



The conductor ideals of maximal subrings in non-commutative rings

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Abstract. Let R be a maximal subring of a ring T , and $(R : T)$, $(R : T)_\ell$ and $(R : T)_r$ denote the largest ideal, left ideal and right ideal of T that are contained in R , respectively. It is shown that both $(R : T)_\ell$ and $(R : T)_r$ are prime ideals of R , and $|\text{Min}_R((R : T))| \leq 2$. We prove that if T_R has a maximal submodule, then $(R : T)_\ell$ is a right primitive ideal of R . We investigate the conditions under which $(R : T)_r$ is a completely prime (right) ideal of R or of T . We show that $\text{Char}(R/(R : T)_\ell) = \text{Char}(R/(R : T)_r)$, and if $\text{Char}(T)$ is neither zero nor a prime number, then $(R : T) \neq 0$. When $|\text{Min}(R)| \geq 3$, both $(R : T)$ and $(R : T)_\ell(R : T)_r$ are nonzero ideals. Assuming R is integrally closed in T , we prove that $(R : T)_\ell$ and $(R : T)_r$ are prime one-sided ideals of T ; moreover $(R : T)$ is a semiprime ideal of T and either $(R : T)$ is a prime ideal of T or $(R : T) = (R : T)_\ell \cap (R : T)_r$ is a semiprime ideal of R . We observe that if $(R : T)_l T = T$, then T is a finitely generated left R -module and $(R : T)_\ell$ is a finitely generated right R -module which is also a right primitive ideal of R . Finally, we study the transfer of the Noetherian and the Artinian properties between R and T .

Keywords: Maximal subrings, conductor ideal, prime ideal, idealizer, integrally closed, finiteness condition.

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1 Introduction

1.1 Motivation In [11], Ferrand and Olivier studied minimal ring extensions of commutative rings. Note that if $R \subseteq T$ is a minimal extension of commutative rings (that is, R is a maximal subring of T), then the integral closure of R in T , say S , is a subring between R and T ; therefore by the minimality of the extension, we infer that either $S = R$ (that is, R is integrally closed in T) or $S = T$ (that is, T is integral over R , equivalently T is a finitely generated R -module). They proved that T is integral over R if and only if $(R : T) \in \text{Max}(R)$, see [11, Proposition 4.1]. Moreover, if R is integrally closed in T , then $(R : T)$ is a prime ideal of T , see [11, Lemma 3.2]. Hence, in all cases we deduce that $(R : T)$ is a prime ideal of R .

In this paper, motivated by these results, we study the conductor (left or right) ideals of minimal ring extensions in non-commutative rings. Specifically, if $R \subseteq T$ is a minimal ring extension of rings (that is, R is a maximal subring of T , where T is an arbitrary ring not necessarily commutative ring), then we study the properties of $(R : T)$, the largest ideal of T contained in R , $(R : T)_\ell$, the largest left ideal of T contained in R , and $(R : T)_r$, the largest right ideal of T contained in R . As we show, these are ideals of R (hence we use "maximal subrings" rather than "minimal ring extensions" throughout). In fact, we proved that both $(R : T)_\ell$ and $(R : T)_r$ are prime ideals of R .

It is interesting to know that each ring R can be realized as a maximal subring of some larger ring T , see [1, Theorem 3.7]. Note that if T is a ring and A be a one-sided ideal of T , then the idealizer of A in T is the largest subring of T , in which A is a two-sided ideal. More exactly, if A is a right ideal of T , then the *idealizer* of A is the subring $\mathbb{I}_T(A) := \{r \in T \mid rA \subseteq A\}$. By this definition we have the following from [1]:

Theorem 1.1. (1) [1, Theorem 4.1] *Let T be a ring and A be a maximal right/left ideal of T that is not a two-sided ideal of T . Then the idealizer of A is a maximal subring of T . In particular, every ring either has a maximal subring or is a quasi duo ring (that is, any maximal left/right ideal is two-sided).*

(2) [1, Proposition 4.2] *Let R be a maximal subring of a ring T that contains a maximal one-sided ideal A of T that is not a two-sided ideal of T . Then R is the idealizer of A in T .*

In [5, 6, 15, 16, 20], it is proved that if R is a finite maximal subring of a ring T , then T is also finite. In [3, Theorem 3.8], it is shown that if R is a maximal subring of a commutative ring T , then R is Artinian if and only if T is Artinian and integral over R ; this immediately implies the aforementioned finiteness results for commutative rings. It is clear that, if R is a maximal subring of a commutative ring T , then T is Noetherian whenever R is Noetherian, since $T = R[\alpha]$, for each $\alpha \in T \setminus R$ and use the Hilbert Basis Theorem. Motivated by these results in commutative case, we are interested to investigate the Artinian and the Noetherian properties in minimal ring extension of non-commutative rings too.

Finally, we refer the reader to [7–9, 12, 13, 24], for minimal ring extension of commutative rings and [10] for non-commutative case. Also, we refer the reader to [2–4, 23] for maximal subrings in commutative rings and [1] for maximal subrings of non-commutative rings.

1.2 Review of the results Assume that R is a maximal subring of a ring T . In Section 2, we obtain some basic fact about the conductor ideals of R . We prove that $(R : T)_\ell$ and $(R : T)_r$ are prime ideals of R . Moreover $\text{Min}_R((R : T)) \subseteq \{(R : T)_\ell, (R : T)_r\}$ and in fact $|\text{Min}_R((R : T))| = 1$ if and only if $(R : T)_\ell$ and $(R : T)_r$ are comparable, equivalently $(R : T)$ is prime ideal of R . We show that if T/R as right R -module has a maximal submodule, then $(R : T)_\ell$ is a right primitive ideal of R . Conversely, we observe that if either $(R : T)_\ell \in \text{Max}_r(R)$ or R is a right Artinian ring, then T/R has a maximal right R -submodule. We investigate conditions under which $(R : T)_r$ is a completely prime (right) ideal of T or of R . In particular, if $(R : T)_r$ is a completely prime right ideal of T , then $(R : T)_r$ is a completely prime ideal of R (that is, $R/(R : T)_r$ is a domain) and $T/(R : T)_r$ is a torsion-free left $R/(R : T)_r$ -module; conversely if $T/(R : T)_r$ is a torsion-free left $R/(R : T)_r$ -module, then either $(R : T)_r$ is an ideal of T or $(R : T)_r$ is a completely prime right ideal of T and R is the idealizer of $(R : T)_r$ in T . Moreover, if R is a right Artinian ring and $(R : T)_r$ is a completely prime right ideal of T , then either R/P or T/P is a division ring. We prove that the ring $\text{End}((T/R)_R)$, and therefore the ring $\text{End}_{\mathbb{Z}}(T/R)$ contain a copy of $R/(R : T)_r$. Consequently, the ring $\text{End}_{\mathbb{Z}}(T/R)$ contains a copies of $R/(R : T)_r$ and $R/(R : T)_\ell$. In particular, the characteristic of the ring $\text{End}_{\mathbb{Z}}(T/R)$ is either zero or

a prime number and thus $\text{Char}(R/(R : T)_\ell) = \text{Char}(R/(R : T)_r)$. We show that if $\text{Char}(T)$ is not a prime number, then either there exists a prime number p such that $pT \subseteq (R : T)$ (and therefore $(R : T) \neq 0$) or $\text{Char}(T) = \text{Char}(\text{End}_{\mathbb{Z}}(T/R)) = 0$. In Section 3, we show that if R is (2-)integrally closed in T , then $(R : T)_\ell$ and $(R : T)_r$ are prime one-sided ideals of T . In particular, in this case, $(R : T)$ is a semiprime ideal of T and either $(R : T)$ is a prime ideal of T or a semiprime ideal of R . Finally in Section 4, we consider some finiteness conditions on the extension $R \subseteq T$. We show that if $(R : T)_l T = T$, then T is a finitely generated left R -module, $(R : T)_\ell$ is a finitely generated right ideal of R and R is the idealizer of $(R : T)_\ell$. In particular, if $(R : T) \in \text{Max}(T) \setminus \text{Spec}(R)$, then ${}_R T$, T_R , $(R : T)_\ell$ as right ideal of R and $(R : T)_r$ as left ideal of R , are all finitely generated. Consequently, if $(R : T) \in \text{Max}(T)$ and R is left Noetherian (respectively, Artinian), then either $(R : T) = (R : T)_\ell$ or T is a right Noetherian (respectively, Artinian) ring. We show that if R is a Noetherian (respectively, an Artinian) ring and $(R : T) \neq 0$, then $T/\text{l.ann}_T((R : T)_\ell)$ and $T/\text{r.ann}_T((R : T)_r)$ are left and right Noetherian (respectively, Artinian) rings, respectively. Moreover, if in addition, T is semiprime, then $T/\text{ann}_T((R : T))$ is a Noetherian (respectively, an Artinian) ring. In particular, if T is a prime ring, then T is finitely generated as left and right R -modules and consequently T is a Noetherian (respectively, an Artinian) ring. Finally, if R is a Noetherian ring, T is not a prime ring and R contains a prime ideal Q of T , then either T is Noetherian or $Q = (R : T)$ (therefore Q is unique), Q is a minimal prime ideal of T and either $Q = (R : T)_\ell$ or $Q = (R : T)_r$.

