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The conductor ideals of maximal subrings in non-commutative rings

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Abstract. Let R be a maximal subring of a ring T, and $(R:T)$, $(R:T)_\ell$ and $(R:T)_r$ denote the largest ideal, left ideal and right ideal of T that are contained in R, respectively. It is shown that both $(R:T)_\ell$ and $(R:T)_r$ are prime ideals of R, and $|\operatorname{Min}_R((R:T))| \leq 2$. We prove that if T_R has a maximal submodule, then $(R:T)_\ell$ is a right primitive ideal of R. We investigate the conditions under which $(R:T)_r$ is a completely prime (right) ideal of R or of T. We show that $\operatorname{Char}(R/(R:T)_\ell) = \operatorname{Char}(R/(R:T)_r)$, and if $\operatorname{Char}(T)$ is neither zero nor a prime number, then $(R:T) \neq 0$. When $|\operatorname{Min}(R)| \geq 3$, both $(R:T)$ and $(R:T)_\ell(R:T)_r$ are nonzero ideals. Assuming R is integrally closed in T, we prove that $(R:T)_\ell$ and $(R:T)_r$ are prime one-sided ideals of T; moreover $(R:T)$ is a semiprime ideal of T and either $(R:T)$ is a prime ideal of T or $(R:T) = (R:T)_\ell \cap (R:T)_r$ is a semiprime ideal of R. We observe that if $(R:T)_l T = T$, then T is a finitely generated left R-module and $(R:T)_\ell$ is a finitely generated right R-module which is also a right primitive ideal of R. Finally, we study the transfer of the Noetherian and the Artinian properties between R and T.

Keywords: Maximal subrings, conductor ideal, prime ideal, idealizer, integrally closed, finiteness condition.

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1 Introduction

1.1 Motivation In [11], Ferrand and Olivier studied minimal ring extensions of commutative rings. Note that if $R \subseteq T$ is a minimal extension of commutative rings (that is, R is a maximal subring of T), then the integral closure of R in T, say S, is a subring between R and T; therefore by the minimality of the extension, we infer that either $S = R$ (that is, R is integrally closed in T) or $S = T$ (that is, T is integral over R, equivalently T is a finitely generated R-module). They proved that T is integral over R if and only if $(R : T) \in \mathrm{Max}(R)$, see [11, Proposition 4.1]. Moreover, if R is integrally closed in T, then $(R : T)$ is a prime ideal of T, see [11, Lemma 3.2]. Hence, in all cases we deduce that $(R : T)$ is a prime ideal of R.

In this paper, motivated by these results, we study the conductor (left or right) ideals of minimal ring extensions in non-commutative rings. Specifically, if $R \subseteq T$ is a minimal ring extension of rings (that is, R is a maximal subring of T, where T is an arbitrary ring not necessarily commutative ring), then we study the properties of $(R : T)$, the largest ideal of T contained in R, $(R : T)_\ell$, the largest left ideal of T contained in R, and $(R : T)_r$, the largest right ideal of Tcontained in R. As we show, these are ideals of R (hence we use "maximal subrings" rather that "minimal ring extensions" throughout). In fact, we proved that both $(R : T)_\ell$ and $(R : T)_r$ are prime ideals of R.

It is interesting to know that each ring R can be realized as a maximal subring of some larger ring T, see [1, Theorem 3.7]. Note that if T is a ring and A be a one-sided ideal of T, then the idealizer of A in T is the largest subring of T, in which A is a two-sided ideal. More exactly, if A is a right ideal of T, then the *idealizer* of A is the subring $\mathbb{I}_T(A) := \{r \in T \mid rA \subseteq A\}$. By this definition we have the following from [1]:

Theorem 1.1. (1) [1, Theorem 4.1] *Let T be a ring and A be a maximal right/left ideal of T that is not a two-sided ideal of T. Then the idealizer of A is a maximal subring of T. In particular, every ring either has a maximal subring or is a quasi duo ring* (*that is, any maximal left/right ideal is two-sided*).

(2) [1, Proposition 4.2] *Let R be a maximal subring of a ring T that contains a maximal one-sided ideal A of T that is not a two-sided ideal of T. Then R is the idealizer of A in T.*

In [5, 6, 15, 16, 20], it is proved that if R is a finite maximal subring of a ring T, then T is also finite. In [3, Theorem 3.8], it is shown that if R is a maximal subring of a commutative ring T, then R is Artinian if and only if T is Artinian and integral over R; this immediately implies the aforementioned finiteness results for commutative rings. It is clear that, if R is a maximal subring of a commutative ring T, then T is Noetherian whenever R is Noetherian, since $T = R[\alpha]$, for each $\alpha \in T \setminus R$ and use the Hilbert Basis Theorem. Motivated by these results in commutative case, we are interested to investigate the Artinian and the Noetherian properties in minimal ring extension of non-commutative rings too.

Finally, we refer the reader to [7–9, 12, 13, 24], for minimal ring extension of commutative rings and [10] for non-commutative case. Also, we refer the reader to [2–4, 23] for maximal subrings in commutative rings and [1] for maximal subrings of non-commutative rings.

1.2 Review of the results Assume that R is a maximal subring of a ring T. In Section 2, we obtain some basic fact about the conductor ideals of R. We prove that $(R:T)_\ell$ and $(R:T)_r$ are prime ideals of R. Moreover $\mathrm{Min}_R((R:T)) \subseteq \{(R:T)_\ell, (R:T)_r\}$ and in fact $|\mathrm{Min}_R((R:T))| = 1$ if and only if $(R:T)_\ell$ and $(R:T)_r$ are comparable, equivalently $(R:T)$ is prime ideal of R. We show that if T/R as right R-module has a maximal submodule, then $(R:T)_\ell$ is a right primitive ideal of R. Conversely, we observe that if either $(R:T)_\ell \in \mathrm{Max_r}(R)$ or R is a right Artinian ring, then T/R has a maximal right R-submodule. We investigate conditions under which $(R:T)_r$ is a completely prime (right) ideal of T or of R. In particular, if $(R:T)_r$ is a completely prime right ideal of T, then $(R:T)_r$ is a completely prime ideal of R (that is, $R/(R:T)_r$ is a domain) and $T/(R:T)_r$ is a torsion-free left $R/(R:T)_r$-module; conversely if $T/(R:T)_r$ is a torsion-free left $R/(R:T)_r$-module, then either $(R:T)_r$ is an ideal of T or $(R:T)_r$ is a completely prime right ideal of T and R is the idealizer of $(R:T)_r$ in T. Moreover, if R is a right Artinian ring and $(R:T)_r$ is a completely prime right ideal of T, then either R/P or T/P is a division ring. We prove that the ring $\mathrm{End}((T/R)_R)$, and therefore the ring $\mathrm{End}_{\mathbb{Z}}(T/R)$ contain a copy of $R/(R:T)_r$. Consequently, the ring $\mathrm{End}_{\mathbb{Z}}(T/R)$ contains a copies of $R/(R:T)_r$ and $R/(R:T)_\ell$. In particular, the characteristic of the ring $\mathrm{End}_{\mathbb{Z}}(T/R)$ is either zero or a prime number and thus $\mathrm{Char}(R/(R:T)_\ell) = \mathrm{Char}(R/(R:T)_r)$. We show that if $\mathrm{Char}(T)$ is not a prime number, then either there exists a prime number p such that $pT \subseteq (R:T)$ (and therefore $(R:T) \neq 0$) or $\mathrm{Char}(T) = \mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R)) = 0$. In Section 3, we show that if R is (2-)integrally closed in T, then $(R:T)_\ell$ and $(R:T)_r$ are prime one-sided ideals of T. In particular, in this case, $(R:T)$ is a semiprime ideal of T and either $(R:T)$ is a prime ideal of T or a semiprime ideal of R. Finally in Section 4, we consider some finiteness conditions on the extension $R \subseteq T$. We show that if $(R:T)_l T = T$, then T is a finitely generated left R-module, $(R:T)_\ell$ is a finitely generated right ideal of R and R is the idealizer of $(R:T)_\ell$. In particular, if $(R:T) \in \mathrm{Max}(T) \setminus \mathrm{Spec}(R)$, then ${}_RT$, T_R, $(R:T)_\ell$ as right ideal of R and $(R:T)_r$ as left ideal of R, are all finitely generated. Consequently, if $(R:T) \in \mathrm{Max}(T)$ and R is left Noetherian (respectively, Artinian), then either $(R:T) = (R:T)_\ell$ or T is a right Noetherian (respectively, Artinian) ring. We show that if R is a Noetherian (respectively, an Artinian) ring and $(R:T) \neq 0$, then $T/\,\mathrm{l.ann}_T((R:T)_\ell)$ and $T/\,\mathrm{r.ann}_T((R:T)_r)$ are left and right Noetherian (respectively, Artinian) rings, respectively. Moreover, if in addition, T is semiprime, then $T/\,\mathrm{ann}_T((R:T))$ is a Noetherian (respectively, an Artinian) ring. In particular, if T is a prime ring, then T is finitely generated as left and right R-modules and consequently T is a Noetherian (respectively, an Artinian) ring. Finally, if R is a Noetherian ring, T is not a prime ring and R contains a prime ideal Q of T, then either T is Noetherian or $Q = (R:T)$ (therefore Q is unique), Q is a minimal prime ideal of T and either $Q = (R:T)_\ell$ or $Q = (R:T)_r$.

