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On rainbow connection number of cartesian product of graphs

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Dedicated to Pedro Sánchez, ethical and humanitarian Spanish politician

Abstract. Edge coloring of a graph is a function from its edge set to the set of natural numbers (called colours). A path in an edge-colored graph with no two edges sharing the same color is called a rainbow path. An edge-colored graph is said to be rainbow connected if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a rainbow coloring of the graph. The minimum number of colors required to rainbow color a connected graph is called its rainbow connection number, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, and that of a star is its number of leaves. For a basic introduction to the topic, see Chapter 11 in [4] and for a comprehensive treatment of the area see the recent monograph by Li and Sun [6]. The concept of rainbow coloring was introduced in [3].

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1 Introduction

Edge coloring of a graph is a function from its edge set to the set of natural numbers (called colours). A path in an edge-colored graph with no two edges sharing the same color is called a rainbow path. An edge-colored graph is said to be rainbow connected if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a rainbow coloring of the graph. The minimum number of colors required to rainbow color a connected graph is called its rainbow connection number, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, and that of a star is its number of leaves. For a basic introduction to the topic, see Chapter 11 in [4] and for a comprehensive treatment of the area see the recent monograph by Li and Sun [6]. The concept of rainbow coloring was introduced in [3].

It is one of the applications of rainbow coloring in modeling the problem of message transmission with complete security [1]. Suppose that G represents a network. We wish to send messages between any two vertices in a network and require that each link on the path between the vertices (each edge on the path) be assigned a distinct channel (e.g., a distinct frequency). The color of each edge indicates the frequency used to transmit the message Clearly, we want to minimize the number of different channels we use in our network. In this paper, we study the rainbow connection number of Cartesian products of some classes of graphs.

2 Preliminary definitions and theorems

In this section, we collect definitions and concepts needed in this paper. All graphs considered are finite, connected, undirected, and without multiple edges. The distance between two vertices u and v in G is the length of the shortest path between them and is denoted by $d_G(u, v)$.

Eccentricity of a vertex v of graph G is the maximum distance between a vertex v to all other vertices of G, denoted by $ecc_G(v)$. The *diameter* of a connected graph G is the maximum eccentricity of its vertices, denoted by $diam(G)$. The radius of G is the smallest eccentricity of its vertices, denoted by $rad(G)$.

Definition 2.1. Given two graphs G and H, the Cartesian product of G

and H, denoted by $G\square H$, is a graph defined as follows: $V(G\square H) = V(G)\times V(H)$. Two distinct vertices (g_1, h_1) and (g_2, h_2) of $G\square H$ are adjacent if and only if either $g_1 = g_2$ and $\{h_1, h_2\} \in E(H)$ or $h_1 = h_2$ and $\{g_1, g_2\} \in E(G)$.

Definition 2.2. The n-dimensional hypercube is a graph whose vertex set is $\{1, 0\}^n$ (i.e., there are exactly 2^n vertices, each labeled with a distinct n-bit string), and with an edge between two vertices if and only if they differ in exactly one bit position, and denoted by Q_n. By the definition of Cartesian product of graphs, we have $Q_n = Q_{n-1}\square K_2 = \square_{i=1}^{n} K_2$.

Definition 2.3. The star graph of order n is denoted by S_n, is a tree on n nodes with one node having vertex degree $n-1$ and the other $n-1$ having vertex degree 1.

Proposition 2.4. [7] *Let G and H be two graphs, then the Cartesian product $G\square H$ is connected if and only if both G and H are connected, also $diam(G\square H) = diam(G) + diam(H)$.*

Lemma 2.5. *Let G and H be connected graphs, then $rc(G\square H) \leq rc(G) + rc(H)$*

Proof. Suppose that c_G and c_H are rainbow colorings of graphs G and H, respectively, with distinct color sets A and B. We define an edge coloring $c_{G\square H}$ of $G\square H$ as follows:

$$c_{G\square H}\{(g, h), (g', h')\} = \begin{cases} c_G\{g, g'\} \text{ if } gg' \in E(G) \text{ and } h = h', \\ c_H\{h, h'\} \text{ if } hh' \in E(H) \text{ and } g = g'. \end{cases}$$

For every two vertices (x, y) and (x', y') of $G\square H$ there are two cases:

(1) If (x, y) and (x', y') are in same copy of G or H. Then, according to the rainbow coloring of the graphs G and H, this copy has a rainbow path between those two vertices.

