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A note on the first nonzero Fitting ideal of a module

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Abstract. Let R be a commutative ring and M be a finitely generated R-module. Let $\mathrm{I}(M)$ be the first nonzero Fitting ideal of M. In this paper we characterize some modules over Noetherian UFDs, whose first nonzero Fitting ideal is a prime ideal. We show that if P is a prime ideal and M is a finitely generated R-module with $\mathrm{I}(M) = P$ and $\mathrm{T}(M_P) \neq 0$, then M is isomorphic to $R/P \oplus N$, for some projective R-module N of constant rank. Also, we investigate some conditions under which $M/\mathrm{T}(M)$ is free.

1 Introduction

Let R be a commutative ring with identity. Given any finitely generated R-module M, we can associate with M a sequence of ideals of R known as the Fitting invariants or Fitting ideals of M. The Fitting ideals are named after H. Fitting who investigated their properties in [5] in 1936. Fitting ideals can provide us with useful information about the structure of a module. We will see that in some cases, if we know the Fitting ideals of a module, then we can determine the structure of the R-module completely. Even when

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this is not the case, the Fitting ideals can still help us to understand some interesting properties of modules.
For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0\ , \tag{1.1}$$

where R^n is a free R-module with basis $\{e_1, \ldots, e_n\}$, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \ldots + a_{n\lambda}e_n$, with λ in some index set Λ. Assume that A be the following matrix:

$$\begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}.$$

We call A the matrix presentation of M with respect to $x_1, ..., x_n$. Let $\mathrm{Fitt}_i(M)$ be an ideal of R generated by the minors of size $n-i$ of matrix A. For $i \geq n$, $\mathrm{Fitt}_i(M)$ is defined R and for $i < 0$, $\mathrm{Fitt}_i(M)$ is defined as the zero ideal. It is known that $\mathrm{Fitt}_i(M)$ is the invariant ideal determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [5]. The ideal $\mathrm{Fitt}_i(M)$ will be called the i-th Fitting ideal of the module M. It follows from the definition that $\mathrm{Fitt}_i(M) \subseteq \mathrm{Fitt}_{i+1}(M)$, for every i. The most important Fitting ideal of M is the first of the $\mathrm{Fitt}_i(M)$ that is nonzero. We shall denote this Fitting ideal by $\mathrm{I}(M)$.

Fitting ideals are also, used in mathematical physics. M. Einsiedler and T. Ward showed how the dynamical properties of the system may be deduced from the Fitting ideals and they proved the entropy and expansiveness related with only the first Fitting ideal. This gives an easy computation instead of computing syzygy modules [4].

D.A. Buchsbaum and D. Eisenbud showed in [2] that if R is a Noetherian ring, then M is a finitely generated projective R-module of constant rank if and only if $\mathrm{I}(M) = R$. A lemma of J. Lipman asserts that if R is a quasilocal ring and $M = R^n/N$, and $\mathrm{I}(M)$ is the jth Fitting ideal of M, then $\mathrm{I}(M)$ is a regular principal ideal if and only if N is finitely generated free and $M/\,\mathrm{T}(M)$ is free of rank j [13]. This result was generalized by J. Ohm [15, Theorem 6.2] and by S. Hadjirezaei [9, Theorem 3.2].

At this point, a natural question arises: if $\mathrm{I}(M)$ is any ideal, for example a maximal ideal, a prime ideal and a primary ideal, how much can we say about the structure of M? In [6]- [11] these questions are answered in some cases. A partial list of important contributors to the theory of Fitting ideals includes H. Fitting, D. A. Buchsbaum, J. Lipman, C. Huneke, D. Katz, D. G. Northcott, D. Eisenbud (for references for each author see [2, 3, 5, 12–14]). Some recent works on Fitting ideals, due to author are [6]-[11].

An element of R is regular if it is a nonzerodivisor and an ideal of R is regular if it contains a regular element. Assume that $\mathrm{T}(M)$, the torsion submodule of M, is the submodule of M consisting of all elements of M that are annihilated by a regular element of R. An R-module M is a torsion module if $M = \mathrm{T}(M)$ and is a torsionfree R-module if $\mathrm{T}(M) = 0$. An R-module M is called a regular module if $\mathrm{I}(M)$ is a regular ideal.

2 Modules whose first nonzero Fitting ideals is prime

Fitting ideals are strong tools to characterize and recognize some properties of modules.

In [6], the authers characterize all modules over local Noetherian UFDs, whose first nonzero Fitting ideal is the maximal ideal.

