characterizations of EQ-algebras, and intention was to define a structure similar to EQ-algebras but without the product. This new logical algebra, the equality algebra, has two connectives, a meet operation and an equivalence, and a constant. Equality algebra is introduced by Jenei [9]. Since equality algebra can be a good alternative to possible algebraic semantics for fuzzy type theory, the study of equality algebra is very valuable. Recently, many mathematicians have studied this algebraic structure in various fields. For instance, the relation between equality algebra with other algebraic structure, filters, ideals, annihilators and co-annihilators are investigated that continued in [1–3, 7, 8, 12, 13, 15].

Lately a lot of mathematicians have been looking into the idea of topology on logical algebraic structures specifically BL-algebras. For instance Borzooei in [6] looked at how separation axioms are connected to (semi) topological quotient BL-algebras. They explored when a (semi)topological quotient BL-algebra can be a T_1-space or Hausdorff or regular or normal. In [5], the authors introduced the concept of quasi-filter neighborhoods in (semi)topological BL-algebras and discussed some of their characteristics through statements and proofs. Furthermore by applying the idea of quasi-filter they discovered certain situations in which a BL-algebra can be turned into a metrizable structure.

In quantale senmatics of linear logic we can see that ! is just a quantic conucleus and ? is a closure operator, but not a quantic nucleus in general. As we all know, the quantic nuclei and the quantic conuclei play an important role in quantale theory, because they determine the quotients and subobjects in the category of quantales. The quantic nucleus and the quantic conucleus are dual, but there is no obvious connection between them in general quantale. So it is an interesting work to study the relationship between the quantic conuclei and the quantic nuclei. In [19], the corresponding relationship between the quantic nuclei and the quantic conuclei is not one-to-one in general quantale. In which kind of quantale, there is a good corresponding relationship between the quantic nucleus and the quantic conucleus.

The concept of a nucleus on a residuated lattice was introduced by Galatos and Tsinakis in [11], and they used nuclei to give some characterizations for a residuated lattice to be an integral generalized MV-algebra.

In this paper, we define the concept of nucleus map on equality algebras and some related results are investigated. Then by using this notion and upsets, a topology is constructed in equality algebra and it is shown that an equality algebra endowed with this topology becomes a topological space. Furthermore, some properties of the topological space are investigated such as compactness and connectedness. Moreover, we study continuities of all the operations with respect to the topology in an equality algebra. Finally, the relationships are revealed between the two topologies on quotient equality algebras.

2 Preliminaries

Definition 2.1. [9] An algebraic structure $(E; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if it satisfies the following conditions, for all $x, y, z \in E$,
$(E1)$ $(E, \wedge, 1)$ is a commutative idempotent integral monoid,
$(E2)$ $x \sim y = y \sim x$,
$(E3)$ $x \sim x = 1$,
$(E4)$ $x \sim 1 = x$,
$(E5)$ $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,

$(E6)$ $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
$(E7)$ $x \sim y \leq (x \sim z) \sim (y \sim z)$.

An equality algebra $(E, \leq)$ with a binary relation defined by "$x \leq y$ if and only if $x \wedge y = x$" is a poset. Also, we define the operation "$\rightarrow$" on E as: $x \rightarrow y = x \sim (x \wedge y)$.

An equality algebra $(E, \sim, \wedge, 0, 1)$ is called a *bounded equality algebra* if for every $x \in E$, $0 \leq x$. In a bounded equality algebra E, we define the negation operation "$'$" by: $x' = x \rightarrow 0 = x \sim 0$, for all $x \in E$.

A lattice equality algebra is an equality algebra which is a lattice.

If for any $x, y, z \in E$, $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$, then E is called a *positive implicative* or a *p-equality algebra*, for short (see [3, 21]).

Note: From now on, let $(E; \wedge, \sim, 1)$ or E be an equality algebra, unless otherwise stated.

Proposition 2.2. [9, 21] *The following conditions hold, for any $x, y, z \in E$:*
(i) $x \rightarrow y = 1$ *if and only if* $x \leq y$,
(ii) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$, *and* $x \rightarrow x = 1$,
(iii) $x \leq y \rightarrow x$,
(iv) $x \leq (x \rightarrow y) \rightarrow y$,
(v) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(vi) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$,
(vii) *If E is a lattice, then* $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.
(viii) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,
(ix) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$ *and* $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Definition 2.3. [9] Let $\emptyset \neq F \subseteq E$. Then $F \in \mathcal{F}(E)$ if and only if, for all $x, y \in E$, $1 \in F$, and if $x \in F$ and $x \rightarrow y \in F$, then $y \in F$.

A filter F is called a *proper filter* of E, if $F \neq E$. Clearly, if E is bounded, then $F \in \mathcal{F}(E)$ is proper if and only if it is not containing 0. Let $F \in \mathcal{F}(E)$. The relation θ_F on E by $(x, y) \in \theta_F$ if and only if $\{x \rightarrow y, y \rightarrow x\} \subseteq F$ is a congruence relation. Let $\frac{E}{F} = \{[x] | \, x \in E\}$, where $[x] = \{y \in E \mid \ (x, y) \in \theta_F\}$. Then $\frac{E}{F}$ is an equality algebra and the binary relation $\leq_F$ on $\frac{E}{F}$ by $[x] \leq_F [y]$ if and only if $x \rightarrow y \in F$ is a poset.

Definition 2.4. [16] Let $\emptyset \neq A \subseteq E$. The smallest filter of E containing A is called *the generated filter by A in E* which is denoted by $\langle A \rangle$. Indeed,

$\langle A\rangle = \bigcap_{A\subseteq F\in\mathcal{F}(E)} F.$

Proposition 2.5. [16] *Let $\emptyset \neq A \subseteq E$. Then*

$$\langle A\rangle = \{x \in E \mid a_1 \to (a_2 \to (\cdots \to (a_n \to x)\cdots,)) = 1,\ \textit{for some } n \in \mathbb{N} \textit{ and } a_1, \cdots, , a_n \in A\}.$$

In particular, for any element $a \in E$, we have

$$\langle a\rangle = \{x \in E \mid a \to^n x = 1,\ \textit{for some } n \in \mathbb{N}\},$$

where $x \to^0 y = y$ and $x \to^n y = x \to (x \to^{n-1} y)$. If $F \in \mathcal{F}(E)$ and $a \in E \setminus F$, then

$$\langle F \cup \{a\}\rangle = \{x \in E \mid a \to^n x \in F,\ \textit{for some } n \in \mathbb{N}\}.$$

If $F, G \in \mathcal{F}(E)$, then

$$\begin{aligned}\langle F \cup G\rangle &= \{x \in E \mid\ g \to x \in F,\ \textit{for some } g \in G\}\\ &= \{x \in E \mid\ f \to x \in G,\ \textit{for some } f \in F\}.\end{aligned}$$

Definition 2.6. [8, 15, 16] Let F be a proper filter of E and $x, y \in E$. Then

(i) F is a *maximal filter* of E if it is not included in any other proper filter of E. The set of all maximal filters of E is denoted by $\mathcal{M}ax(E)$.

(ii) F is a *prime filter* of E if for any $x, y \in E$, $x \to y \in F$ or $y \to x \in F$. The set of all prime filters of E is denoted by $\mathcal{P}rime(E)$.

(iii) If E is a lattice, then F is called a $\vee$-*irreducible* if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for any $x, y \in E$. The set of all $\vee$-irreducible filters of E is denoted by $\mathcal{S}pec(E)$.

Proposition 2.7. [15] (i) *Each maximal filter of E is prime. Indeed, $\mathcal{M}ax(E) \subseteq \mathcal{P}rime(E)$.*
Suppose E is a lattice. Then
(ii) *$F \in \mathcal{S}pec(E)$ if and only if $\frac{E}{F}$ is a chain.*
(iii) *$\mathcal{P}rime(E) \subseteq \mathcal{S}pec(E)$.*

Definition 2.8. [20] Let E be an equality algebra. The mapping $f : E \to E$ is called a *modal operator*, if it satisfies the following conditions, for all $x, y \in E$:
(f1) $x \leq f(x)$;
(f2) $f(f(x)) = f(x)$;
(f3) $f(x \wedge y) = f(x) \wedge f(y)$;
(f4) $f(x \sim y) \leq f(x) \sim f(y)$.