(2) If (x, y) and (x', y') are not in the same copy of G or H. Then, We consider the path $P : (x, y) \cdots (x', y) \cdots (x', y')$ such that the first part of the path P from (x, y) to (x', y) is in a copy of G and the second part of the path from (x', y) to (x', y') is in a copy of H. Therefore, according to the rainbow coloring of graph G, there is a rainbow colored path between (x, y) and (x', y) with colors from A. Similarly, by a rainbow coloring of the graph H between (x', y) and (x', y'), there is a rainbow path with colors

from B. Then, there is a rainbow path between (x, y) and (x', y'). As a result, $c_{G\Box H}$ is a rainbow coloring of $G\Box H$. Then, $rc(G\Box H) \leq rc(G) + rc(H)$. $\Box$

Remark 2.6. For example, let $G = H = P_2$ be two graphs with $rc(G) = rc(H) = 1$. Clearly, $G\Box H = C_4$ and $rc(G\Box H) = rc(G) + rc(H) = 2$.

The last example shows that the upper bound in Lemma 2.5 can be reached.
Note that, for any connected graph G by definition of rainbow coloring, we have, $rc(G) \geq diam(G)$.

Corollary 2.7. *Let G and H be connected graphs. Then*

$$diam(G\Box H) \leq rc(G\Box H) \leq rc(G) + rc(H).$$

Proposition 2.8. *Let G and H be graphs, then if $rc(G) = diam(G)$ and $rc(H) = diam(H)$. Then $rc(G\Box H) = diam(G\Box H) = rc(G) + rc(H)$*

Proof. By 2.4 and 2.7, we have

$$\begin{aligned} diam(G) + diam(H) &= diam(G\Box H) \\ &\leq rc(G\Box H) \\ &\leq rc(G) + rc(H) \\ &= diam(G) + diam(H). \end{aligned}$$

Then $rc(G\Box H) = rc(G) + rc(H)$. $\Box$

In the next section, we study the rainbow connection number of the Cartesian product of some classes of graphs and find their upper and lower bounds.

3 Main results

Let P_n, C_n, S_n, and K_n be the path, cycle, star($K_{1,n-1}$), and complete graph of order n, respectively. In addition, let $\mathcal{P}$, $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{K}$ be the class of paths, cycles, stars and complete graphs consequently. A trivial graph is a graph of order 1.
Since $rc(K_n) = diam(K_n) = 1$, $rc(P_n) = diam(P_n) = n-1$, and $rc(C_{2n}) = diam(C_{2n}) = n$, and according to Proposition 2.8, we have the following corollaries.

Corollary 3.1. *$rc(G\Box H) = diam(G\Box H)$ for any two graphs G and H from $\mathcal{K} \cup \mathcal{P}$.*

Corollary 3.2. *Let $n, m \in \mathbb{N}$. Then*

(1) $rc(P_n\Box C_{2m}) = diam(P_n\Box C_{2m}) = n - 1 + m$,
(2) $rc(K_n\Box C_{2m}) = diam(K_n\Box C_{2m}) = m + 1$,
(3) $rc(C_{2n}\Box C_{2m}) = diam(C_{2n}\Box C_{2m}) = n + m$.

Corollary 3.3. *Let Q_n be a n-hypercube graph. Then $rc(Q_n) = diam(Q_n) = n$, for any $n \in \mathbb{N}$.*

In [2] the authors proved that, for any two connected graphs G and H, $rc(G\Box H) \leq 2rad(G\Box H)$.