1.3 Notations and definitions All rings in this paper are unital with $1 \neq 0$. All subrings, modules and homomorphisms are also unital. If $R \subsetneq T$ is a ring extension and there exists no other subring properly between R and T , then R is called a maximal subring of T , or the extension $R \subseteq T$ is called a minimal ring extension. If T is a ring, R and S are subrings of T , then clearly T is a (R, S) -bimodule and therefore we can consider (R, S) -subbimodules of T . In particular, if $t \in T$, then the (R, S) -subbimodule of T which is generated by t is denoted by $RtS = \{\sum_{i=1}^n r_i t s_i \mid n \geq 1, r_i \in R, s_i \in S, 1 \leq i \leq n\}$. It is clear that if $R \subseteq S$ and $t \in S$, then $R + RtS$ and $R + StR$ are also subrings of T that contain R . In particular, for each $t \in T$,

the subrings $R + RtT$ and $R + TtR$ of T contain R . More generally, if I is left (respectively, right) ideal of T , then $R + IR$ (respectively, $R + RI$) is a subring of T that contains R . If T is a ring, I is an ideal of T and M is a left (respectively, right) T -module, then $\text{Min}_T(I)$, $\text{Max}_r(T)$, $\text{Max}_\ell(T)$, $\text{Max}(T)$, $\text{Spec}(T)$, $\text{l. ann}_T(M)$ (respectively, $\text{r. ann}_T(M)$) and $\text{ann}_T(I)$ denote the set of all minimal prime ideals of I in T , the set of all maximal right ideals of T , the set of all maximal left ideals of T , the set of all maximal ideals of T , the set of all prime ideals of T , the left annihilator of M in T (respectively, the right annihilator of M in T) and the annihilator of I in T (that is, $\text{ann}_T(I) = \text{l. ann}_T(I) \cap \text{r. ann}_T(I)$), respectively. We use $\text{Min}(T)$ for $\text{Min}_T(0)$. $J(T)$ denotes the Jacobson radical of a ring T and for an ideal I of T , we denote the prime radical of I by $\text{rad}_T(I)$. The characteristic of a ring T is denoted by $\text{Char}(T)$. If T is a ring, then $\text{dim}(T)$ denotes the classical Krull dimension of T (that is, the supremum of the lengths of all chains of prime ideals of T). If X is a subset of a ring T , then $C_T(X)$ is the centralizer of X in T , in particular, $C(T) = C_T(T)$ is the center of T . If T is a ring, then $\mathbb{M}_n(T)$ denote the ring of all $n \times n$ square matrices over T . If M is a left (respectively, right) module over a ring T , then the ring of all T -module homomorphisms of M is denoted by $\text{End}_T(M)$ (respectively, $\text{End}(M_T)$); when $T = \mathbb{Z}$ (that is, M is an abelian group), then we use $\text{End}_{\mathbb{Z}}(M)$. A ring T is called left (respectively, right) quasi duo if each maximal left (respectively, right) ideal of T is a two-sided ideal of T , see [19]. T is called quasi duo if T is left and right quasi duo ring. A ring T is called left (respectively, right) duo if each left (respectively, right) ideal of T is two-sided. Similarly duo rings are defined. If R is a subring of a ring T and $t \in T$, then we say that t is left (respectively, right) n -integral over R , if t is a root of a left (respectively, right) monic polynomial of degree n over R ($n \geq 1$). R is called left (respectively, right) n -integrally closed in T , if every left (respectively, right) n -integral element of T over R belongs to R . R is called n -integrally closed in T whenever R is left and right n -integrally closed in T . R is called left (respectively, right) integrally closed in T , if R is left (respectively, right) n -integrally closed in T , for each n . R is integrally closed in T , if R is left and right integrally closed in T . For other notations and definitions we refer the reader to [14, 17, 18, 22, 26].

2 The conductor ideals of maximal subrings

Let $R \subseteq T$ be a ring extension. We define $(R : T) := \{x \in T \mid TxT \subseteq R\}$, $(R : T)_\ell := \{x \in T \mid Tx \subseteq R\}$ and $(R : T)_r := \{x \in T \mid xT \subseteq R\}$. In other words, $(R : T)$ is the largest ideal of T contained in R , $(R : T)_\ell$ (respectively, $(R : T)_r$) is the largest left (respectively, right) ideal of T contained in R . It is not hard to see that $(R : T)_\ell = \text{r.ann}_R(T/R)$ and therefore $(R : T)_\ell$ is an ideal of R ; similarly, $(R : T)_r = \text{l.ann}_R(T/R)$ is an ideal of R (note, clearly T/R is a left/right R -module). Hence $(R : T)_\ell$, $(R : T)_r$ and $(R : T)$ are ideals of R , called the left conductor ideal, the right conductor ideal and the conductor ideal of the extension $R \subseteq T$ (or of T in R), respectively. It is clear that $(R : T)_\ell(R : T)_r \subseteq (R : T) \subseteq (R : T)_\ell \cap (R : T)_r$. In particular, if $R \neq T$, then $(R : T)_\ell + (R : T)_r \subseteq R$ and therefore $(R : T)_\ell + (R : T)_r \subsetneq T$. Now we want to prove some generalization of the results in [11]. Let us first review some facts from [11]. If T is a commutative ring and R is a maximal subring of R , then $(R : T)$ is a prime ideal of R . In fact, since R' , the integral closure of R in T , is a ring between R and T , then by maximality of R either $R' = T$, that is, T is integral over R (equivalently, T is a finitely generated R -module) or $R' = R$, that is, R is integrally closed in T . Moreover, T is integral over R if and only if $(R : T) \in \text{Max}(R)$ (and thus is a prime ideal of R); and if R is integrally closed in T , then $(R : T)$ is a prime ideal of T ; consequently, is a prime ideal of R (furthermore, for any $x, y \in T$, if $xy \in R$, then $x \in R$ or $y \in R$). Hence, in all cases, $(R : T)$ is a prime ideal of R . Now the following is in order.

Lemma 2.1. *Let R be a maximal subring of a ring T . Then $(R : T)_\ell$ and $(R : T)_r$ are prime ideals of R .*

Proof. Let $a, b \in R$ and $aRb \subseteq (R : T)_\ell$. Thus $TaRb \subseteq R$. Now assume that $a \notin (R : T)_\ell$, that is, $Ta \not\subseteq R$. Thus $TaR \not\subseteq R$. Since R is a maximal subring of T and $R + TaR$ is a subring of T that properly contains R , we conclude that $R + TaR = T$. Thus $Tb = Rb + TaRb \subseteq R$, that is, $b \in (R : T)_\ell$. Hence $(R : T)_\ell$ is a prime ideal of R . Similarly, $(R : T)_r$ is a prime ideal of R . \square

From the previous result, we make the following observation regarding conductor ideal of maximal subrings.

Proposition 2.2. *Let R be a maximal subring of a ring T . Then the following hold:*

- (1) $\text{rad}_R((R : T)) = (R : T)_\ell \cap (R : T)_r$.
- (2) $(R : T)$ is a prime ideal of R if and only if either $(R : T) = (R : T)_\ell$ or $(R : T) = (R : T)_r$, equivalently $(R : T)_\ell$ and $(R : T)_r$ are comparable.
- (3) $|\text{Min}_R((R : T))| \leq 2$. In fact, $|\text{Min}_R((R : T))| = 1$ if and only if $(R : T)_\ell$ and $(R : T)_r$ are comparable. Otherwise, $\text{Min}_R((R : T)) = \{(R : T)_\ell, (R : T)_r\}$ and either $(R : T)$ is a prime ideal of T or there exist Q_l and Q_r in $\text{Min}_T((R : T))$ such that $Q_l \cap R = (R : T)_\ell$, $Q_r \cap R = (R : T)_r$, $R/(R : T)_\ell \cong T/Q_l$ and $R/(R : T)_r \cong T/Q_r$;
- (4) $(R : T)$ is a semiprime ideal of R if and only if $(R : T) = (R : T)_\ell \cap (R : T)_r$.
- (5) If R is a zero dimensional ring, then $(R : T)_\ell$ and $(R : T)_r$ are maximal ideals of R . Hence either $(R : T) = (R : T)_\ell = (R : T)_r$ or $(R : T)_\ell + (R : T)_r = R$.
- (6) $(R : T) = \text{r. ann}_T(T/(R : T)_r) = \text{l. ann}_T(T/(R : T)_\ell) = \text{l. ann}_R(T/(R : T)_\ell) = \text{r. ann}_R(T/(R : T)_r)$. In particular, if $(R : T)_\ell \in \text{Max}_\ell(T)$ (respectively, $(R : T)_r \in \text{Max}_r(T)$), then $(R : T)$ is a left (respectively, right) primitive ideal of T .
- (7) If $(R : T) \in \text{Max}(R)$, then $(R : T) = (R : T)_\ell = (R : T)_r$.
- (8) If $J(R)$ is nilpotent (in particular, if R is a one-sided Artinian ring), then $J(R) \subseteq (R : T)_\ell \cap (R : T)_r$ and $J(R)^2 \subseteq (R : T)$.
- (9) If $\text{Nil}^*(R)$ is nilpotent (in particular, if R is a one-sided Noetherian ring), then $\text{Nil}^*(R) \subseteq (R : T)_\ell \cap (R : T)_r$ and $(\text{Nil}^*(R))^2 \subseteq (R : T)$.

Proof. (1) Since $((R : T)_\ell \cap (R : T)_r)^2 \subseteq (R : T)_\ell(R : T)_r \subseteq (R : T) \subseteq (R : T)_\ell \cap (R : T)_r$ and $(R : T)_\ell, (R : T)_r$ are prime ideals of R , we conclude that $\text{rad}_R((R : T)) = (R : T)_\ell \cap (R : T)_r$.

(2) The first part of (2) is evident by (1). It is clear that if either $(R : T) = (R : T)_\ell$ or $(R : T) = (R : T)_r$, then $(R : T)_\ell$ and $(R : T)_r$ are comparable. Conversely, suppose that $(R : T)_\ell$ and $(R : T)_r$ are comparable. For instance, let $(R : T)_\ell \subseteq (R : T)_r$. Let $x \in (R : T)_\ell$, therefore $Tx \subseteq (R : T)_\ell \subseteq (R : T)_r$. Thus $TxT \subseteq R$. Hence $x \in (R : T)$. Consequently, $(R : T) = (R : T)_\ell$.

