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with Applications

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Measurability in the category of structural topological spaces

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Dedicated to Prof. S.N. Hosseini

Abstract. In this paper, we first show that the category of measurable spaces is isomorphic to a particular category of structural topological spaces. Next, we define structural measurable space and we introduce a notion of structural outer measure adapted to a topological structure, along with a corresponding concept of structural measure for objects in the category of structural topological spaces. These concepts are formulated in terms of functions, transformations and natural transformations. Next, we illustrate these concepts with various examples, including several fuzzy topological spaces. Finally, under certain conditions we prove a generalization of Carathéodory's Extension and Carathéodory's Criterion showing that each notion -structural outer measure and structural measure- induces the other.

1 Introduction

One of the central paths in category theory is the exploration of categorical versions of various mathematical concepts. A rich and diverse array of such

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generalizations has been developed, as referenced in [1, 7, 10]. Among these is the definition of a structural topology on an object within a category, a categorical analogue of the classical notion of topology on a set. This concept was first introduced in [6], systematically studied in [8]. The studies in [8] not only offer various examples of topological structures but also present categorical versions of several classical topological concepts by using this framework.

These developments naturally lead to the question: can similar categorical formulations be made for other structures that are either topologically flavored or intimately connected with topology? One such closely related structure is that of measure and integration theory. Interestingly, we have discovered that by leveraging the concept of structural topology in a category, a categorical analogue of measure can indeed be defined on objects possessing such structure. In this article, we introduce this categorical version of measure, and develop categorical formulations of several key results from classical measure and integration theory. More specifically:

In Section 2, we provide some of the results needed in the subsequent sections. In Section 3, we prove that the category of measurable spaces can be realized as a category of structural topological spaces. In Section 4, first we define structural measurable space and define functions on a topological structure $\mathsf{S} = (b, t, \wedge, \vee)$ relative to a base $(\mathbb{P}, P)$ on $(\mathcal{E}, \mathcal{C})$ from the objects of $\mathcal{E}$ to Set and from the objects of the category of structural topological spaces $\mathsf{S}Top$, respectively, as a structural outer measure and a structural measure. It is worth noting that one can form transformations and natural transformations of both structural outer measures and structural measures. Also, we explore several examples to illustrate the theory, including cases arising from fuzzy topological spaces. At the end we generalize Carathéodory's Extension Theorem and Carathéodory's Criterion and under suitable conditions, we show that each of these structural outer measure and structural measure can be derived from the other.

2 Preliminairs and tools

In this section, we state the required definitions and some previous results that will be used as tools for our study.

Below, we provide the definitions of the objects and morphisms of the category $\mathsf{S}Top$, as given in [8]. For categorical notions, we refer the reader to [1, 7, 10].

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{C}$ be categories.

(a) A pair of functors $\mathcal{E}^{op} \xrightarrow{\mathbb{P}} \mathcal{C} \xrightarrow{P} \mathcal{C}$, whith $\mathcal{C}$ finitely complete, is called a base on $(\mathcal{E}, \mathcal{C})$.

(b) A topological structure relative to a base $(\mathbb{P}, P)$ or a $(\mathbb{P}, P)$-topological structure, is a quadruple $\mathsf{S} = (b, t, \wedge, \vee)$, where

$$1 \xrightarrow{b} \mathbb{P},\ 1 \xrightarrow{t} \mathbb{P},\ S \circ \mathbb{P} \xrightarrow{\wedge} \mathbb{P} \text{ and } P \circ \mathbb{P} \xrightarrow{\vee} \mathbb{P}$$

are natural transformations and $\mathcal{E}^{op} \xrightarrow{1} \mathcal{C}$ is the constant functor with value the terminal object of $\mathcal{C}$, and $S : \mathcal{C} \longrightarrow \mathcal{C}$ is the squaring functor.

(c) A structural topology on an object X of $\mathcal{E}$, relative to a given structure S, or simply an S-topology on X, is defined as a $\mathcal{C}$-monomorphism $T_X \overset{\tau_X}{\rightarrowtail} \mathbb{P}(X)$ such that there exist morphisms

$$1 \xrightarrow{b_X} T_X,\ 1 \xrightarrow{t_X} T_X,\ T_X \times T_X \xrightarrow{\wedge_X} T_X \text{ and } P(T_X) \xrightarrow{\vee_X} T_X$$

rendering commutative the following diagrams:

$$\begin{array}{ccc} & & T_X \\ & \overset{b_X}{\nearrow} \quad /// & \downarrow{\scriptstyle \tau_X} \\ 1 & \xrightarrow[b_X]{} & \mathbb{P}(X) \end{array} \qquad \begin{array}{ccc} & & T_X \\ & \overset{t_X}{\nearrow} \quad /// & \downarrow{\scriptstyle \tau_X} \\ 1 & \xrightarrow[t_X]{} & \mathbb{P}(X) \end{array}$$

$$\begin{array}{ccc} T_X \times T_X & \xrightarrow{\wedge_X} & T_X \\ {\scriptstyle \tau_X \times \tau_X}\downarrow & /// & \downarrow{\scriptstyle \tau_X} \\ \mathbb{P}(X) \times \mathbb{P}(X) & \xrightarrow[\wedge_X]{} & \mathbb{P}(X) \end{array} \qquad \begin{array}{ccc} P(T_X) & \xrightarrow{\vee_X} & T_X \\ {\scriptstyle P(\tau_X)}\downarrow & /// & \downarrow{\scriptstyle \tau_X} \\ P(\mathbb{P}(X)) & \xrightarrow[\vee_X]{} & \mathbb{P}(X) \end{array}$$

In this case, the pair (X, τ_X) is called a structural topological space, or simply an S-topological space. We sometimes write (X, T_X) to denote the structural topological space (X, τ_X) .