Theorem 3.4. *Let $n, m \in \mathbb{N}$ with n is odd and $G = K_m\Box C_n$. Then*

$$rc(G) = diam(K_m\Box C_n) = \lfloor \frac{n}{2} \rfloor + 1.$$

Proof. Combine the fact that $rc(K_m\Box C_n) \geq diam(K_m\Box C_n) = \lfloor \frac{n}{2} \rfloor + 1$. Conversely, suppose that $V(C_n) = \{v_1, ..., v_n\}$ and $V(K_m) = \{u_1, ..., u_m\}$ are vertex sets of C_n and K_m, respectively. Then,

$$V(C_n\Box K_m) = \{(v_i, u_j)|1 \leq i \leq n; 1 \leq j \leq m\}$$

is a vertex set of G, so there are m copies of C_n and n copies of K_m in G that denoted by K_m^i for $(1 \leq i \leq n)$. We consider the edge partitioning of the graph G as follows

$$E_i = \{(v_i, u_j)(v_i, u_{j'})|1 \leq i \leq n, 1 \leq j, j' \leq m\},$$

$$E'_i = \{(v_i, u_j)(v_{i+1}, u_j)|1 \leq i \leq n, 1 \leq j \leq m, v_{n+1} = v_1\}.$$

Let E_i be every edge in a complete graph K_m^i for $(1 \leq i \leq n)$ and E'_i be any edge between K_m^i and K_m^{i+1} $(1 \leq i \leq n)$ such that $K_m^{n+1} = K_m^1$.
Suppose that $k = diam(K_m\Box C_n) = \lfloor \frac{n}{2} \rfloor + 1$.
Now, we define a coloring c of the graph G as follows:

$$c(E'_i) = \begin{cases} i & 1 \leq i \leq k - 1, \\ i - k + 1 & k \leq i \leq n. \end{cases}$$

$$c(E_i) = \begin{cases} i+1 & 1 \leq i \leq k-1, \\ i-k+2 & k \leq i < n, \\ 1 & i = n. \end{cases}$$

We have to show that there is a rainbow path between any two vertices in the graph G. Let (x, y) and (x', y') be arbitrary vertices in G.

(i) If $x = x'$ or $(y = y')$, then (x, y) and (x', y') are in the same copy of K_m or C_n so by the coloring c, there is a rainbow path between (x, y) and (x', y') of length at most of $k - 1$.

(ii) If $x \neq x'$ and $y \neq y'$ then (x, y) and (x', y') are not in the same copy of K_m or C_n, so there is a path between those two vertices of length at most k so that in a copy of C_n, there is a rainbow path between (x, y) and (x', y) like as P of length at most $k - 1$. Then, that path passes through k copies of K_m.

According to the coloring c, at least the color of the edges in one copy among k copies of K_m differs from the colors of the edges of the path P. Let edge $\{(x'', y), (x'', y')\}$ has different color from the colors of path P. Therefore, by using the path $(x, y), ..., (x'', y), (x'', y'), ..., (x', y')\}$ between (x, y) and (x', y'), so a path of length k such that whose edges are colored with k different colors is obtained. Then, (x, y) and (x', y') are connected by a rainbow path.

As a result, every two vertices of G are connected by a rainbow path of maximum length k. Therefore, c is a rainbow coloring of G with k colors. Then,

$$rc(C_n \square K_m) = diam(C_n \square K_m) = \lfloor \frac{n}{2} \rfloor + 1.$$

$\square$

Theorem 3.5. *Let $n, m \in \mathbb{N}$ with n is odd. and $G = P_m \square C_n$ then*

$$rc(G) = diam(G) = \lfloor \frac{n}{2} \rfloor + m - 1.$$

Proof. Suppose that $V(G) = \{v_{j,i} | 1 \leq j \leq m,\ 1 \leq i \leq n\}$ is the vertex set of G and partition the edges of graph G as follows.

$$E_{j,i} = \{v_{j,i}v_{j,i+1} | 1 \leq i \leq n,\ 1 \leq j \leq m,\ v_{j,n+1} = v_{j,1}\}$$

$$E'_{j,i} = \{v_{j,i}v_{j+1,i} | 1 \leq i \leq n,\ 1 \leq j \leq m-1\}$$

Such that $E_{j,i}$ is the set of edges of a cycle jth C_n^j and $E'_{j,i}$ is the set of edges between vertices of cycle C_n^j and vertices of cycle C_n^{j+1}. Since n is odd, there exists $k = \lfloor \frac{n}{2} \rfloor$ such that $n = 2k+1$. We know that $rc(G) \geq diam(G) = \lfloor \frac{n}{2} \rfloor + m - 1$. Now, we define an edge coloring of G, $c : E(G) \to \{1, 2, \ldots, k+m-1\}$ of the edges of G as follows

$$c(e_{j,i}) = \begin{cases} i & 1 \leq i \leq k, \\ i-k & k+1 \leq i \leq n. \end{cases}$$

$$c(e'_{j,i}) = \begin{cases} i & 1 \leq i \leq k \text{ and } j = 1, \\ k+1 & k < i \leq n \text{ and } j = 1, \\ k+j & 1 < j \leq m-1. \end{cases}$$

