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The dual-classical Krull dimension of rings via topology

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Abstract. Let R be a ring and $\mathcal{X} = \mathcal{SH}(R) - \{0\}$ be the set all of non-zero strongly hollow ideals (briefly, *sh*-ideals) of R. We first study the concept SH-topology and investigate some of the basic properties of a topological space with this topology. It is shown that, if $\mathcal{X}$ is with SH-topology, then $\mathcal{X}$ is Noetherian if and only if every subset of $\mathcal{X}$ is quasi-compact if and only if R has *dcc* on semi-*sh*-ideals. Finally, the relation between the dual-classical Krull dimension of R and the derived dimension of $\mathcal{X}$ with a certain topology has been studied. It is proved that, if $\mathcal{X}$ has derived dimension, then R has the dual-classical Krull dimension and in case R is a D-ring (i.e., the lattice of ideals of R is distributive), then the converse is true. Moreover these two dimension differ by at most 1.

1 Introduction and Preliminaries

The classical Krull dimension for a ring have been studied by several authors and for commutative Noetherian rings, it coincides with the Krull dimension

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as introduced by Gabriel and Rentschler, see [2]. In 1972, Krause extended the concepts of both dimensions to arbitrary ordinals and investigated their relationship. He also, showed that a ring R has the classical Krull dimension if and only if it has *acc* on prime ideals. In 1980, Karamzedeh [4] studied the *derived dimension* with respect to a certain topology, which he defined on $X = \sum(R)$, the set of all prime ideals of R and showed that R has classical Krull dimension if and only if X has derived dimension and these two dimensions differ by at most 1. Recently, in [3], we introduced the concept of *dual-classical Krull dimension* of rings for any ordinal number, on the set of strongly hollow ideals. Our definition is in the same vein, as definition of the *classical Krull dimension* by Krause, on the set of prime ideals, see [2]. Also, we dualized almost all the basic results of classical Krull dimension. At the end of [3], we raised a question namely, what is the topological expression for the algebraic concept of dual-classical Krull dimension. In this paper, we answer this question. For this purpose, we first define SH-topology and investigate some of the basic properties of a topological space with this topology. Among various findings, it is proved that, R has *dcc* on sh-ideals if and only if R has *dcc* on finite sum of non-comparable sh-ideals. Also, it is shown that if $\mathcal{X}$ is a topological space with SH-topology, then $\mathcal{X}$ is Noetherian if and only if every subset of $\mathcal{X}$ is quasi-compact if and only if R has *dcc* on semi-sh-ideals. Next, we define *W-topology* and the connection between *derived dimension* and *dual-classical dimension* is verified. It is proved that, if R is a D-ring and $\mathcal{X} = \mathcal{SH}(R) - \{0\}$ be with *W-topology*, then $\mathcal{X}$ has derived dimension if and only if R has dual-classical Krull dimension. Moreover, these two dimension differ by at most 1. In essence, we dualize almost all of the results which Karamzadeh obtained in [4]. It is convenient, when we are dealing with classical and dual-classical Krull dimension, to begin our list of ordinals with 1.
Throughout this paper, all rings are associative with $1 \neq 0$ and by an ideal we mean a two-sided one. An ideal S of a ring R is small, if $S + A \neq R$, for every proper ideal A of R. In what follows, we recall some definitions and facts from [3], which are needed. For more details and some of the basic facts about sh-ideals and the dual-classical Krull dimension of a ring, the reader is referred to [3].

Definition 1.1. An ideal L of a ring R is *strongly hollow* (briefly, sh-ideal), if $L \subseteq A + B$ implies that $L \subseteq A$ or $L \subseteq B$, for every ideals $A,\ B$ of R.

Remark 1.2. We note that every strongly hollow ideal is a hollow ideal. Also, it is clear that every atomic module is a hollow module, but the converse is not true in general. We provide examples of hollow ideals and hollow modules, which are not atomic.

(1) Every uniserial Noetherian ring is hollow which is not atomic. For example, ring $R = \mathbb{Z}_{p^n}$, for some prime number of p and $n \in \mathbb{N}$, is hollow and non-atomic. Since n-dim$(R) = n$-dim$(I) = 0$, for every ideal of R.

(2) Every proper ideal of an uniserial 1-atomic ring, is hollow and non-atomic.

(3) Every proper submodule of an uniserial 1-atomic module is hollow and non-atomic, for example $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module.