Definition 2.9. [14] A map $c : E \to E$ is a *closure operator*, if for any $x, y \in E$ we have
(c_1) $x \leq c(x)$,
(c_2) If $x \leq y$, then $c(x) \leq c(y)$,
(c_3) $c(c(x)) = c(x)$.

3 Nucleus map on equality algebras

In this section, we introduce the concept of nucleus map on equality algebra and some related properties are investigated. Also, by using properties of equality algebra we make some nucleus map on equality algebra.

Definition 3.1. The map $\mu : E \to E$ is called *nucleus* on E, if for any $x, y \in E$:

(N1) $x \leq \mu(x)$.

(N2) $\mu(\mu(x)) = \mu(x)$.

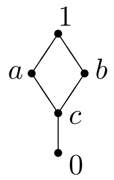
(N3) $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.

The set of all nucleus maps on E is denoted by $\mathcal{NU}(E)$.

Example 3.2. (i) Clearly, Id_E and the fixed function $\mu : E \to E$ such that for any $x \in E$, $\mu(x) = 1$ are nucleus maps on E.

(ii) Suppose $E = \{0, a, b, c, 1\}$ is a poset such that $0 \leq c \leq a, b \leq 1$.

Define "$\sim$" on E as follows:



$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra. Define $\mu : E \to E$ where $\mu(0) = 0$, $\mu(a) = \mu(c) = a$ and $\mu(b) = \mu(1) = 1$. Clearly, $\mu \in \mathcal{NU}(E)$.

Proposition 3.3. *Let $\mu \in \mathcal{NU}(E)$. Then for any $x, y \in E$, we have*

(i) $\mu(1) = 1$.

(ii) *If $x \leq y$, then $\mu(x) \leq \mu(y)$.*

(iii) $\mu(x) \to \mu(y) \leq x \to \mu(y)$.

(iv) $\mu(x \wedge y) = \mu(\mu(x) \wedge y) = \mu(x \wedge \mu(y)) = \mu(\mu(x) \wedge \mu(y))$.

Proof. (i) Clearly, by (N1), we have $1 \leq \mu(1) \leq 1$. Hence, $\mu(1) = 1$.

(ii) If $x \leq y$, then $x = x \wedge y$. Hence, by (N3), $\mu(x) = \mu(x \wedge y) = \mu(x) \wedge \mu(y)$. Thus, $\mu(x) \leq \mu(y)$.

(iii) By (N1), we have $x \leq \mu(x)$ and by Proposition 2.2(vi), $\mu(x) \to \mu(y) \leq x \to \mu(y)$.

(iv) By (N1), $x \leq \mu(x)$ and so $x \wedge y \leq \mu(x) \wedge y$ and by (ii), we get

$$\mu(x \wedge y) \leq \mu(\mu(x) \wedge y). \tag{3.1}$$

In addition, by (N1), $y \leq \mu(y)$, and so $\mu(x) \wedge y \leq \mu(x) \wedge \mu(y)$. By (ii), we have

$$\mu(\mu(x) \wedge y) \leq \mu(\mu(x) \wedge \mu(y)). \tag{3.2}$$

Moreover, by (N3), $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$, and so $\mu(\mu(x \wedge y)) = \mu(\mu(x) \wedge \mu(y))$. By (N2), we get

$$\mu(x \wedge y) = \mu(\mu(x) \wedge \mu(y)). \tag{3.3}$$

Hence, by Equations (3.1), (3.2) and (3.3), we have

$$\mu(x \wedge y) \leq \mu(\mu(x) \wedge y) \leq \mu(\mu(x) \wedge \mu(y)) = \mu(x \wedge y).$$

The proof of other case is similar. $\square$

Note. It is obvious that every modal operator on E is nucleus on E. Moreover, by (N1), (N2) and Proposition 3.3(ii), we get any nucleus is a closure operator on E.

In the following, by using the properties of equality algebra we try to construct some special nucleus maps on equality algebras.

Theorem 3.4. *Let E be a lattice and $x, y, z \in E$. If E satisfies the condition $(x \wedge y) \to z = (x \to z) \vee (y \to z)$, then*

(i) *the map $\omega_a : E \to E$; by $\omega_a(x) = (x \to a) \to a$, is nucleus on E, for any $a \in E$. Specially, if E is a bounded, take $a = 0$, then $\omega_a ='' $ is nucleus on E.*

(ii) *any map $\mu : E \to E$ such that $\mu(x) \to \mu(y) = x \to \mu(y)$ is a nucleus.*

Proof. (i) By Proposition 2.2(iv), $x \leq (x \to a) \to a = \omega_a(x)$, so (N1) holds. Also, by Proposition 2.2(viii), $\omega_a(\omega_a(x)) = \omega_a((x \to a) \to a) = (((x \to a) \to a) \to a) \to a = (x \to a) \to a = \omega_a$. Thus, (N2) holds, too. Finally, by assumption and Proposition 2.2(vii), we have

$$\begin{aligned}\omega_a(x \wedge y) &= ((x \wedge y) \to a) \to a = ((x \to a) \vee (y \to a)) \to a \\ &= ((x \to a) \to a) \wedge ((y \to a) \to a) = \omega_a(x) \wedge \omega_a(y).\end{aligned}$$

Hence, (N3) holds. Therefore, $\omega_a \in \mathcal{NU}(E)$.

(ii) Assume $x \in E$. Since $x \to \mu(x) = \mu(x) \to \mu(x) = 1$, we get $x \leq \mu(x)$, and so (N1) holds. By (N1), $\mu(x) \leq \mu(\mu(x))$. For converse, by assumption we have $\mu(\mu(x)) \to \mu(x) = \mu(x) \to \mu(x) = 1$. Thus, $\mu(\mu(x)) \leq \mu(x)$, and so $\mu(\mu(x)) = \mu(x)$. Hence, (N2) holds. Suppose $x, y \in E$ such that $x \leq y$. Since by (N1), $x \leq y \leq \mu(y)$ we have $\mu(x) \to \mu(y) = x \to \mu(y) = 1$, and so $\mu(x) \leq \mu(y)$. Hence, from $x \wedge y \leq x, y$, we have $\mu(x \wedge y) \leq \mu(x) \wedge \mu(y)$. Moreover, by assumption we have:

$$\begin{aligned}(\mu(x) \wedge \mu(y)) \to \mu(x \wedge y) &= (\mu(x) \to \mu(x \wedge y)) \vee (\mu(y) \to \mu(x \wedge y)) \\ &= (x \to \mu(x \wedge y)) \vee (y \to \mu(x \wedge y))\end{aligned}$$

$$\begin{aligned} &= (x \wedge y) \to \mu(x \wedge y) \\ &= 1. \qquad \text{by (N1)} \end{aligned}$$

Thus, $\mu(x) \wedge \mu(y) \leq \mu(x \wedge y)$, and so $\mu(x) \wedge \mu(y) = \mu(x \wedge y)$. Hence, (N3) holds. Therefore, $\mu \in \mathcal{NU}(E)$. □

Theorem 3.5. *Let $\mu \in \mathcal{NU}(E)$ be a homomorphism. Then $\mu \leq \omega_a$ if and only if $a \in \mathcal{F}ix_\mu(E)$, where $\mathcal{F}ix_\mu(E) = \{x \in E \mid \mu(x) = x\}$.*

Proof. Let $\mu \leq \omega_a$. Then $\mu(x) \leq \omega_a(x)$, for any $x \in E$. Take $x = a$. Hence,

$$\mu(a) \leq \omega_a(a) = (a \to a) \to a = a.$$

Thus, $\mu(a) \leq a$. By (N1) we get $\mu(a) = a$, and so $a \in \mathcal{F}ix_\mu(E)$.

Conversely, by (N1) and Proposition 2.2(vi), respectively, we have

$$\mu(x \to a) \to a \leq (x \to a) \to a \tag{3.4}$$

Now, assume $a \in \mathcal{F}ix_\mu(E)$ and $x \in E$. Then

$$\begin{aligned} \mu(x) &\leq (\mu(x) \to a) \to a \qquad \text{by Proposition 2.2(iv)} \\ &= (\mu(x) \to \mu(a)) \to a \qquad \text{since } a \in \mathcal{F}ix_\mu(E) \\ &= (\mu(x \to a)) \to a \qquad \text{by homomorphism} \\ &\leq (x \to a) \to a \qquad \text{by (3.4)} \\ &= \omega_a(x). \end{aligned}$$

Hence, for any $x \in E$, $\mu(x) \leq \omega_a(x)$. Therefore, $\mu \leq \omega_a$. □

Note: A non-unite element $a \in E$ is said to be a *co-atom* of E if whenever $a \leq b \leq 1$, then $b = a$ or $b = 1$.