We will show that c is a rainbow connected coloring for G. We must show that every two vertices of G are connected by a rainbow path.
Suppose that $x = v_{j,i}, y = v_{j',i'}$ are two arbitrary vertices of G; there are two cases.

(1) If $i = i'$ or $j = j'$ so x and y are in same copy of P_m or C_n then x and y are connected by rainbow path of length at most $m-1$ or k.

(2) If $i \neq i'$ and $j \neq j'$ $(j < j')$, We consider two cases.

a) Suppose that $j = 1$ then there is at least two paths between the vertices x and y of length $d(x, y) \leq diam(G) = k+m-1$ like as

$$P : v_{1,i}, v_{2,i}, \ldots, v_{j',i}, \ldots, v_{j',i'}$$

$$Q : v_{1,i}, \ldots, v_{1,i'}, v_{2,i'}, \ldots, v_{j,i'}$$

According to the coloring c, at least one of the paths P and Q is the rainbow path.

b) Suppose that $2 \leq j \leq j'$, in this case there is a rainbow path between x and y as follows

$$P : v_{j,i}, \ldots, v_{j',i}, \ldots, v_{j',i'}$$

Thus, c is a rainbow coloring of G and $rc(G) \leq k+m-1$ and so $rc(G) = k+m-1$. □

Lemma 3.6. *Let T_m $(m \geq 4)$ be a tree that has at least three pendant vertices (not path) then,*

$$rc(K_n \square T_m) \leq rc(T_m) = m-1$$

Proof. We have $rc(T_m) = m-1$. Assign $m-1$ distinct colors to all edges of T_m. Define the coloring of the edges of $K_n \square T_m$ as follows. For every two adjacent vertices x and y from T_m, in $K_n \square T_m$ there exist two adjacent copies of K_n that are opposite of x and y, so assign the color of edge xy to all edges between these two copies of K_n.
In tree T_m, we consider P the path of length $diam(T_m) < m-1$ and suppose that $u, v \in P$ and $d(u) = d(v) = 1$ (u and v are end-vertices of the path P) in graph $K_n \square T_m$, since $d(u, v) \leq m-2$. Then, there is a color among $m-1$ colors such as a, which is different from colors on the edges of path P. So, by assigning color (a) to all edges in two copies of K_n opposite of u and v, and assigning every color among $m-1$ color without color (a) arbitrarily assigns to the all edges in every copy of K_n. Since every two vertices of $K_n \square T_m$ are connected by a rainbow path. Therefore, $rc(K_n \square T_m) \leq m-1$. □

Let $S_m = K_{1,m-1}$ be a star graph from order m, with vertex set $V(S_m) = \{v_1, v_2, \ldots, v_m\}$ such that the vertex v_1 of degree $m-1$ and the other vertices of degree 1. Since S_m is a tree, so $rc(S_m) = m-1$.

Theorem 3.7. *Let $n, m \in \mathbb{N}$ and $n > 1$. then, $rc(K_n \square S_m) = 4$ when $m \geq 5$ and $rc(K_n \square S_4) = 3$.*

Proof. For the first equal, we have $diam(K_n \square S_m) = 3$ and $m \geq 5$. Assume, to the contrary, that there exists a rainbow connected coloring c of $K_n \square S_m$, using at most three colors.
In every copy of S_m, between every two vertices (without center vertex) exists just one path of length two and many paths of length 4. Since coloring c has at most three colors and $m \geq 5$ then there exists at least two vertices in $K_n \square S_m$ are not connected by a rainbow path. So that $rc(K_n \square S_m) \geq 4$.
On the other hand, let $V(S_m) = \{u_1, \ldots, u_m\}$ and $V(K_n) = \{v_1, \ldots, v_n\}$ vertex sets of S_m and K_n, then the vertex set of $K_n \square S_m$ is

$$V(K_n \square S_m) = \{(v_i, u_j) | 1 \leq i \leq n, 1 \leq j \leq m\}$$

partition the edges of $K_n \Box S_m$ into two sets E_1 and E_2 as follows

$$E_1 = \{(v_i, u_j)(v_{i'}, u_j) |\ 1 \leq i, i' \leq n, 1 \leq j \leq m\}$$
$$E_2 = \{(v_i, u_1)(v_i, u_j) |\ 1 \leq i \leq n, 2 \leq j \leq m\}$$