1.3 Notations and definitions All rings in this paper are unital with $1 \neq 0$. All subrings, modules and homomorphisms are also unital. If $R \subsetneq T$ is a ring extension and there exists no other subring properly between R and T, then R is called a maximal subring of T, or the extension $R \subseteq T$ is called a minimal ring extension. If T is a ring, R and S are subrings of T, then clearly T is a (R,S)-bimodule and therefore we can consider (R,S)-subbimodules of T. In particular, if $t \in T$, then the (R,S)-subbimodule of T which is generated by t is denoted by $RtS = \{\sum_{i=1}^{n} r_i t s_i \mid n \geq 1,\ r_i \in R,\ s_i \in S,\ 1 \leq i \leq n\}$. It is clear that if $R \subseteq S$ and $t \in S$, then $R + RtS$ and $R + StR$ are also subrings of T that contain R. In particular, for each $t \in T$, the subrings $R + RtT$ and $R + TtR$ of T contain R. More generally, if I is left (respectively, right) ideal of T, then $R + IR$ (respectively, $R + RI$) is a subring of T that contains R. If T is a ring, I is an ideal of T and M is a left (respectively, right) T-module, then $\mathrm{Min}_T(I)$, $\mathrm{Max_r}(T)$, $\mathrm{Max}_\ell(T)$, $\mathrm{Max}(T)$, $\mathrm{Spec}(T)$, $\mathrm{l.ann}_T(M)$ (respectively, $\mathrm{r.ann}_T(M)$) and $\mathrm{ann}_T(I)$ denote the set of all minimal prime ideals of I in T, the set of all maximal right ideals of T, the set of all maximal left ideals of T, the set of all maximal ideals of T, the set of all prime ideals of T, the left annihilator of M in T (respectively, the right annihilator of M in T) and the annihilator of I in T (that is, $\mathrm{ann}_T(I) = \mathrm{l.ann}_T(I) \cap \mathrm{r.ann}_T(I)$), respectively. We use $\mathrm{Min}(T)$ for $\mathrm{Min}_T(0)$. $J(T)$ denotes the Jacobson radical of a ring T and for an ideal I of T, we denote the prime radical of I by $\mathrm{rad}_T(I)$. The characteristic of a ring T is denoted by $\mathrm{Char}(T)$. If T is a ring, then $\dim(T)$ denotes the classical Krull dimension of T (that is, the supremum of the lengths of all chains of prime ideals of T). If X is a subset of a ring T, then $C_T(X)$ is the centralizer of X in T, in particular, $C(T) = C_T(T)$ is the center of T. If T is a ring, then $\mathbb{M}_n(T)$ denote the ring of all $n \times n$ square matrices over T. If M is a left (respectively, right) module over a ring T, then the ring of all T-module homomorphisms of M is denoted by $\mathrm{End}(_TM)$ (respectively, $\mathrm{End}(M_T)$); when $T = \mathbb{Z}$ (that is, M is an abelian group), then we use $\mathrm{End}_{\mathbb{Z}}(M)$. A ring T is called left (respectively, right) quasi duo if each maximal left (respectively, right) ideal of T is a two-sided ideal of T, see [19]. T is called quasi duo if T is left and right quasi duo ring. A ring T is called left (respectively, right) duo if each left (respectively, right) ideal of T is two-sided. Similarly duo rings are defined. If R is a subring of a ring T and $t \in T$, then we say that t is left (respectively, right) n-integral over R, if t is a root of a left (respectively, right) monic polynomial of degree n over R ($n \geq 1$). R is called left (respectively, right) n-integrally closed in T, if every left (respectively, right) n-integral element of T over R belongs to R. R is called n-integrally closed in T whenever R is left and right n-integrally closed in T. R is called left (respectively, right) integrally closed in T, if R is left (respectively, right) n-integrally closed in T, for each n. R is integrally closed in T, if R is left and right integrally closed in T. For other notations and definitions we refer the reader to [14, 17, 18, 22, 26].

2 The conductor ideals of maximal subrings

Let $R \subseteq T$ be a ring extension. We define $(R:T) := \{x \in T \mid TxT \subseteq R\}$, $(R:T)_\ell := \{x \in T \mid Tx \subseteq R\}$ and $(R:T)_r := \{x \in T \mid xT \subseteq R\}$. In other words, $(R:T)$ is the largest ideal of T contained in R, $(R:T)_\ell$ (respectively, $(R:T)_r$) is the largest left (respectively, right) ideal of T contained in R. It is not hard to see that $(R:T)_\ell = \mathrm{r.ann}_R(T/R)$ and therefore $(R:T)_\ell$ is an ideal of R; similarly, $(R:T)_r = \mathrm{l.ann}_R(T/R)$ is an ideal of R (note, clearly T/R is a left/right R-module). Hence $(R:T)_\ell$, $(R:T)_r$ and $(R:T)$ are ideals of R, called the left conductor ideal, the right conductor ideal and the conductor ideal of the extension $R \subseteq T$ (or of T in R), respectively. It is clear that $(R:T)_\ell(R:T)_r \subseteq (R:T) \subseteq (R:T)_\ell \cap (R:T)_r$. In particular, if $R \neq T$, then $(R:T)_\ell + (R:T)_r \subseteq R$ and therefore $(R:T)_\ell + (R:T)_r \subsetneq T$. Now we want to prove some generalization of the results in [11]. Let us first review some facts from [11]. If T is a commutative ring and R is a maximal subring of R, then $(R:T)$ is a prime ideal of R. In fact, since R', the integral closure of R in T, is a ring between R and T, then by maximality of R either $R' = T$, that is, T is integral over R (equivalently, T is a finitely generated R-module) or $R' = R$, that is, R is integrally closed in T. Moreover, T is integral over R if and only if $(R:T) \in \mathrm{Max}(R)$ (and thus is a prime ideal of R); and if R is integrally closed in T, then $(R:T)$ is a prime ideal of T; consequently, is a prime ideal of R (furthermore, for any $x, y \in T$, if $xy \in R$, then $x \in R$ or $y \in R$). Hence, in all cases, $(R:T)$ is a prime ideal of R. Now the following is in order.

Lemma 2.1. *Let R be a maximal subring of a ring T. Then $(R:T)_\ell$ and $(R:T)_r$ are prime ideals of R.*

Proof. Let $a, b \in R$ and $aRb \subseteq (R:T)_\ell$. Thus $TaRb \subseteq R$. Now assume that $a \notin (R:T)_\ell$, that is, $Ta \nsubseteq R$. Thus $TaR \nsubseteq R$. Since R is a maximal subring of T and $R + TaR$ is a subring of T that properly contains R, we conclude that $R + TaR = T$. Thus $Tb = Rb + TaRb \subseteq R$, that is, $b \in (R:T)_\ell$. Hence $(R:T)_\ell$ is a prime ideal of R. Similarly, $(R:T)_r$ is a prime ideal of R. □

From the previous result, we make the following observation regarding conductor ideal of maximal subrings.

Proposition 2.2. *Let R be a maximal subring of a ring T. Then the following hold:*

(1) $\mathrm{rad}_R((R:T)) = (R:T)_\ell \cap (R:T)_r$.

(2) *$(R:T)$ is a prime ideal of R if and only if either $(R:T) = (R:T)_\ell$ or $(R:T) = (R:T)_r$, equivalently $(R:T)_\ell$ and $(R:T)_r$ are comparable.*

(3) *$|\mathrm{Min}_R((R:T))| \leq 2$. In fact, $|\mathrm{Min}_R((R:T))| = 1$ if and only if $(R:T)_\ell$ and $(R:T)_r$ are comparable. Otherwise, $\mathrm{Min}_R((R:T)) = \{(R:T)_\ell, (R:T)_r\}$ and either $(R:T)$ is a prime ideal of T or there exist Q_l and Q_r in $\mathrm{Min}_T((R:T))$ such that $Q_l \cap R = (R:T)_\ell$, $Q_r \cap R = (R:T)_r$, $R/(R:T)_\ell \cong T/Q_l$ and $R/(R:T)_r \cong T/Q_r$;*

(4) *$(R:T)$ is a semiprime ideal of R if and only if $(R:T) = (R:T)_\ell \cap (R:T)_r$.*

(5) *If R is a zero dimensional ring, then $(R:T)_\ell$ and $(R:T)_r$ are maximal ideals of R. Hence either $(R:T) = (R:T)_\ell = (R:T)_r$ or $(R:T)_\ell + (R:T)_r = R$.*

(6) *$(R:T) = \mathrm{r.ann}_T(T/(R:T)_r) = \mathrm{l.ann}_T(T/(R:T)_\ell) = \mathrm{l.ann}_R(T/(R:T)_\ell) = \mathrm{r.ann}_R(T/(R:T)_r)$. In particular, if $(R:T)_\ell \in \mathrm{Max}_\ell(T)$ (respectively, $(R:T)_r \in \mathrm{Max_r}(T)$), then $(R:T)$ is a left (respectively, right) primitive ideal of T.*

(7) *If $(R:T) \in \mathrm{Max}(R)$, then $(R:T) = (R:T)_\ell = (R:T)_r$.*

(8) *If $J(R)$ is nilpotent (in particular, if R is a one-sided Artinian ring), then $J(R) \subseteq (R:T)_\ell \cap (R:T)_r$ and $J(R)^2 \subseteq (R:T)$.*

(9) *If $Nil^*(R)$ is nilpotent (in particular, if R is a one-sided Noetherian ring), then $Nil^*(R) \subseteq (R:T)_\ell \cap (R:T)_r$ and $(Nil^*(R))^2 \subseteq (R:T)$.*

Proof. (1) Since $((R:T)_\ell \cap (R:T)_r)^2 \subseteq (R:T)_\ell(R:T)_r \subseteq (R:T) \subseteq (R:T)_\ell \cap (R:T)_r$ and $(R:T)_\ell$, $(R:T)_r$ are prime ideals of R, we conclude that $\mathrm{rad}_R((R:T)) = (R:T)_\ell \cap (R:T)_r$.

(2) The first pard of (2) is evident by (1). It is clear that if either $(R:T) = (R:T)_\ell$ or $(R:T) = (R:T)_r$, then $(R:T)_\ell$ and $(R:T)_r$ are comparable. Conversely, suppose that $(R:T)_\ell$ and $(R:T)_r$ are comparable. For instance, let $(R:T)_\ell \subseteq (R:T)_r$. Let $x \in (R:T)_\ell$, therefore $Tx \subseteq (R:T)_\ell \subseteq (R:T)_r$. Thus $TxT \subseteq R$. Hence $x \in (R:T)$. Consequently, $(R:T) = (R:T)_\ell$.