The notation $L \subseteq_{sh} R$ means that L is an sh-ideal of R. Also, $\mathcal{SH}(R)$ denote the set of all sh-ideals of R. Note that always, $0 \in \mathcal{SH}(R)$. Also, if R is a simple ring, then $\mathcal{SH}(R) = \{0, R\}$. R is an sh-ring, if it is an sh-ideal or equivalently, $A + B \neq R$, for any proper ideals A and B of R. Every simple ring is an sh-ring.

Definition 1.3. An ideal I is called a *semi-sh-ideal* if it is equal to the sum of all sh-ideals contained in itself. A *semi-sh-ring* is a non-zero ring R, for which R is a *semi-sh-ideal* or equivalently, $R = \sum \mathcal{SH}(R)$.

Definition 1.4. Let R be a ring and $\mathcal{Y} = \mathcal{Y}(R) = \mathcal{SH}(R)$. Set $\mathcal{Y}_{-1} = \{0\}$ and for each ordinal number $\alpha \geq 0$, let $\mathcal{Y}_\alpha$ be the set of all $L \in \mathcal{Y}$ such that for every $L' \in \mathcal{Y}$ strictly contained in L, there exists an ordinal $\beta < \alpha$ such that $L' \in \mathcal{Y}_\beta$. Then $\mathcal{Y}_0 \subsetneq \mathcal{Y}_1 \subsetneq \mathcal{Y}_2 \subsetneq \dots$. The smallest ordinal α, for which $\mathcal{Y}_\alpha = \mathcal{Y}$ is called dual-classical Krull dimension of R and denoted by $d.cl.k$-dim R.

Note that, $d.cl.k$-dim $R = -1$ if and only if $\mathcal{Y} = \{0\}$, that is R has no any non-zero sh-ideal. Also, $\mathcal{Y}_0$ consists of 0 and all minimal sh-ideals.

Proposition 1.5. *Let R be a ring in which the intersection of maximal ideals is zero. Then $d.cl.k$-dim $R \leq 0$.*

Theorem 1.6. *Let R be a ring.*

(1) *If R satisfies dcc on sh-ideals, then R has dual-classical Krull dimension.*

(2) *If R is a D-ring, the converse of* (1) *holds.*

2 SH-topology

Let $\mathcal{X} = \mathcal{X}(R)$ be the set of all non-zero sh-ideals of R, that is $\mathcal{X}(R) = \mathcal{SH}(R) - \{0\}$. In this section, we introduce and study the concept of SH-topology.

We begin with the following definition.

Definition 2.1. Let R be a ring and I be an ideal of R. We define $V(I) = \{L \in \mathcal{X} : L \subseteq I\}$ and $\underline{I} = \Sigma V(I) = \Sigma_{L \in V(I)} L$, that is $\underline{I}$ equals to the sum of all sh-ideals, which contained in I.

In what follows, we give the basic properties of these concepts.

Lemma 2.2. *Let R be a ring and I, J and $\{I_\gamma\}_{\gamma \in \Gamma}$ be ideals of R. Then*

(1) $V(I) = \emptyset$ *if and only if I does not contain any non-zero sh-ideal of R.*

(2) $V(I) = \{I\}$ *if and only if I is a minimal sh-ideal of R.*

(3) $V(I) = \mathcal{X}$ *if and only if I contains all of the sh-ideals of R.*

(4) $V(\cap I_\gamma) = \cap V(I_\gamma)$.

(5) $I = \underline{I}$ *if and only if I is a semi-sh-ideal.*

(6) *If* $V(I) = V(J)$, *then* $\underline{I} = \underline{J}$.

(7) $\underline{\cap I_\gamma} = \cap \underline{I_\gamma}$.

(8) $V(I) = V(\underline{I})$.

(9) $V(I + J) = V(I) \cup V(J)$.

Proof. The proof items (1) up to (7) are obvious.

(8) Since $\underline{I} \subseteq I$, we have $V(\underline{I}) \subseteq V(I)$. Conversely, if $L \in V(I)$, then L is a sh-ideal contained in I, hence $L \subseteq \underline{I}$ and so $L \in V(\underline{I})$. Consequently, $V(I) \subseteq V(\underline{I})$ and we are done.