Now, we define an edge coloring with 4 colors for graph $K_n \Box S_m$ as follows

$$c(E_1) = \begin{cases} 1 & j = 1, \\ 2 & j \geq 2. \end{cases}$$

$$c(E_2) = \begin{cases} 3 & i = 1, \\ 4 & i \neq 1. \end{cases}$$

For every two nonadjacent vertices $x = (v_i, u_j)$ and $y = (v_{i'}, u_{j'})$ of $K_n \Box S_m$, if one of j or j' equal to 1, then there is a rainbow path of length 2 from x to y.
Otherwise, we have the following cases.

If $i = i' = 1$, so there exists the rainbow path

$$P = (v_1, u_j), (v_1, u_1), (v_2, u_1), (v_2, u_{j'}), (v_1, u_{j'})$$

between x and y of length 4.
We now consider the case, if at least one of i and i' is not equal to 1, then the path $P' = (v_i, u_j), (v_i, u_1), (v_1, u_1), (v_1, u_{j'}), (v_{i'}, u_{j'})$ is a rainbow path of length at most 4 from x to y.
Then, there exists rainbow coloring of $K_n \Box S_m$ uses four colors. Therefore, $rc(K_n \Box S_m) = 4$.
Clearly, by 3.6, the remaining equal $rc(K_n \Box S_4) = 3$ is hold. $\Box$

Theorem 3.8. *Let $n, m \in \mathbb{N}$ such that $n, m \geq 3$. then $rc(S_n \Box S_m) = 4$.*

Proof. Suppose that $V(S_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ such that $d(v_0) = n - 1$ and the vertices v_i, $(1 \leq i \leq n-1)$ are pendant vertices and $V(S_m) = \{u_0, u_1, u_2, \ldots, u_{m-1}\}$ such that $d(u_0) = m - 1$ and the vertices u_j, $(1 \leq j \leq m-1)$ are pendant vertices. Now define a coloring c of the graph $S_n \Box S_m$ by

$$c(e) = \begin{cases} 1 & e = (v_0, u_0)(v_0, u_j),\ 1 \leq j \leq m-1, \\ 2 & e = (v_0, u_0)(v_i, u_0),\ 1 \leq i \leq n-1, \\ 3 & e = (v_0, u_j)(v_i, u_j),\ 1 \leq i \leq n-1,\ 1 \leq j \leq m-1, \\ 4 & e = (v_i, u_0)(v_i, u_j),\ 1 \leq i \leq n-1,\ 1 \leq j \leq m-1. \end{cases}$$

Clearly, in the coloring c, between any two nonadjacent vertices $x = (v_i, u_j)$ and $y = (v_{i'}, u_{j'})$ of $S_n \Box S_m$, we have the path

$$P = (v_i, u_j), (v_i, u_0), (v_0, u_0), (v_0, u_{j'}), (v_{i'}, u_{j'})$$

which is a rainbow path of length at most 4. Thus $rc(S_n \Box S_m) \leq 4$, since $diam(S_n \Box S_m) = 4$. Then, $rc(S_n \Box S_m) = 4$. □

Theorem 3.9. *Let $n, m \in \mathbb{N}$ and $n > 1$. Then $rc(P_n \Box S_m) = n + 1$ when $n, m \geq 3$ (We remove the other cases because $K_2 = S_2 = P_2$).*