Theorem 2.1. [6, Theorem 2.2] *Let (R, P) be a Noetherian local UFD and let M be a finitely generated R-module. Then $\mathrm{I}(M) = P$ if and only if*

(1) If M is torsionfree then M is isomorphic to $R^n/\langle (a_1, \ldots, a_n)^t \rangle$, where $P = \langle a_1, \ldots, a_n \rangle$ and n is a positive integer.

(2) If M is not torsionfree then M is isomorphic to $R^n \oplus R/P$, for some positive integer n.

In the following, we partly generalize this theorem to prime ideals.

Theorem 2.2. *Let R be a Noetherian UFD and P be a prime ideal of R. Let M be a finitely generated R-module with $\mathrm{I}(M) = P$ and $\mathrm{T}(M_P) \neq 0$. Then M is isomorphic to $R/P \oplus N$, for some projective R-module N of constant rank.*

Proof. Assume that $P = \langle a_1, a_2, ..., a_k \rangle$, for some positive integer k. We have $\mathrm{I}(M_P) = PR_P$. So by [6, Theorem 2.2], since M_P is not torsion-free, $M_P \cong \frac{R_P}{PR_P} \oplus R_P^{t-1}$, for some positive integer t. Let $\mathrm{T}(M_P) = \langle \frac{x_1}{1} \rangle$.

By [7, Lemma 2.5] $\mathrm{Ann}(\frac{x_1}{1}) = PR_P$, so $\frac{a_i}{1}\frac{x_1}{1} = 0$ and therefore there exist elements $s_i \in R \setminus P$ such that $a_i s_i x_1 = 0$, for all $1 \leq i \leq k$. Put $s = s_1 ... s_k$. Hence $P \subseteq \mathrm{Ann}(sx_1)$. It is easily seen that $\mathrm{Ann}(sx_1) = P$. Replacing x_1 by sx_1, we can assume that $\mathrm{Ann}(x_1) = P$. Assume that $M = \langle x_1, x_2, ..., x_n\rangle$, for some $x_2, ..., x_n \in M$. Let $R^m \xrightarrow{\varphi} R^n \xrightarrow{\psi} M \longrightarrow 0$ be a free presentation of M such that $(a_{ij}) \in M_{n\times m}(R)$ is a matrix presentation of φ. Therefore $R_P^m \xrightarrow{\varphi_P} R_P^n \xrightarrow{\psi_P} M_P \longrightarrow 0$ is a free presentation of M_P. On the other hand, $M_P \cong \frac{R_P}{PR_P} \oplus R_P^{t-1}$, so we have the minimal free presentation $R_P^k \xrightarrow{\varphi'} R_P^t \xrightarrow{\psi'} M_P \longrightarrow 0$, where

$$\varphi' = \begin{pmatrix} a_1 & \dots & a_k \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

By [3, Theorem 20.2], φ_P may be put in the form of $\varphi_P = \begin{pmatrix} \varphi' & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, where 1 is the $(n-t)\times(n-t)$ identity matrix. So we conclude that $a_{11}, a_{12}, ..., a_{1m}$ all belong to P. Let $r_1x_1 \in \langle x_1\rangle \cap \langle x_2, ..., x_n\rangle$, then there exist some $r_2, ..., r_n \in R$ such that $r_1x_1 + \cdots + r_nx_n = 0$. So $(r_1, ..., r_n)^t$ is a linear combination of columns of (a_{ij}). Thus $r_1 \in P = \mathrm{Ann}(x_1)$. This means $\langle x_1\rangle \cap \langle x_2, ..., x_n\rangle = 0$. Hence $M = \langle x_1\rangle \oplus \langle x_2, ..., x_n\rangle$. By [1, Exercise 15], we have $P = P\,\mathrm{I}(\langle x_2, ..., x_n\rangle)$. Thus there exists an element $a \in \mathrm{I}(\langle x_2, ..., x_n\rangle)$ such that $(1 + a)P = 0$ and since R is a domain, $a = -1$. Therefore $\mathrm{I}(\langle x_2, ..., x_n\rangle) = R$ and by [2, Lemma 1], $\langle x_2, ..., x_n\rangle$ is projective of constant rank. So $M = \langle x_1\rangle \oplus \langle x_2, ..., x_n\rangle \cong \frac{R}{P} \oplus N$, for some projective module N of constant rank. □

In what follows, $\mathrm{Min}(R)$ is the set of all minimal prime ideals of R.

Proposition 2.3. Let R be a ring and let M be a finitely generated regular R-module such that $\mathrm{I}(M)$ has a primary decomposition. Then $\sqrt{\mathrm{I}(M_P)} = PR_P$ for every minimal prime ideal P of R if and only if $\sqrt{\mathrm{I}(M)} = \mathbb{N}(R)$ and $\mathrm{Min}(R)$ is finite.