Definition 2.2. Let (X, τ_X) and (Y, τ_Y) be S-topological spaces. An $\mathcal{E}$-morphism $f : X \longrightarrow Y$ is said to be structurally continuous (or S-continuous), if there exists a $\mathcal{C}$-morphism $T_f : T_Y \longrightarrow T_X$ such that the following diagram commutes.

$$\begin{array}{ccc} T_X & \overset{\tau_X}{\rightarrowtail} & \mathbb{P}(X) \\ {\scriptstyle T_f}\Big\uparrow & /// & \Big\uparrow{\scriptstyle \mathbb{P}(f)} \\ T_Y & \underset{\tau_Y}{\rightarrowtail} & \mathbb{P}(Y). \end{array}$$

In this case we write $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$.

Proposition 2.3. [8, Proposition 2.1] *The* S*-topologiacl spaces together with* S*-continuous maps form a category.*

The category of S-topological spaces and S-continuous maps is denoted by $\mathsf{S}Top$. It is straightforward to verify the following result.

Theorem 2.4. [8, Theorem 2.1] *The mapping* $U : \mathsf{S}Top \longrightarrow \mathcal{E}$ *taking* $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ *to* $f : X \longrightarrow Y$ *is a faithful functor.*

By the above Theorem, the category $\mathsf{S}Top$ is concrete over $\mathcal{E}$.

We assume that the category $\mathcal{C}$ admits intersections and, since it is also assumed to be finitely complete, it follows by the dual of Theorem 14.17 in [1] that is $(\mathsf{ExtEpi}, \mathsf{Mono})$-structured, where ExtEpi denotes the class of extremal epimorphisms. Moreover, we assume that the square functor preserves ExtEpi-morphisms, and that the functor P preserves both ExtEpi-morphisms and monomorphisms.

Theorem 2.5. [8, Section 4] *Suppose that* $\mathcal{C}$ *has binary coproducts. If* $\mathcal{E}$ *has equalizers, binary products or finite limits, then so does the category* $\mathsf{S}Top$*.*

3 Measurable spaces as a structural topological spaces

In this section, we establish that every measurable space can be regarded as a structural topological space. Moreover, we prove that the category of measurable spaces is isomorphic to a category of structural topological spaces defined with respect to a special topological structure over a fixed base $(\mathbb{P}, P)$. We proceed with the following definitions and results:

Definition 3.1.
(1) Let X be a set and $\mathcal{F} \subseteq \mathbb{P}(X)$ be a subset of the power set of X. Then $\mathcal{F}$ is called a σ- algebra and the pair $(X, \mathcal{F})$ is said to be a measurable space if $\mathcal{F}$ contains X and is closed under set difference and countable unions.
(2) Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. A function $f : X \to Y$ is said to be measurable function, denoted by $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$, if for every $A \in \mathcal{G}$, the preimage $f^{-1}(A) \in \mathcal{F}$. Equivalently, $f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$. See [2].

Clearly, identity functions are measurable, and measurability is preserved under composition. Therefore:

Theorem 3.2. *The class of measurable spaces and measurable functions forms a category denoted by* $Meas$.

Proposition 3.3. *Consider functors* $Set^{op} \xrightarrow{\mathbb{P}} Set \xrightarrow{P} Set$ *where* $\mathbb{P}$ *is the powerset functor and* P *is the functor that takes a set* X *to* $P(X) = X^\omega$ *and a function* f *to* $P(f) = f^\omega$. *Define a* $(\mathbb{P}, P)$*-topological structure* $\mathsf{S} = (b, t, \wedge, \vee)$ *over the set* X *by*

$$\begin{aligned} b_X &: 1 \to \mathbb{P}(X), \text{ given by} && 1 \mapsto \emptyset \\ t_X &: 1 \to \mathbb{P}(X), \text{ given by} && 1 \mapsto X \\ \wedge_X &: \mathbb{P}(X) \times \mathbb{P}(X) \to \mathbb{P}(X), \text{ defined by} && (A, B) \mapsto A \setminus B \\ \vee_X &: P(\mathbb{P}(X)) = (\mathbb{P}(X))^\omega \to \mathbb{P}(X), \text{ defined by} && (A_i)_{i=1}^\infty \mapsto \bigcup_{i=1}^\infty A_i. \end{aligned}$$

The inclusion map $\tau_X : T_X \hookrightarrow \mathbb{P}(X)$ *is a structural topology over* X *if and only if* (X, T_X) *is a measurable space and thus* T_X *is a* σ*-algebra.*

Proof. It is obvious the actions $\wedge$ and $\vee$ are natural transformations. Indeed, for every function $f : X \to Y$ and two subsets $A, B \in \mathbb{P}(Y)$ and any collection $\{A_i\}_{i\in I} \subseteq \mathbb{P}(Y)$, we have