(3) By the proof of (1) note that for each prime ideal Q of R that contains $(R:T)$, we have $(R:T)_\ell \subseteq Q$ or $(R:T)_r \subseteq Q$. Therefore

$\mathrm{Min}_R((R:T)) \subseteq \{(R:T)_\ell, (R:T)_r\}$. Hence by (1), we obtain that $|\mathrm{Min}_R((R:T))| = 1$ if and only if $(R:T)_\ell$ and $(R:T)_r$ are comparable. For the next part of (3), assume that $(R:T)_\ell$ and $(R:T)_r$ are incomparable and therefore $\mathrm{Min}_R((R:T)) = \{(R:T)_\ell, (R:T)_r\}$. Note that if $A \subseteq B$ is a minimal ring extension with $(A:B) = 0$, and P is a minimal prime ideal of A, then either B is a prime ring or P is a contraction of a minimal prime ideal of B. To see this, note that $A \setminus P$, is a m-system in A and therefore in B, thus there exists a prime ideal Q of B such that $Q \cap A \subseteq P$. If $Q = 0$, then B is prime, otherwise $A + Q = B$, as A is a maximal subring of B and $(A:B) = 0$. Thus $A/(A \cap Q) \cong B/Q$ as rings. Hence $A \cap Q$ is a prime ideal of A and then $Q \cap A = P$, because P is minimal. Clearly we may assume that Q is a minimal prime of B. Applying this fact to the minimal ring extension $R/(R:T) \subseteq T/(R:T)$, we deduce that either $(R:T) \in \mathrm{Spec}(T)$ or there exist minimal prime ideals Q_l and Q_r of $\mathrm{Min}_T((R:T))$ such that $Q_l \cap R = (R:T)_\ell$ and $Q_r \cap R = (R:T)_r$. If $Q_l \subseteq R$, then $(R:T)_\ell \subseteq Q_l \subseteq (R:T) \subseteq (R:T)_r$ which is absurd. Thus Q_l, and similarly Q_r, are not contained in R. Hence by maximality of R we infer that $R + Q_l = T = R + Q_r$. Consequently, we have the ring isomorphisms $R/(R:T)_\ell \cong T/Q_l$ and $R/(R:T)_r \cong T/Q_r$.

(4) is evident by (1). (5) is clear.

(6) Let $x \in T$ (or $x \in R$), then $x \in (R:T) \iff TxT \subseteq R \iff xT \subseteq (R:T)_\ell \iff x \in \mathrm{l.ann}_T(T/(R:T)_\ell)$ (or $x \in \mathrm{l.ann}_R(T/(R:T)_\ell)$). Hence $(R:T) = \mathrm{l.ann}_T(T/(R:T)_\ell) = \mathrm{l.ann}_R(T/(R:T)_\ell)$. The proof of the other equalities of (6) are similar. The final part of (6) is obvious, because whenever $(R:T)_\ell \in \mathrm{Max}_\ell(T)$, then $T/(R:T)_\ell$ is a simple left T-module and then $\mathrm{l.ann}_T(T/(R:T)_\ell) = (R:T)$ is a left primitive ideal of T, and consequently a prime ideal of T.

(7) is clear, because $(R:T)_\ell$ and $(R:T)_r$ are proper ideals of R that contains $(R:T)$.

(8) and (9) Let I be a nilpotent one-sided ideal of R, then clearly $I \subseteq (R:T)_\ell \cap (R:T)_r$, for $(R:T)_\ell$ and $(R:T)_r$ are prime ideals of R. Consequently, $I^2 \subseteq (R:T)_\ell (R:T)_r \subseteq (R:T)$. □

Example 2.3. Let T be a ring and M be a maximal left ideal of T that is not an ideal of T. If R is the idealizer of M in T, then by Theorem 1.1(1), R is a maximal subring of T and clearly $(R:T)_\ell = M$. Moreover, the following hold:

1. T/R is a simple left R-module.
2. $R \subseteq T$ is a left integral extension.
3. R/M is a division ring.
4. $(R : T)$ is a left primitive ideal of T.
5. $(R : T)_r$ is a left primitive ideal of R.

For (1) note that since M is a maximal left ideal of T which is not an ideal of T, for each $x \in T \setminus R$ we infer that $M + Mx = T$ and therefore $R + Rx = T$. This shows that T/R is a simple left R-module. Hence by $(R : T)_r = \mathrm{l.ann}_R(T/R)$ we deduce (5). Also note that $x^2 \in T = R + Rx$ implies that $R \subseteq T$ is a left integral extension, hence (2) holds. (3) is obvious, because T/M is a simple left T-module and we have the rings isomorphism $R/M \cong \mathrm{End}({}_T(T/M))$, by [21, Lemma 1.3]. Finally, for (4) note that by Proposition 2.2(6), $(R : T) = \mathrm{l.ann}_T(T/M)$ and consequently $(R : T)$ is a left primitive ideal of T.

It is easy to see that in the previous example, the inclusion $(R : T) \subsetneq (R : T)_\ell$ holds. Consequently, $(R : T)_\ell \nsubseteq (R : T)_r$. In the next example, we present a maximal subring R of a ring T such that $(R : T)_\ell \neq (R : T)_r$ and $(R : T) \neq (R : T)_\ell \cap (R : T)_r$.

Example 2.4. Let D be a division ring and let $T = \mathbb{M}_2(D)$. It is easy to see that $R = \begin{pmatrix} D & 0 \\ D & D \end{pmatrix}$ is a maximal subring of T. One can check that $(R : T)_\ell = \begin{pmatrix} D & 0 \\ D & 0 \end{pmatrix}$ and $(R : T)_r = \begin{pmatrix} 0 & 0 \\ D & D \end{pmatrix}$. Hence $(R : T)_\ell \cap (R : T)_r = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \neq (R : T) = 0$. Also note that $(R : T) = 0$ is a prime (in fact maximal) ideal of T, but is not a prime ideal in R.

Now we have the following generalization of [11, Proposition 4.1].

Corollary 2.5. *Let R be a maximal subring of a ring T. If T/R as right R-module has a maximal submodule, then $(R : T)_\ell$ is a right primitive ideal of R. In particular, if T/R (that is, T_R) is a finitely generated right R-module, then $(R : T)_\ell$ is a right primitive ideal of R.*

Proof. First we show that, if M is a proper right R-submodule of T that contains R, then $\mathrm{r.ann}_R(T/M) = (R : T)_\ell$. Since $R \subseteq M$, we conclude that $(R : T)_\ell \subseteq \mathrm{r.ann}_R(T/M)$. Now assume that $x \in \mathrm{r.ann}_R(T/M)$, but $x \notin (R : T)_\ell$. Thus $Tx \nsubseteq R$ and then $R + TxR = R$, because R is a maximal subring of T. Since $Tx \subseteq M$ and M is a right R-submodule of T, we deduce that $TxR \subseteq M$. Thus $T = R + TxR \subseteq M$, which is absurd. Thus $\mathrm{r.ann}_R(T/M) \subseteq (R : T)_\ell$ and therefore the equality holds. Now by the assumption, let M/R be a maximal right R-submodule of T/R, then M is a maximal right R-submodule of T and therefore T/M is a simple right R-module. Consequently, $\mathrm{r.ann}_R(T/M)$ is a right primitive ideal of R. Hence by the first part of the proof we are done. The final part is evident. □

Remark 2.6. If T is a commutative ring and R is a maximal subring of T, then the following are equivalent:

1. T is integral over R (that is, T is a finitely generated R-module).
2. $(R : T) \in \mathrm{Max}(R)$.
3. T has a maximal R-submodule that contains R.
4. T/R is a semisimple $R/(R : T)$-module.

To see this, note that (1) and (2) are equivalent by [11, Proposition 4.1]. Clearly, (3) follows from (1); conversely (3) implies (2), by Corollary 2.5, and the fact that in a commutative ring an ideal is maximal if and only if it is primitive. Now assume that (2) holds. Thus T/R is a nonzero $R/(R : T)$-module. Since $R/(R : T)$ is a field, we obtain that T/R is a semisimple $R/(R : T)$-submodule, consequently (4) holds. Finally, suppose that T/R is a semisimple $R/(R : T)$-module, then T/R has a maximal R-submodule and hence (3) holds.

In the next two results we see that the converse of Corollary 2.5, holds (similar to the previous remark) under certain conditions.

Corollary 2.7. *Let R be a maximal subring of a ring T. If $(R : T)_\ell \in \mathrm{Max_r}(R)$, then T has a maximal right R-submodule that contains R.*

Proof. Since $(R : T)_\ell \in \mathrm{Max_r}(R)$, we conclude that $R/(R : T)_\ell$ is a division ring. Clearly, T/R is a nonzero right $R/(R : T)_\ell$-module, and therefore has a maximal submodule. □

In [3, Theorem 3.8], it is proved that for a maximal subring R of a commutative ring T, R is Artinian if and only if T is Artinian and integral over R. A key step in the proof of this result, by Remark 2.6, relies on the fact that in this setting, $(R:T)$ is a maximal ideal of R. This leads us to the following natural generalization.

Proposition 2.8. *Let R be a right Artinian ring that is a maximal subring of a ring T. Then $(R:T)_\ell$ and $(R:T)_r$ are maximal ideals of R. T/R as right $R/(R:T)_\ell$-modules (resectively, $R/(R:T)_r$) is isotypic semisimple (and therefore has a maximal submodule). In particular, if additionally R is local, then $(R:T) = (R:T)_\ell = (R:T)_r$.*

Proof. By Lemma 2.1, $P := (R:T)_\ell$ is a prime ideal of R and since R is a right Artinian ring we conclude that $R/P \cong \mathbb{M}_n(D)$, for some division ring D and some natural number n. Hence R/P is a simple ring and consequently P is a maximal ideal of R. Also note that T/R is a nonzero right R/P-module. Thus T/R is a semisimple R/P-module. Since $R/P \cong \mathbb{M}_n(D)$, it follows that each simple component of T/R is isomorphic to the other one. The proof for $(R:T)_r$ is similar. The final part is evident. □

Next, we investigate conditions under which the conductor ideals of a maximal subring R of a ring T are completely prime (right/left) ideals in either R or T. We recall that a proper right ideal P of a ring S is said to be completely prime right ideal, if for all $a, b \in S$ with $aP \subseteq P$ and $ab \in P$, we have $a \in P$ or $b \in P$, see [25]. A prime ideal of a ring S is called completely prime if S/P is a domain.