(9) Clearly, $V(I_1) \cup V(I_2) \subseteq V(I_1 + I_2)$. Now, let $L \in V(I_1 + I_2)$, then L is a non-zero sh-ideal and $L \subseteq I_1 + I_2$. If $L \notin V(I_1)$, then $L \nsubseteq I_1$ and so $L \subseteq I_2$, that is, $L \in V(I_2)$. Hence, $V(I_1 + I_2) \subseteq V(I_1) \cup V(I_2)$. Thus, $V(I_1) \cup V(I_2) = V(I_1 + I_2)$ □

Theorem 2.3. *Let R be a ring. The sets $V(I)$, where I is an ideal of R, satisfy the axioms for closed sets in a topological space.*

Proof. Clearly, $V(0) = \emptyset$, so the empty set is closed. Also, $V(R) = \mathcal{X}$, so the entire space is closed. Now, let I_1, I_2 be two ideals of R. By Lemma 2.2(9), $V(I_1) \cup V(I_2) = V(I_1 + I_2)$, so $V(I_1) \cup V(I_2)$ is closed. is a closed set. Finally, for any collection $\{I_\gamma\}$ of ideals, Lemma 2.2(4) implies that $\cap V(I_\gamma) = V(\cap I_\gamma)$, hence $\cap V(I_\gamma)$ is closed. □

Definition 2.4. The collection of complements of the sets $V(I)$, denoted by $V^c(I)$, is a topology on $\mathcal{X}$ and we call it SH-topology.

Recall that a topological space X is a T_0-space if and only if for every distinct points $x, y \in X$, there is an open set containing one and not the other. Also, X is a T_1-space if and only if for every distinct points $x, y \in X$, there are open sets containing one and not the other. Finally, X is a T_2-space (Hausdorff space) if and only if for every distinct points $x, y \in X$, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

Remark 2.5. The SH-topology on $\mathcal{X}$ is trivial if and only if it is single point. For this, let $\mathcal{X}$ is trivial, then, for all ideal I either $V(I) = \emptyset$ or $V(I) = \mathcal{X}$. Now, if $L_1 \neq L_2$ are *sh*-ideals of R, then $V(L_1) = \mathcal{X} = V(L_2)$. So $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$, thus $L_1 = L_2$, and we are done. Conversely, if $\mathcal{X} = \{L\}$ then for all ideal I we have $V(I) = \emptyset$ or $V(I) = \{L\} = \mathcal{X}$. Hence, $\mathcal{X}$ is a trivial space.

Lemma 2.6. *Let R be a ring and $\mathcal{X} = \mathcal{X}(R)$ be with SH-topology. Then*

(1) *$\mathcal{X}$ is a T_0-space.*

(2) *$\mathcal{X}$ is a T_1-space if and only if every non-zero sh-ideal of R is a minimal sh-ideal if and only if every non-trivial sh-ideal of R is a maximal sh-ideal.*

(3) *If $\mathcal{X}$ is Hausdorff, then there exist ideals I_1 and I_2, such that every sh-ideal of R is contained either in I_1 or I_2.*

Proof. (1) Assume that $L_1 \neq L_2$, so $L_1 \nsubseteq L_2$ or $L_2 \nsubseteq L_1$. If $L_1 \nsubseteq L_2$, then $L_1 \in V^c(L_2)$ while $L_2 \notin V^c(L_2)$. Similarly, if $L_2 \nsubseteq L_1$, then $L_2 \in V^c(L_1)$ while $L_1 \notin V^c(L_1)$ and so $\mathcal{X}$ is a T_0-space.

(2) Suppose that $\mathcal{X}$ is a T_1-space and L and L' are non-zero sh-ideals of R. If $L' \subseteq L$, then any open set $G = V^c(I)$ that contains L' will also contains L. Because, $L \notin G$ implies that $L \subseteq I$ and so $L' \subseteq I$, the contradiction required. Conversely, let any sh-ideal of R be a minimal sh-ideal, then for different sh-ideals of L_1, L_2, we can easy to see that $L_1 \in V^c(L_2)$ and $L_2 \notin V^c(L_2)$ and vice versa. This means that $\mathcal{X}$ is a T_1-space.