(3) By the proof of (1) note that for each prime ideal Q of R that contains $(R : T)$, we have $(R : T)_\ell \subseteq Q$ or $(R : T)_r \subseteq Q$. Therefore

$\text{Min}_R((R : T)) \subseteq \{(R : T)_\ell, (R : T)_r\}$. Hence by (1), we obtain that $|\text{Min}_R((R : T))| = 1$ if and only if $(R : T)_\ell$ and $(R : T)_r$ are comparable. For the next part of (3), assume that $(R : T)_\ell$ and $(R : T)_r$ are incomparable and therefore $\text{Min}_R((R : T)) = \{(R : T)_\ell, (R : T)_r\}$. Note that if $A \subseteq B$ is a minimal ring extension with $(A : B) = 0$, and P is a minimal prime ideal of A , then either B is a prime ring or P is a contraction of a minimal prime ideal of B . To see this, note that $A \setminus P$, is a m -system in A and therefore in B , thus there exists a prime ideal Q of B such that $Q \cap A \subseteq P$. If $Q = 0$, then B is prime, otherwise $A + Q = B$, as A is a maximal subring of B and $(A : B) = 0$. Thus $A/(A \cap Q) \cong B/Q$ as rings. Hence $A \cap Q$ is a prime ideal of A and then $Q \cap A = P$, because P is minimal. Clearly we may assume that Q is a minimal prime of B . Applying this fact to the minimal ring extension $R/(R : T) \subseteq T/(R : T)$, we deduce that either $(R : T) \in \text{Spec}(T)$ or there exist minimal prime ideals Q_l and Q_r of $\text{Min}_T((R : T))$ such that $Q_l \cap R = (R : T)_\ell$ and $Q_r \cap R = (R : T)_r$. If $Q_l \subseteq R$, then $(R : T)_\ell \subseteq Q_l \subseteq (R : T) \subseteq (R : T)_r$ which is absurd. Thus Q_l , and similarly Q_r , are not contained in R . Hence by maximality of R we infer that $R + Q_l = T = R + Q_r$. Consequently, we have the ring isomorphisms $R/(R : T)_\ell \cong T/Q_l$ and $R/(R : T)_r \cong T/Q_r$.

(4) is evident by (1). (5) is clear.

(6) Let $x \in T$ (or $x \in R$), then $x \in (R : T) \iff TxT \subseteq R \iff xT \subseteq (R : T)_\ell \iff x \in \text{l.ann}_T(T/(R : T)_\ell)$ (or $x \in \text{l.ann}_R(T/(R : T)_\ell)$). Hence $(R : T) = \text{l.ann}_T(T/(R : T)_\ell) = \text{l.ann}_R(T/(R : T)_\ell)$. The proof of the other equalities of (6) are similar. The final part of (6) is obvious, because whenever $(R : T)_\ell \in \text{Max}_\ell(T)$, then $T/(R : T)_\ell$ is a simple left T -module and then $\text{l.ann}_T(T/(R : T)_\ell) = (R : T)$ is a left primitive ideal of T , and consequently a prime ideal of T .

(7) is clear, because $(R : T)_\ell$ and $(R : T)_r$ are proper ideals of R that contains $(R : T)$.

(8) and (9) Let I be a nilpotent one-sided ideal of R , then clearly $I \subseteq (R : T)_\ell \cap (R : T)_r$, for $(R : T)_\ell$ and $(R : T)_r$ are prime ideals of R . Consequently, $I^2 \subseteq (R : T)_\ell(R : T)_r \subseteq (R : T)$. \square

Example 2.3. Let T be a ring and M be a maximal left ideal of T that is not an ideal of T . If R is the idealizer of M in T , then by Theorem 1.1(1), R is a maximal subring of T and clearly $(R : T)_\ell = M$. Moreover, the following hold:

1. T/R is a simple left R -module.
2. $R \subseteq T$ is a left integral extension.
3. R/M is a division ring.
4. $(R : T)$ is a left primitive ideal of T .
5. $(R : T)_r$ is a left primitive ideal of R .

For (1) note that since M is a maximal left ideal of T which is not an ideal of T , for each $x \in T \setminus R$ we infer that $M + Mx = T$ and therefore $R + Rx = T$. This shows that T/R is a simple left R -module. Hence by $(R : T)_r = \text{l.ann}_R(T/R)$ we deduce (5). Also note that $x^2 \in T = R + Rx$ implies that $R \subseteq T$ is a left integral extension, hence (2) holds. (3) is obvious, because T/M is a simple left T -module and we have the rings isomorphism $R/M \cong \text{End}_T(T/M)$, by [21, Lemma 1.3]. Finally, for (4) note that by Proposition 2.2(6), $(R : T) = \text{l.ann}_T(T/M)$ and consequently $(R : T)$ is a left primitive ideal of T .

It is easy to see that in the previous example, the inclusion $(R : T) \subsetneq (R : T)_\ell$ holds. Consequently, $(R : T)_\ell \not\subseteq (R : T)_r$. In the next example, we present a maximal subring R of a ring T such that $(R : T)_\ell \neq (R : T)_r$ and $(R : T) \neq (R : T)_\ell \cap (R : T)_r$.

Example 2.4. Let D be a division ring and let $T = \mathbb{M}_2(D)$. It is easy to see that $R = \begin{pmatrix} D & 0 \\ D & D \end{pmatrix}$ is a maximal subring of T . One can check that $(R : T)_\ell = \begin{pmatrix} D & 0 \\ D & 0 \end{pmatrix}$ and $(R : T)_r = \begin{pmatrix} 0 & 0 \\ D & D \end{pmatrix}$. Hence $(R : T)_\ell \cap (R : T)_r = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \neq (R : T) = 0$. Also note that $(R : T) = 0$ is a prime (in fact maximal) ideal of T , but is not a prime ideal in R .

Now we have the following generalization of [11, Proposition 4.1].

Corollary 2.5. *Let R be a maximal subring of a ring T . If T/R as right R -module has a maximal submodule, then $(R : T)_\ell$ is a right primitive ideal of R . In particular, if T/R (that is, T_R) is a finitely generated right R -module, then $(R : T)_\ell$ is a right primitive ideal of R .*

Proof. First we show that, if M is a proper right R -submodule of T that contains R , then $\text{r.ann}_R(T/M) = (R : T)_\ell$. Since $R \subseteq M$, we conclude that $(R : T)_\ell \subseteq \text{r.ann}_R(T/M)$. Now assume that $x \in \text{r.ann}_R(T/M)$, but $x \notin (R : T)_\ell$. Thus $Tx \not\subseteq R$ and then $R + TxR = R$, because R is a maximal subring of T . Since $Tx \subseteq M$ and M is a right R -submodule of T , we deduce that $TxR \subseteq M$. Thus $T = R + TxR \subseteq M$, which is absurd. Thus $\text{r.ann}_R(T/M) \subseteq (R : T)_\ell$ and therefore the equality holds. Now by the assumption, let M/R be a maximal right R -submodule of T/R , then M is a maximal right R -submodule of T and therefore T/M is a simple right R -module. Consequently, $\text{r.ann}_R(T/M)$ is a right primitive ideal of R . Hence by the first part of the proof we are done. The final part is evident. \square

Remark 2.6. If T is a commutative ring and R is a maximal subring of T , then the following are equivalent:

1. T is integral over R (that is, T is a finitely generated R -module).
2. $(R : T) \in \text{Max}(R)$.
3. T has a maximal R -submodule that contains R .
4. T/R is a semisimple $R/(R : T)$ -module.

To see this, note that (1) and (2) are equivalent by [11, Proposition 4.1]. Clearly, (3) follows from (1); conversely (3) implies (2), by Corollary 2.5, and the fact that in a commutative ring an ideal is maximal if and only if it is primitive. Now assume that (2) holds. Thus T/R is a nonzero $R/(R : T)$ -module. Since $R/(R : T)$ is a field, we obtain that T/R is a semisimple $R/(R : T)$ -submodule, consequently (4) holds. Finally, suppose that T/R is a semisimple $R/(R : T)$ -module, then T/R has a maximal R -submodule and hence (3) holds.

In the next two results we see that the converse of Corollary 2.5, holds (similar to the previous remark) under certain conditions.

Corollary 2.7. *Let R be a maximal subring of a ring T . If $(R : T)_\ell \in \text{Max}_r(R)$, then T has a maximal right R -submodule that contains R .*

Proof. Since $(R : T)_\ell \in \text{Max}_r(R)$, we conclude that $R/(R : T)_\ell$ is a division ring. Clearly, T/R is a nonzero right $R/(R : T)_\ell$ -module, and therefore has a maximal submodule. \square

In [3, Theorem 3.8], it is proved that for a maximal subring R of a commutative ring T , R is Artinian if and only if T is Artinian and integral over R . A key step in the proof of this result, by Remark 2.6, relies on the fact that in this setting, $(R : T)$ is a maximal ideal of R . This leads us to the following natural generalization.

Proposition 2.8. *Let R be a right Artinian ring that is a maximal subring of a ring T . Then $(R : T)_\ell$ and $(R : T)_r$ are maximal ideals of R . T/R as right $R/(R : T)_\ell$ -modules (respectively, $R/(R : T)_r$) is isotypic semisimple (and therefore has a maximal submodule). In particular, if additionally R is local, then $(R : T) = (R : T)_\ell = (R : T)_r$.*

Proof. By Lemma 2.1, $P := (R : T)_\ell$ is a prime ideal of R and since R is a right Artinian ring we conclude that $R/P \cong \mathbb{M}_n(D)$, for some division ring D and some natural number n . Hence R/P is a simple ring and consequently P is a maximal ideal of R . Also note that T/R is a nonzero right R/P -module. Thus T/R is a semisimple R/P -module. Since $R/P \cong \mathbb{M}_n(D)$, it follows that each simple component of T/R is isomorphic to the other one. The proof for $(R : T)_r$ is similar. The final part is evident. \square

Next, we investigate conditions under which the conductor ideals of a maximal subring R of a ring T are completely prime (right/left) ideals in either R or T . We recall that a proper right ideal P of a ring S is said to be completely prime right ideal, if for all $a, b \in S$ with $aP \subseteq P$ and $ab \in P$, we have $a \in P$ or $b \in P$, see [25]. A prime ideal of a ring S is called completely prime if S/P is a domain.