$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B), \qquad f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i).$$

By definition, a subset $T_X \subseteq \mathbb{P}(X)$ is a structural topology iff it is closed under quadruple $\mathsf{S} = (b, t, \wedge, \vee)$ of natural transformations. In other words $T_X \subseteq \mathbb{P}(X)$ is a structural topology iff $X, \emptyset \in T_X$, and it is closed under set difference and countable union. Hence (X, T_X) is a structural topological space whenever (X, T_X) is a measurable space and thus $T_X \subseteq \mathbb{P}(X)$ is a σ-algebra. □

Theorem 3.4. *Let the functors* $Set^{op} \xrightarrow{\mathbb{P}} Set \xrightarrow{P} Set$ *and the* $(\mathbb{P}, P)$*-topological structure of* $\mathsf{S} = (b, t, \wedge, \vee)$ *be as in 3.3. Then the categories* $\mathsf{S}Top$ *and* $Meas$ *are isomorphic.*

Proof. By proposition 3.3 for any set X and any subset $T_X \subseteq \mathbb{P}(X)$, the pair (X, T_X) is a measurable space if and only if (X, T_X) is a structural topological space. Moreover, given any function $f : X \to Y$, f is structural continuous if and only if it is a measurable function. Therefore, the functor $I : \mathsf{S}Top \to Meas$ by $I_{(X,\tau_X)} = (X, T_X)$ and $I(f) = f$ is an isomorphism and so the categories $\mathsf{S}Top$ and $Meas$ are isomorphic. □

Corollary 3.5. *The category* $Meas$ *of measurable spaces, has limits.*

Proof. By Theorem 3.4 and the fact that the category Set of sets has limits and coproducts, the result is obtained. See [8, Theorem 4.2.]. □

4 Measure and measurability

In this section, we introduce the notion of structural measueable space and define structural (transformation and natural transformation) outer measure, and structural (transformation and natural transformation) measure. Several examples are presented to illustrate the theory, including fuzzy topological spaces. Finally, we generalize
Carathéodory's Extension and Carathéodory's Criterion theorems and, under suitable conditions, show that each of these notions - structural outer measure and measure- can be derived from the other.

Definition 4.1. Let the pair of functors $\mathcal{E}^{op} \xrightarrow{\mathbb{P}} \mathcal{C} \xrightarrow{P} \mathcal{C}$, with $\mathcal{C}$ complete and cocomplete, where P is an infinite countable product functor taking an object X to the product $P(X) = X^\omega$ and a morphism f to $P(f) = f^\omega$ be the base on $(\mathcal{E}, \mathcal{C})$ and let $\mathsf{S} = (b, t, \wedge, \vee)$ be a topological structure relative to the base $(\mathbb{P}, P)$. Suppose that $T_X \overset{\tau_X}{\rightarrowtail} \mathbb{P}(X)$ is a $\mathcal{C}$-monomorphism. The pair (X, T_X) is called structural measurable space if (X, T_X) is a structural topological space. In this case, we say T_X is a structural σ-algebra.

In the rest of the paper, we assume that P is infinite countable product covariant functor and we say $\mathsf{S} = (b, t, \wedge, \vee)$ is a measurable structure relative to the base $(\mathbb{P}, P)$.

Proposition 4.2. *Let $\mathcal{C}$ have intersections. For $X \in \mathcal{E}$, the intersection of any collection of structural σ-algebras on X is a σ-algebra.*

Proof. It immediately follows from [8, Proposition 3.1]. □

Corollary 4.3. *Let $\mathcal{C}$ have intersections. Given a subobject $m : M \rightarrowtail \mathbb{P}(X)$, there is a smallest σ-algebra on X containing m.*

Proof. It immediately follows from [8, Corollary 3.1]. □

Corollary 4.4. *Let $\mathcal{C}$ have intersections. For $X \in \mathcal{E}$, the collection of structural σ-algebras on X is a complete lattice.*

Proof. By [8, Corollary 3.2], the functor $U : \mathsf{S}Top \longrightarrow \mathcal{E}$ has complete fibres, so $\{(X, T_X) : T_X \textit{ is a } \sigma\text{-algebra} \textit{ on } X\}$ is a complete lattice, as desired. □

Let $\mathcal{E}^{op} \xrightarrow{1} \mathcal{C}^{op}$ be the constant functor with value the terminal object of $\mathcal{C}$, and $\mathcal{E}^{op} \underset{\mathcal{C}(1, P\circ\mathbb{P})}{\overset{\mathcal{C}(1,\mathbb{P})}{\rightrightarrows}} Set$ be the composition of the functors,

$$\mathcal{E}^{op} \underset{\langle 1, P\circ\mathbb{P}\rangle}{\overset{\langle 1,\mathbb{P}\rangle}{\rightrightarrows}} \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathcal{C}(-,-)} Set \; .$$