Proposition 2.9. *Let R be a maximal subring of a ring T and $P = (R:T)_r$. If P is a completely prime right ideal of T, then R/P is a domain (that is, P is a completely prime ideal of R) and T/P is a torsion-free left R/P-module. Conversely, if P is a completely prime ideal of R and T/P is a torsion-free left R/P-module, then either P is an ideal of T (and therefore $(R:T) = P$) or P is a completely prime right ideal of T and $R = \mathbb{I}_T(P)$.*

Proof. First assume that P is a completely prime right ideal of T. Let $a, b \in R$ and $ab \in P$. Since P is an ideal of R, we conclude that $aP \subseteq P$. Therefore $a \in P$ or $b \in P$, because P is a completely prime right ideal of T. Thus R/P is a domain. Since $P = (R:T)_r$ is a right ideal of T and P

is an ideal of R (that is, $PT \subseteq R$ and therefore $P(T/R) = 0$), we deduce that T/P is a left R/P-module. Now we show that T/P is a torsion-free left R/P-module. Suppose that $(r+P)(t+P) = 0$, where $r \in R$ and $t \in T$. This implies that $rt \in P$ and $rP \subseteq P$. Since P is a completely prime right ideal of T, we infer that either $r \in P$ or $t \in P$. Consequently, T/P is a torsion-free left R/P-module. Conversely, suppose that R/P is a domain and T/P is a torsion-free left R/P-module. Let $a, b \in T$ such that $aP \subseteq P$ and $ab \in P$. We have two cases, either P is an ideal of T (and therefore $P = (R : T)$) or P is not an ideal of T. In the latter case, since P is an ideal of R and R is a maximal subring of T, we obtain that $R = \mathbb{I}_T(P)$. It follows that $a \in R$ and since $ab \in P$, we have $(a+P)(b+P) = 0$. So $a \in P$ or $b \in P$, because T/P is a torsion-free left R/P-module. Consequently, P is a completely prime right ideal of T. □

We obtain the following quick corollaries now.

Corollary 2.10. *Let R be a maximal subring of a ring T and $P = (R : T)_r \in \mathrm{Max_r}(T)$. Then P is a completely prime ideal of R (that is, R/P is a domain) and T/P is a torsion-free left R/P-module.*

Proof. Note that by [25, Corollary 2.10(A)], P is a completely prime right ideal of T and therefore we are done by the previous proposition. □

Assuming additionally that T is a right Artinian ring in the first part of Proposition 2.9, yields the following stronger conclusion.

Corollary 2.11. *Let R be a maximal subring of a right Artinian ring T and $P = (R : T)_r$ be a completely prime right ideal of T. The either T/P or R/P is a division ring.*

Proof. We have two cases, either P is an ideal of T or is not. First assume that P is an ideal of T. By [25, Corollary 2.10(B)], we conclude that $P \in \mathrm{Max_r}(T)$. So T/P is a division ring. If P is not an ideal of T, then $R = \mathbb{I}_T(P)$, because R is a maximal subring of T and P is an ideal of R. Again by [25, Corollary 2.10(B)], $P \in \mathrm{Max_r}(R)$ and hence R/P is a division ring. □

When T is a (left/right) duo ring and R is a maximal subring of T, then $(R : T)$ is a completely prime ideal of R. This follows directly from our next result.

Proposition 2.12. *Let R be a maximal subring of a left duo ring T. Then $(R:T)=(R:T)_\ell$ is a completely prime ideal of R. Moreover, if T is a duo ring, then $(R:T)=(R:T)_\ell=(R:T)_r$ is a completely prime ideal of R.*

Proof. Since T is a left duo ring, we conclude that $(R:T)=(R:T)_\ell$. Now assume that $a,b\in R$ such that $ab\in(R:T)$ and $a\notin(R:T)=(R:T)_\ell$. Therefore $Tab\subseteq R$ and $Ta\nsubseteq R$. Since R is a maximal subring of T and Ta is an ideal of T, we infer that $R+Ta=T$. Hence $Tb=(R+Ta)b=Rb+Tab\subseteq R$, that is, $b\in(R:T)=(R:T)_\ell$. Consequently, $(R:T)=(R:T)_\ell$ is a completely prime ideal of R. The final part is evident. □

The next proposition establishes a relationship between the centralizer of a maximal subring R of a ring T and the conductor ideals.

Proposition 2.13. *Let R be a maximal subring of a ring T. One of the following holds:*

(1) *$C_T(R)\subseteq R$. In particular, R contains the center of T.*

(2) *$(R:T)=(R:T)_\ell=(R:T)_r$, and there exists $\alpha\in T$ such that $T=R[\alpha]$, where $\alpha r=r\alpha$, for each $r\in R$.*

Proof. Assume that R does not contain $C_T(R)$. If $R=\mathbb{I}_T((R:T)_r)$, then clearly $C_T(R)\subseteq C_T((R:T)_r)\subseteq\mathbb{I}_T((R:T)_r)=R$, which is absurd. Thus $(R:T)_r$ is an ideal of T. Similarly, $(R:T)_\ell$ is an ideal of T and hence the first part of (2) holds. The second part is evident for $C_T(R)\nsubseteq R$. □

As seen in Proposition 2.9, and in the proofs of Corollary 2.11, and the preceding proposition, a maximal subring R of a ring T can be expressed as the idealizer of a one-sided ideal of T. In the next remark we precisely determine when R is of the form of an idealizer of a one-sided ideal of T.

Remark 2.14. Let R be a maximal subring of a ring T. Then exactly one of the following holds:

1. $(R:T)=(R:T)_\ell\subsetneq(R:T)_r$ and $R=\mathbb{I}_T((R:T)_r)$.
2. $(R:T)=(R:T)_r\subsetneq(R:T)_\ell$ and $R=\mathbb{I}_T((R:T)_\ell)$.
3. $(R:T)\subsetneq(R:T)_r,(R:T)_\ell$ and $R=\mathbb{I}_T((R:T)_r)=\mathbb{I}_T((R:T)_\ell)$.
4. $(R:T)=(R:T)_\ell=(R:T)_r$.

Moreover, if $(R : T)$ is not a prime ideal of R, then (3) holds. To see this, note that $(R : T) \subseteq (R : T)_\ell \cap (R : T)_r$. Hence, if either $(R : T)_\ell$ or $(R : T)_r$ is an ideal of T, then we obtain that $(R : T) = (R : T)_\ell$ or $(R : T) = (R : T)_r$. For instance, if $(R : T)_\ell$ is an ideal of T, but $(R : T)_r$ is not an ideal of T, then $(R : T) = (R : T)_\ell$ and since $(R : T)_r$ is an ideal of R, we conclude that $R = \mathbb{I}_T((R : T)_r)$, because R is a maximal subring of T that is contained in the proper subring $\mathbb{I}_T((R : T)_r)$ of T. Thus (1) holds. By a similar argument we see (2). If $(R : T)_\ell$ and $(R : T)_r$ are ideals of T, then clearly (4) holds. Finally, assume that $(R : T)_\ell$ and $(R : T)_r$ are not ideals of T. Since these are ideals of R and R is a maximal subring of T, we deduce that $R = \mathbb{I}_T((R : T)_\ell) = \mathbb{I}_T((R : T)_r)$. The final part is evident by Lemma 2.1.

Remark 2.15. Let T be a ring and A a right ideal of T. Then the map $\phi : \mathbb{I}(A)/A \longrightarrow \mathrm{End}((T/A)_T)$ defined by $\phi(r + A) = f_{r+A}$ where $f_{r+A}(x + A) = rx + A$, for each $r \in \mathbb{I}(A)$ and for all $x \in T$, is a ring isomorphism, see [21, Lemma 1.3]. In particular, if R is a maximal subring of T and $A = (R : T)_r$, then either $R/A \cong \mathrm{End}((T/A)_T)$ (and hence $\mathrm{End}((T/A)_T)$ is a prime ring) or $\phi(R/A)$ is a maximal subring of $\mathrm{End}((T/A)_T) \cong T/A$.

As shown in the preceding remarks, there is a closed relationship between the idealizer of a one-sided ideal A of a ring T and the endomorphism ring of T/A. In subsequent results of this section, we focus on endomorphism rings to derive further properties of maximal subrings. Specifically, we will explore algebraic connections between the quotient rings of a maximal subring R of a ring T with respect to the conductor ideals, and the endomorphism rings of T/R as left/right R-modules and as a $\mathbb{Z}$-module. First we require the following lemma, which is analogous to Remark 2.15.

Lemma 2.16. *Let R be a subring of a ring T and set $A = (R : T)_r$. Then the map $\psi : R/A \longrightarrow \mathrm{End}((T/R)_R)$, defined by $\psi(r + A) = g_{r+A}$ where $g_{r+A}(t + R) = rt + R$, for each $r \in R$ and for all $t \in T$, is a ring monomorphism. Consequently, up to ring isomorphism, R/A is a subring of $\mathrm{End}((T/R)_R)$. In particular, up to ring isomorphism, R/A is a subring of $\mathrm{End}_{\mathbb{Z}}(T/R)$.*

Proof. The proof of ψ is a ring homomorphism follows the same reasoning as [21, Lemma 1.3]. It remains to show that $\mathrm{Ker}(\psi) = 0$. Note that $r + A \in$

$\mathrm{Ker}(\psi) \Longleftrightarrow g_{r+A} = 0 \Longleftrightarrow g_{r+A}(t+R) = 0$ for each $t \in T$, $\Longleftrightarrow rt + R = 0$, for each $t \in T$, that is, $rT + R = 0 \Longleftrightarrow rT \subseteq R \Longleftrightarrow r \in A$. □

We recall that since we adopt the convention of writing maps on the left, the statement of Remark 2.15 (for left ideals) and Lemma 2.16 (for $(R:T)_\ell$) require appropriate modification. Specifically, if T is a ring and B is a left ideal of T, then $\mathbb{I}(B)/B \cong (\mathrm{End}(_T(T/B)))^{\mathrm{op}}$, where the superscript "op" denote the opposite ring. Moreover, if R is a subring of T and $B = (R : T)_\ell$, then the ring R/B embeds into $(\mathrm{End}(_R(T/R)))^{\mathrm{op}}$. Therefore, $(R/B)^{\mathrm{op}}$ embeds into the ring $\mathrm{End}(_R(T/R))$, and hence into the ring $\mathrm{End}_{\mathbb{Z}}(T/R)$. It is clear that for any ring S, we have $\mathrm{Char}(S) = \mathrm{Char}(S^{\mathrm{op}})$, and S is a prime ring if and only if S^{op} is a prime ring. This leads us to the following.