Proposition 2.9. *Let R be a maximal subring of a ring T and $P = (R : T)_r$. If P is a completely prime right ideal of T , then R/P is a domain (that is, P is a completely prime ideal of R) and T/P is a torsion-free left R/P -module. Conversely, if P is a completely prime ideal of R and T/P is a torsion-free left R/P -module, then either P is an ideal of T (and therefore $(R : T) = P$) or P is a completely prime right ideal of T and $R = \mathbb{I}_T(P)$.*

Proof. First assume that P is a completely prime right ideal of T . Let $a, b \in R$ and $ab \in P$. Since P is an ideal of R , we conclude that $aP \subseteq P$. Therefore $a \in P$ or $b \in P$, because P is a completely prime right ideal of T . Thus R/P is a domain. Since $P = (R : T)_r$ is a right ideal of T and P

is an ideal of R (that is, $PT \subseteq R$ and therefore $P(T/R) = 0$), we deduce that T/P is a left R/P -module. Now we show that T/P is a torsion-free left R/P -module. Suppose that $(r+P)(t+P) = 0$, where $r \in R$ and $t \in T$. This implies that $rt \in P$ and $rP \subseteq P$. Since P is a completely prime right ideal of T , we infer that either $r \in P$ or $t \in P$. Consequently, T/P is a torsion-free left R/P -module. Conversely, suppose that R/P is a domain and T/P is a torsion-free left R/P -module. Let $a, b \in T$ such that $aP \subseteq P$ and $ab \in P$. We have two cases, either P is an ideal of T (and therefore $P = (R : T)$) or P is not an ideal of T . In the latter case, since P is an ideal of R and R is a maximal subring of T , we obtain that $R = \mathbb{I}_T(P)$. It follows that $a \in R$ and since $ab \in P$, we have $(a+P)(b+P) = 0$. So $a \in P$ or $b \in P$, because T/P is a torsion-free left R/P -module. Consequently, P is a completely prime right ideal of T . \square

We obtain the following quick corollaries now.

Corollary 2.10. *Let R be a maximal subring of a ring T and $P = (R : T)_r \in \text{Max}_r(T)$. Then P is a completely prime ideal of R (that is, R/P is a domain) and T/P is a torsion-free left R/P -module.*

Proof. Note that by [25, Corollary 2.10(A)], P is a completely prime right ideal of T and therefore we are done by the previous proposition. \square

Assuming additionally that T is a right Artinian ring in the first part of Proposition 2.9, yields the following stronger conclusion.

Corollary 2.11. *Let R be a maximal subring of a right Artinian ring T and $P = (R : T)_r$ be a completely prime right ideal of T . The either T/P or R/P is a division ring.*

Proof. We have two cases, either P is an ideal of T or is not. First assume that P is an ideal of T . By [25, Corollary 2.10(B)], we conclude that $P \in \text{Max}_r(T)$. So T/P is a division ring. If P is not an ideal of T , then $R = \mathbb{I}_T(P)$, because R is a maximal subring of T and P is an ideal of R . Again by [25, Corollary 2.10(B)], $P \in \text{Max}_r(R)$ and hence R/P is a division ring. \square

When T is a (left/right) duo ring and R is a maximal subring of T , then $(R : T)$ is a completely prime ideal of R . This follows directly from our next result.

Proposition 2.12. *Let R be a maximal subring of a left duo ring T . Then $(R : T) = (R : T)_\ell$ is a completely prime ideal of R . Moreover, if T is a duo ring, then $(R : T) = (R : T)_\ell = (R : T)_r$ is a completely prime ideal of R .*

Proof. Since T is a left duo ring, we conclude that $(R : T) = (R : T)_\ell$. Now assume that $a, b \in R$ such that $ab \in (R : T)$ and $a \notin (R : T) = (R : T)_\ell$. Therefore $Tab \subseteq R$ and $Ta \not\subseteq R$. Since R is a maximal subring of T and Ta is an ideal of T , we infer that $R + Ta = T$. Hence $Tb = (R + Ta)b = Rb + Tab \subseteq R$, that is, $b \in (R : T) = (R : T)_\ell$. Consequently, $(R : T) = (R : T)_\ell$ is a completely prime ideal of R . The final part is evident. \square

The next proposition establishes a relationship between the centralizer of a maximal subring R of a ring T and the conductor ideals.

Proposition 2.13. *Let R be a maximal subring of a ring T . One of the following holds:*

- (1) $C_T(R) \subseteq R$. In particular, R contains the center of T .
- (2) $(R : T) = (R : T)_\ell = (R : T)_r$, and there exists $\alpha \in T$ such that $T = R[\alpha]$, where $\alpha r = r\alpha$, for each $r \in R$.

Proof. Assume that R does not contain $C_T(R)$. If $R = \mathbb{I}_T((R : T)_r)$, then clearly $C_T(R) \subseteq C_T((R : T)_r) \subseteq \mathbb{I}_T((R : T)_r) = R$, which is absurd. Thus $(R : T)_r$ is an ideal of T . Similarly, $(R : T)_\ell$ is an ideal of T and hence the first part of (2) holds. The second part is evident for $C_T(R) \not\subseteq R$. \square

As seen in Proposition 2.9, and in the proofs of Corollary 2.11, and the preceding proposition, a maximal subring R of a ring T can be expressed as the idealizer of a one-sided ideal of T . In the next remark we precisely determine when R is of the form of an idealizer of a one-sided ideal of T .

Remark 2.14. Let R be a maximal subring of a ring T . Then exactly one of the following holds:

1. $(R : T) = (R : T)_\ell \subsetneq (R : T)_r$ and $R = \mathbb{I}_T((R : T)_r)$.
2. $(R : T) = (R : T)_r \subsetneq (R : T)_\ell$ and $R = \mathbb{I}_T((R : T)_\ell)$.
3. $(R : T) \subsetneq (R : T)_r, (R : T)_\ell$ and $R = \mathbb{I}_T((R : T)_r) = \mathbb{I}_T((R : T)_\ell)$.
4. $(R : T) = (R : T)_\ell = (R : T)_r$.

Moreover, if $(R : T)$ is not a prime ideal of R , then (3) holds. To see this, note that $(R : T) \subseteq (R : T)_\ell \cap (R : T)_r$. Hence, if either $(R : T)_\ell$ or $(R : T)_r$ is an ideal of T , then we obtain that $(R : T) = (R : T)_\ell$ or $(R : T) = (R : T)_r$. For instance, if $(R : T)_\ell$ is an ideal of T , but $(R : T)_r$ is not an ideal of T , then $(R : T) = (R : T)_\ell$ and since $(R : T)_r$ is an ideal of R , we conclude that $R = \mathbb{I}_T((R : T)_r)$, because R is a maximal subring of T that is contained in the proper subring $\mathbb{I}_T((R : T)_r)$ of T . Thus (1) holds. By a similar argument we see (2). If $(R : T)_\ell$ and $(R : T)_r$ are ideals of T , then clearly (4) holds. Finally, assume that $(R : T)_\ell$ and $(R : T)_r$ are not ideals of T . Since these are ideals of R and R is a maximal subring of T , we deduce that $R = \mathbb{I}_T((R : T)_\ell) = \mathbb{I}_T((R : T)_r)$. The final part is evident by Lemma 2.1.

Remark 2.15. Let T be a ring and A a right ideal of T . Then the map $\phi : \mathbb{I}(A)/A \rightarrow \text{End}((T/A)_T)$ defined by $\phi(r + A) = f_{r+A}$ where $f_{r+A}(x + A) = rx + A$, for each $r \in \mathbb{I}(A)$ and for all $x \in T$, is a ring isomorphism, see [21, Lemma 1.3]. In particular, if R is a maximal subring of T and $A = (R : T)_r$, then either $R/A \cong \text{End}((T/A)_T)$ (and hence $\text{End}((T/A)_T)$ is a prime ring) or $\phi(R/A)$ is a maximal subring of $\text{End}((T/A)_T) \cong T/A$.

As shown in the preceding remarks, there is a closed relationship between the idealizer of a one-sided ideal A of a ring T and the endomorphism ring of T/A . In subsequent results of this section, we focus on endomorphism rings to derive further properties of maximal subrings. Specifically, we will explore algebraic connections between the quotient rings of a maximal subring R of a ring T with respect to the conductor ideals, and the endomorphism rings of T/R as left/right R -modules and as a \mathbb{Z} -module. First we require the following lemma, which is analogous to Remark 2.15.

Lemma 2.16. *Let R be a subring of a ring T and set $A = (R : T)_r$. Then the map $\psi : R/A \rightarrow \text{End}((T/R)_R)$, defined by $\psi(r + A) = g_{r+A}$ where $g_{r+A}(t + R) = rt + R$, for each $r \in R$ and for all $t \in T$, is a ring monomorphism. Consequently, up to ring isomorphism, R/A is a subring of $\text{End}((T/R)_R)$. In particular, up to ring isomorphism, R/A is a subring of $\text{End}_{\mathbb{Z}}(T/R)$.*

Proof. The proof of ψ is a ring homomorphism follows the same reasoning as [21, Lemma 1.3]. It remains to show that $\text{Ker}(\psi) = 0$. Note that $r + A \in$

$\text{Ker}(\psi) \iff g_{r+A} = 0 \iff g_{r+A}(t + R) = 0$ for each $t \in T$, $\iff rt + R = 0$, for each $t \in T$, that is, $rT + R = 0 \iff rT \subseteq R \iff r \in A$. \square

We recall that since we adopt the convention of writing maps on the left, the statement of Remark 2.15 (for left ideals) and Lemma 2.16 (for $(R : T)_\ell$) require appropriate modification. Specifically, if T is a ring and B is a left ideal of T , then $\mathbb{I}(B)/B \cong (\text{End}_T(T/B))^{\text{op}}$, where the superscript "op" denote the opposite ring. Moreover, if R is a subring of T and $B = (R : T)_\ell$, then the ring R/B embeds into $(\text{End}_R(T/R))^{\text{op}}$. Therefore, $(R/B)^{\text{op}}$ embeds into the ring $\text{End}_R(T/R)$, and hence into the ring $\text{End}_{\mathbb{Z}}(T/R)$. It is clear that for any ring S , we have $\text{Char}(S) = \text{Char}(S^{\text{op}})$, and S is a prime ring if and only if S^{op} is a prime ring. This leads us to the following.