(3) If R has at most one sh-ideal, we are done. Now, let $L_1 \neq L_2$ be two sh-ideals of R. Since $\mathcal{X}$ is Hausdorff, then there exist ideals I_1 and I_2 such that, $L_1 \subseteq V^c(I_1)$ and $L_2 \subseteq V^c(I_2)$ and $V^c(I_1) \cap V^c(I_2) = \emptyset$. This implies that $\mathcal{X} = V(I_1) \cup V(I_2) = V(I_1 + I_2)$, hence every sh-ideal of R is contained either in I_1 or I_2. □

Proposition 2.7. *The following statements are equivalent for any ring R.*

(1) *R has dcc on sh-ideals.*

(2) *R has dcc on finite sum of non-comparable sh-ideals.*

Proof. (2) ⇒ (1) It is evident.

(1) ⇒ (2) Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \dots$ be an infinite descending chain of ideals, each of which is of the form $I_n = \Sigma_{L_i \in \Delta_n} L_i$, where Δ_n is a finite set of non-comparable sh-ideals. With out loss of generality, we can assume that (i) $\Delta_n \cap \Delta_{n+1} = \emptyset$ (note, if $L \in \Delta_n \cap \Delta_{n+1}$, then since Δ_{n+1} is a set of non-comparable sh-ideals, then L may not contain any other sh-ideal and we can remove it) and also (ii) For every $L \in \Delta_n$, there exists $L' \in \Delta_{n+1}$ such that $L \supsetneq L'$ (note, otherwise L' can be omitted). Hence, we get an infinite chain $L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_n \supsetneq \dots$ of sh-ideals where $L_i \in \Delta_i$ for each i, this is a contradiction. □

Corollary 2.8. *Let R be a D-ring with dual classical Krull dimension. Then, R has dcc on finite sum of non-comparable sh-ideals.*

Recall that, a topological space X is called Noetherian if it satisfies the ascending chain condition for open subsets.

Proposition 2.9. *Let R be a ring and* $\mathcal{X} = \mathcal{X}(R)$ *be with SH-topology. The following statements are equivalent.*

(1) $\mathcal{X}$ *is Noetherian.*

(2) *Every subset of* $\mathcal{X}$ *is quasi-compact.*

(3) *R has dcc on semi-sh-ideals.*

Proof. (1) ⇒ (2) Let A be a subset of $\mathcal{X}$ and $\{O_\lambda\}$ be an open cover for A. Then $A \subseteq \cup_\lambda O_\lambda$. If $\{O_\lambda\}$ has not a finite subcover of A, then there exists a sequence $\lambda_1, \lambda_2, \lambda_3, \dots$ such that $O_{\lambda_1} \subsetneq O_{\lambda_1} \cup O_{\lambda_2} \subsetneq \cdots \subsetneq \cup_{i=1}^{n} O_{\lambda_i} \subsetneq \dots$, which is a contradiction.

(2) ⇒ (3) Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \dots$ be an infinite descending chain of semi-sh-ideals, each of which is of the form $I_n = \Sigma\Delta_n$, where $\Delta_n \subseteq \mathcal{X}$. Then $V(I_1) \supseteq V(I_2) \supseteq \cdots \supseteq V(I_n) \supseteq \dots$ and so $\mathcal{X} - V(I_1) \subseteq \mathcal{X} - V(I_2) \subseteq \cdots \subseteq \mathcal{X} - V(I_n) \subseteq \dots$. Let $A = \cup(\mathcal{X} - V(I_n))$. Then by hypothesis, there exists $i_1, \dots i_r \in \mathbb{N}$ such that $A = \cup_{j=1}^{r}(\mathcal{X} - V(I_{i_i}))$, so $A = \mathcal{X} - V(I_m)$, where $m = max\{i_1, \dots, i_r\}$. It follows that $V(I_m) = V(I_k)$ for all $k \geq m$. Since I_m and I_k are semi-sh-ideals, we have $I_m = \Sigma V(I_m) = \Sigma V(I_k) = I_k$ for all $k \geq m$ and hence we are done.