Proposition 2.17. *Let R be a maximal subring of a ring T. Then* $\mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R))$ *is either* 0 *or is a prime number. Moreover,* $\mathrm{Char}(R/(R : T)_\ell) = \mathrm{Char}(R/(R : T)_r) = \mathrm{Char}(R/(R : T))$.

Proof. By Lemma 2.1, Lemma 2.16 and preceding comments, $(R/(R : T)_\ell)^{\mathrm{op}}$ and $R/(R : T)_r$ are prime rings, that are subrings of $\mathrm{End}_{\mathbb{Z}}(T/R)$. Hence, $\mathrm{Char}(R/(R : T)_\ell) = \mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R)) = \mathrm{Char}(R/(R : T)_r)$. Now note that, the center of a prime ring is an integral domain and the characteristic of an integral domain is either 0 or is a prime number; consequently, we obtain the first part of the proposition. For the final part, we have two cases: If $\mathrm{Char}(R/(R : T)_\ell) = \mathrm{Char}(R/(R : T)_r) = 0$, then it is clear that $\mathrm{Char}(R/(R : T)) = 0$, because $R/(R : T)_\ell$ and $R/(R : T)_r$ are quotients rings of $R/(R : T)$. Hence assume that $\mathrm{Char}(R/(R : T)_\ell) = \mathrm{Char}(R/(R : T)_r) = p$, where p is a prime number. Thus $pT = Tp \subseteq R$ and a fortiori $pT = Tp = TpT \subseteq R$. Thus $\mathrm{Char}(R/(R : T)) = p$. □

Now we have the following immediate corollary.

Corollary 2.18. *Let R be a maximal subring of a ring T. Assume that the map* $\mathrm{Char} : \mathrm{Spec}(R) \longrightarrow \mathbb{N} \cup \{0\}$, *defined by* $P \longmapsto \mathrm{Char}(R/P)$ *is one-to-one. Then $(R : T)_\ell = (R : T)_r$, in particular $(R : T) = (R : T)_\ell = (R : T)_r$.*

We now present several examples of rings satisfying the property assumed in the previous corollary.

Example 2.19. (1) Assume that $R := \mathbb{M}_n(\mathbb{Z})$, where $n > 1$ is a natural number. Then for each prime number q, it is clear that $\mathbb{Z}$ is a maximal subring of $S := \mathbb{Z}[1/q]$ and therefore R is a maximal subring of $T := \mathbb{M}_n(S)$. It is obvious that, the map Char mentioned in the previous corollary is one-to-one. Hence, $(R : T) = (R : T)_\ell = (R : T)_r$.

(2) Let $R = R_1 \times \cdots \times R_n$, where $n > 1$ and each R_i is a simple ring with $\mathrm{Char}(R_i) = p_i$ is a prime number. Assume that $p_i \neq p_j$ for $i \neq j$. If R is a maximal subring of a ring T, then $(R : T) = (R : T)_\ell = (R : T)_r = R_1 \times \cdots \times R_{i-1} \times 0 \times R_{i+1} \times \cdots \times R_n$, for some i.

(3) Let R be a one-sided Artinian ring that is a maximal subring of a ring T. Assume that $R/J(R) \cong \mathbb{M}_{n_1}(D_1) \times \ldots \mathbb{M}_{n_k}(D_k)$, where n_i and k are natural numbers and D_i is a division ring for each i. Then either $(R : T) = (R : T)_\ell = (R : T)_r$ or there exist $i \neq j$ such that $\mathrm{Char}(D_i) = \mathrm{Char}(D_j)$.

In the next two corollary, we derive further results concerning the relationship between the characteristic, conductor ideals, and associated endomorphism rings.

Corollary 2.20. *Let R be a maximal subring of a ring T with* $\mathrm{Char}(T)$ *is not a prime number. Then either there exists a prime number p such that $pT \subseteq (R : T)$ or* $\mathrm{Char}(T) = \mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R)) = 0$.

Proof. By Proposition 2.17, $\mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R))$ is either 0 or a prime number. If $\mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R)) = p$, where p is a prime number, then $p.1_{T/R} = 0$ (note that $1_{T/R}$ denotes the identity map on T/R; that is, $1_{T/R}(t + R) = t + R$, for all $t \in T$). Hence, for any $t \in T$, $pt + R = R$, that is, $pT \subseteq R$. Thus $0 \neq pT \subseteq (R : T)$, (note $\mathrm{Char}(T)$ is not a prime number, therefore $pT \neq 0$ and it is clear that $pT = Tp$ is an ideal of T). Otherwise, assume that $\mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R)) = 0$. If $\mathrm{Char}(T) = m > 0$, then for each $\phi \in \mathrm{End}_{\mathbb{Z}}(T/R)$, we have $m\phi = 0$, because $m\phi(t) = \phi(mt) = \phi(0) = 0$, for any $t \in T$. It follows that $\mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R)) > 0$, which is absurd. Consequently, $\mathrm{Char}(T) = \mathrm{Char}(\mathrm{End}_{\mathbb{Z}}(T/R)) = 0$, as desired. □

The following is immediate now.

Corollary 2.21. *Let R be a maximal subring of a ring T. Assume that* $\mathrm{Char}(T)$ *is neither zero nor a prime number. Then* $(R : T) \neq 0$.

Finally, in this section, we present some observations concerning the annihilators of conductor ideals of maximal subrings.

Proposition 2.22. *Let R be a maximal subring of a ring T. If* $\mathrm{l.ann}_T((R:T)) + \mathrm{r.ann}_T((R:T)) = T$, *then* $(R:T)^2 = 0$.

Proof. By assumption, there exist elements $a \in \mathrm{l.ann}_T((R:T))$ and $b \in \mathrm{r.ann}_T((R:T))$ such that $a+b=1$. Consequently, $(R:T)a = (R:T)$ and $(R:T) = b(R:T)$. Now note that $ab \in \mathrm{l.ann}_T((R:T)) \cap \mathrm{r.ann}_T((R:T))$. Thus $(R:T)^2 = (R:T)(R:T) = (R:T)ab(R:T) = 0$. □

In particular, if R is a maximal subring of a semiprime ring T, and the characteristic of T is neither zero nor a prime number, then $\mathrm{l.ann}_T((R:T)) + \mathrm{r.ann}_T((R:T))$ is a proper ideal of T, by Corollary 2.21 and the previous result. As shown in the next conclusion, for certain rings the left (respectively, right) conductor ideal of a maximal subring coincides with the annihilator of the corresponding right (respectively, left) conductor ideal.

Theorem 2.23. *Let R be a maximal subring of a ring T. Then the following hold:*

(1) *If $(R:T)_\ell$ and $(R:T)_r$ are incomparable, then* $\mathrm{l.ann}_R((R:T)_\ell) + \mathrm{r.ann}_R((R:T)_\ell) \subseteq (R:T)_r$ *and* $\mathrm{l.ann}_R((R:T)_r) + \mathrm{r.ann}_R((R:T)_r) \subseteq (R:T)_\ell$.

(2) *If $(R:T)_\ell(R:T)_r = 0$ (in particular, if $(R:T) = 0$), then* $\mathrm{Min}(R) \subseteq \{(R:T)_\ell, (R:T)_r\}$. *In particular, if $(R:T)_\ell$ and $(R:T)_r$ are incomparable, then* $(R:T)_\ell = \mathrm{l.ann}_R((R:T)_r)$ *and* $(R:T)_r = \mathrm{r.ann}_R((R:T)_\ell)$ *are precisely minimal prime ideals of R.*

(3) *If $(R:T)_\ell \cap (R:T)_r = 0$ and $(R:T)_\ell \neq 0 \neq (R:T)_r$, then* $(R:T)_\ell = \mathrm{l.ann}_R((R:T)_r) = \mathrm{r.ann}_R((R:T)_r)$ *and* $(R:T)_r = \mathrm{r.ann}_R((R:T)_\ell) = \mathrm{l.ann}_R((R:T)_\ell)$ *are precisely minimal prime ideals of R. In particular, T is not a prime ring.*

(4) *If R is a reduced ring, $(R:T)_\ell \cap (R:T)_r = 0$ and $(R:T)_\ell \neq 0 \neq (R:T)_r$, then $(R:T)_\ell$ and $(R:T)_r$ are completely prime ideals of R. In particular, R embeds into a product of two domains.*

Proof. (1) is clear, because $\mathrm{l.ann}_R((R:T)_\ell)(R:T)_\ell = (R:T)_\ell\, \mathrm{r.ann}_R((R:T)_\ell) = 0 \subseteq (R:T)_r$ and $(R:T)_r$ is a prime ideal of R, which does not contain $(R:T)_\ell$ by our assumption. For (2), first note that $(R:T)_\ell(R:T)_r \subseteq (R:T)$, hence if $(R:T) = 0$, we conclude that $(R:T)_\ell(R:T)_r = 0$.