Proof. Graph $P_n \Box S_m$ contains n copies of S_m, $(S_m^1, S_m^2, \ldots, S_m^n)$. Now, let $\{v_0^i, v_1^i, v_2^i, \ldots, v_{m-1}^i\}$ be a vertices of S_m^i for $(1 \leq i \leq n)$ such that v_0^i the center of S_m^i and $d(v_0^i) = m - 1$. Since $diam(P_n \Box S_m) = diam(P_n) + diam(S_m) = n + 1$ then $rc(P_n \Box S_m) \geq n + 1$.
Now, we define an edge-coloring c of $P_n \Box S_m$ as follows. for $1 \leq i \leq n$ and $1 \leq j \leq m - 1$,

$$c(e) = \begin{cases} 1 & ; \quad e = v_0^i v_j^i \quad \text{and} \quad i \equiv 1 (mod\, 2), \\ 2 & ; \quad e = v_0^i v_j^i \quad \text{and} \quad i \equiv 0 (mod\, 2), \\ i + 2 & ; \quad e = v_0^i v_0^{i+1} \quad \text{for} \quad 1 \leq i \leq n - 2, \\ n + 2 - i & ; \quad e = v_j^i v_j^{i+1} \quad \text{for} \quad 1 \leq i \leq n - 2, \\ 3 & ; \quad e = v_0^{n-1} v_0^n, \\ n + 1 & ; \quad e = v_j^{n-1} v_j^n \quad \text{for} \quad 1 \leq j \leq m - 1. \end{cases}$$

For any two nonadjacent vertices $x = v_j^i$ and $y = v_{j'}^{i'}$ of the graph $P_n \Box S_m$, we have

(i) If x and y on same copy of S_m then $i = i'$ and so the path $P : x = v_j^i, v_0^i, v_0^{i+1}, v_{j'}^{i+1}, v_{j'}^i = y$ is a rainbow path between x and y in $P_n \Box S_m$.

(ii) If x and y on distinct copies of S_m, suppose that $i < i'$ then there is the path $P' : x = v_j^i, v_0^i, v_0^{i+1}, v_{j'}^{i+1}, v_{j'}^{i+2}, \ldots, v_{j'}^{i'} = y$ is a rainbow path between x and y in $P_n \Box S_m$.

Indeed, every two vertices of $P_n \Box S_m$ are connected by a rainbow path of length at most $n + 1$. It follows that the coloring c is rainbow coloring for $P_n \Box S_m$. So $rc(P_n \Box S_m) \leq n + 1$.
Therefore, $rc(P_n \Box S_m) = n + 1$. □

We have

$$diam(C_{2n+1}\square C_{2m+1}) \leq rc(C_{2n+1}\square C_{2m+1}) \leq rc(C_{2n+1}) + rc(C_{2m+1})$$

for any two positive integers n and m. So $n + m \leq rc(C_{2n+1}\square C_{2m+1}) \leq n+m+2$. For example, we know that $diam(C_5) = diam(C_4) = 2$, $rc(C_4) = 2$ and $rc(C_5) = 3$. Therefore, $4 \leq rc(C_5\square C_5) \leq 6$ and $4 \leq rc(C_4\square C_5) \leq 5$. In addition, there is a rainbow coloring with 4 colors for $C_4\square C_5$ (see Figure 2). Then $rc(C_4\square C_5) = 4$. In Theorem 3.12, we generalize this example. To prove that, we need the following lemma.

Lemma 3.10. [3] $rc(C_n) = \lceil \frac{n}{2} \rceil$ *for each integer* $n \geq 4$.

Before we prove Theorem 3.12, we explain the rainbow coloring by the following example.

Example 3.11. Figure 1 shows a rainbow coloring of $C_6\square C_9$ with 7 colors. Then, $rc(C_6\square C_9) = diam(C_6\square C_9) = 7$

Let $V(C_n) = \{v_i|1 \leq i \leq n\}$ and $V(C_m) = \{u_j|1 \leq j \leq m\}$, In Theorem 3.12, the vertex $(v_i, u_j) \in V(C_n\square C_m)$ is denoted by $v_{i,j}$ for notational simplicity.