(3) ⇒ (1) Let $V(I_1) \supseteq V(I_2) \supseteq \cdots \supseteq V(I_n) \supseteq \dots$ is an infinite descending chain of closed subsets of $\mathcal{X}$. Hence, we have $\Sigma V(I_1) \supseteq \Sigma V(I_2) \supseteq \cdots \supseteq \Sigma V(I_n) \supseteq \dots$. and so $\underline{I_1} \supseteq \underline{I_2} \supseteq \cdots \supseteq \underline{I_n} \supseteq \dots$. By (3), there exists $m \in \mathbb{N}$ such that $\underline{I_m} = \underline{I_k}$, thus $\Sigma V(I_m) = \Sigma V(I_k)$ for all $k \geq m$. Hence, $V(I_m) = V(I_k)$ for all $k \geq m$. □

Corollary 2.10. *Let R be a ring and $\mathcal{X} = \mathcal{X}(R)$ be with SH-topology. If $\mathcal{X}$ is quasi-compact, then R has dual-classical-Krull dimension.*

Proof. Since $\mathcal{X}$ is quasi-compact, then R has *dcc* on semi-sh-ideals, by the part (3) of the previous proposition. Thus, it also has *dcc* on sh-ideals and by Theorem 1.6, it has dual-classical Krull dimension. □

3 Derived dimension versus dual-classical Krull dimension

Recall that if A is a subset of a topological space X, then an element $x \in X$ is called a limit point for A if every open set containing A intersects A in at least one point of A distinct of x. The set of all limit points of A is called the derived set of A and is denoted by A'. Every $x \in A - A'$ is called an isolated point of A. Also, the α-derivative of X is defined by transfinite induction as follows: $X_0 = X$ and $X_{\alpha+1} = X'_\alpha$ and if α is a limit ordinal, $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$. Note that, the sets X_α need not to be closed in general, however, in case X is Hausdorff, every X_α is a closed. X is called scattered, if $X_\alpha = \emptyset$ for some ordinal α. If X is scattered, then the smallest ordinal α

sucah that $X_\alpha = \emptyset$ is called the derived dimension of X and is denoted by $d(X) = \alpha$, see [4].

Let R be a ring and $\mathcal{B} = \{V(I) : I \text{ is an ideal of } R\}$. For each $L \in \mathcal{X}$, clearly $L \in V(L)$, so $\mathcal{X} = \cup\mathcal{B}$. Also, by Lemma 2.2(4), we have $V(I_1 \cap I_2) = V(I_1) \cap V(I_2)$. Hence, $\mathcal{B}$ is a base for some topology on $\mathcal{X} = \mathcal{X}(R) = \mathcal{SH}(R) - \{0\}$. The topology on $\mathcal{X}$ with $\mathcal{B}$ as a base, is called *W-topology.*

Lemma 3.1. *Let R be a ring, $\mathcal{X} = \mathcal{X}(R)$ be with W-topology and $S \subseteq \mathcal{X}$. Then an element $L \in S$ is an isolated point of S if and only if it is a minimal element of S.*

Proof. If $L \in S$ is a minimal element, then $V(L) \cap S = \{L\}$, hence L is an isolated point of S. Conversely, let $L \in S$ be an isolated point. Then, there exists an open subset G of $\mathcal{X}$ such that $G \cap S = \{L\}$. But there exists $V(L')$ such that $L \in V(L') \subseteq G$. This implies that $V(L') \cap S = \{L\}$. Now, let $L'' \in S$ and $L'' \subseteq L$. Then $L'' \in V(L') \cap S = \{L\}$ and so $L'' = L$. Thus, L is a minimal element of S. □

Corollary 3.2. *Let R be a ring, $\mathcal{X} = \mathcal{X}(R)$ be with W-topology and $S \subseteq \mathcal{X}$. The set of all isolated points of S is open.*

Proof. Suppose that $L \in S$. By the previous lemma, L is a minimal element of S and then $V(L) = \{L\} \subseteq S$. According to W-topology, $V(L)$ is open and so S is a open set. □

Let R be a ring and $\mathcal{X} = \mathcal{X}(R)$. We set $\mathcal{X}_0 = \mathcal{X}$ and for every β, by transfinite induction, we define $\mathcal{X}_{\beta+1} = \mathcal{X}'_\beta$, the set of limit points of $\mathcal{X}_\beta$ and $\mathcal{X}_\beta = \cap_{\gamma<\beta}\mathcal{X}_\gamma$, for a limit ordinal β. Note that, $\mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_n \supseteq \ldots$. Something to see this, it suffices to show that every $\mathcal{X}_\beta$ is a closed set. For this manner, we proceed by transfinite induction on β. Clearly, $\mathcal{X}_0 = \mathcal{X}$ is closed. Now, let $\mathcal{X}_\gamma$ be closed for every $\gamma < \beta$. If $\beta = \gamma + 1$, then $\mathcal{X}_\beta = \mathcal{X}_\gamma - \mathcal{S}_\gamma = \mathcal{X}_\gamma \cap \mathcal{S}^c_\gamma$, where $\mathcal{S}_\gamma$ is the set of all isolated points of $\mathcal{X}_\gamma$ which is open, by Corollary 3.2. Hence, $\mathcal{S}^c_\gamma$ is closed and by induction hypothesis, so is $\mathcal{X}_\beta$. If β is a limit ordinal, since the intersection of any family of closed sets is closed, $\mathcal{X}_\beta = \cap_{\gamma<\beta}\mathcal{X}_\gamma$ is closed.