Theorem 3.6. *If E is a p-equality algebra such that $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, then*

(i) *the map $\mu_u : E \to E$ defined by $\mu_u(x) = u \to x$ is a nucleus on E, for any $u \in E$.*

(ii) *If $u \in E$ is a co-atom and $a \in \mathcal{F}ix_{\mu_u}(E)$, then $a \to c \in \mathcal{F}ix_{\mu_u}(E)$, for any $c \in E$.*

(iii) *the structure $(\mathcal{F}ix_{\mu_u}(E), \wedge, \approx, 1)$ is an equality algebra where for any $x, y \in \mathcal{F}ix_{\mu_u}(E)$,*

$$x \approx y := (x \to y) \wedge (y \to x).$$

Proof. (i) For any $x \in E$, by Proposition 2.2 (iii), $x \leq u \to x = \mu_u(x)$, so (N1) holds. Also, since E is a p-equality algebra, we have $\mu_u(\mu_u(x)) = u \to (u \to x) = (u \to u) \to (u \to x) = 1 \to (u \to x) = u \to x = \mu_u(x)$. Thus, (N2) holds. In addition, by assumption, we have

$$\mu_u(x \wedge y) = u \to (x \wedge y) = (u \to x) \wedge (u \to y) = \mu_u(x) \wedge \mu_u(y).$$

Hence, (N3) holds. Therefore, $\mu_u \in \mathcal{NU}(E)$.

(ii) By assumption, we have $\mu_u(a \to c) = a \to c$, and so $a \to c \in \mathcal{F}ix_{\mu_u}(E)$.

(iii) Consider $x, y \in \mathcal{F}ix_{\mu}(E)$. Since μ is nucleus, by (N3) we have $\mu(x \wedge y) = \mu(x) \wedge \mu(y) = x \wedge y$. Thus, $x \wedge y \in \mathcal{F}ix_{\mu}(E)$. Moreover, by (ii), $\mathcal{F}ix_{\mu_u}(E)$ is closed under $\to$, respectively. Thus, by assumption $(\mathcal{F}ix_{\mu_u}(E), \wedge, \approx, 1)$ is an equality algebra. $\square$

Example 3.7. (i) Let $E = \{0, a, b, 1\}$ be a chain where $0 \leq a \leq b \leq 1$. Define "$\sim$" on E as follows:

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then E is a p-equality algebra with condition $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, for all $x, y, z \in E$.

(ii) Suppose $E = \{0, a, b, c, d, 1\}$ is a poset such that $0 \leq a, b \leq c \leq d \leq 1$. Define "$\sim$" on E as follows:

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra. Obviously, E satisfies in condition $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, but is not a p-equality algebra, since

$$1 = a \to d = a \to (c \to b) \neq (a \to c) \to (a \to b) = 1 \to d = d.$$

4 Nucleus topology on equality algebras

In this section, a topology is constructed by nucleus and upsets in equality algebra and it is shown that an equality algebra endowed with this topology becomes a topological space. Furthermore, some properties of the topological space are investigated such as compactness and connectedness. Moreover, we study continuities of all the operations with respect to the topology in an equality algebra.

Let $(X; \leq)$ be a poset. Then for any $Y \subseteq X$, we define $\uparrow Y = \{x \in X \mid \exists\, y \in Y,\ \text{s.t.},\ y \leq x\}$. The set Y is called *upset* if $\uparrow Y = Y$. The set of all upsets of X is denoted by $\Gamma(X)$.
An upset Y is called *finitely generated* if there exists $n \in \mathbb{N}$ such that $Y =\uparrow \{y_1, y_2, \ldots, y_n\}$, for some $y_1, y_2, \ldots, y_n \in Y$. Obviously, $Y \subseteq\uparrow Y$, for any $Y \subseteq X$.

Proposition 4.1. *Suppose $(X; \leq)$ is a poset and $\{Y_i\}_{i\in I} \subseteq \mathcal{P}(X)$. Then*

(i) *If $Y_i \subseteq Y_j$, then $\uparrow Y_i \subseteq\uparrow Y_j$.*

(ii) *If $\{Y_i\}_{i\in I} \subseteq \Gamma(X)$, then $\bigcup\limits_{i\in I} Y_i \in \Gamma(X)$.*

(iii) *If $\{Y_i\}_{i\in I} \subseteq \Gamma(X)$, then $\bigcap\limits_{i\in I} Y_i \in \Gamma(X)$.*

(iv) *If $Y_i \in \Gamma(X)$, then $Y_i = \bigcup\limits_{x\in Y_i} \uparrow x$.*

Proof. The proof is straightforward. □

Definition 4.2. Assume $a \in E$ and $\mu \in \mathcal{NU}(E)$. For any $X \in \Gamma(X)$, define

$$\mathcal{D}^{\mu}_{a}(X) = \{x \in E \mid a \overset{n}{\to} \mu(x) \in X,\ \text{for some } n \in \mathbb{N}\},$$

where for any $n \in \mathbb{N}$, $a \overset{n}{\to} \mu(x) = a \to (a \to (\cdots (a \to \mu(x)) \cdots,))$.

Note. If E is bounded, then $\mathcal{D}^{\mu}_{0}(X) = E$. So, from now on, we always consider $a \neq 0$ even if E is bounded, unless otherwise stated.

Example 4.3. Consider the nucleus map in Example 3.2(ii) and $X = \{b, 1\}$. Then $\mathcal{D}_a^\mu(X) = \{a, b, 1\}$.

Proposition 4.4. *Let $\emptyset \neq X, Y \in \Gamma(E)$, $\mu \in \mathcal{NU}(E)$ and $a \in E$. Then we have the following properties:*

(i) *$a \in \mathcal{D}_a^\mu(X)$ and $1 \in \mathcal{D}_a^\mu(X)$.*

(ii) *$\mathcal{D}_a^\mu(X) \in \Gamma(E)$.*

(iii) *$X \subseteq \mathcal{D}_a^\mu(X)$.*

(iv) *If $X \subseteq Y$, then $\mathcal{D}_a^\mu(X) \subseteq \mathcal{D}_a^\mu(Y)$.*

(v) *If $F \in \mathcal{F}(E)$ and μ is a homomorphism, then $\mathcal{D}_a^\mu(F) \in \mathcal{F}(E)$.*

Proof. (i) By (N1), $a \leq \mu(a)$ and so $a \overset{n}{\to} \mu(a) = 1 \in X$, for any $n \in \mathbb{N}$. Thus, $a \in \mathcal{D}_a^\mu(X)$. Since $X \in \Gamma(E)$ and $a \to \mu(1) = 1 \in X$, we get $1 \in \mathcal{D}_a^\mu(X)$.

(ii) Assume $x \leq y$ and $x \in \mathcal{D}_a^\mu(X)$. Then there is $n \in \mathbb{N}$ such that $a \overset{n}{\to} \mu(x) \in X$. Since $\mu \in \mathcal{NU}(E)$, by Proposition 3.3(ii), we have $\mu(x) \leq \mu(y)$. By Proposition 2.2(vi), $a \overset{n}{\to} \mu(x) \leq a \overset{n}{\to} \mu(y)$. Since $a \overset{n}{\to} \mu(x) \in X$ and $X \in \Gamma(E)$, we get $a \overset{n}{\to} \mu(y) \in X$, and so $y \in \mathcal{D}_a^\mu(X)$. Hence, $\mathcal{D}_a^\mu(X) \in \Gamma(E)$.

(iii) Suppose $x \in X$. By Proposition 2.2(iii) and (N1) we have

$$x \leq \mu(x) \leq a \to \mu(x) \leq a \overset{2}{\to} \mu(x) \leq \cdots \leq a \overset{n}{\to} \mu(x).$$

Since $x \in X$ and $X \in \Gamma(E)$ we have $a \overset{n}{\to} \mu(x) \in X$, for any $n \in \mathbb{N}$. Hence, $x \in \mathcal{D}_a^\mu(X)$, and so $X \subseteq \mathcal{D}_a^\mu(X)$.