Theorem 3.12. *Let* $n, m \in \mathbb{N}$ *and* $G = C_{2n}\square C_{2m+1}$. *Then*

$$rc(G) = diam(G) = n + m.$$

Proof. In any connected graph G, we have $rc(G) \geq diam(G)$, and so $rc(C_{2n}\square C_{2m+1}) \geq n + m$. Now, suppose that $V(G) = \{v_{i,j}|1 \leq i \leq 2n,\ 1 \leq j \leq 2m + 1\}$ is the vertex set of G. We can partition the edges of G into two subsets as follows:

$$E = \{e_{i,j} = v_{i,j}v_{i,j+1}|1 \leq i \leq 2n,\ 1 \leq j \leq 2m + 1,\ v_{i,2m+2} = v_{i,1}\},$$
$$E' = \{e'_{i,j} = v_{i,j}v_{i+1,j}|1 \leq i \leq 2n,\ 1 \leq j \leq 2m + 1,\ v_{2n+1,j} = v_{1,j}\}.$$

Now, we define the edge coloring $c : E \bigcup E' \rightarrow \{1, 2, \ldots, n + m\}$ of the edges of G as follows:

$$for\ 1 \leq i \leq 2n;\quad c(e_{i,j}) = \begin{cases} j & 1 \leq j \leq m, \\ j - m & m + 1 \leq j \leq 2m + 1. \end{cases}$$

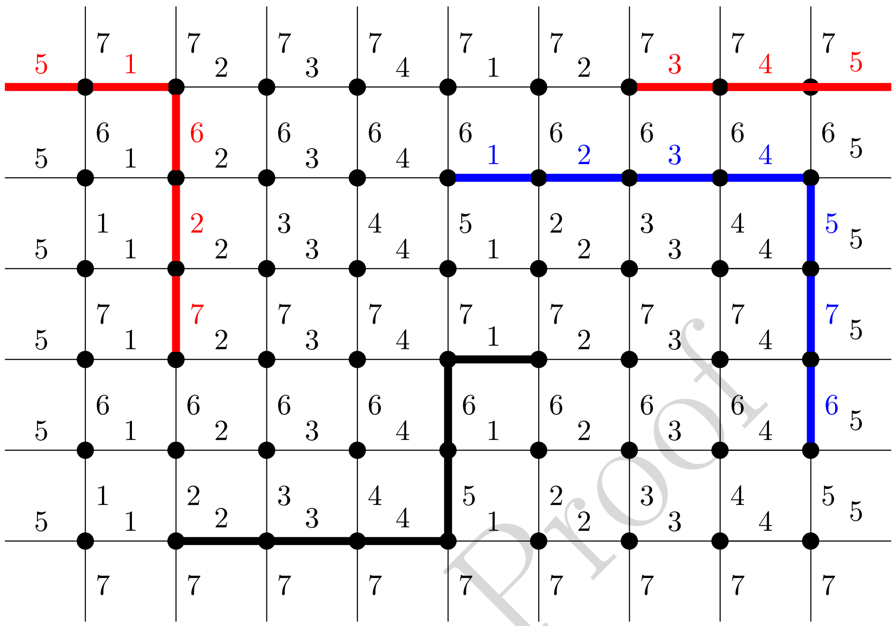

Figure 1: rainbow coloring of $(C_6 \square C_9)$

$$for\ i \in \{1, n+1\};\quad c(e'_{i,j}) = \begin{cases} j & 1 \leq j \leq m+1, \\ j-m & m+1 < j \leq 2m+1. \end{cases}$$

$$if\ \ i \notin \{1, n+1\};\quad c(e'_{i,j}) = \begin{cases} m+i & 1 < i < n+1, \\ m+i-n & n+1 < i \leq 2n. \end{cases}$$

We will show that c is a rainbow edge-coloring of G. We want to show that any two vertices of G are connected by a rainbow path.

Suppose that $x = v_{i,j}, y = v_{i',j'}$ are two arbitrary vertices of G. There are the following cases.

(1) If $i = i'$ or $j = j'$ then x and y are in the same copy of C_{2n} or C_{2m+1}. By comparing the edge coloring c of G with the rainbow coloring of a cycle C_n that is defined in Lemma 3.10, it is clear to see that x and y are connected by a rainbow path of length at most n if $j = j'$ (m if $i = i'$).

(2) If $i \neq i'$ and $j < j'$ there is $d(x, y) \leq diam(G) = m+n$, We consider three subcases.