One can easily verify that the natural transformations $\mathsf{S} \circ \mathbb{P} \xrightarrow{\wedge} \mathbb{P}$ and $P \circ \mathbb{P} \xrightarrow{\vee} \mathbb{P}$ yield the natural transformations $\mathcal{C}(1, \mathsf{S} \circ \mathbb{P}) \xrightarrow{\wedge} \mathcal{C}(1, \mathbb{P})$

and $\mathcal{C}(1, P\circ\mathbb{P}) \xrightarrow{\vee} \mathcal{C}(1,\mathbb{P})$ defined for each $X \in \mathcal{E}$ and $a, b \in \mathcal{C}(1,\mathbb{P}(X))$ by $\wedge_X(a,b) = \wedge_X\langle a,b\rangle$, and for each $\{a_i\}_1^\infty \subseteq \mathcal{C}(1,\mathbb{P}(X))$ by $\vee_X\{a_i\}_1^\infty = \vee_X\langle a_i\rangle_1^\infty$ which we also respectively denote by $a \wedge_X b$ and $\vee a_i$, see [9]. We assume that $\vee_X\langle a_i\rangle_1^\infty = a_{i_0}$ whenever $a_i = b_X$ for all $i \neq i_0$. In this case we have $a_1 \vee a_2 = \vee_X\langle a_i\rangle$ when $a_i = b_X$ for all $i \geq 3$.

Definition 4.5. Let $\mathsf{S} = (b, t, \wedge, \vee)$ be a structure relative to a base $(\mathbb{P}, P)$, and $a : 1 \rightarrowtail \mathbb{P}(X)$ and $b : 1 \rightarrowtail \mathbb{P}(X)$ be subobjects of $\mathbb{P}(X)$. We say that

(a) a is less than or equal to b, written $a \leq b$, if $a \vee b = b$.

(b) a contains b whenever $b \leq a$.

To proceed, we assume that for each X, $\vee_X$ is commutative, associative, and idempotent, so that $\mathcal{C}(1,\mathbb{P}(X))$ is a $\vee_X$-semi lattice (see [5, 12]) and $b_X \leq a \leq t_X$ for all $a \in \mathcal{C}(1,\mathbb{P}(X))$.

The following lemma collects several structural properties of the $\vee_X$-semi lattice $\mathcal{C}(1,\mathbb{P}(X))$ that will be used throughout this paper.

Lemma 4.6. *Let $a, b, c, d \in \mathcal{C}(1,\mathbb{P}(X))$. Then the following assertions hold:*

(a) *$a \vee b$ is supremum of $\{a, b\}$.*

(b) *The relation $\leq$ defined on $\mathcal{C}(1,\mathbb{P}(X))$ is a partial order.*

(c) *If $a \leq b$ and $c \leq d$, then $a \vee c \leq b \vee d$.*

Proof. Proof is straightforward. □

The class of partially ordered commutative monoids and order preserving homomorphisms forms a category denoted by $POCMon$. See [3], [4].

Definition 4.7. Let $F' : \mathcal{E}^{op} \to POCMon$ be a functor such that for each X, every non-negative series converges in $F(X)$. Consider $F = V \circ F' : \mathcal{E}^{op} \to Set$ where $V : POCMon \to Set$ is the forgetful functor.

(1) For each $X \in \mathcal{E}$, a function $\mu_X : \mathcal{C}(1,\mathbb{P}(X)) \to F(X)$ is called a structural outer measure (in short S-outer measure) if it satisfies:

(a) $\mu_X(b_X) = 0$.

(b) If $a_1 \leq a_2$, then $\mu_X(a_1) \leq \mu_X(a_2)$.

(c) For each countable collection of $\{a_i\}_1^\infty \subseteq \mathcal{C}(1, \mathbb{P}(X))$, we have

$$\mu_X(\vee_X\langle a_i\rangle) \leq \sum_{i=1}^{\infty} \mu_X(a_i).$$

(2) A structural transformation outer measure (in short ST-outer measure) is a collection $\mu = \{\mu_X : X \in \mathcal{E}\}$ of structural outer measures.

(3) A structural natural transformation outer measure (in short SNT-outer measure) is a structural transformation outer measure $\mu : \mathcal{C}(1, \mathbb{P}(-)) \to F : \mathcal{E}^{op} \to Set$ that is a natural transformation.

Remark 4.8. (i) Let $W : \mathsf{S}Top^{op} \to \mathcal{C}$ be a functor taking an object (X, T_X) to T_X and a morphism $f : (X, T_X) \to (Y, T_Y)$ to $T_f : T_Y \to T_X$. The composition of functors $\mathcal{C}(1, -) : \mathcal{C} \to Set$ and W is denoted by $\mathcal{C}(1, T_-)$, so that $\mathcal{C}(1, T_-) = \mathcal{C}(1, W(-))$.

(ii) a and b are disjoint if $a \wedge_X b = a$ and so $b \wedge_X a = b$.

Definition 4.9. Let $F' : \mathsf{S}Top^{op} \to POCMon$ be a functor such that for each X, every non-negative series converges in $F(X, T_X)$. Consider $F = V \circ F' : \mathsf{S}Top^{op} \to Set$ where $V : POMon \to Set$ is the forgetful functor.

(1) For each $(X, T_X) \in \mathsf{S}Top$, a function $\mu_{(X,T_X)} : \mathcal{C}(1, T_X) \to F(X, T_X)$ is called a structural measure (in short S-measure) if it satisfies:

(a) $\mu_{(X,T_X)}(b_X) = 0$.

(b) For each pairwise disjoint countable set of $\{a_i\}_1^\infty \subseteq \mathcal{C}(1, T_X)$, we have

$$\mu_{(X,T_X)}(\vee_X\langle a_i\rangle_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mu_X(a_i).$$

In this case we say $(X, T_X, \mu_{(X,T_X)})$ is a structural measure space.