Proposition 2.17. *Let R be a maximal subring of a ring T . Then $\text{Char}(\text{End}_{\mathbb{Z}}(T/R))$ is either 0 or is a prime number. Moreover, $\text{Char}(R/(R : T)_\ell) = \text{Char}(R/(R : T)_r) = \text{Char}(R/(R : T))$.*

Proof. By Lemma 2.1, Lemma 2.16 and preceding comments, $(R/(R : T)_\ell)^{\text{op}}$ and $R/(R : T)_r$ are prime rings, that are subrings of $\text{End}_{\mathbb{Z}}(T/R)$. Hence, $\text{Char}(R/(R : T)_\ell) = \text{Char}(\text{End}_{\mathbb{Z}}(T/R)) = \text{Char}(R/(R : T)_r)$. Now note that, the center of a prime ring is an integral domain and the characteristic of an integral domain is either 0 or is a prime number; consequently, we obtain the first part of the proposition. For the final part, we have two cases: If $\text{Char}(R/(R : T)_\ell) = \text{Char}(R/(R : T)_r) = 0$, then it is clear that $\text{Char}(R/(R : T)) = 0$, because $R/(R : T)_\ell$ and $R/(R : T)_r$ are quotient rings of $R/(R : T)$. Hence assume that $\text{Char}(R/(R : T)_\ell) = \text{Char}(R/(R : T)_r) = p$, where p is a prime number. Thus $pT = Tp \subseteq R$ and a fortiori $pT = Tp = TpT \subseteq R$. Thus $\text{Char}(R/(R : T)) = p$. \square

Now we have the following immediate corollary.

Corollary 2.18. *Let R be a maximal subring of a ring T . Assume that the map $\text{Char} : \text{Spec}(R) \rightarrow \mathbb{N} \cup \{0\}$, defined by $P \mapsto \text{Char}(R/P)$ is one-to-one. Then $(R : T)_\ell = (R : T)_r$, in particular $(R : T) = (R : T)_\ell = (R : T)_r$.*

We now present several examples of rings satisfying the property assumed in the previous corollary.

Example 2.19. (1) Assume that $R := \mathbb{M}_n(\mathbb{Z})$, where $n > 1$ is a natural number. Then for each prime number q , it is clear that \mathbb{Z} is a maximal subring of $S := \mathbb{Z}[1/q]$ and therefore R is a maximal subring of $T := \mathbb{M}_n(S)$. It is obvious that, the map Char mentioned in the previous corollary is one-to-one. Hence, $(R : T) = (R : T)_\ell = (R : T)_r$.

(2) Let $R = R_1 \times \cdots \times R_n$, where $n > 1$ and each R_i is a simple ring with $\text{Char}(R_i) = p_i$ is a prime number. Assume that $p_i \neq p_j$ for $i \neq j$. If R is a maximal subring of a ring T , then $(R : T) = (R : T)_\ell = (R : T)_r = R_1 \times \cdots \times R_{i-1} \times 0 \times R_{i+1} \times \cdots \times R_n$, for some i .

(3) Let R be a one-sided Artinian ring that is a maximal subring of a ring T . Assume that $R/J(R) \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_k}(D_k)$, where n_i and k are natural numbers and D_i is a division ring for each i . Then either $(R : T) = (R : T)_\ell = (R : T)_r$ or there exist $i \neq j$ such that $\text{Char}(D_i) = \text{Char}(D_j)$.

In the next two corollary, we derive further results concerning the relationship between the characteristic, conductor ideals, and associated endomorphism rings.

Corollary 2.20. *Let R be a maximal subring of a ring T with $\text{Char}(T)$ is not a prime number. Then either there exists a prime number p such that $pT \subseteq (R : T)$ or $\text{Char}(T) = \text{Char}(\text{End}_{\mathbb{Z}}(T/R)) = 0$.*

Proof. By Proposition 2.17, $\text{Char}(\text{End}_{\mathbb{Z}}(T/R))$ is either 0 or a prime number. If $\text{Char}(\text{End}_{\mathbb{Z}}(T/R)) = p$, where p is a prime number, then $p \cdot 1_{T/R} = 0$ (note that $1_{T/R}$ denotes the identity map on T/R ; that is, $1_{T/R}(t + R) = t + R$, for all $t \in T$). Hence, for any $t \in T$, $pt + R = R$, that is, $pT \subseteq R$. Thus $0 \neq pT \subseteq (R : T)$, (note $\text{Char}(T)$ is not a prime number, therefore $pT \neq 0$ and it is clear that $pT = Tp$ is an ideal of T). Otherwise, assume that $\text{Char}(\text{End}_{\mathbb{Z}}(T/R)) = 0$. If $\text{Char}(T) = m > 0$, then for each $\phi \in \text{End}_{\mathbb{Z}}(T/R)$, we have $m\phi = 0$, because $m\phi(t) = \phi(mt) = \phi(0) = 0$, for any $t \in T$. It follows that $\text{Char}(\text{End}_{\mathbb{Z}}(T/R)) > 0$, which is absurd. Consequently, $\text{Char}(T) = \text{Char}(\text{End}_{\mathbb{Z}}(T/R)) = 0$, as desired. \square

The following is immediate now.

Corollary 2.21. *Let R be a maximal subring of a ring T . Assume that $\text{Char}(T)$ is neither zero nor a prime number. Then $(R : T) \neq 0$.*

Finally, in this section, we present some observations concerning the annihilators of conductor ideals of maximal subrings.

Proposition 2.22. *Let R be a maximal subring of a ring T . If $\text{l.ann}_T((R : T)) + \text{r.ann}_T((R : T)) = T$, then $(R : T)^2 = 0$.*

Proof. By assumption, there exist elements $a \in \text{l.ann}_T((R : T))$ and $b \in \text{r.ann}_T((R : T))$ such that $a + b = 1$. Consequently, $(R : T)a = (R : T)$ and $(R : T)b = 0$. Now note that $ab \in \text{l.ann}_T((R : T)) \cap \text{r.ann}_T((R : T))$. Thus $(R : T)^2 = (R : T)(R : T) = (R : T)ab(R : T) = 0$. \square

In particular, if R is a maximal subring of a semiprime ring T , and the characteristic of T is neither zero nor a prime number, then $\text{l.ann}_T((R : T)) + \text{r.ann}_T((R : T))$ is a proper ideal of T , by Corollary 2.21 and the previous result. As shown in the next conclusion, for certain rings the left (respectively, right) conductor ideal of a maximal subring coincides with the annihilator of the corresponding right (respectively, left) conductor ideal.

Theorem 2.23. *Let R be a maximal subring of a ring T . Then the following hold:*

- (1) *If $(R : T)_\ell$ and $(R : T)_r$ are incomparable, then $\text{l.ann}_R((R : T)_\ell) + \text{r.ann}_R((R : T)_\ell) \subseteq (R : T)_r$ and $\text{l.ann}_R((R : T)_r) + \text{r.ann}_R((R : T)_r) \subseteq (R : T)_\ell$.*
- (2) *If $(R : T)_\ell(R : T)_r = 0$ (in particular, if $(R : T) = 0$), then $\text{Min}(R) \subseteq \{(R : T)_\ell, (R : T)_r\}$. In particular, if $(R : T)_\ell$ and $(R : T)_r$ are incomparable, then $(R : T)_\ell = \text{l.ann}_R((R : T)_r)$ and $(R : T)_r = \text{r.ann}_R((R : T)_\ell)$ are precisely minimal prime ideals of R .*
- (3) *If $(R : T)_\ell \cap (R : T)_r = 0$ and $(R : T)_\ell \neq 0 \neq (R : T)_r$, then $(R : T)_\ell = \text{l.ann}_R((R : T)_r) = \text{r.ann}_R((R : T)_r)$ and $(R : T)_r = \text{r.ann}_R((R : T)_\ell) = \text{l.ann}_R((R : T)_\ell)$ are precisely minimal prime ideals of R . In particular, T is not a prime ring.*
- (4) *If R is a reduced ring, $(R : T)_\ell \cap (R : T)_r = 0$ and $(R : T)_\ell \neq 0 \neq (R : T)_r$, then $(R : T)_\ell$ and $(R : T)_r$ are completely prime ideals of R . In particular, R embeds into a product of two domains.*

Proof. (1) is clear, because $\text{l.ann}_R((R : T)_\ell)(R : T)_\ell = (R : T)_\ell \text{r.ann}_R((R : T)_\ell) = 0 \subseteq (R : T)_r$ and $(R : T)_r$ is a prime ideal of R , which does not contain $(R : T)_\ell$ by our assumption. For (2), first note that $(R : T)_\ell(R : T)_r \subseteq (R : T)$, hence if $(R : T) = 0$, we conclude that $(R : T)_\ell(R : T)_r = 0$.