(iv) Let $x \in \mathcal{D}_a^\mu(X)$. Then for $n \in \mathbb{N}$, $a \overset{n}{\to} \mu(x) \in X$. Since $X \subseteq Y$, we get $a \overset{n}{\to} \mu(x) \in Y$, and so $x \in \mathcal{D}_a^\mu(Y)$.

(v) By Proposition 3.3(i), $\mu(1) = 1$ and clearly, $1 \in \mathcal{D}_a^\mu(F)$. Suppose $x, x \to y \in \mathcal{D}_a^\mu(F)$. Then there exists $n, m \in \mathbb{N}$ such that $a \overset{m}{\to} \mu(x) \in F$ and $a \overset{n}{\to} \mu(x \to y) \in F$. So, by Proposition 2.2(x) and (v), we have:

$$\begin{aligned} a \overset{n}{\to} \mu(x \to y) &= a \overset{n}{\to} (\mu(x) \to \mu(y)) \\ &\leq a \overset{n}{\to} \Big((a \overset{m}{\to} \mu(x)) \to (a \overset{m}{\to} \mu(y))\Big) \\ &= (a \overset{m}{\to} \mu(x)) \to \Big(a \overset{n}{\to} (a \overset{m}{\to} \mu(y))\Big) \end{aligned}$$

Since $a \overset{n}{\to} \mu(x \to y) \in F$, $a \overset{m}{\to} \mu(x) \in F$ and $F \in \mathcal{F}(E)$, we have $a \overset{n+m}{\to} \mu(y) \in F$. Hence, $y \in \mathcal{D}_a^\mu(F)$, and so $\mathcal{D}_a^\mu(F) \in \mathcal{F}(E)$. $\square$

Theorem 4.5. *Let $\emptyset \neq X \in \Gamma(E)$ and $a, x, y \in E$. If $\mu \in \mathcal{NU}(E)$ such that $\mu(x \to y) = x \to \mu(y)$, then*

(i) $\mathcal{D}^{\mu}_{a}(\mathcal{D}^{\mu}_{a}(X)) = \mathcal{D}^{\mu}_{a}(X)$.

(ii) $\mathcal{D}^{\mu}_{a} : \Gamma(E) \to \Gamma(E)$ *is a closure operator on* $\Gamma(E)$.

Proof. (i) By Proposition 4.4(iii), $X \subseteq \mathcal{D}^{\mu}_{a}(X)$ and by Proposition 4.4(iv), we get $\mathcal{D}^{\mu}_{a}(X) \subseteq \mathcal{D}^{\mu}_{a}(\mathcal{D}^{\mu}_{a}(X))$. Conversely, $x \in \mathcal{D}^{\mu}_{a}(\mathcal{D}^{\mu}_{a}(X))$, then for $n \in \mathbb{N}$, $a \overset{n}{\to} \mu(x) \in \mathcal{D}^{\mu}_{a}(X)$. Thus, there exists $m \in \mathbb{N}$ such that $a \overset{m}{\to} \mu(a \overset{n}{\to} \mu(x)) \in X$. By assumption, we have $a \overset{m}{\to} (a \overset{n}{\to} \mu(\mu(x))) \in X$. Since $\mu \in \mathcal{NU}(E)$, by (N2), $a \overset{m}{\to} (a \overset{n}{\to} \mu(x)) \in X$, and so $a \overset{n+m}{\to} \mu(x) \in X$. Hence, $x \in \mathcal{D}^{\mu}_{a}(X)$. Hence, $\mathcal{D}^{\mu}_{a}(\mathcal{D}^{\mu}_{a}(X)) \subseteq \mathcal{D}^{\mu}_{a}(X)$. Therefore, $\mathcal{D}^{\mu}_{a}(\mathcal{D}^{\mu}_{a}(X)) = \mathcal{D}^{\mu}_{a}(X)$.

(ii) By (i) and Proposition 4.4(ii), (iii) and (iv), we get $\mathcal{D}^{\mu}_{a}$ is a closure operator. □

Proposition 4.6. *Let $a \in E$, $\mu \in \mathcal{NU}(E)$ and $\emptyset \neq X_\alpha \in \Gamma(E)$ for $\alpha \in I$. Then*

(i) $\mathcal{D}^{\mu}_{a}(\emptyset) = \emptyset$ *and* $\mathcal{D}^{\mu}_{a}(E) = E$.

(ii) $\mathcal{D}^{\mu}_{a}\left(\bigcup_{\alpha \in I} X_\alpha\right) = \displaystyle\bigcup_{\alpha \in I} \mathcal{D}^{\mu}_{a}(X_\alpha)$.

(iii) $\bigcap_{\alpha \in I} \mathcal{D}^{\mu}_{a}(X_\alpha) \subseteq \mathcal{D}^{\mu}_{a}\left(\bigcap_{\alpha \in I} X_\alpha\right)$.

(iv) *If* $\mid I \mid < \infty$, *then* $\mathcal{D}^{\mu}_{a}\left(\bigcap_{\alpha \in I} X_\alpha\right) = \bigcap_{\alpha \in I} \mathcal{D}^{\mu}_{a}(X_\alpha)$.

Proof. (i) It is clear.

(ii) If $\{\mathcal{D}^{\mu}_{a}(X_\alpha)\}_{\alpha \in I} \subseteq \Gamma(E)$, then by Propositions 4.1(ii), we have $\bigcup_{\alpha \in I} X_\alpha \in \Gamma(E)$. If $x \in \mathcal{D}^{\mu}_{a}\left(\bigcup_{\alpha \in I} X_\alpha\right)$, then there is $n \in \mathbb{N}$ such that $a \overset{n}{\to} \mu(x) \in \bigcup_{\alpha \in I} X_\alpha$. Thus, there exists $\eta \in I$ such that $a \overset{n}{\to} \mu(x) \in X_\eta$, and so $x \in \mathcal{D}^{\mu}_{a}(X_\eta)$. Hence, $x \in \bigcup_{\alpha \in I} \mathcal{D}^{\mu}_{a}(X_\alpha)$, and so $\mathcal{D}^{\mu}_{a}\left(\bigcup_{\alpha \in I} X_\alpha\right) \subseteq \displaystyle\bigcup_{\alpha \in I} \mathcal{D}^{\mu}_{a}(X_\alpha)$. Now, since $X_\alpha \subseteq \bigcup_{\alpha \in I} X_\alpha$ for any $\alpha \in I$, by (iii) we get $\mathcal{D}^{\mu}_{a}(X_\alpha) \subseteq \mathcal{D}^{\mu}_{a}\left(\bigcup_{\alpha \in I} X_\alpha\right)$. So $\bigcup_{\alpha \in I} \mathcal{D}^{\mu}_{a}(X_\alpha) \subseteq \mathcal{D}^{\mu}_{a}(\bigcup_{\alpha \in I} X_\alpha)$. Hence, $\mathcal{D}^{\mu}_{a}\left(\bigcup_{\alpha \in I} X_\alpha\right) = \bigcup_{\alpha \in I} \mathcal{D}^{\mu}_{a}(X_\alpha)$.

(iii) If $x \in \bigcap_{\alpha} \mathcal{D}^{\mu}_{a}(X_\alpha)$, then $x \in \mathcal{D}^{\mu}_{a}(X_\alpha)$ for any $\alpha \in I$. Hence, there is $n_\alpha \in \mathbb{N}$ for any $\alpha \in I$ such that $a \overset{n_\alpha}{\to} \mu(x) \in X_\alpha$. We put $t = \max\{n_\alpha\}_{\alpha \in I}$. By Proposition 2.2(iii), $a \overset{n_\alpha}{\to} \mu(x) \leq a \overset{t}{\to} \mu(x)$ for any $\alpha \in I$. Since

$X_\alpha \in \Gamma(E)$ for $\alpha \in I$, we obtain $a \xrightarrow{t} \mu(x) \in X_\alpha$ for any $\alpha \in I$. Hence, $a \xrightarrow{t} \mu(x) \in \bigcap_{\alpha\in I} X_\alpha$, and so $x \in \mathcal{D}_a^\mu\left(\bigcap_{\alpha\in I} X_\alpha\right)$. Hence, $\bigcap_{\alpha\in I} \mathcal{D}_a^\mu(X_\alpha) \subseteq \mathcal{D}_a^\mu\left(\bigcap_{\alpha\in I} X_\alpha\right)$.