(2) A structural transformation measure (in short ST-measure) is a collection $\mu = \{\mu_{X,T_X} : (X, T_X) \in \mathsf{S}Top\}$ of structural measures.

(3) A structural natural transformation measure (in short SNT-measure) is a structural transformation measure $\mu : \mathcal{C}(1, T_-) \to F : \mathsf{S}Top^{op} \to Set$ that is a natural transformation.

Example 4.10. Consider the functor F as a constant functor with value $([0,\infty],+,\leq)$. Since the category Set is a topos and $Set(1,\mathbb{P}(X))\cong\mathbb{P}(X)$ for all X (see [7], [10]), and every σ-algebra is a structural topology, then all classic measure (or outer measure) $\mu_X : T_X \to [0,\infty]$ $(\mu_X : \mathbb{P}(X) \to [0,\infty])$ induces an S-measure (or S-outer measure).

In general case μ is an SNT-outer measure whenever $\mu_X(f^{-1}(B)) = \mu_Y(B)$ for all function $f : X \to Y$ and $B \in \mathbb{P}(Y)$.

Example 4.11. In Example 4.10, we can consider $F(X) = ([1,\infty],\times,\leq)$.

Example 4.12. In Example 4.10, we can also consider $F(X) = (Card,+,\leq)$ where $Card$ is the set of cardinal numbers.

Example 4.13. With the base as in Theorem 3.3, let F be defined by $F(X) = (\mathbb{P}(X),\cup,\subseteq)$ for each X and $F(f) = f^{-1}$ for each function f. The identity natural transformation $\mu = I_{\mathbb{P}} : Set(1,\mathbb{P}(-)) \to F : Set^{op} \to Set$ is an SNT-outer measure, since $Set(1,\mathbb{P}(X)) \cong \mathbb{P}(X)$ for all X. Let $F : \mathsf{S}Top^{op} \to Set$ be the functor taking a structural topological space (X,T_X) to $F(X,T_X) = (T_X,\cup,\subseteq)$ and a continuous map $f : (X,T_X) \to (Y,T_Y)$ to $F(f) = T_f : T_Y \to T_X$. The identity natural transformation $\mu = I_{T_-} : Set(1,T_-) \to F : \mathsf{S}Top^{op} \to Set$ is an SNT-measure.

Suppose that Set_p denotes the category of set pairs (X,A), where $A \subseteq X$ and morphisms $f : (X,A) \to (Y,B)$ where $f : X \to Y$ is a function such that $f(A) \subseteq B$. Since the category Set of sets is isomorphic to a full subcategory of Set_p, the above Examples can be generalized to:

Example 4.14. Let $Set_p^{op} \xrightarrow{\mathbb{P}} Set$ be a functor taking a pair (X,A) to the powerset $\mathbb{P}(X)$ and a morphism $f : (X,A) \to (Y,B)$ to $\mathbb{P}(f) = f^{-1}(-)$. Define $(\mathbb{P},P)$-measurable structure $\mathsf{S} = (b,t,\wedge,\vee)$ over a pair (X,A) as follows:

$$b_{(X,A)}(1) = \emptyset, \qquad t_{(X,A)}(1) = X, \qquad \wedge_{(X,A)}(C,B) = C \setminus B,$$

$$\vee_{(X,A)}(A_i)_{i=1}^{\infty} = \bigcup_{i=1}^{\infty} A_i.$$

The inclusion map $\tau_X : T_X \hookrightarrow \mathbb{P}(X)$ is a structural topology over X if and only if (X, T_X) is a measurable space and T_X is a σ-algebra. All of measures and outer measures defined in 4.10, 4.11, 4.12 and 4.13 are instances of this example.

Example 4.15. Using the frame $(\mathbb{P}(U), \subseteq)$ as a complete Boolean Algebra, let $(\mathbb{P}(U), \subseteq)^{op} \xrightarrow{\mathbb{P}} Set$ be the powerset functor. Define $(\mathbb{P}, P)$-measurable structure $\mathsf{S} = (b, t, \wedge, \vee)$ over a subset $X \subseteq U$ as follows:

$$b_X(1) = \emptyset, \quad t_X(1) = X, \quad \wedge_X(C, B) = C \setminus B, \quad \vee_X(A_i)_{i=1}^{\infty} = \bigcup_{i=1}^{\infty} A_i.$$

The inclusion map $\tau_X : T_X \hookrightarrow \mathbb{P}(X)$ is a structural topology over X if and only if (X, T_X) is a measurable space or T_X is a σ-algebra. Every (natural) transformation $\mu : \mathcal{C}(1, \mathbb{P}(-)) \to F : (\mathbb{P}(U), \subseteq)^{op} \to Set$ (or $\mu : \mathcal{C}(1, T_{-}) \to F : \mathsf{S}Top^{op} \to Set$) such that μ_X satisfies the conditions of 4.7 (or 4.9) is an (ST- SNT-) outer measure (or an (ST- SNT-) measure). All of measures and outer measures defined in 4.10, 4.11, 4.12 and 4.13 are instances of this example.