Thus assume that $(R : T)_\ell(R : T)_r = 0$. This implies that $\text{Min}(R) \subseteq \{(R : T)_\ell, (R : T)_r\}$, by Lemma 2.1. Now suppose that $(R : T)_\ell$ and $(R : T)_r$ are incomparable. Let $P = (R : T)_\ell$ and $I = (R : T)_r$. Thus $PI = 0$ and $I \neq 0$ (by incomparability). It is clear that $P \subseteq \text{l.ann}_R(I)$. Since $\text{l.ann}_R(I)I = 0 \subseteq P$ and P is a prime ideal of R that does not contain I , we deduce that $P = \text{l.ann}_R(I)$. Finally, if P is not a minimal ideal of R , let Q be a prime ideal properly contained in P . Then $\text{l.ann}_R(I)I = 0 \subseteq Q \subsetneq P = \text{l.ann}_R(I)$ implies that $I \subseteq Q \subseteq P$, which is absurd, because I and P are incomparable. Consequently, P is a minimal prime ideal of R . Similarly, $(R : T)_r = \text{r.ann}_R((R : T)_\ell)$ is a minimal prime ideal of R and hence (2) holds. For (3), note that since $(R : T)_\ell$ and $(R : T)_r$ are ideals of R , then we infer that $(R : T)_\ell(R : T)_r$ and $(R : T)_r(R : T)_\ell$ are subsets of $(R : T)_\ell \cap (R : T)_r$. Hence if $(R : T)_\ell \cap (R : T)_r = 0$, we conclude that $(R : T)_\ell(R : T)_r = 0$ and $(R : T)_r(R : T)_\ell = 0$. By a similar argument of (2), the conclusion of (3) can be proved. For the final part of (3), since $0 \neq (R : T)_\ell = \text{r.ann}_R((R : T)_r) \subseteq \text{r.ann}_T((R : T)_r)$, we obtain that T is not a prime ring. For (4), it is clear that $(R : T)_\ell$ and $(R : T)_r$ are incomparable and by (2) or (3), are minimal prime ideal of R . Since R is reduced, by [17, Lemma 12.6], $(R : T)_\ell$ and $(R : T)_r$ are completely prime ideals of R , that is, $R/(R : T)_\ell$ and $R/(R : T)_r$ are domains and clearly R embeds in product of them. \square

By Theorem 2.23(2) we have the following immediate result.

Corollary 2.24. *Let R be a ring which is a maximal subring of a ring T with $|\text{Min}(R)| \geq 3$. Then both ideals $(R : T)$ and $(R : T)_\ell(R : T)_r$ are nonzero.*

3 Integrally closed maximal subrings

As noted in the introduction, if R is a maximal subring of a commutative ring T and R is integrally closed of T , then $(R : T)$ is a prime ideal in T . We now investigate the analogous question in non-commutative setting: When are the one-sided conductor ideals $(R : T)_r$ and $(R : T)_\ell$ prime right or left ideals? Our study begins with the following key result.

Theorem 3.1. *Let R be a maximal subring of a ring T , $P = (R : T)_r$, $a, b \in T$ such that $aTb \subseteq P$. If $a, b \notin P$, then the following hold:*

- (1) a is right (2-)integral over R and b is left (2-)integral over R .
- (2) If $a \notin R$, then $T = R + RaR$.
- (3) If $b \notin R$, then $T = R + RbR$.

In particular, if $b \notin R$, then $a \notin R$; and if $a \notin R$, then either $b \notin R$ or $b \in (R : T)_\ell$.

Proof. First note that by our assumption, $aTbT \subseteq R$, $aT \not\subseteq R$ and $bT \not\subseteq R$. Hence by maximality of R we deduce that $R + RaT = T = R + RbT$. Thus there exist $r_i \in R$ and $s_i \in T$ such that $a = r_0 + r_1bs_1 + \cdots + r_nbs_n$. Multiplying this equation from left by a , we have $a^2 = ar_0 + ar_1bs_1 + \cdots + ar_nbs_n$. Now for each i , $ar_ibs_i \in aTbT \subseteq R$. Therefore $r := ar_1bs_1 + \cdots + ar_nbs_n \in R$ and $a^2 = ar_0 + r$ that shows a is right (2-)integral over R . By a similar argument (from $aTb \subseteq P \subseteq R$ and $T = R + RaT$) we deduce that b is left (2-)integral over R . Hence (1) holds. Now we prove (2). Assume that $a \notin R$. We claim that $S := R + RaR$ is a subring of T . To see this, it suffices to show that if $x, y, u, v \in R$, then $(xay)(uav) \in S$. We have $(xay)(uav) = xayur_0v + xayur_1bs_1v + \cdots + xayur_nbs_nv$. The first term of the sum clearly lies in RaR and at once in S . For the other terms of sum, note that since $ayur_ibs_i \in aTbT \subseteq R$, we obtain that $xayur_ibs_iv \in R$. Consequently, $xayur_1bs_1v + \cdots + xayur_nbs_nv \in R$. This implies that $(xay)(uav) \in S$ and thus S is a subring of T properly contains R (because $a \notin R$). So $S = T$, as R is a maximal subring of T . The proof of (3) is similar (using the inclusion $aTb \subseteq P \subseteq R$). Finally, for the last part of the theorem, assume that $b \notin R$, but $a \in R$. By (3), we have $T = R + RbR$. Consequently, $aT = aR + aRbR \subseteq R + aTbT \subseteq R$, that is, $a \in P$, which is absurd. Now assume that $a \notin R$, but $b \in R$. It follows from (2) that $T = R + RaR$ and then $Tb = Rb + RaRb \subseteq R$. Thus $b \in (R : T)_\ell$. \square

Remark 3.2. Similar to the previous result and its proof, one can prove that, if R is a maximal subring of a ring T , $P = (R : T)_\ell$, $a, b \in T$ such that $aTb \subseteq P$ and $a, b \notin P$, then the following hold:

1. a is right (2-)integral over R and b is left (2-)integral over R .
2. If $a \notin R$, then $T = R + RaR$.
3. If $b \notin R$, then $T = R + RbR$.

In particular, if $a \notin R$, then $b \notin R$; and if $b \notin R$, then either $a \notin R$ or $a \in (R : T)_r$.

Let T be a ring and P be a proper one-sided ideal of T . We call P prime if for all $a, b \in T$, the inclusion $aTb \subseteq P$ implies that either $a \in P$ or $b \in P$. For instance, any maximal one-sided ideal M of a ring T is prime. To verify this, suppose M is a maximal left ideal of T and $a, b \in T$ satisfy $aTb \subseteq M$. If $b \in M$, we are done. Otherwise, $b \notin M$, the maximality of M yields that $M + Tb = T$. Consequently, $aT = aM + aTb \subseteq M$, whence $a \in M$. Thus, M is prime. This leads us to the following.

Corollary 3.3. *Let R be a maximal subring of a ring T which is (2-)integrally closed in T , then $(R : T)_\ell$ and $(R : T)_r$ are prime one-sided ideals of T .*

Regarding the conductor ideal $(R : T)$ of an integrally closed maximal subring R of a ring T , we make the following observation.

Remark 3.4. Let R be a maximal subring of a ring T that is (2-)integrally closed in T . If $a, b \in T$ satisfy $aTb \subseteq (R : T)$, but $a, b \notin (R : T)$, then the following hold:

1. $a, b \in R$.
2. $RaT \subseteq R$, but $TaR \not\subseteq R$. In particular, $T = R + TaR$.
3. $TbR \subseteq R$, but $RbT \not\subseteq R$. In particular, $T = R + RbT$.
4. $a \in (R : T)_r$, $b \in (R : T)_\ell$, $a \notin (R : T)_\ell$ and $b \notin (R : T)_r$.

To see these, for (1) we show that $a \in R$, the proof for $b \in R$ is similar. Since $a, b \notin (R : T)$, we infer that TaT and TbT are not contained in R . Therefore by maximality of R , we deduce that $R + TaT = T$ and $R + TbT = T$. Consequently, $a \in R + TbT$. It follows that there exist $n \in \mathbb{N}$, $x_i, y_i \in T$ for $1 \leq i \leq n$ and $r \in R$ such that $a = r + x_1by_1 + \cdots + x_nby_n$. Multiplying by a from left, we obtain that $a^2 = ar + ax_1by_1 + \cdots + ax_nby_n$. Now note that for all i , $ax_ib \in aTb \subseteq (R : T)$; and $(R : T)$ is an ideal of T contained in R , whence $ax_iby_i \in (R : T) \subseteq R$ for all i . This implies that $s := ax_1by_1 + \cdots + ax_nby_n \in R$ and $a^2 = ar + s$, that is, a is right (2-)integral over R and therefore $a \in R$. Thus (1) holds. (4) is obvious by (2) and (3). We prove $TaR \not\subseteq R$ and $TbR \subseteq R$, the proofs of the other parts of (2) and (3) are similar. Since $b \notin (R : T)$, we conclude that

$TbT \not\subseteq R$. By maximality of R , we deduce that $R + TbT = T$. It follows that $aR + aTbT = aT$. Consequently, $TaR + TaTbT = TaT$. From $a \notin (R : T)$, we have $TaT \not\subseteq R$; and from $aTb \subseteq (R : T)$ we obtain that $TaTbT \subseteq R$. Thus by $TaR + TaTbT = TaT$, we conclude that $TaR \not\subseteq R$. So $Ta \not\subseteq R$, a fortiori $a \notin (R : T)_\ell$. Since $aTb \subseteq (R : T) \subseteq (R : T)_\ell$, we infer that $b \in (R : T)_\ell$, by the previous corollary. Consequently, $Tb \subseteq R$ and then $TbR \subseteq R$.

We conclude this section by proving that if R is an integrally closed maximal subring of a ring T , then $(R : T)$ is a semiprime ideal of T . First, we require the following lemma.

Lemma 3.5. *Let R be a maximal subring of a ring T and $x \in (R : T)_\ell \cup (R : T)_r$. If $xTx \subseteq (R : T)$, then $x \in (R : T)$.*

Proof. Assume that $x \in (R : T)_\ell$. Thus $Tx \subseteq R$ and a fortiori $TxR \subseteq R$. Suppose $x \notin (R : T)$. It follows that $TxT \not\subseteq R$ and consequently $R + TxT = T$, by maximality of R . Since $xTx \subseteq (R : T)$, we infer that $TxTxT \subseteq R$. This implies that $TxT = TxR + TxTxT \subseteq R$, which is absurd. Hence, $x \in (R : T)$. The proof for the case $x \in (R : T)_r$ is similar. \square

By the above observation we have the following key result.