Thus assume that $(R:T)_\ell(R:T)_r=0$. This implies that $\mathrm{Min}(R)\subseteq\{(R:T)_\ell,(R:T)_r\}$, by Lemma 2.1. Now suppose that $(R:T)_\ell$ and $(R:T)_r$ are incomparable. Let $P=(R:T)_\ell$ and $I=(R:T)_r$. Thus $PI=0$ and $I\neq 0$ (by incomparability). It is clear that $P\subseteq \mathrm{l.ann}_R(I)$. Since $\mathrm{l.ann}_R(I)I=0\subseteq P$ and P is a prime ideal of R that does not contain I, we deduce that $P=\mathrm{l.ann}_R(I)$. Finally, if P is not a minimal ideal of R, let Q be a prime ideal properly contained in P. Then $\mathrm{l.ann}_R(I)I=0\subseteq Q\subsetneq P=\mathrm{l.ann}_R(I)$ implies that $I\subseteq Q\subseteq P$, which is absurd, because I and P are incomparable. Consequently, P is a minimal prime ideal of R. Similarly, $(R:T)_r=\mathrm{r.ann}_R((R:T)_\ell)$ is a minimal prime ideal of R and hence (2) holds. For (3), note that since $(R:T)_\ell$ and $(R:T)_r$ are ideals of R, then we infer that $(R:T)_\ell(R:T)_r$ and $(R:T)_r(R:T)_\ell$ are subsets of $(R:T)_\ell\cap(R:T)_r$. Hence if $(R:T)_\ell\cap(R:T)_r=0$, we conclude that $(R:T)_\ell(R:T)_r=0$ and $(R:T)_r(R:T)_\ell=0$. By a similar argument of (2), the conclusion of (3) can be proved. For the final part of (3), since $0\neq(R:T)_\ell=\mathrm{r.ann}_R((R:T)_r)\subseteq\mathrm{r.ann}_T((R:T)_r)$, we obtain that T is not a prime ring. For (4), it is clear that $(R:T)_\ell$ and $(R:T)_r$ are incomparable and by (2) or (3), are minimal prime ideal of R. Since R is reduced, by [17, Lemma 12.6], $(R:T)_\ell$ and $(R:T)_r$ are completely prime ideals of R, that is, $R/(R:T)_\ell$ and $R/(R:T)_r$ are domains and clearly R embeds in product of them. □

By Theorem 2.23(2) we have the following immediate result.

Corollary 2.24. *Let R be a ring which is a maximal subring of a ring T with $|\mathrm{Min}(R)|\geq 3$. Then both ideals $(R:T)$ and $(R:T)_\ell(R:T)_r$ are nonzero.*

3 Integrally closed maximal subrings

As noted in the introduction, if R is a maximal subring of a commutative ring T and R is integrally closed of T, then $(R:T)$ is a prime ideal in T. We now investigate the analogous question in non-commutative setting: When are the one-sided conductor ideals $(R:T)_r$ and $(R:T)_\ell$ prime right or left ideals? Our study begins with the following key result.

Theorem 3.1. *Let R be a maximal subring of a ring T, $P=(R:T)_r$, $a,b\in T$ such that $aTb\subseteq P$. If $a,b\notin P$, then the following hold:*

(1) *a is right* $(2-)$*integral over R and b is left* $(2-)$*integral over R.*

(2) *If $a \notin R$, then $T = R + RaR$.*

(3) *If $b \notin R$, then $T = R + RbR$.*

In particular, if $b \notin R$, then $a \notin R$; and if $a \notin R$, then either $b \notin R$ or $b \in (R : T)_\ell$.

Proof. First note that by our assumption, $aTbT \subseteq R$, $aT \nsubseteq R$ and $bT \nsubseteq R$. Hence by maximality of R we deduce that $R + RaT = T = R + RbT$. Thus there exist $r_i \in R$ and $s_i \in T$ such that $a = r_0 + r_1bs_1 + \cdots + r_nbs_n$. Multiplying this equation from left by a, we have $a^2 = ar_0 + ar_1bs_1 + \cdots + ar_nbs_n$. Now for each i, $ar_ibs_i \in aTbT \subseteq R$. Therefore $r := ar_1bs_1 + \cdots + ar_nbs_n \in R$ and $a^2 = ar_0 + r$ that shows a is right $(2-)$integral over R. By a similar argument (from $aTb \subseteq P \subseteq R$ and $T = R + RaT$) we deduce that b is left $(2-)$integral over R. Hence (1) holds. Now we prove (2). Assume that $a \notin R$. We claim that $S := R + RaR$ is a subring of T. To see this, it suffices to show that if $x, y, u, v \in R$, then $(xay)(uav) \in S$. We have $(xay)(uav) = xayu(r_0 + r_1bs_1 + \cdots + r_nbs_n)v = xayur_0v + xayur_1bs_1v + \cdots + xayur_nbs_nv$. The first term of the sum clearly lies in RaR and at once in S. For the other terms of sum, note that since $ayur_ibs_i \in aTbT \subseteq R$, we obtain that $xayur_ibs_iv \in R$. Consequently, $xayur_1bs_1v + \cdots + xayur_nbs_nv \in R$. This implies that $(xay)(uav) \in S$ and thus S is a subring of T properly contains R (because $a \notin R$). So $S = T$, as R is a maximal subring of T. The proof of (3) is similar (using the inclusion $aTb \subseteq P \subseteq R$). Finally, for the last part of the theorem, assume that $b \notin R$, but $a \in R$. By (3), we have $T = R + RbR$. Consequently, $aT = aR + aRbR \subseteq R + aTbT \subseteq R$, that is, $a \in P$, which is absurd. Now assume that $a \notin R$, but $b \in R$. It follows from (2) that $T = R + RaR$ and then $Tb = Rb + RaRb \subseteq R$. Thus $b \in (R : T)_\ell$. □

Remark 3.2. Similar to the previous result and its proof, one can prove that, if R is a maximal subring of a ring T, $P = (R : T)_\ell$, $a, b \in T$ such that $aTb \subseteq P$ and $a, b \notin P$, then the following hold:

1. a is right $(2-)$integral over R and b is left $(2-)$integral over R.
2. If $a \notin R$, then $T = R + RaR$.
3. If $b \notin R$, then $T = R + RbR$.

In particular, if $a \notin R$, then $b \notin R$; and if $b \notin R$, then either $a \notin R$ or $a \in (R:T)_r$.

Let T be a ring and P be a proper one-sided ideal of T. We call P prime if for all $a, b \in T$, the inclusion $aTb \subseteq P$ implies that either $a \in P$ or $b \in P$. For instance, any maximal one-sided ideal M of a ring T is prime. To verify this, suppose M is a maximal left ideal of T and $a, b \in T$ satisfy $aTb \subseteq M$. If $b \in M$, we are done. Otherwise, $b \notin M$, the maximality of M yields that $M + Tb = T$. Consequently, $aT = aM + aTb \subseteq M$, whence $a \in M$. Thus, M is prime. This leads us to the following.

Corollary 3.3. *Let R be a maximal subring of a ring T which is $(2-)$integrally closed in T, then $(R:T)_\ell$ and $(R:T)_r$ are prime one-sided ideals of T.*

Regarding the conductor ideal $(R:T)$ of an integrally closed maximal subring R of a ring T, we make the following observation.

Remark 3.4. Let R be a maximal subring of a ring T that is $(2-)$integrally closed in T. If $a, b \in T$ satisfy $aTb \subseteq (R:T)$, but $a, b \notin (R:T)$, then the following hold:

1. $a, b \in R$.
2. $RaT \subseteq R$, but $TaR \nsubseteq R$. In particular, $T = R + TaR$.
3. $TbR \subseteq R$, but $RbT \nsubseteq R$. In particular, $T = R + RbT$.
4. $a \in (R:T)_r$, $b \in (R:T)_\ell$, $a \notin (R:T)_\ell$ and $b \notin (R:T)_r$.

To see these, for (1) we show that $a \in R$, the proof for $b \in R$ is similar. Since $a, b \notin (R:T)$, we infer that TaT and TbT are not contained in R. Therefore by maximality of R, we deduce that $R + TaT = T$ and $R + TbT = T$. Consequently, $a \in R + TbT$. It follows that there exist $n \in \mathbb{N}$, $x_i, y_i \in T$ for $1 \leq i \leq n$ and $r \in R$ such that $a = r + x_1by_1 + \cdots + x_nby_n$. Multiplying by a from left, we obtain that $a^2 = ar + ax_1by_1 + \cdots + ax_nby_n$. Now note that for all i, $ax_ib \in aTb \subseteq (R:T)$; and $(R:T)$ is an ideal of T contained in R, whence $ax_iby_i \in (R:T) \subseteq R$ for all i. This implies that $s := ax_1by_1 + \cdots + ax_nby_n \in R$ and $a^2 = ar + s$, that is, a is right $(2-)$integral over R and therefore $a \in R$. Thus (1) holds. (4) is obvious by (2) and (3). We prove $TaR \nsubseteq R$ and $TbR \subseteq R$, the proofs of the other parts of (2) and (3) are similar. Since $b \notin (R:T)$, we conclude that

$TbT \nsubseteq R$. By maximality of R, we deduce that $R+TbT = T$. It follows that $aR + aTbT = aT$. Consequently, $TaR + TaTbT = TaT$. From $a \notin (R:T)$, we have $TaT \nsubseteq R$; and form $aTb \subseteq (R:T)$ we obtain that $TaTbT \subseteq R$. Thus by $TaR + TaTbT = TaT$, we conclude that $TaR \nsubseteq R$. So $Ta \nsubseteq R$, a fortiori $a \notin (R:T)_\ell$. Since $aTb \subseteq (R:T) \subseteq (R:T)_\ell$, we infer that $b \in (R:T)_\ell$, by the previous corollary. Consequently, $Tb \subseteq R$ and then $TbR \subseteq R$.

We conclude this section by proving that if R is an integrally closed maximal subring of a ring T, then $(R:T)$ is a semiprime ideal of T. First, we require the following lemma.

Lemma 3.5. *Let R be a maximal subring of a ring T and $x \in (R:T)_\ell \cup (R:T)_r$. If $xTx \subseteq (R:T)$, then $x \in (R:T)$.*

Proof. Assume that $x \in (R:T)_\ell$. Thus $Tx \subseteq R$ and a fortiori $TxR \subseteq R$. Suppose $x \notin (R:T)$. It follows that $TxT \nsubseteq R$ and consequently $R+TxT = T$, by maximality of R. Since $xTx \subseteq (R:T)$, we infer that $TxTxT \subseteq R$. This implies that $TxT = TxR + TxTxT \subseteq R$, which is absurd. Hence, $x \in (R:T)$. The proof for the case $x \in (R:T)_r$ is similar. □

By the above observation we have the following key result.