This example can be generalized to:

Example 4.16. Let $(X, \leq)$ be a complete Boolean algebra in which the binary meet distributes over countable joins, and $\mathbb{P}(x) = x \downarrow = \{a \in X | a \leq x\}$. Define $(\mathbb{P}, P)$-measurable structure $\mathsf{S} = (b, t, \wedge, \vee)$ over a member x as follows:

$$b_x(1) = 0, \quad t_x(1) = x, \quad \wedge_x(a, b) = a \wedge_x b = a \wedge b^c, \quad \vee_x(a_i)_{i=1}^{\infty} = \bigvee_{i=1}^{\infty} a_i.$$

Same arguments about measures as in Example 4.14 work here as well.

Example 4.17. Let $Set^{op} \xrightarrow{\mathbb{P}} Set$ be a functor taking $\mathbb{P}(X) = I^X$(the set of all fuzzy sets in X). Define $(\mathbb{P}, P)$-measurable structure $\mathsf{S} = (b, t, \wedge, \vee)$ over a set X as follows:

$$b_X(1) = \underline{0}, \quad t_X(1) = \underline{1}, \quad \wedge_X(f, g) = f \wedge (1 - g), \quad \vee_X(f_i)_{i=1}^{\infty} = \bigvee_{i=1}^{\infty} f_i.$$

The inclusion map $\tau_X : T_X \hookrightarrow \mathbb{P}(X)$ is a structural topology over X if and only if (X, T_X) is a fuzzy measurable space and T_X is a σ-algebra. Every (natural) transformation $\mu : \mathcal{C}(1, \mathbb{P}(-)) \to F : \mathcal{E}^{op} \to Set$ (or $\mu : \mathcal{C}(1, T_{-}) \to F : \mathsf{S}Top^{op} \to Set$) such that μ_X satisfies the conditions of 4.7 (or 4.9) is an (ST- SNT-) outer measure (or an (ST- SNT-) measure), which is called fuzzy set outer measure (or fuzzy set measure).
As an example, with the constant functor F by $F(X) = [0, \infty]$ for each object X and $F(f) = I$ for each morphism f, define $\mu_X : I^X \to [0, \infty]$ by

$$\mu_X(f) = sup\{f(x) : x \in X\},$$

which gives a simple outer measure. As a particular case, one may also define $\mu_X(f) = \int_X f(x) d\mu_X$ which corresponds to the classical outer measure.

Example 4.18. Let U be a set, and $I^U \xrightarrow{\mathbb{P}} Set$ be the functor taking X to $\mathbb{P}(X) = \mathcal{F}_X$(the set of fuzzy subsets in fuzzy set X on U) and for $f : X \to Y$(i.e. $X \leq Y$) $\mathbb{P}(f) = - \wedge X$. Define $(\mathbb{P}, P)$-measurable structure $\mathsf{S} = (b, t, \wedge, \vee)$ over a fuzzy set X as follows:

$$b_X(1) = \underline{0}, \quad t_X(1)X, \quad \wedge_X(f, g) = f \wedge (X - g), \quad \vee_X(f_i)_{i=1}^{\infty} = \bigvee_{i=1}^{\infty} f_i.$$

The inclusion map $\tau_X : T_X \hookrightarrow \mathbb{P}(X)$ is a structural topology over X if and only if the pair (X, T_X) is a fuzzy measurable space and T_X is a σ-algebra. Every (natural) transformation $\mu : \mathcal{C}(1, \mathbb{P}(-)) \to F : I^U \to Set$ (or $\mu : \mathcal{C}(1, T_{-}) \to F : \mathsf{S}Top^{op} \to Set$) such that μ_X satisfies the conditions of 4.7 (or 4.9) is an (ST- SNT-) outer measure (or an (ST- SNT-) measure) which is called fuzzy subset outer measure (or fuzzy subset measure).
As an example, consider the constant functor$F(X) = ([0, \infty], +, \leq)$ and identity map. Define $\mu_X : \mathcal{F}_X \to [0, \infty]$ by

$$\mu_X(f) = sup\{f(x) : x \in U\},$$

which gives a simple outer measure. As a particular case such as $U \subseteq \mathbb{R}$, one may also define

$$\mu_X(f) = \int_U f(t) dl(t)$$

which corresponds to the classical outer measure where l is the lebesgue measure.

Example 4.19. Let $Set^{op} \xrightarrow{\mathbb{P}} Set$ be the functor taking X to

$$\mathbb{P}(X) = II^X = \{(U,V) \in I^X \times I^X : \ \forall x \in X \ \ 0 \leq U(x) + V(x) \leq 1\}$$

(intuitionistic fuzzy sets on X). Define $(\mathbb{P}, P)$-measurable structure $\mathsf{S} = (b, t, \wedge, \vee)$ over a set X as follows:

$$b_X(1) = (\underline{0}, \underline{1}), \qquad \wedge_X((U_1, V_1), (U_2, V_2)) = (U_1 \wedge V_2, V_1 \vee U_2),$$

$$t_X(1) = (\underline{1}, \underline{0}), \qquad \vee_X((U_i, V_i))_{i=1}^{\infty} = \bigvee_{i=1}^{\infty} (U_i, V_i) = (\bigvee_{i=1}^{\infty} U_i, \bigwedge_{i=1}^{\infty} V_i).$$