Theorem 3.6. *Let R be a maximal subring of a ring T that is (2-)integrally closed in T (or whenever $x \in T$ and $x^2 \in R$, then $x \in R$). Then $(R : T)$ is a semiprime ideal of T . Moreover, either $(R : T)$ is a prime ideal of T or $(R : T) = (R : T)_\ell \cap (R : T)_r$ (is a semiprime ideal of R).*

Proof. We must prove that if $x \in T$ and $xTx \subseteq (R : T)$, then $x \in (R : T)$. Since $xTx \subseteq R$, we infer that $x^2 \in R$. It follows that $x \in R$, by our assumption. So, $xRx \subseteq xTx \subseteq (R : T) \subseteq (R : T)_\ell \cap (R : T)_r$. This implies that $xRx \subseteq (R : T)_\ell$, and consequently $x \in (R : T)_\ell$, because $(R : T)_\ell$ is a prime ideal of R , by Lemma 2.1. Thus $x \in (R : T)$, by the previous lemma. Hence $(R : T)$ is a semiprime ideal of T , that is, there exists a family Q_i , $i \in I$, of prime ideals of T , such that $(R : T) = \bigcap_{i \in I} Q_i$. Now we have two cases. If there exists $i \in I$, such that $Q_i \subseteq R$, then $Q_i \subseteq (R : T)$, and at once $(R : T) = Q_i$ is a prime ideal of T . Hence, assume that for all $i \in I$, $Q_i \not\subseteq R$. Consequently, for all $i \in I$, we obtain that $R + Q_i = T$, by maximality of R . This implies that $R/(Q_i \cap R) \cong T/Q_i$, as rings. Thus $Q_i \cap R$ is a prime

ideal of R , for all $i \in I$. It is clear that $(R : T) = \bigcap_{i \in I} (Q_i \cap R)$, that is, $(R : T)$ is a semiprime ideal of R . The final part is evident by Proposition 2.2(4). \square

4 Finiteness conditions

As noted in the introduction, if R is a finite maximal subring of a ring T , then T is also finite, see [5, 6, 15, 16, 20]. In commutative rings, for a maximal subring R of a ring T , it is known that R is an Artinian if and only if T is Artinian and T integral over R , see [3, Theorem 3.8]. furthermore, if R is a Noetherian maximal subring of a commutative ring T , then the Hilbert Basis Theorem implies that T is Noetherian. Conversely, if R is a maximal subring of a commutative Noetherian ring T and T is integral over R , then the Eakin-Nagata-Eisenbud Theorem (see [18, Theorem 3.98]) guaranties R is Noetherian. In this section, we obtain analogous observations for non-commutative rings under certain finiteness condition on maximal subrings. We begin with the following result.

Proposition 4.1. *Let R be a maximal subring of a ring T and $P = (R : T)_\ell$. If $PT = T$, then $R = \mathbb{I}_T(P)$, T is a finitely generated left R -module, P is a finitely generated right R -module which is a right primitive ideal of R and T is not a left quasi duo ring. Moreover, if in addition R is a left Artinian/Noetherian ring, then so is T .*

Proof. Since P is a left ideal of T which is a two sided ideal of R , and P is not an ideal of T , we conclude that $R \subseteq \mathbb{I}_T(P) \subsetneq T$. By maximality of R , we deduce that $R = \mathbb{I}_T(P)$. From $PT = T$, we obtain that if M is a maximal left ideal of T that contains P , then M is not an ideal of T . Consequently, T is not a left quasi duo ring. Since $T = PT$, we deduce that $1 = y_1 t_1 + \cdots + y_n t_n$, for some $y_i \in P$, $t_i \in T$ and $n \in \mathbb{N}$. Thus for any $x \in T$ we have $x = x1 = (xy_1)t_1 + \cdots + (xy_n)t_n$. Now note that $y_i \in P$ that is a left ideal of T , a fortiori $xy_i \in P \subseteq R$. This shows that $T = Rt_1 + \cdots + Rt_n$. Also note that for any $p \in P$, we have $p = 1p = y_1 t_1 p + \cdots + y_n t_n p$. Now since $t_i p \in T p \subseteq P \subseteq R$, we have $P = y_1 R + \cdots + y_n R$, that is, P is a finitely generated right R -module. Since ${}_R T$ is finitely generated, Corollary 2.5 implies that P is a right primitive ideal of R . The final part is evident, because ${}_R T$ is finitely generated. \square

Clearly, an analogous statement hold for the right conductor ideal $(R : T)_r$. Hence we obtain the following result.

Corollary 4.2. *Let R be a maximal subring of a ring T and $(R : T) \in \text{Max}(T)$. If $(R : T) \subsetneq (R : T)_\ell, (R : T)_r$ (that is, $(R : T)_\ell$ and $(R : T)_r$ are not ideals of T), then the following hold:*

- (1) ${}_R T$ and T_R are finitely generated.
- (2) $(R : T)_\ell$ is a finitely generated right R -module which is a right primitive ideal of R . In particular, $\text{l.ann}_T((R : T)_\ell) = \text{l.ann}_T(p_1) \cap \cdots \cap \text{l.ann}_T(p_m)$ for some $p_1, \dots, p_m \in (R : T)_\ell$.
- (3) $(R : T)_r$ is a finitely generated left R -module which is a left primitive ideal of R . In particular, $\text{r.ann}_T((R : T)_r) = \text{r.ann}_T(q_1) \cap \cdots \cap \text{r.ann}_T(q_n)$ for some $q_1, \dots, q_n \in (R : T)_r$.

In particular, if $(R : T) \in \text{Max}(T) \setminus \text{Spec}(R)$, then (1) – (3) hold.

Proof. Since $(R : T) \subsetneq (R : T)_\ell, (R : T)_r$, we conclude that $(R : T)_l T = T$ and $T(R : T)_r = T$, for $(R : T)$ is a maximal ideal of T . Hence we are done by the previous proposition and its proof. For the final part note that if $(R : T)$ is not a prime ideal of R , then by Lemma 2.1, it cannot be equal to either $(R : T)_\ell$ or $(R : T)_r$. \square

In particular, if we strengthen the hypotheses of the previous corollary by assuming R is left/right Noetherian or Artinian, we obtain the following.

Corollary 4.3. *Let R be a maximal subring of a ring T and $(R : T) \in \text{Max}(T)$. Then the following hold:*

- (1) *If R is a left Noetherian (respectively, Artinian) ring, then either T is a left Noetherian (respectively, Artinian) ring or $(R : T) = (R : T)_\ell$. In particular, $(R : T)_\ell \subseteq (R : T)_r$ (respectively, $(R : T) = (R : T)_\ell = (R : T)_r$).*
- (2) *If R is a right Noetherian (respectively, Artinian) ring, then either T is a right Noetherian (respectively, Artinian) ring or $(R : T) = (R : T)_r$. In particular, $(R : T)_r \subseteq (R : T)_\ell$ (respectively, $(R : T) = (R : T)_\ell = (R : T)_r$).*

Proof. For (1), note that either $(R : T)_\ell = (R : T) \subseteq (R : T)_r$ or $(R : T) \subsetneq (R : T)_\ell$. Hence, by Proposition 4.1, ${}_R T$ is finitely generated. Consequently, T is a left Noetherian (respectively, Artinian) R -module. It follows that T is a left Noetherian (respectively, Artinian) ring. Also note that, if R is a left Artinian ring, then by Lemma 2.1, $(R : T)_\ell$ and $(R : T)_r$ are maximal ideals of R . Thus in this case, $(R : T)_\ell = (R : T) \subseteq (R : T)_r$, a fortiori $(R : T) = (R : T)_\ell = (R : T)_r$. The proof of (2) is similar. \square

Consequently, if R is an integrally closed maximal subring of a zero-dimensional ring T and $(R : T)$ is not a semiprime ideal of R , then Theorem 3.6 together with the previous corollary implies that T is a left/right Noetherian (respectively, Artinian) ring, whenever R is a left/right Noetherian (respectively, Artinian) ring.

For the next result, observe that in a simple ring T , the conductor ideal $(R : T)$ is zero for every maximal subring R of T . Hence, applying the preceding result yields the following conclusion.

Corollary 4.4. *Let R be a maximal subring of a simple ring T . Then the following hold:*

- (1) *If R is a left Noetherian (respectively, Artinian) ring, then either T is a left Noetherian (respectively, Artinian) ring or $(R : T)_\ell = 0$ (respectively, $(R : T)_\ell = (R : T)_r = 0$, $R = \mathbb{M}_n(D)$, and $T = \mathbb{M}_n(S)$, where D is a division ring, S is a simple ring, and D is a maximal subring of S).*
- (2) *If R is a right Noetherian (respectively, Artinian) ring, then either T is a right Noetherian (respectively, Artinian) ring or $(R : T)_r = 0$ (respectively, $(R : T)_\ell = (R : T)_r = 0$, $R = \mathbb{M}_n(D)$, and $T = \mathbb{M}_n(S)$, where D is a division ring, S is a simple ring, and D is a maximal subring of S).*

Proof. For (1), the Noetherian part is clear by the previous result. Hence assume that R is a left Artinian ring, if T is not a left Artinian ring, then by the previous corollary $(R : T)_\ell = 0$. Since $(R : T)_\ell$ is a prime ideal of R , we conclude that $R = \mathbb{M}_n(D)$, for some division ring D and $n \geq 1$ (hence R is a right Artinian too). Thus $T = \mathbb{M}_n(S)$, for a simple ring S , because R is a subring of T and T is a simple ring. Since R is a maximal subring of T , we deduce that D is a maximal subring of S . The proof of (2) is similar. \square

Analogous to Proposition 2.13, we establish a relationship involving the centralizer of a left Noetherian maximal subring R of a ring T , the left Noetherian property of T , and the conductor ideals, as follows.

Proposition 4.5. *Let R be a left Noetherian maximal subring of a ring T . Then either $C_T(R) \subseteq R$ or T is a left Noetherian ring and $(R : T) = (R : T)_\ell = (R : T)_r$.*

Proof. Assume that $C_T(R)$ is not contained in R and $\alpha \in C_T(R) \setminus R$. Since α commutes with any element of R , we conclude that $R[\alpha]$ is a left Noetherian ring, by Hilbert Basis Theorem. Clearly $T = R[\alpha]$, for R is a maximal subring of T and $\alpha \in T \setminus R$. Hence T is a left Noetherian ring. Finally, if $(R : T)_\ell$ is not an ideal of T , then by Remark 2.14, we deduce that $R = \mathbb{I}_T((R : T)_\ell)$. Consequently, R contains $C_T(R)$, which is absurd. It follows that $(R : T)_\ell$ is an ideal of T . Similarly, $(R : T)_r$ is an ideal of T . Hence $(R : T) = (R : T)_\ell = (R : T)_r$ as desired. \square

We now state the main result of this section.