Theorem 3.6. *Let R be a maximal subring of a ring T that is* (2-)*integrally closed in T (or whenever $x \in T$ and $x^2 \in R$, then $x \in R$). Then $(R:T)$ is a semiprime ideal of T. Moreover, either $(R:T)$ is a prime ideal of T or $(R:T) = (R:T)_\ell \cap (R:T)_r$ (is a semiprime ideal of R).*

Proof. We must prove that if $x \in T$ and $xTx \subseteq (R:T)$, then $x \in (R:T)$. Since $xTx \subseteq R$, we infer that $x^2 \in R$. It follows that $x \in R$, by our assumption. So, $xRx \subseteq xTx \subseteq (R:T) \subseteq (R:T)_\ell \cap (R:T)_r$. This implies that $xRx \subseteq (R:T)_\ell$, and consequently $x \in (R:T)_\ell$, because $(R:T)_\ell$ is a prime ideal of R, by Lemma 2.1. Thus $x \in (R:T)$, by the previous lemma. Hence $(R:T)$ is a semiprime ideal of T, that is, there exists a family Q_i, $i \in I$, of prime ideals of T, such that $(R:T) = \bigcap_{i\in I} Q_i$. Now we have two cases. If there exists $i \in I$, such that $Q_i \subseteq R$, then $Q_i \subseteq (R:T)$, and at once $(R:T) = Q_i$ is a prime ideal of T. Hence, assume that for all $i \in I$, $Q_i \nsubseteq R$. Consequently, for all $i \in I$, we obtain that $R + Q_i = T$, by maximality of R. This implies that $R/(Q_i \cap R) \cong T/Q_i$, as rings. Thus $Q_i \cap R$ is a prime ideal of R, for all $i \in I$. It is clear that $(R : T) = \bigcap_{i \in I}(Q_i \cap R)$, that is, $(R : T)$ is a semiprime ideal of R. The final part is evident by Proposition 2.2(4). □

4 Finiteness conditions

As noted in the introduction, if R is a finite maximal subring of a ring T, then T is also finite, see [5, 6, 15, 16, 20]. In commutative rings, for a maximal subring R of a ring T, it is known that R is an Artinian if and only if T is Artinian and T integral over R, see [3, Theorem 3.8]. furthermore, if R is a Noetherian maximal subring of a commutative ring T, then the Hilbert Basis Theorem implies that T is Noetherian. Conversely, if R is a maximal subring of a commutative Noetherian ring T and T is integral over R, then the Eakin-Nagata-Eisenbud Theorem (see [18, Theorem 3.98]) guaranties R is Noetherian. In this section, we obtain analogous observations for non-commutative rings under certain finiteness condition on maximal subrings. We begin with the following result.

Proposition 4.1. *Let R be a maximal subring of a ring T and $P = (R : T)_\ell$. If $PT = T$, then $R = \mathbb{I}_T(P)$, T is a finitely generated left R-module, P is a finitely generated right R-module which is a right primitive ideal of R and T is not a left quasi duo ring. Moreover, if in addition R is a left Artinian/Noetherian ring, then so is T.*

Proof. Since P is a left ideal of T which is a two sided ideal of R, and P is not an ideal of T, we conclude that $R \subseteq \mathbb{I}_T(P) \subsetneq T$. By maximality of R, we deduce that $R = \mathbb{I}_T(P)$. From $PT = T$, we obtain that if M is a maximal left ideal of T that contains P, then M is not an ideal of T. Consequently, T is not a left quasi duo ring. Since $T = PT$, we deduce that $1 = y_1t_1 + \cdots + y_nt_n$, for some $y_i \in P$, $t_i \in T$ and $n \in \mathbb{N}$. Thus for any $x \in T$ we have $x = x1 = (xy_1)t_1 + \cdots + (xy_n)t_n$. Now note that $y_i \in P$ that is a left ideal of T, a fortiori $xy_i \in P \subseteq R$. This shows that $T = Rt_1 + \cdots + Rt_n$. Also note that for any $p \in P$, we have $p = 1p = y_1t_1p + \cdots + y_nt_np$. Now since $t_ip \in Tp \subseteq P \subseteq R$, we have $P = y_1R + \cdots + y_nR$, that is, P is a finitely generated right R-module. Since ${}_RT$ is finitely generated, Corollary 2.5 implies that P is a right primitive ideal of R. The final part is evident, because ${}_RT$ is finitely generated. □

Clearly, an analogous statement hold for the right conductor ideal $(R : T)_r$. Hence we obtain the following result.

Corollary 4.2. *Let R be a maximal subring of a ring T and $(R : T) \in \mathrm{Max}(T)$. If $(R : T) \subsetneq (R : T)_\ell, (R : T)_r$ (that is, $(R : T)_\ell$ and $(R : T)_r$ are not ideals of T), then the following hold:*

1. *${}_RT$ and T_R are finitely generated.*
2. *$(R : T)_\ell$ is a finitely generated right R-module which is a right primitive ideal of R. In particular, $\mathrm{l.ann}_T((R : T)_\ell) = \mathrm{l.ann}_T(p_1) \cap \cdots \cap \mathrm{l.ann}_T(p_m)$ for some $p_1, \ldots p_m \in (R : T)_\ell$.*
3. *$(R : T)_r$ is a finitely generated left R-module which is a left primitive ideal of R. In particular, $\mathrm{r.ann}_T((R : T)_r) = \mathrm{r.ann}_T(q_1) \cap \cdots \cap \mathrm{r.ann}_T(q_n)$ for some $q_1, \ldots q_n \in (R : T)_r$.*

In particular, if $(R : T) \in \mathrm{Max}(T) \setminus \mathrm{Spec}(R)$, then $(1) - (3)$ hold.

Proof. Since $(R : T) \subsetneq (R : T)_\ell, (R : T)_r$, we conclude that $(R : T)_l T = T$ and $T(R : T)_r = T$, for $(R : T)$ is a maximal ideal of T. Hence we are done by the previous proposition and its proof. For the final part note that if $(R : T)$ is not a prime ideal of R, then by Lemma 2.1, it cannot be equal to either $(R : T)_\ell$ or $(R : T)_r$. □

In particular, if we strengthen the hypotheses of the previous corollary by assuming R is left/right Noetherian or Artinian, we obtain the following.

Corollary 4.3. *Let R be a maximal subring of a ring T and $(R : T) \in \mathrm{Max}(T)$. Then the following hold:*

1. *If R is a left Noetherian (respectively, Artinian) ring, then either T is a left Noetherian (respectively, Artinian) ring or $(R : T) = (R : T)_\ell$. In particular, $(R : T)_\ell \subseteq (R : T)_r$ (respectively, $(R : T) = (R : T)_\ell = (R : T)_r$).*
2. *If R is a right Noetherian (respectively, Artinian) ring, then either T is a right Noetherian (respectively, Artinian) ring or $(R : T) = (R : T)_r$. In particular, $(R : T)_r \subseteq (R : T)_\ell$ (respectively, $(R : T) = (R : T)_\ell = (R : T)_r$).*

Proof. For (1), note that either $(R:T)_\ell = (R:T) \subseteq (R:T)_r$ or $(R:T) \subsetneq (R:T)_\ell$. Hence, by Proposition 4.1, ${}_RT$ is finitely generated. Consequently, T is a left Noetherian (respectively, Artinian) R-module. It follows that T is a left Noetherian (respectively, Artinian) ring. Also note that, if R is a left Artinian ring, then by Lemma 2.1, $(R:T)_\ell$ and $(R:T)_r$ are maximal ideals of R. Thus in this case, $(R:T)_\ell = (R:T) \subseteq (R:T)_r$, a fortiori $(R:T) = (R:T)_\ell = (R:T)_r$. The proof of (2) is similar. □

Consequently, if R is an integrally closed maximal subring of a zero-dimensional ring T and $(R:T)$ is not a semiprime ideal of R, then Theorem 3.6 together with the previous corollary implies that T is a left/right Noetherian (respectively, Artinian) ring, whenever R is a left/right Noetherian (respectively, Artinian) ring.

For the next result, observe that in a simple ring T, the conductor ideal $(R:T)$ is zero for every maximal subring R of T. Hence, applying the preceding result yields the following conclusion.

Corollary 4.4. *Let R be a maximal subring of a simple ring T. Then the following hold:*

(1) *If R is a left Noetherian* (*respectively, Artinian*) *ring, then either T is a left Noetherian* (*respectively, Artinian*) *ring or $(R:T)_\ell = 0$* (*respectively, $(R:T)_\ell = (R:T)_r = 0$, $R = \mathbb{M}_n(D)$, and $T = \mathbb{M}_n(S)$, where D is a division ring, S is a simple ring, and D is a maximal subring of S*).

(2) *If R is a right Noetherian* (*respectively, Artinian*) *ring, then either T is a right Noetherian* (*respectively, Artinian*) *ring or $(R:T)_r = 0$* (*respectively, $(R:T)_\ell = (R:T)_r = 0$, $R = \mathbb{M}_n(D)$, and $T = \mathbb{M}_n(S)$, where D is a division ring, S is a simple ring, and D is a maximal subring of S*).

Proof. For (1), the Noetherian part is clear by the previous result. Hence assume that R is a left Artinian ring, if T is not a left Artinian ring, then by the previous corollary $(R:T)_\ell = 0$. Since $(R:T)_\ell$ is a prime ideal of R, we conclude that $R = \mathbb{M}_n(D)$, for some division ring D and $n \geq 1$ (hence R is a right Artinian too). Thus $T = \mathbb{M}_n(S)$, for a simple ring S, because R is a subring of T and T is a simple ring. Since R is a maximal subring of T, we deduce that D is a maximal subring of S. The proof of (2) is similar. □

Analogous to Proposition 2.13, we establish a relationship involving the centralizer of a left Noetherian maximal subring R of a ring T, the left Noetherian property of T, and the conductor ideals, as follows.

Proposition 4.5. *Let R be a left Noetherian maximal subring of a ring T. Then either $C_T(R) \subseteq R$ or T is a left Noetherian ring and $(R : T) = (R : T)_\ell = (R : T)_r$.*

Proof. Assume that $C_T(R)$ is not contained in R and $\alpha \in C_T(R) \setminus R$. Since α commutes with any element of R, we conclude that $R[\alpha]$ is a left Noetherian ring, by Hilbert Basis Theorem. Clearly $T = R[\alpha]$, for R is a maximal subring of T and $\alpha \in T \setminus R$. Hence T is a left Noetherian ring. Finally, if $(R : T)_\ell$ is not an ideal of T, then by Remark 2.14, we deduce that $R = \mathbb{I}_T((R : T)_\ell)$. Consequently, R contains $C_T(R)$, which is absurd. It follows that $(R : T)_\ell$ is an ideal of T. Similarly, $(R : T)_r$ is an ideal of T. Hence $(R : T) = (R : T)_\ell = (R : T)_r$ as desired. □

We now state the main result of this section.