We cite the following well-known fact from [4, Lemma 3].

Lemma 3.3. *The following are equivalent for any toplogical space X.*

(1) *Every nonempty subset of X contains an isolated point.*

(2) *There is an ordinal α such that $X_\alpha = \emptyset$.*

It follows by the above lemma that if every non-empty subset of X, which has an isolated point, then X has derived dimension. The next result is now immediate.

Corollary 3.4. *Let R be a ring and $\mathcal{X} = \mathcal{X}(R)$ be with W-toplology. Then the following statements are equivalent.*

(1) *R has dcc on sh-ideals.*

(2) *$\mathcal{X}$ has derived dimension.*

We need the following result, too.

Theorem 3.5. *Let R be a ring and $\mathcal{X} = \mathcal{X}(R)$ be with W-topology. If $\mathcal{X}$ has derived dimension, then R has dual-classical Krull dimension.*

Proof. By Theorem 1.6 and Corollary 3.4, it is evident. □

We are now ready to prove the following proposition, which is a crucial step towads proving our main result.

Proposition 3.6. *Let R be a ring, $\mathcal{X} = \mathcal{X}(R)$ be with W-topology and $\alpha \geq 0$ be an ordinal. Then $\mathcal{X}_\alpha(R) = \cup_{\beta\leq\alpha}\mathcal{S}_\beta$, where $\mathcal{S}_\beta$ is the set of all isolated points of $\mathcal{X}_\beta$.*

Proof. We proceed by induction on α. For $\alpha = 0$, since $\mathcal{X}_0(R)$ consists of all minimal sh-ideals of R, by Lemma 3.1, we have $\mathcal{X}_0(R) = \mathcal{S}_0$. Let us assume that $\mathcal{X}_\gamma(R) = \cup_{\beta\leq\gamma}\mathcal{S}_\beta$ for all $\gamma < \alpha$. We show that $\mathcal{X}_\alpha(R) = \cup_{\beta\leq\alpha}\mathcal{S}_\beta$. For this, let $L \in \cup_{\beta\leq\alpha}\mathcal{S}_\beta$. Then, $0 \neq L \in \mathcal{S}_\beta$, for some $\beta \leq \alpha$. If $L \in \mathcal{S}_\alpha$, by Lemma 3.1, L is a minimal element of $\mathcal{X}_\alpha$. Hence, if $L' \in \mathcal{X} = \mathcal{SH}(R) - \{0\}$ and $L' \subsetneq L$, then $L' \notin \mathcal{X}_\alpha = \cap_{\beta<\alpha}\mathcal{X}_\beta = \mathcal{X} - \cup_{\beta<\alpha}\mathcal{S}_\beta$. This implies that $L' \in \mathcal{S}_\beta$ for some $\beta < \alpha$. Hence, $L' \in \cup_{\gamma\leq\beta}\mathcal{S}_\gamma$. By induction hypothesis, we have $L' \in \mathcal{X}_\beta(R)$. This shows that $L \in \mathcal{X}_\alpha(R)$. Now, let $L \notin \mathcal{S}_\alpha$. Then $L \in \mathcal{S}_\beta$, for some $\beta < \alpha$. This implies that $L \in \cup_{\gamma\leq\beta}\mathcal{S}_\gamma = \mathcal{X}_\beta(R) \subseteq \mathcal{X}_\alpha(R)$. Therefore, $\mathcal{X}_\alpha(R) \supseteq \cup_{\beta\leq\alpha}\mathcal{S}_\beta$.