(iv) By Propositions 4.1(iii), $\bigcap_{\alpha\in I} X_\alpha \in \Gamma(E)$. If $\mid I \mid= k$, and $x \in \mathcal{D}_a^\mu\left(\bigcap_{\alpha=1}^k X_\alpha\right)$, then there is $n \in \mathbb{N}$ such that $a \xrightarrow{n} \mu(x) \in \bigcap_{\alpha=1}^k X_\alpha$. Then $a \xrightarrow{n} \mu(x) \in X_\alpha$ for $1 \leq \alpha \leq k$. Thus, $x \in \mathcal{D}_a^\mu(X_\alpha)$ for $1 \leq \alpha \leq k$ and so $x \in \bigcap_{\alpha=1}^k \mathcal{D}_a^\mu(X_\alpha)$. Conversely, holds by (iii). Therefore, $\mathcal{D}_a^\mu\left(\bigcap_{\alpha\in I} X_\alpha\right) = \bigcap_{\alpha\in I} \mathcal{D}_a^\mu(X_\alpha)$. □

Theorem 4.7. *For any $a \in E$ and $\mu \in \mathcal{NU}(E)$, $\tau_a = \{\mathcal{D}_a^\mu(X) \mid X \in \Gamma(E)\}$ forms a topology on E and (E, τ_a) is a topological space.*

Proof. By Proposition 4.6, it is straightforward to see that (E, τ_a) is a topological space. □

Example 4.8. Let E and μ be as in Example 3.2(iii). Then $\Gamma(E) = \{\emptyset, \{1\}, \{b, 1\}, \{a, 1\}, \{c, a, b, 1\}, E\}$ and $\tau_a = \{\emptyset, \{a, b, c, 1\}, E\}$.

Proposition 4.9. (i) *For any $a \in E$, $\{a\}$ is dense in topological space (E, τ_a).*

(ii) *If E is bounded, then τ_0 is a trivial topology on E.*

(iii) *The set $\mathcal{B}_a = \{\mathcal{D}_a^\mu(\uparrow x) \mid x \in E\}$ is a base for the topology τ_a on E.*

Proof. (i) For proving that $\{a\}$ is dense, we have to show that $\overline{\{a\}} = E$. Suppose U is a closed subset of E such that $\{a\} \subseteq U$. So, U^c is open, thus there exists $X \in \Gamma(E)$ such that $U^c = \mathcal{D}_a^\mu(X)$. By Proposition 4.4(ii), $a \in \mathcal{D}_a^\mu(X)$, and so $a \in U \cap U^c = \emptyset$, which is an absurd. Hence, $U^c = \emptyset$, and so $U = E$. Therefore, $\{a\}$ is dense.

(ii) Let $X \in \Gamma(E)$. If $X = \emptyset$, then $\mathcal{D}_0^\mu(X) = \emptyset$. If $X \neq \emptyset$, since $X \in \Gamma(E)$, we get $1 \in X$. Also, for any $n \in \mathbb{N}$ and $x \in E$, $0 \xrightarrow{n} \mu(x) = 1 \in X$, and so $x \in \mathcal{D}_0^\mu(X)$. Hence, $\tau_0 = \{\emptyset, E\}$ is a trivial topology.

(iii) Suppose U is an open subset of (E, τ_a). Then there is $X \in \Gamma(E)$ such that $U = \mathcal{D}_a^\mu(X)$. Since $X \in \Gamma(E)$, we have $X = \bigcup\{\uparrow x \mid x \in X\}$. By Proposition 4.4(vii), $\mathcal{D}_a^\mu(X) = \bigcup\{\mathcal{D}_a^\mu(\uparrow x) \mid x \in X\}$. Thus $\mathcal{B}_a$ is a base for the topology τ_a on E. □

Theorem 4.10. *Assume $\mu \in \mathcal{NU}(E)$, $a, b \in E$ and $\emptyset \neq X \subseteq E$.*

(i) *If* $a \leq b$, *then* $\mathcal{D}^{\mu}_{b}(X) \subseteq \mathcal{D}^{\mu}_{a}(X)$.
Suppose μ *satisfies the condition* $\mu(x \to y) = x \to \mu(y)$, *for all* $x, y \in E$. *Then*

(ii) *If* $a \leq b$, *then the topology* τ_b *is finer than the topology* τ_a.

(iii) X *is compact in* (E, τ_a) *if and only if* $X \subseteq \mathcal{D}^{\mu}_{a}(\uparrow \{x_{i_1}, x_{i_2}, \cdots, x_{i_n}\})$, *for some* $x_{i_1}, x_{i_2}, \cdots, x_{i_n} \in X$.

Proof. (i) Assume $x \in \mathcal{D}^{\mu}_{b}(X)$. Then there exists $n \in \mathbb{N}$ such that $b \overset{n}{\to} \mu(x) \in X$. Since $a \leq b$, by Proposition 2.2(vi) we get $b \to \mu(x) \leq a \to \mu(x)$. In addition, by Proposition 2.2(vi), we get

$$b \to (b \to \mu(x)) \leq b \to (a \to \mu(x)) \leq a \to (a \to \mu(x)).$$

Thus, $b \overset{2}{\to} \mu(x) \leq a \overset{2}{\to} \mu(x)$. By repeating this method, we have $b \overset{n}{\to} \mu(x) \leq a \overset{n}{\to} \mu(x)$. Since $b \overset{n}{\to} \mu(x) \in X$ and $X \in \Gamma(E)$, we get $a \overset{n}{\to} \mu(x) \in X$. Hence, $x \in \mathcal{D}^{\mu}_{a}(X)$, and so $\mathcal{D}^{\mu}_{b}(X) \subseteq \mathcal{D}^{\mu}_{a}(X)$.

(ii) Let $\mathcal{B}_a$ and $\mathcal{B}_b$ be the bases for the topologies τ_a and τ_b, respectively. If $\mathcal{D}^{\mu}_{a}(\uparrow x)$ is an element of the base $\mathcal{B}_a$ such that $z \in \mathcal{D}^{\mu}_{a}(\uparrow x)$, for $z \in E$, then there is $n \in \mathbb{N}$ such that $a \overset{n}{\to} \mu(z) \in \uparrow x$, that is, $x \leq a \overset{n}{\to} \mu(z)$. Since by Proposition 2.2(iii) and (N1), $z \leq \mu(z) \leq b \overset{n}{\to} \mu(z)$, and $z \in \uparrow z$, by Proposition 4.4(ii), $z \in \mathcal{D}^{\mu}_{b}(\uparrow z) \in \mathcal{B}_b$. From Proposition 4.4(v) and $a \leq b$ it follows that $\mathcal{D}^{\mu}_{b}(\uparrow z) \subseteq \mathcal{D}^{\mu}_{a}(\uparrow z)$. We will prove that $\mathcal{D}^{\mu}_{a}(\uparrow z) \subseteq \mathcal{D}^{\mu}_{a}(\uparrow x)$. Let $u \in \mathcal{D}^{\mu}_{a}(\uparrow z)$. Then there is $m \in \mathbb{N}$ such that $a \overset{m}{\to} \mu(u) \in \uparrow z$ and so $z \leq a \overset{m}{\to} \mu(u)$. By Proposition 3.3(ii) and (N2),

$$\mu(z) \leq \mu(a \overset{m}{\to} \mu(u)) = a \overset{m}{\to} \mu(\mu(u)) = a \overset{m}{\to} \mu(u).$$

By Proposition 2.2(v),

$$a \overset{n}{\to} \mu(z) \leq a \overset{n}{\to} \left(a \overset{m}{\to} \mu(u)\right) = a \overset{m+n}{\to} \mu(u).$$

Thus $x \leq a \overset{m+n}{\to} \mu(u)$, that is, $u \in \mathcal{D}^{\mu}_{a}(\uparrow x)$. Hence, $\mathcal{D}^{\mu}_{a}(\uparrow z) \subseteq \mathcal{D}^{\mu}_{a}(\uparrow x)$, and so $\mathcal{D}^{\mu}_{b}(\uparrow z) \subseteq \mathcal{D}^{\mu}_{a}(\uparrow x)$. Therefore, the topology τ_b is finer than the topology τ_a.