Theorem 4.6. *Let R be a Noetherian (respectively, Artinian) maximal subring of a ring T and $(R : T) \neq 0$. Then the following hold:*

(1) *$T/\operatorname{l.ann}_T((R : T)_\ell)$ and $T/\operatorname{l.ann}_T((R : T))$, are left Noetherian (respectively, Artinian) rings.*

(2) *$T/\operatorname{r.ann}_T((R : T)_r)$ and $T/\operatorname{r.ann}_T((R : T))$, are right Noetherian (respectively, Artinian) rings.*

(3) *If T is semiprime, then $T/\operatorname{ann}_T((R : T))$ is a Noetherian (respectively, an Artinian) ring.*

(4) *If T is a prime ring, then T is finitely generated as left and right R-modules. In particular, T is a Noetherian (respectively, an Artinian) ring. Moreover, $(R : T)_\ell$ and $(R : T)_r$ are right and left primitive ideals of R, respectively.*

Proof. (1) Since R is a right Noetherian ring, there exist $x_1, \ldots, x_n \in (R : T)_\ell$ such that $(R : T)_\ell = x_1R + \cdots + x_nR$. Thus $\operatorname{l.ann}_T((R : T)_\ell) = \operatorname{l.ann}_T(x_1) \cap \cdots \cap \operatorname{l.ann}_T(x_n)$. Therefore $T/\operatorname{l.ann}_T((R : T)_\ell)$ embeds into $Tx_1 \times \cdots \times Tx_n$ as a left T-module, and as a left R-module too. Since $Tx_i \subseteq R$, we conclude that $T/\operatorname{l.ann}_T((R : T)_\ell)$ embeds into R^n as a left

R-module, and consequently $T/\operatorname{l.ann}_T((R:T)_\ell)$ is left Noetherian (respectively, Artinian), because R is left Noetherian (respectively, Artinian). Also note that, $(R:T) \subseteq (R:T)_\ell$ and a fortiori $\operatorname{l.ann}_T((R:T)_\ell) \subseteq \operatorname{l.ann}_T((R:T))$. This implies that $T/\operatorname{l.ann}_T((R:T))$ is left Noetherian (respectively, Artinian). Similarly (2) holds. For (3), first note that since T is a semiprime ring, we obtain that the left and right annihilators of $(R:T)$ are coincided. Thus by (1) and (2), we deduce that $T/\operatorname{ann}_T((R:T))$ is Noetherian (respectively, Artinian). Finally, for (4), since $0 \neq (R:T) \subseteq (R:T)_\ell$, we infer that $\operatorname{l.ann}_T((R:T)_\ell) = 0$. By the proof of (1), T is isomorphic to a left R-submodule of R^n. It follows that T is a finitely generated left R-module. Similarly, T is a finitely generated right R-module. This shows that T is Noetherian (respectively, Artinian). The final part is evident by Corollary 2.5. □

Note that in the previous theorem, the assumption $(R:T) \neq 0$ is not required for parts $(1) - (3)$. However, since the conclusions in the case $(R:T) = 0$ are trivial for these part, we include the condition $(R:T) \neq 0$ in the theorem statement for brevity and to focus on nontrivial case.

In [15] and [16], It was shown that if a finite ring R is a maximal subring of a ring T, then T must also be finite. In the following remark, we deduce a special case of this fact by the previous theorem.

Remark 4.7. Let T be a prime ring with a finite maximal subring R. If $(R:T)_\ell \neq 0$, then T is finite. To see this, note that by the proof of (4) in the previous theorem, T embeds into R^n and consequently, T is finite.

Also we have the following observations.

Proposition 4.8. *Let R be a left Noetherian (respectively, Artinian) maximal subring of a ring T. Assume that $(R:T)_\ell$ is a finitely generated as right ideal of R and $(R:T)$ contains a prime ideal Q of T. Then either $(R:T) = (R:T)_\ell \subseteq (R:T)_r$ (respectively, $(R:T) = (R:T)_\ell = (R:T)_r$) or T is a left Noetherian (respectively, Artinian) ring.*

Proof. Similar to the proof of the previous theorem, $T/\operatorname{l.ann}_T((R:T)_\ell)$ is a left Noetherian (respectively, Artinian) R-module. Since $\operatorname{l.ann}_T((R:T)_\ell)(R:T)_\ell = 0 \subseteq Q$ and Q is a prime ideal of T, we have two cases: either $(R:T)_\ell \subseteq Q$ or $\operatorname{l.ann}_T((R:T)_\ell) \subseteq Q$. If $(R:T)_\ell \subseteq Q$, then

$(R:T)_\ell \subseteq (R:T)$ and therefore $(R:T)_\ell = (R:T) \subseteq (R:T)_r$ (respectively, $(R:T)_\ell = (R:T) = (R:T)_r$, because R is a left Artinian ring and $(R:T)_\ell$ and $(R:T)_r$ are prime ideals of R, by Lemma 2.1). Hence assume that $\mathrm{l.ann}_T((R:T)_\ell) \subseteq Q$ and consequently $\mathrm{l.ann}_T((R:T)_\ell) \subseteq (R:T) \subseteq R$. Thus $\mathrm{l.ann}_T((R:T)_\ell)$ is a left Noetherian (respectively, Artinian) R-module. It follows that T is a left Noetherian (respectively, Artinian) R-module and a fortiori is a left Noetherian (respectively, Artinian) ring. □

Proposition 4.9. *Let T be a ring which is not prime and R be a Noetherian maximal subring of T with $(R:T) \neq 0$. If R contains a prime ideal Q of T, then either T is Noetherian or $Q = (R:T) = (R:T)_\ell$ or $Q = (R:T) = (R:T)_r$ is a minimal prime ideal of T (thus Q is unique). Moreover, if T is neither left Noetherian nor right Noetherian, then $Q = (R:T) = (R:T)_\ell = (R:T)_r$.*

Proof. First note that $Q \subseteq (R:T)$. Now we have two cases: (a) $Q \neq (R:T)$. This implies that $\mathrm{l.ann}_T((R:T))$ and $\mathrm{r.ann}_T((R:T))$ are contained in Q. By (1) and (2) of Theorem 4.6, we conclude that T/Q is Noetherian. Since $Q \subseteq R$, we deduce that T is Noetherian too. (b) Hence suppose that $Q = (R:T)$. Let $Q' \in \mathrm{Min}(T)$ and $Q' \subsetneq Q$, thus $Q' \subsetneq (R:T)$. By the first case we obtain that T is Noetherian and we are done. Consequently, $Q = (R:T)$ is a minimal prime ideal of T. Now assume that $(R:T)_\ell$ and $(R:T)_r$ are not contained in $Q = (R:T)$. Then we observe that $\mathrm{l.ann}_T((R:T)_\ell)$ and $\mathrm{r.ann}_T((R:T)_r)$ are contained in Q, because Q is a prime ideal. By (1) and (2) of Theorem 4.6, T/Q is a left/right Noetherian ring. This implies that T is a Noetherian ring, because $Q \subseteq R$. By a similar argument and using (1) and (2) of Theorem 4.6 and the fact that Q is prime, we conclude the final part. □

Corollary 4.10. *Let R be an Artinian maximal subring of a prime ring T. Then either $R \cong \mathbb{M}_n(D)$ for some division ring D (in particular, $T = \mathbb{M}_n(S)$, for some ring S, where D is a maximal subring of S) or $T \cong \mathbb{M}_n(D')$ for some division ring D'.*

Proof. If R is a prime ring, then clearly the first part of the statement of the theorem holds. Hence assume that R is not a prime ring. Consequently, $(R:T)_\ell \neq 0$ and a fortiori $\mathrm{l.ann}_T((R:T)_\ell) = 0$, because T is prime. Thus by Theorem 4.6(1), we conclude that T is a left Artinian ring and we are done (note T is prime). $\square$

Finally we conclude this paper by the following result.

Proposition 4.11. *Let T be a ring which is not prime and R be an Artinian maximal subring of T with $(R:T) \neq 0$. Then either T is Artinian or $\dim(T) = 0$ or R contains a unique (minimal) prime ideal of T, say Q, and $T/Q \cong \mathbb{M}_n(S)$, for some ring S, where S has a maximal subring D that is a division ring.*

Proof. Assume that T is not Artinian. We have two cases: (a) each prime ideal Q of T is not contained in R. Thus $R + Q = T$, because $Q \neq 0$ (note T is not prime) and R is a maximal subring of T. It follows that $R/(R \cap Q) \cong T/Q$, as rings. This shows that Q is a maximal ideal of T, that is, $\dim(T) = 0$. (b) Assume that there exists a prime Q of T such that $Q \subseteq R$. Consequently, $Q \subseteq (R:T) \subseteq (R:T)_\ell \cap (R:T)_r$. Now, if $\mathrm{l.ann}_T((R:T))$ is not contained in Q, we conclude that $(R:T)_\ell \subseteq Q \subseteq (R:T) \subseteq (R:T)_r$, because Q is a prime ideal of T. Since R is an Artinian ring and $(R:T)_\ell$, $(R:T)_r$ are prime ideals in R, we deduce that $(R:T)_\ell = Q = (R:T) = (R:T)_r$. Similarly if $\mathrm{r.ann}_T((R:T)_r)$ is not contained in Q, the previous equalities hold. Thus assume that Q contains $\mathrm{l.ann}_T((R:T)_\ell)$ and $\mathrm{r.ann}_T((R:T)_r)$. By (1) and (2) of Theorem 4.6, we deduce that T/Q is an Artinian ring. Since $Q \subseteq R$, we obtain that T is Artinian, which is absurd. Consequently, $Q = (R:T) = (R:T)_\ell = (R:T)_r$ is a maximal ideal of R. It follows that $R/Q \cong \mathbb{M}_n(D')$ for a division ring D' and a natural number n. Since R/Q is a maximal subring of T/Q, we conclude that $T/Q \cong \mathbb{M}_n(S)$, for some ring S, where S has a maximal subring $D \cong D'$, as ring. $\square$

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