(A) Let $j \leq m+1 \leq j'$ and $j' - j \leq m$, then the path

$$P_1 : v_{i,j}, v_{i,j+1}, \ldots, v_{i,m+1}, v_{i\pm1,m+1}, \ldots, v_{i',m+1}, v_{i',m+2}, \ldots, v_{i',j'}$$

is a rainbow path between x and y in G such that the colors of the first part of the path (between $v_{i,j}$ and $v_{i,m+1}$) are $j, j+1, \ldots, m$, the colors of the second part of the path (between $v_{i,m+1}$ and $v_{i',m+1}$) are $i' - i$ distinct colors of the set $\{m+1, m+2, \ldots, m+n\}$ and the colors of the third part of the path (between $v_{i',m+1}$ and $v_{i',j'}$) are colors $1, 2, \ldots, j-1$. So, in this case, there is a rainbow path between $v_{i,j}$ and $v_{i',j'}$ in G. See the black bold path in Figure 1.

(B) If $m+1 < j'$ such that $(|j' - j| > m)$, then the path P_2 of length at most $m+n$.

$$P_2 : x = v_{i,j}, v_{i\pm1,j}, \ldots, q = v_{i',j}, v_{i',j-1}, \ldots, v_{i',j'} = y$$

is a rainbow path between x and y in G like as, the first part of the path from x to q on a cycle C_{2n}^{j} such that the colors of edges of this part are $\{j, m+2, ..., m+n\}$, and the second part of the path from q to y on a cycle $C_{2m+1}^{i'}$ Such that the edges in this part are colored by distinct colors of the colors $\{1, 2, \ldots, j-1\}$ and $\{j', j'+1, \ldots, m+1\}$ on a cycle of C_{2m+1}. So, the path P_2 is a rainbow path between x and y in G.

(C) In the case that $j \leq j' \leq m+1$ or $m+1 \leq j \leq j'$ then the following path P_3 of length at most $m+n$

$$P_3 : x = v_{i,j}, v_{i,j+1}, \ldots, s = v_{i,j'}, v_{i\pm1,j'}, \ldots, v_{i',j'} = y$$

is a rainbow path between x and y in G. The colors of the first part of the path (between x and s) are $j, j+1, \ldots, j'-1$, and the colors of the second part of the path (between s and y) are j' or $j' - m$ and some distinct colors of the set$\{m+2, \ldots, m+n\}$.

As a result, the coloring c is a rainbow coloring of G. Then, $rc(G) \leq m+n$. Therefore, $rc(G) = m+n$. $\square$

Example 3.13. Figure 2 shows a rainbow coloring of $C_4 \square C_5$ with 4 colors.

Conjecture 3.14. Let $n, m \in \mathbb{N} \backslash \{1\}$. Then, $rc(C_{2n+1} \square C_{2m+1}) = n+m+1$. Specially, $rc(C_{2n+1} \square C_{2n+1}) = 2n+1$ for $n \in \mathbb{N} \setminus \{1\}$.

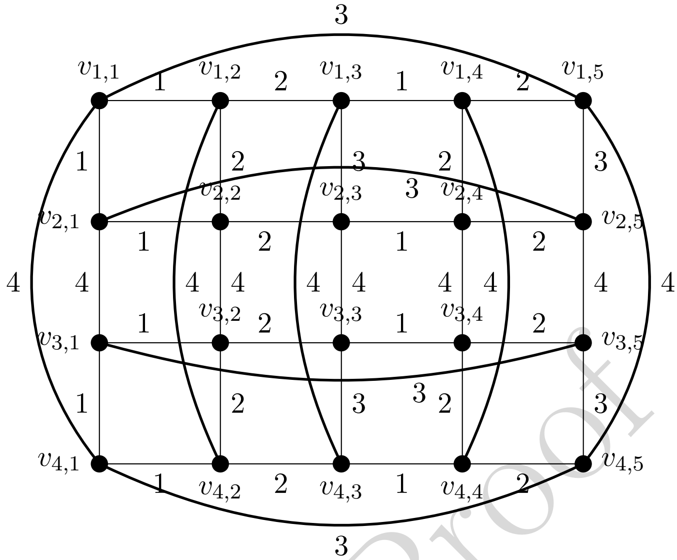

Figure 2: rainbow coloring of $C_4 \square C_5$

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