(iii) Suppose $X \subseteq \mathcal{D}^{\mu}_{a}(\uparrow \{x_{i_1}, x_{i_2}, \cdots, x_{i_n}\})$, for some $x_{i_1}, x_{i_2}, \cdots, x_{i_n} \in X$. If $\{\mathcal{D}^{\mu}_{a}(X_i) \mid i \in I\}$ is a family of open subsets of E such that $X \subseteq \bigcup\limits_{i \in I} \mathcal{D}^{\mu}_{a}(X_i)$, then there is $t_j \in I$ such that $x_{i_j} \in \mathcal{D}^{\mu}_{a}(X_{t_j})$, for $j \in \{1, 2, \cdots, n\}$

and so by Proposition 4.4(i), (iii) and Proposition 4.5(i), we obtain $\mathcal{D}_a^\mu(\uparrow x_{i_j}) \subseteq \mathcal{D}_a^\mu(X_{t_j})$. By using Proposition 4.6(ii) we have

$$X \subseteq \mathcal{D}_a^\mu(\uparrow \{x_{i_1}, x_{i_2}, \cdots, , x_{i_n}\}) = \mathcal{D}_a^\mu(\uparrow x_{i_1}) \cup \mathcal{D}_a^\mu(\uparrow x_{i_2}) \cup \cdots \cup \mathcal{D}_a^\mu(\uparrow x_{i_n})$$
$$\subseteq \mathcal{D}_a^\mu(X_{t_1}) \cup \mathcal{D}_a^\mu(X_{t_2}) \cup \cdots \cup \mathcal{D}_a^\mu(X_{t_n}).$$

So, X is a compact subset of (E, τ_a).

Conversely, suppose that X is a compact subset of E. Since $X \subseteq \bigcup\limits_{x\in X} (\uparrow x)$, by Proposition 4.4(iii) and Proposition 4.6(ii) we have $X \subseteq \bigcup\limits_{x\in X} \mathcal{D}_a^\mu(\uparrow x)$. Hence $\{\mathcal{D}_a^\mu(\uparrow x) \mid x \in X\}$ is a family of open subsets of E whose union contains X. By hypothesis, there are $x_{i_1}, x_{i_2}, \cdots, , x_{i_n} \in X$ such that $X \subseteq \mathcal{D}_a^\mu(\uparrow \{x_{i_1}, x_{i_2}, \cdots, , x_{i_n}\})$. By Proposition 4.6(ii) we get $X \subseteq \mathcal{D}_a^\mu(\uparrow \{x_1, x_2, \cdots, , x_n\})$. □

Remark 4.11. (i) According to Proposition 4.4(i), $\mathcal{D}_a^\mu(X) \in \Gamma(E)$, thus $1 \in \mathcal{D}_a^\mu(X)$. If there is a subset of E such as V such that $1 \notin V$, then V is not an open subset of τ_a. Hence, τ_a is not the power set of E.

(ii) If E is bounded, then by Proposition 4.4(i), it is clear that the topological space (E, τ_a) is compact.

(iii) The topological space (E, τ_a) is an irreducible topological space, since for any two arbitrary closed subsets of E, such as U_1 and U_2, where $E = U_1 \cup U_2$, we get that there exist X_1 and X_2 such that $U_1^c = \mathcal{D}_a^\mu(X_1)$ and $U_2^c = \mathcal{D}_a^\mu(X_2)$, respectively. By Proposition 4.4(ii), $a \in U_1^c \cap U_2^c$, and so $a \notin U_1 \cup U_2 = E$, a contradiction.

(iv) Since for any $X \in \Gamma(E)$ and $a \in E$, $1 \in \mathcal{D}_a^\mu(X)$, it is clear that (E, τ_a) is not T_0, T_1, T_2, T_3 and T_4.

Theorem 4.12. *Consider E is bounded. Then the topological space (E, τ_a) is connected.*

Proof. Let $\emptyset \neq U \subseteq E$ which is both closed and open. Then there exists an upset X of E such that $U = \mathcal{D}_a^\mu(X)$. Since E is bounded, if $0 \in U$, then by Proposition 4.4(i), $U = E$. Otherwise assume $0 \in E \setminus U$. Since U is closed, we get $E \setminus U$ is open, thus, $E \setminus U$ is an upset. From $0 \in E \setminus U$ we have $E \setminus U = E$, so $U = \emptyset$, a contradiction. We conclude that $\{\emptyset, E\}$ is the set of all closed and open subsets of E. Hence E is connected. □

Remark 4.13. Let $\mu \in \mathcal{NU}(E)$ be a homomorphism. Then by (N1) and (N2), we have

$$x \to \mu(y) \leq \mu(x \to \mu(y)) = \mu(x) \to \mu(\mu(y)) = \mu(x) \to \mu(y).$$

Also, by Proposition 3.3(iii), $\mu(x) \to \mu(y) \leq x \to \mu(y)$. Hence, $\mu(x \to y) = x \to \mu(y)$, for any $x, y \in E$.

Theorem 4.14. *Let $a, b \in E$ and $\mu \in \mathcal{NU}(E)$ be a homomorphism.*

(i) *If $b \in \mathcal{F}ix_\mu(E)$, then $(E, \to_b, \tau_a)$ is a left topological equality algebra.*

(ii) *If E and satisfies in $(x \wedge y) \to z = (x \to z) \vee (y \to z)$, then $(E, \wedge, \tau_a)$ is a semi-topological equality algebra.*

Proof. (i) Consider the map $g_b : E \to E$ such that for any $x \in E$, $g_b(x) = b \to x$ and $b \in \mathcal{F}ix_\mu(E)$. We prove that $g_b^{-1}(\mathcal{D}_a^\mu(\uparrow x)) \in \tau_a$. We know that

$$\begin{aligned} g_b^{-1}(\mathcal{D}_a^\mu(\uparrow x)) &= \{u \in E \mid g_b(u) \in \mathcal{D}_a^\mu(\uparrow x)\} \\ &= \underbrace{\{u \in E \mid \exists n \in \mathbb{N}\ s.t\ x \leq a \overset{n}{\to} \mu(b \to u)\}}_{:=\mathcal{G}}. \end{aligned}$$

We show that $\mathcal{G} \in \Gamma(E)$. Suppose $u \leq v$ such that $u \in \mathcal{G}$. Then there exists $n \in \mathbb{N}$ such that $x \leq a \overset{n}{\to} \mu(b \to u)$. Since $u \leq v$, by Proposition 2.2(vi) and Proposition 3.3(ii), we have $b \to u \leq b \to v$ and so $\mu(b \to u) \leq \mu(b \to v)$. Again, by Proposition 2.2(vi), $x \leq a \overset{n}{\to} \mu(b \to u) \leq a \overset{n}{\to} \mu(b \to v)$. Thus, $v \in \mathcal{G}$. Now, we prove that $\mathcal{G} = \mathcal{D}_a^\mu(\mathcal{G})$. By Proposition 4.4(ii), we have $\mathcal{G} \subseteq \mathcal{D}_a^\mu(\mathcal{G})$. For converse, suppose $u \in \mathcal{D}_a^\mu(\mathcal{G})$. Then there exists $m \in \mathbb{N}$ such that $a \overset{m}{\to} \mu(u) \in \mathcal{G}$. Thus,

$$a \overset{m}{\to} \mu(u) \in \mathcal{G} \Rightarrow \exists n \in \mathbb{N}\ s.t\ x \leq a \overset{n}{\to} \mu(b \to (a \overset{m}{\to} \mu(u))).$$

In addition, we have

$$\begin{aligned} &\left[a \overset{n}{\to} \mu(b \to (a \overset{m}{\to} \mu(u)))\right] \to \left[a \overset{n+m}{\to} \mu(b \to u)\right] \\ \geq &\left[\mu(b \to (a \overset{m}{\to} \mu(u)))\right] \to \left[a \overset{m}{\to} \mu(b \to u)\right] \quad \text{by Proposition 2.2(x)} \\ = a &\overset{m}{\to} \left[\left[\mu(b \to (a \overset{m}{\to} \mu(u)))\right] \to \mu(b \to u)\right] \quad \text{by Proposition 2.2(v)} \end{aligned}$$

$$
\begin{aligned}
&= a \overset{m}{\to} \left[(b \to (a \overset{m}{\to} \mu(u))) \to \mu(b \to u)\right] \quad \text{by Remark 4.13}\\
&= \left[(b \to (a \overset{m}{\to} \mu(u)))\right] \to \left[a \overset{m}{\to} \mu(b \to u)\right] \quad \text{by Proposition 2.2(v)}\\
&= \left[(a \overset{m}{\to} (b \to \mu(u)))\right] \to \left[a \overset{m}{\to} \mu(b \to u)\right] \quad \text{by Proposition 2.2(v)}\\
&\geq (b \to \mu(u))) \to \mu(b \to u) \quad \text{by Proposition 2.2(x)}\\
&= (b \to \mu(u))) \to (\mu(b) \to \mu(u)) \quad \text{by assumption}\\
&= (b \to \mu(u))) \to (b \to \mu(u)) \quad \text{by Remark 4.13}\\
&= 1
\end{aligned}
$$