Conversely, let $L \in \mathcal{X}_\alpha(R)$. If $L \notin \cup_{\beta<\alpha}\mathcal{S}_\beta$, we show that $L \in \mathcal{S}_\alpha$. To this end, let $0 \neq L' \in \mathcal{X}(R)$ and $L' \subsetneq L$. Then $L' \in \mathcal{X}_\beta(R) - \{0\} = \cup_{\gamma\leq\beta}\mathcal{S}_\gamma$

for some $\beta < \alpha$. This implies that $L' \notin \mathcal{X}_\alpha = \mathcal{X} - \cup_{\gamma<\alpha}\mathcal{S}_\gamma$. But, $L \in \mathcal{X}_\alpha$ and so L is a minimal sh-ideal of $\mathcal{X}_\alpha$. By Lemma 3.1, $L \in \mathcal{S}_\alpha$. Therefore, $\mathcal{X}_\alpha(R) \subseteq \bigcup_{\beta\leq\alpha}\mathcal{S}_\beta$ and this complete the proof. □

Finally, we conclude the article with the following result, which we were after.

Theorem 3.7. *Let R be a D-ring and $\mathcal{X} = \mathcal{X}(R)$ be with W-topology. Then*

(1) *$\mathcal{X}$ has derived dimension if and only if R has dual-classical Krull dimension.*

(2) $d.cl.k\text{-}\dim R \leq d(\mathcal{X}) \leq d.cl.k\text{-}\dim R + 1$. *Moreover, if $d(\mathcal{X})$ is a limit ordinal, then $d(\mathcal{X}) = d.cl.k\text{-}\dim R$, otherwise, $d(\mathcal{X}) = d.cl.k\text{-}\dim R+1$*

Proof. (1) By Theorem 1.6 and Corollary 3.4, is evident.

(2) Let $d.cl.k\text{-}\dim R = \alpha$. Then $\mathcal{Y} = \mathcal{Y}_\alpha(R)$ and by Proposition 3.6 $\mathcal{X} = \cup_{\beta\leq\alpha}\mathcal{S}_\beta$. Hence, $\mathcal{X}_{\alpha+1} = \mathcal{X} - \cup_{\beta\leq\alpha}S_\beta = \emptyset$. Thus $d(\mathcal{X}) \leq \alpha + 1$. Now, let $d(\mathcal{X}) < \alpha$. Then, there exists $\gamma < \alpha$ such that $\mathcal{X}_\gamma = \emptyset$. This implies that $\mathcal{S}_\gamma = \mathcal{S}_{\gamma+1} = \cdots = \mathcal{S}_\alpha = \emptyset$. Hence, $\mathcal{X} = \cup_{\beta\leq\alpha}\mathcal{S}_\beta = \cup_{\beta\leq\gamma}\mathcal{S}_\beta$. Consequently, $\mathcal{Y}(R) = \mathcal{Y}_\gamma(R)$ and so $d.cl.k\text{-}\dim R \leq \gamma < \alpha$, which is a contradiction. Thus, $\alpha \leq d(\mathcal{X})$. Therefor $\alpha \leq d(\mathcal{X}) \leq \alpha + 1$. For the last part, we note that if $d(\mathcal{X})$ is a limit ordinal, then clearly $d(\mathcal{X}) \neq \alpha+1$ and so $d(\mathcal{X}) = \alpha$. Finally, if $d(X) = \gamma + 1$, then $\mathcal{X}_{\gamma+1} = \mathcal{X} - \cup_{\beta\leq\gamma}S_\beta = \emptyset$. Hence, $\mathcal{X} = \cup_{\beta\leq\gamma}S_\beta$. By Proposition 3.6, $\mathcal{Y}_\gamma(R) = \mathcal{Y}$ and so $d.cl.k\text{-}\dim R = \alpha \leq \gamma$. This implies that $\alpha + 1 \leq \gamma + 1 = d(\mathcal{X})$. Thus $d(\mathcal{X}) = \alpha + 1$ and this complete the proof. □

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4 Conclusion

At the end of [3], we raised a question to dualize the work of Karamzadeh concerning the classical Krull dimension, in [4], in order to find a topological concept equivalent to our algebraic concept of dual-classical Krull dimension. In this article, we answer this question. As we mentioned in the introduction, in addition to all the results discussed in this article, this article describes the relationship between a dimension in algebra on non-zero strongly hollow ideals and a dimension in topology defined on these ideals. It can be the beginning of a new research for those interested in the relationship between two algebraic properties of a ring and the topology defined on certain ideals of that ring.

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