Every (natural) transformation $\mu : \mathcal{C}(1, \mathbb{P}(-)) \to F : Set^{op} \to Set$ (or $\mu : \mathcal{C}(1, T_{-}) \to F : \mathsf{S}Top^{op} \to Set$) such that μ_X satisfies the conditions of 4.7 (or 4.9) is an (ST- SNT-) outer measure (or an (ST- SNT-) measure) which is called intuitionistic fuzzy set outer measure (or intuitionistic fuzzy set measure).
As an example, consider the constant functor $F(X) = ([0, \infty] \times [0, \infty], +, (\leq, \geq))$ and define $\mu_X : II^X \to [0, \infty] \times [0, \infty]$ by

$$\mu_X(U, V) = sup\{(U(x), V(x)) : x \in X\}$$

which yields a simple outer measure. Alternatively, as a particular case, one may define $\mu_X(U, V) = (\int_X U(x)dl(x), \int_X V(x)dl(x))$, where l is the lebesgue measure.

In all the aforementioned examples, the natural difference $\wedge$ can be interpreted as the meet within the categorical framework. In each of these cases, a structural σ-algebra is obtained; however under these conditions, we no longer deal with classical complement and set-theoretic difference or fuzzy difference. In these general framework, numerous illustrative examples can be provided, particularly those involving **well-powered categories** and **toposes**. However, to maintain conceptual coherence and avoid unnecessary digression from our primary research focus, an exhaustive treatment of these instances has been omitted.

In the following we prove a generalization of Carathéodory's Extension and Criterion, see [2, 11].

Theorem 4.20. (Carathéodory's Extension) *Let* $\mathcal{E}^{op} \xrightarrow{\mathbb{P}} \mathcal{C}$ *be a functor and* $\mathsf{S} = (b, t, \wedge, \vee)$ *be a measurable structure on the base* $(\mathbb{P}, P)$*,* $F : \mathcal{E}^{op} \to POCMon$ *be the constant functor with value* $([0, \infty], +, \leq)$*, and for* $(X, T_X) \in \mathsf{S}Top$*, the function* $\mu_{(X,T_X)} : \mathcal{C}(1, T_X) \to [0, \infty]$ *be an S-measure. Then the map* $\mu^*_X : \mathcal{C}(1, \mathbb{P}(X)) \to [0, \infty]$ *defined by;*

$$\mu^*_X(a) = inf\Big\{ \sum_{i=1}^{\infty} \mu_X(a_i) | a \leq \vee_X \langle a_i \rangle, a_i \in \mathcal{C}(1, T_X) \Big\}.$$

is an S-outer measure such that the following left diagram commutes. Furthermore μ^*_X *is the maximum of structural outer measures* ν *making the following rhight diagram commutative.*

$$\begin{array}{ccc} \mathcal{C}(1, T_X) & \xrightarrow{\mu_{(X,T_X)}} & M \\ \big\downarrow{\scriptstyle inc} & \nearrow_{\mu^*_X} & \\ \mathcal{C}(1, \mathbb{P}(X)) & & \end{array} \quad (1) \qquad \begin{array}{ccc} \mathcal{C}(1, T_X) & \xrightarrow{\mu_{(X,T_X)}} & M \\ \big\downarrow{\scriptstyle inc} & \nearrow_{\nu} & \\ \mathcal{C}(1, \mathbb{P}(X)) & & \end{array} \quad (2)$$

Proof. Since $\tau_X : T_X \rightarrowtail \mathbb{P}(X)$ is a mono, so is the function $\mathcal{C}(1, T_X) \to \mathcal{C}(1, \mathbb{P}(X))$ which is therefore isomorphic to the inclusion $\mathcal{C}(1, T_X) \subseteq \mathcal{C}(1, \mathbb{P}(X))$. By the definition of μ^* it is immediate that:

- $\mu^*(a) = \mu(a)$ for all $a \in \mathcal{C}(1, T_X)$,
- $\mu^*(b_X) = 0 = Min(M)$,
- $\mu^*(a) \leq \mu^*(b)$ for $a \leq b$,
- $\mu^*(a) = \infty = Max(M)$ if for all $\{a_i\}_{i=1}^{i=\infty} \subseteq \mathcal{C}(1, T_X)$ where $a \leq \vee a_i$ we have $\displaystyle\sum_{i=1}^{i=\infty} \mu(a_i) = \infty$.

To prove the countable subadditivity, suppose $\{b_j\}_{j=1}^{\infty} \subseteq \mathcal{C}(1, \mathbb{P}(X))$ and $\varepsilon > 0$. For each j there exists $\{a^j_i\}_{i=1}^{i=\infty} \subseteq \mathcal{C}(1, T_X)$ such that

$$b_j \leq \vee_{i=1}^{i=\infty} a^j_i \quad \text{and} \quad \sum_{i=1}^{i=\infty} \mu(a^j_i) \leq \mu^*(b_j) + 2^{-j}\varepsilon.$$

Since $\vee_{j=1}^{\infty} b_j \leq \vee_{i,j=1}^{\infty} a_i^j$, it follows that

$$\mu^*(\vee b_j) \leq \sum_{j=1}^{j=\infty} \mu^*(b_j) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this shows that μ^* is countably subadditive. □

Corollary 4.21. *If* $\mu = \{\mu_{(X,T_X)} : \mathcal{C}(1, T_X) \to ([0,\infty], +, \leq) | (X, T_X) \in \mathsf{S}Top\}$ *is an ST-measure, then there is an ST-outer measure* $\mu^* = \{\mu_X^* : \mathcal{C}(1, \mathbb{P}(X)) \to ([0,\infty], +, \leq) | X \in \mathcal{E}\}$ *such that for each* $(X, T_X) \in \mathsf{S}Top$*,* $\mu_X^* \circ inc(X, T_X) = \mu_{(X,T_X)}$.