Thus, $x \leq a \overset{n}{\to} \mu(b \to (a \overset{m}{\to} \mu(u))) \leq a \overset{n+m}{\to} \mu(b \to u)$, and so $u \in \mathcal{G}$. Hence, $\mathcal{D}_a^\mu(\mathcal{G}) \subseteq \mathcal{G}$, so $\mathcal{G} = \mathcal{D}_a^\mu(\mathcal{G})$. It shows that g_b is continuous. Therefore, $(E, \to, \tau_a)$ is a left topological equality algebra.

(ii) Consider the map $f_b : E \to E$, defined by $f_b(x) = b \wedge x$, for all $x \in E$. We show that $f_b^{-1}(\mathcal{D}_a^\mu(\uparrow x)) \in \tau_a$. For this, we have

$$
\begin{aligned}
f_b^{-1}(\mathcal{D}_a^\mu(\uparrow x)) &= \{u \in E \mid f_b(u) \in \mathcal{D}_a^\mu(\uparrow x)\}\\
&= \{u \in E \mid u \wedge b \in \mathcal{D}_a^\mu(\uparrow x)\}\\
&= \underbrace{\{u \in E \mid \exists n \in \mathbb{N}\ s.t\ x \leq a \overset{n}{\to} \mu(b \wedge u)\}}_{:=\mathcal{C}}.
\end{aligned}
$$

We show that $\mathcal{C} \in \Gamma(E)$. For this, assume $u \leq v$ such that $u \in \mathcal{C}$. There exists $n \in \mathbb{N}$ such that $x \leq a \overset{n}{\to} \mu(b \wedge u)$. Since $u \leq v$, we have $u \wedge b \leq v \wedge b$ and by Proposition 3.3(ii), $\mu(u \wedge b) \leq \mu(v \wedge b)$ and by Proposition 2.2(vi), for $n \in \mathbb{N}$, we have $x \leq a \overset{n}{\to} \mu(b \wedge u) \leq a \overset{n}{\to} \mu(b \wedge v)$. Hence, $v \in \mathcal{C}$, and so $\mathcal{C} \in \Gamma(E)$. Now, we prove that $\mathcal{D}_a^\mu(\mathcal{C}) = \mathcal{C}$. By Proposition 4.4(ii), we have $\mathcal{C} \subseteq \mathcal{D}_a^\mu(\mathcal{C})$. Conversely, suppose $u \in \mathcal{D}_a^\mu(\mathcal{C})$. Then there exists $n \in \mathbb{N}$ such that $a \overset{n}{\to} \mu(u) \in \mathcal{C}$. Then

$$
a \overset{n}{\to} \mu(u) \in \mathcal{C} \Rightarrow \exists m \in \mathbb{N}\ s.t\ x \leq a \overset{m}{\to} \mu(b \wedge (a \overset{n}{\to} \mu(u)))
$$

Thus,

$$
\begin{aligned}
&\left[a \overset{m}{\to} \mu(b \wedge (a \overset{n}{\to} \mu(u)))\right] \to \left[a \overset{m+n}{\to} \mu(b \wedge u)\right]\\
&\geq \left[\mu(b \wedge (a \overset{n}{\to} \mu(u)))\right] \to \left[a \overset{n}{\to} \mu(b \wedge u)\right] \quad \text{by Proposition 2.2(x)}
\end{aligned}
$$

$$
\begin{aligned}
&= a \overset{n}{\to} \left[\mu\left(b \wedge (a \overset{n}{\to} \mu(u))\right) \to \mu(b \wedge u)\right] \quad \text{by Proposition 2.2(v)}\\
&= a \overset{n}{\to} \left[\left(b \wedge (a \overset{n}{\to} \mu(u))\right) \to \mu(b \wedge u)\right] \quad \text{by Remark 4.13}\\
&= \left(b \wedge (a \overset{n}{\to} \mu(u))\right) \to \left[a \overset{n}{\to} \mu(b \wedge u)\right] \quad \text{by Proposition 2.2(v)}\\
&= \left[b \to \left(a \overset{n}{\to} \mu(b \wedge u)\right)\right] \vee \left[(a \overset{n}{\to} \mu(u)) \to \left(a \overset{n}{\to} \mu(b \wedge u)\right)\right]\\
&\geq [b \to \mu(b \wedge u)] \vee [\mu(u) \to (\mu(b \wedge u))] \quad \text{by Proposition 2.2(x)}\\
&= [b \to \mu(b \wedge u)] \vee [u \to (\mu(b \wedge u))] \quad \text{by Remark 4.13}\\
&= (b \wedge u) \to \mu(b \wedge u) \qquad \text{by (N1)}\\
&= 1
\end{aligned}
$$

Thus, $x \leq a \overset{m}{\to} \mu(b \wedge (a \overset{n}{\to} \mu(u))) \leq a \overset{m+n}{\to} \mu(b \wedge u)$, and so $x \leq a \overset{m+n}{\to} \mu(b \wedge u)$. Hence, $u \in \mathcal{C}$, and so $\mathcal{D}^{\mu}_{a}(\mathcal{C}) \subseteq \mathcal{C}$. Thus, $\mathcal{D}^{\mu}_{a}(\mathcal{C}) = \mathcal{C}$, which means that f_b is continuous. Therefore, $(E, \wedge, \tau_a)$ is a semi-topological. □

In the following, quotient topology and topology on quotient structure of equality algebra are studied.

Let $F \in \mathcal{F}(E)$ and $\pi : E \to \frac{E}{F}$ the canonical homomorphism defined by $\pi(x) = [x]$. Define $\hat{\mu} : \frac{E}{F} \to \frac{E}{F}$, such that $\hat{\mu}(\pi(x)) = \pi(\mu(x))$, that is, $\hat{\mu}([x]) = [\mu(x)]$, for all $x \in E$. One can easily check that $\hat{\mu}$ is a nucleus on $\frac{E}{F}$ induced by a nucleus μ on E. It is clear that $\left(\frac{E}{F}, \tau_{[a]}\right)$ is a topological space, where the set $\tau_{[a]} = \left\{\mathcal{D}^{\hat{\mu}}_{[a]}(\frac{U}{F}) \mid F \subseteq U \in \Gamma(E)\right\}$ is a topology on $\frac{E}{F}$ and the set $\left\{\mathcal{D}^{\hat{\mu}}_{[a]}(\uparrow [x]) \mid [x] \in \frac{E}{F}\right\}$ is a base for the topology $\tau_{[a]}$ on $\frac{E}{F}$. Moreover, the set $\hat{\tau}_a = \{[\mathcal{D}^{\mu}_{a}(U)] \mid F \subseteq U \in \Gamma(E)\}$ is the quotient topology on $\frac{E}{F}$ and $\left(\frac{E}{F}, \hat{\tau}_a\right)$ is the quotient topological space. The set $\{[\mathcal{D}^{\mu}_{a}(\uparrow x)] \mid x \in E\}$ is a base for the quotient topology $\hat{\tau}_a$.