Theorem 4.6. *Let R be a Noetherian (respectively, Artinian) maximal subring of a ring T and $(R : T) \neq 0$. Then the following hold:*

- (1) $T/\text{l.ann}_T((R : T)_\ell)$ and $T/\text{l.ann}_T((R : T))$, are left Noetherian (respectively, Artinian) rings.
- (2) $T/\text{r.ann}_T((R : T)_r)$ and $T/\text{r.ann}_T((R : T))$, are right Noetherian (respectively, Artinian) rings.
- (3) If T is semiprime, then $T/\text{ann}_T((R : T))$ is a Noetherian (respectively, an Artinian) ring.
- (4) If T is a prime ring, then T is finitely generated as left and right R -modules. In particular, T is a Noetherian (respectively, an Artinian) ring. Moreover, $(R : T)_\ell$ and $(R : T)_r$ are right and left primitive ideals of R , respectively.

Proof. (1) Since R is a right Noetherian ring, there exist $x_1, \dots, x_n \in (R : T)_\ell$ such that $(R : T)_\ell = x_1R + \dots + x_nR$. Thus $\text{l.ann}_T((R : T)_\ell) = \text{l.ann}_T(x_1) \cap \dots \cap \text{l.ann}_T(x_n)$. Therefore $T/\text{l.ann}_T((R : T)_\ell)$ embeds into $Tx_1 \times \dots \times Tx_n$ as a left T -module, and as a left R -module too. Since $Tx_i \subseteq R$, we conclude that $T/\text{l.ann}_T((R : T)_\ell)$ embeds into R^n as a left

R -module, and consequently $T/\text{l.ann}_T((R : T)_\ell)$ is left Noetherian (respectively, Artinian), because R is left Noetherian (respectively, Artinian). Also note that, $(R : T) \subseteq (R : T)_\ell$ and a fortiori $\text{l.ann}_T((R : T)_\ell) \subseteq \text{l.ann}_T((R : T))$. This implies that $T/\text{l.ann}_T((R : T))$ is left Noetherian (respectively, Artinian). Similarly (2) holds. For (3), first note that since T is a semiprime ring, we obtain that the left and right annihilators of $(R : T)$ are coincided. Thus by (1) and (2), we deduce that $T/\text{ann}_T((R : T))$ is Noetherian (respectively, Artinian). Finally, for (4), since $0 \neq (R : T) \subseteq (R : T)_\ell$, we infer that $\text{l.ann}_T((R : T)_\ell) = 0$. By the proof of (1), T is isomorphic to a left R -submodule of R^n . It follows that T is a finitely generated left R -module. Similarly, T is a finitely generated right R -module. This shows that T is Noetherian (respectively, Artinian). The final part is evident by Corollary 2.5. \square

Note that in the previous theorem, the assumption $(R : T) \neq 0$ is not required for parts (1) – (3). However, since the conclusions in the case $(R : T) = 0$ are trivial for these part, we include the condition $(R : T) \neq 0$ in the theorem statement for brevity and to focus on nontrivial case.

In [15] and [16], It was shown that if a finite ring R is a maximal subring of a ring T , then T must also be finite. In the following remark, we deduce a special case of this fact by the previous theorem.

Remark 4.7. Let T be a prime ring with a finite maximal subring R . If $(R : T)_\ell \neq 0$, then T is finite. To see this, note that by the proof of (4) in the previous theorem, T embeds into R^n and consequently, T is finite.

Also we have the following observations.

Proposition 4.8. *Let R be a left Noetherian (respectively, Artinian) maximal subring of a ring T . Assume that $(R : T)_\ell$ is a finitely generated as right ideal of R and $(R : T)$ contains a prime ideal Q of T . Then either $(R : T) = (R : T)_\ell \subseteq (R : T)_r$ (respectively, $(R : T) = (R : T)_\ell = (R : T)_r$) or T is a left Noetherian (respectively, Artinian) ring.*

Proof. Similar to the proof of the previous theorem, $T/\text{l.ann}_T((R : T)_\ell)$ is a left Noetherian (respectively, Artinian) R -module. Since $\text{l.ann}_T((R : T)_\ell)(R : T)_\ell = 0 \subseteq Q$ and Q is a prime ideal of T , we have two cases: either $(R : T)_\ell \subseteq Q$ or $\text{l.ann}_T((R : T)_\ell) \subseteq Q$. If $(R : T)_\ell \subseteq Q$, then

$(R : T)_\ell \subseteq (R : T)$ and therefore $(R : T)_\ell = (R : T) \subseteq (R : T)_r$ (respectively, $(R : T)_\ell = (R : T) = (R : T)_r$, because R is a left Artinian ring and $(R : T)_\ell$ and $(R : T)_r$ are prime ideals of R , by Lemma 2.1). Hence assume that $\text{l.ann}_T((R : T)_\ell) \subseteq Q$ and consequently $\text{l.ann}_T((R : T)_\ell) \subseteq (R : T) \subseteq R$. Thus $\text{l.ann}_T((R : T)_\ell)$ is a left Noetherian (respectively, Artinian) R -module. It follows that T is a left Noetherian (respectively, Artinian) R -module and a fortiori is a left Noetherian (respectively, Artinian) ring. \square

Proposition 4.9. *Let T be a ring which is not prime and R be a Noetherian maximal subring of T with $(R : T) \neq 0$. If R contains a prime ideal Q of T , then either T is Noetherian or $Q = (R : T) = (R : T)_\ell$ or $Q = (R : T) = (R : T)_r$ is a minimal prime ideal of T (thus Q is unique). Moreover, if T is neither left Noetherian nor right Noetherian, then $Q = (R : T) = (R : T)_\ell = (R : T)_r$.*

Proof. First note that $Q \subseteq (R : T)$. Now we have two cases: (a) $Q \neq (R : T)$. This implies that $\text{l.ann}_T((R : T))$ and $\text{r.ann}_T((R : T))$ are contained in Q . By (1) and (2) of Theorem 4.6, we conclude that T/Q is Noetherian. Since $Q \subseteq R$, we deduce that T is Noetherian too. (b) Hence suppose that $Q = (R : T)$. Let $Q' \in \text{Min}(T)$ and $Q' \subsetneq Q$, thus $Q' \subsetneq (R : T)$. By the first case we obtain that T is Noetherian and we are done. Consequently, $Q = (R : T)$ is a minimal prime ideal of T . Now assume that $(R : T)_\ell$ and $(R : T)_r$ are not contained in $Q = (R : T)$. Then we observe that $\text{l.ann}_T((R : T)_\ell)$ and $\text{r.ann}_T((R : T)_r)$ are contained in Q , because Q is a prime ideal. By (1) and (2) of Theorem 4.6, T/Q is a left/right Noetherian ring. This implies that T is a Noetherian ring, because $Q \subseteq R$. By a similar argument and using (1) and (2) of Theorem 4.6 and the fact that Q is prime, we conclude the final part. \square

Corollary 4.10. *Let R be an Artinian maximal subring of a prime ring T . Then either $R \cong \mathbb{M}_n(D)$ for some division ring D (in particular, $T = \mathbb{M}_n(S)$, for some ring S , where D is a maximal subring of S) or $T \cong \mathbb{M}_n(D')$ for some division ring D' .*

Proof. If R is a prime ring, then clearly the first part of the statement of the theorem holds. Hence assume that R is not a prime ring. Consequently, $(R : T)_\ell \neq 0$ and a fortiori $\text{l.ann}_T((R : T)_\ell) = 0$, because T is prime. Thus

by Theorem 4.6(1), we conclude that T is a left Artinian ring and we are done (note T is prime). \square

Finally we conclude this paper by the following result.

Proposition 4.11. *Let T be a ring which is not prime and R be an Artinian maximal subring of T with $(R : T) \neq 0$. Then either T is Artinian or $\dim(T) = 0$ or R contains a unique (minimal) prime ideal of T , say Q , and $T/Q \cong \mathbb{M}_n(S)$, for some ring S , where S has a maximal subring D that is a division ring.*

Proof. Assume that T is not Artinian. We have two cases: (a) each prime ideal Q of T is not contained in R . Thus $R + Q = T$, because $Q \neq 0$ (note T is not prime) and R is a maximal subring of T . It follows that $R/(R \cap Q) \cong T/Q$, as rings. This shows that Q is a maximal ideal of T , that is, $\dim(T) = 0$. (b) Assume that there exists a prime Q of T such that $Q \subseteq R$. Consequently, $Q \subseteq (R : T) \subseteq (R : T)_\ell \cap (R : T)_r$. Now, if $\text{l.ann}_T((R : T))$ is not contained in Q , we conclude that $(R : T)_\ell \subseteq Q \subseteq (R : T) \subseteq (R : T)_r$, because Q is a prime ideal of T . Since R is an Artinian ring and $(R : T)_\ell, (R : T)_r$ are prime ideals in R , we deduce that $(R : T)_\ell = Q = (R : T) = (R : T)_r$. Similarly if $\text{r.ann}_T((R : T)_r)$ is not contained in Q , the previous equalities hold. Thus assume that Q contains $\text{l.ann}_T((R : T)_\ell)$ and $\text{r.ann}_T((R : T)_r)$. By (1) and (2) of Theorem 4.6, we deduce that T/Q is an Artinian ring. Since $Q \subseteq R$, we obtain that T is Artinian, which is absurd. Consequently, $Q = (R : T) = (R : T)_\ell = (R : T)_r$ is a maximal ideal of R . It follows that $R/Q \cong \mathbb{M}_n(D')$ for a division ring D' and a natural number n . Since R/Q is a maximal subring of T/Q , we conclude that $T/Q \cong \mathbb{M}_n(S)$, for some ring S , where S has a maximal subring $D \cong D'$, as ring. \square

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