In the above mentioned case, $([0,\infty], +, \leq)$ may be extended to $(M, +, \leq) \in POCMon$; however, this extension requires certain conditions that would divert us from the fundamental subject under consideration.

Definition 4.22. Let $a, b \in \mathcal{C}(1, \mathbb{P}(X))$. we say that b is the complement of a, denoted by $b = a^c$, if $a \vee b = t_X$ and a, b are disjoint. We say that $\mathcal{C}(1, \mathbb{P}(X))$ has complements if every $a \in \mathcal{C}(1, \mathbb{P}(X))$ has a complement.

Assume that $(\mathcal{C}(1, \mathbb{P}(X)), \wedge', \vee)$ is a Boolean algebra with the binary meet distributing over countable joins and $a \wedge' b = a \wedge_X b^c$ we have:

Theorem 4.23. (Carathéodory's Criterion) *Let* $\mathcal{E}^{op} \xrightarrow{\mathbb{P}} \mathcal{C}$ *be a functor and* $\mathsf{S} = (b, t, \wedge, \vee)$ *be a measurable structure on the base* $(\mathbb{P}, P)$*,* $F : \mathcal{E}^{op} \to POCMon$ *be the constant functor with value* $(M, +, \leq)$ *and for* $X \in \mathcal{E}$*, let* $\mu_X^* : \mathcal{C}(1, \mathbb{P}(X)) \to (M, +, \leq)$ *be an S-outer measure. Then there is a structural* σ*-algebra* $T_X \xrightarrow{\tau_X} \mathbb{P}(X)$ *such that* $\mathcal{C}(1, T_X)$ *is a* σ*-algebra, and an S-measure* $\mu_{(X,T_X)} : \mathcal{C}(1, T_X) \to (M, +, \leq)$ *defined by restricting* μ_X^* *to* $\mathcal{C}(1, T_X)$*. So that we have the following commutative diagram.*

$$\begin{array}{ccc} \mathcal{C}(1, T_X) & \xrightarrow{\mu_{(X,T_X)}} & M \\ \downarrow{\scriptstyle inc} & \nearrow{\scriptstyle \mu_X^*} & \\ \mathcal{C}(1, \mathbb{P}(X)) & & \end{array}$$

Proof. Form the following equalizer in $\mathcal{C}$,

$$E_X \xrightarrow{e_X} \mathbb{P}(X) \times \mathbb{P}(X) \underset{\mu_X^* \pi_1}{\overset{+(\mu_X^* \times \mu_X^*)\langle \wedge_X, \wedge'_X \rangle}{\rightrightarrows}} M$$

let $S = \{ S_X \overset{s_X}{\rightarrowtail} \mathbb{P}(X) \ : 1 \times s_X \ \ factors \ \ through \ \ e_X \}$. $S \neq \emptyset$ as it contains b_X and t_X. Let $\tau_X : T_X \rightarrowtail \mathbb{P}(X)$ be $\tau_X = \bigvee_{s_X \in S} s_X$. Then $1 \times \tau_X = \bigvee (1 \times s_X)$ factors through e_X and so it belongs to S and is maximum of S. Since join of $\mathsf{S}Top$ objects, is an $\mathsf{S}Top$ object (see [8]), τ_X is in $\mathsf{S}Top$ and thus T_X is a structural σ-algebra. It easily follows that $\mathcal{C}(1, T_X)$ is a σ-algebra.

Now let $\mu_{(X,T_X)}$ be defined by restricting μ^*_X to $\mathcal{C}(1, T_X)$, so that we have the following commutative triangle.

$$\begin{array}{ccc} \mathcal{C}(1, T_X) & \xrightarrow{\mu_{(X,T_X)}} & M \\ {\scriptstyle inc}\downarrow & \nearrow_{\mu^*_X} & \\ \mathcal{C}(1, \mathbb{P}(X)) & & \end{array}$$

It easily follows that $\mu_{(X,T_X)}$ is an S-measure and it is easy to see an element $b \in \mathcal{C}(1, T_X)$ which is called μ^*-measurable, for every $a \in \mathcal{C}(1, \mathbb{P}(X))$, satisfies in:

$$\mu^*_X(a) = \mu^*(a \wedge' b) + \mu^*(a \wedge b).$$

□

Corollary 4.24. *If* $\mu^* = \{\mu^*_X : \mathcal{C}(1, \mathbb{P}(X)) \to (M, +, \leq) | X \in \mathcal{E}\}$ *is an ST-outer measure, then there is an ST-measure* $\mu = \{\mu_{(X,T_X)} : \mathcal{C}(1, T_X) \to (M, +, \leq) | (X, T_X) \in \mathsf{S}Top\}$ *such that for each* $(X, T_X) \in \mathsf{S}Top$*,* $\mu^*_X \circ inc(X, T_X) = \mu_{(X,T_X)}$.

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