Proposition 4.15. *Consider $F, G \in \mathcal{F}(E)$ such that $F \subseteq G$. Then $\mathcal{D}^{\hat{\mu}}_{[a]}\left(\frac{G}{F}\right) = \frac{\mathcal{D}^{\mu}_{a}(G)}{F}$.*

Proof. Let $x \in E$. Then

$$
\begin{aligned}
\mathcal{D}^{\hat{\mu}}_{[a]}\left(\frac{G}{F}\right) &= \left\{[x] \in \frac{E}{F} \mid \exists n \in \mathbb{N}\ s.t\ [a] \overset{n}{\to} \hat{\mu}([x]) \in \frac{G}{F}\right\}\\
&= \left\{[x] \in \frac{E}{F} \mid \exists n \in \mathbb{N}\ s.t\ [a] \overset{n}{\to} [\mu(x)] \in \frac{G}{F}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{[x] \in \frac{E}{F} \mid \exists n \in \mathbb{N} \ s.t \ [a \xrightarrow{n} \mu(x)] \in \frac{G}{F}\right\} \\
&= \left\{[x] \in \frac{E}{F} \mid \exists n \in \mathbb{N} \ s.t \ a \xrightarrow{n} \mu(x) \in G\right\} \\
&= \left\{[x] \in \frac{E}{F} \mid x \in \mathcal{D}_a^{\mu}(G)\right\} \\
&= \frac{\mathcal{D}_a^{\mu}(G)}{F}.
\end{aligned}
$$

□

Theorem 4.16. *Assume $F \in \mathcal{F}(E)$. Then*

(i) $[\mathcal{D}_a^{\mu}(\uparrow x)] \subseteq \mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x])$.

(ii) *The quotient topology $\hat{\tau}_a$ is finer than the topology $\tau_{[a]}$.*

Proof. (i) Let $[u] \in [\mathcal{D}_a^{\mu}(\uparrow x)] = \{[v] \mid v \in \mathcal{D}_a^{\mu}(\uparrow x)\}$, for any $u \in E$. Then there is $v \in \mathcal{D}_a^{\mu}(\uparrow x)$ such that $[u] = [v]$, so there is $n \in \mathbb{N}$ such that $x \leq a \xrightarrow{n} \mu(v)$, and so $x \to (a \xrightarrow{n} \mu(v)) = 1$. Thus, $[x] \to ([a] \xrightarrow{n} [\mu(v)]) = [1]$ From $[u] = [v]$ we have $[x] \to ([a] \xrightarrow{n} [\mu(u)]) = 1$. Hence, $[u] \in \mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x])$.

(ii) As we know, $\left\{\mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x]) \mid [x] \in \frac{E}{F}\right\}$ is a base for the topology $\tau_{[a]}$. In addition $\{[\mathcal{D}_a^{\mu}(\uparrow x)] \mid x \in E\}$ is a basis for the topology $\hat{\tau}_a$ on $\frac{E}{F}$. By (i), it is clear that the quotient topology $\hat{\tau}_a$ is finer than the topology $\tau_{[a]}$. □

Theorem 4.17. *Suppose $\mu \in \mathcal{NU}(E)$ is a homomorphism. Then for $F \in \mathcal{F}(E)$, we have*

(i) $[\mathcal{D}_a^{\mu}(\uparrow x)] = \mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x])$.

(ii) *The quotient topology $\hat{\tau}_a$ and the topology $\tau_{[a]}$ are same.*

Proof. By Theorem 4.16, we have $[\mathcal{D}_a^{\mu}(\uparrow x)] \subseteq \mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x])$. Conversely, suppose $[u] \in \mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x])$. Then there is $n \in \mathbb{N}$ such that $[x] \leq [a] \xrightarrow{n} \mu([u])$. Thus, $x \to (a \xrightarrow{n} \mu(u)) \in F$. So, there is $v \in F$ such that $v = x \to (a \xrightarrow{n} \mu(u))$. Then by Proposition 2.2(v) and assumption, we have

$$1 = v \to (x \to (a \xrightarrow{n} \mu(u))) = x \to (a \xrightarrow{n} (v \to \mu(u)))$$

$$= x \to (a \overset{n}{\to} \mu(v \to u)) \qquad \text{by Remark 4.13}$$

Hence, $v \to u \in \mathcal{D}_a^{\mu}(\uparrow x)$. Since $v \in F$, we get $[u] = [1] \to [u] = [v] \to [u] \in [\mathcal{D}_a^{\mu}(\uparrow x)]$, and so $\mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x]) \subseteq [\mathcal{D}_a^{\mu}(\uparrow x)]$. Therefore, $[\mathcal{D}_a^{\mu}(\uparrow x)] = \mathcal{D}_{[a]}^{\hat{\mu}}(\uparrow [x])$.
(ii) By (i) the proof is clear. □

5 Conclusions and future works

In this paper, the concept of nucleus map on equality algebras is defined and some related results are investigated. Then by using this notion and upsets in equality algebra, a topology is constructed and it is shown that an equality algebra endowed with this topology becomes a topological space. Furthermore, some properties of the topological space are investigated such as compactness and connectedness. Moreover, continuities of all the operations with respect to the topology in an equality algebra is studied. Finally, the relationships are revealed between the two topologies on quotient equality algebras.

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References

[1] Aaly Kologani, M., Mohseni Takallo, M., Borzooei, R.A., and Jun, Y.B., *Commutative equality algebras and &-equality algebras*, Ann. Univ. Craiova Math. Comput. Sci. Ser. 47(2) (2020), 331-345.

[2] Aaly Kologani, M., Mohseni Takallo, M., Borzooei, R.A., and Jun, Y.B., *Implicative equality algebras and annihilators in equality algebras*, J. Discrete Math. Sci. Cryptogr. (2021), DOI:10.1080/09720529.2021.1876293.

[3] Aaly Kologani, M., Mohseni Takallo, M., Borzooei, R.A., and Jun, Y.B., *Right and left mappings in equality algebras*, Kragujev. J. Math. 45(5) (2022), 815-832.

[4] Aaly Kologani, M. and Niazian, S., *Some different representations of equality algebras*, Soft Comput. 28 (2024), 13071-13082.

[5] Borzooei, R.A., Kouhestani, N. and Aaly Kologani, M., *On (semi)topological hoop algebras*, Quasigroups Related Syst. 27 (2019), 161-174.

[6] Borzooei, R.A., Rezaei, G.R., and Kouhestani, N., *Separation axioms in (semi)topological quotient BL-algebras*, Soft Comput. 16(7) (2012), 1219-1227.

[7] Borzooei, R.A., Zarean, M. and Zahiri, O., *Involutive equality algebras*, Soft Comput. 22 (2018), 7505-7517.

[8] Borzooei, R.A., Zebardast, F., and Aaly Kologani, M., *Some type filters in equality algebras*, Categ. Gen. Algebr. Struct. Appl. 7 (2017), 33-55.

[9] Jenei, S., *Equality algebras*, Studia Logica 100(6) (2012), 1201-1209.

[10] Jenei, S. and Kóródi, L., *On the variety of equality algebras*, Fuzzy Log. Technol. (2011), 153-155.

[11] Galatos, N. and Tsinakis, C., *Generalized MV-algebras*, J. Algebra 283 (2005), 254-291.

[12] Gao, X.L. and Xin, X.L., *Equality propositional logic and its extensions*, Iran. J. Fuzzy Syst. 16(5) (2019), 125-137.

[13] Mohseni Takallo, M. and Aaly Kologani, M., *MBJ-neutrosophic filters of equality algebras*, J. Algebr. Hyperstruct. Log. Algebr. 1(2) (2020), 57-75.

[14] Munkres, J.R., "Topology: A First Course", Prentice-Hall, 1974.

[15] Niazian, S., *Prime filters and Zariski topology on equality algebras*, Int. J. Ind. Math. 15(2) (2023), 137-150. DOI:10.30495/ijim.2023.22548.

[16] Niazian, S., Aaly Kologani, M., and Borzooei, R.A., *On co-annihilators on equality algebras*, Miskolc Math. Notes 24(2) (2023), 933-951.

[17] Niazian, S., Aaly Kologani, M., and Borzooei, R.A., *Relative co-annihilators in lattice equality algebras*, Math. Bohem. (2024), 1-18. DOI:10.21136/MB.2024.0120-23.

[18] Novák, V. and De Baets, B., *EQ-algebras*, Fuzzy Sets Syst. 160(20) (2009), 2956-2978.

[19] Pan, F.F. and Han, S.W., *Ideal and congruence on quantale*, J. Chongqing Univ. (Natural Science Edition) 30(12) (2007), 89-94.

[20] Song, P., Xin, A.L., and Wang, J., *Modal operators on equality algebras*, Ital. J. Pure Appl. Math. 44 (2020), 4265-281.

[21] Zebardast, F., Borzooei, R.A., and Aaly Kologani, M., *Results on equality algebras*, Inform. Sci. 381 (2017), 270-282.

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