Proof. Let $\sqrt{\mathrm{I}(M)}_P = \sqrt{\mathrm{I}(M)_P} = \sqrt{\mathrm{I}(M_P)} = PR_P$, for every minimal prime ideal P of R. So $\sqrt{\mathrm{I}(M)} \subseteq \mathbb{N}(R)$. Assume that $\mathrm{I}(M) = \bigcap_{i=1}^t I_i$

be a reduced primary decomposition of $\mathrm{I}(M)$. Let P be an arbitrary and fixed minimal prime ideal of R. Since $\sqrt{\mathrm{I}(M_P)} = \bigcap_{i=1}^{t}(\sqrt{I_i})_P = PR_P$, so there exists some j, $1 \leq j \leq t$, such that $\sqrt{I_j} \subseteq P$. Since P is a minimal prime ideal of R and I_j is a primary ideal, hence $\sqrt{I_j} = P$. This implies that $\mathrm{Min}(R)$ has at most t elements. So $\mathrm{Min}(R)$ is finite. It is clear that $\mathbb{N}(R) \subseteq \bigcap_{i=1}^{t}\sqrt{I_i} = \sqrt{\mathrm{I}(M)}$. Hence $\sqrt{\mathrm{I}(M)} = \mathbb{N}(R)$. Conversely, let $\mathrm{Min}(R) = \{P_1, ..., P_s\}$ and let $\sqrt{\mathrm{I}(M)} = \mathbb{N}(R) = \bigcap_{i=1}^{s} P_i$. Since M is regular, for $k = 1, ..., s$, we have $\sqrt{\mathrm{I}(M_{P_k})} = \sqrt{\mathrm{I}(M)}_{P_k} = (\bigcap_{i=1}^{s} P_i)_{P_k} = P_kR_{P_k}$. □

3 T(M) splits

Let $M = R^n/N$ be a finitely generated R-module, where N is a submodule of R^n generated by $\{\mathbf{a}_\lambda; \lambda \in \Lambda\}$. Thus $A = (\mathbf{a}_\lambda)_{\lambda\in\Lambda}$ is a matrix presentation of M. For $\mu = \{j_1, ..., j_q\} \subseteq \Lambda$, $\mathrm{I}_\mu(N)$ is the ideal generated by subdeterminants of size q of the matrix $(a_{ij} : 1 \leq i \leq n, j \in \mu)$.

A lemma of Lipman asserts that if R is a local ring and $M = R^n/N$, and $\mathrm{I}(M)$ is the $(n - q)$th Fitting ideal of M, then $\mathrm{I}(M)$ is a regular principal ideal if and only if N is finitely generated free and $M/\mathrm{T}(M)$ is free of rank $n - q$ [13, Lemma 1].

In [9, Theorem 3.2], the following result is proved.

Theorem 3.1. *Let R be a local ring and let N be a submodule of R^n consisting of elements $\mathbf{a}_\lambda$ with λ in some index set Λ. Let $M \cong R^n/N$ and let $\mathrm{I}(M)$ be the i-th Fitting ideal of M. The following conditions are equivalent:*

(1) $M/\mathrm{T}(M)$ *is free of rank* i*;*

(2) $\mathrm{I}_\mu(N)$ *is a principal regular ideal, for some* $\mu = \{j_1, ..., j_{n-i}\} \subseteq \Lambda$.

Now we want to generalize this theorem to global case. First we need the following Lemmas.

Lemma 3.2. [9, Theorem 2.1] *Let M be a regular R-module generated by elements $x_1, \ldots, x_n$ and let A be the matrix presentation of M with respect to $x_1, ..., x_n$. Then*

$$\mathrm{T}(M) = \{\sum_{i=1}^{n} b_i x_i; \quad \mathrm{rank}((b_1, \ldots, b_n)^t|A) = \mathrm{rank}(A)\}.$$

Lemma 3.3. *Let $M = R^n/N$ be a finitely generated R-module, where N is a submodule of R^n generated by $\{\boldsymbol{a}_\lambda; \lambda \in \Lambda\}$ and $A = (\boldsymbol{a}_\lambda)_{\lambda\in\Lambda}$ be the matrix presentation of M. Assume that $\mathrm{I}(M)$ is the ith Fitting ideal of M and $\mathrm{I}_\mu(N)$ is a regular ideal, for some $\mu = \{j_1, ..., j_{n-i}\} \subseteq \Lambda$. Put $M' = R^n/\langle \boldsymbol{a}_{j_1}, ..., \boldsymbol{a}_{j_{n-i}}\rangle$. Then $\frac{M}{\mathrm{T}(M)} \cong \frac{M'}{\mathrm{T}(M')}$.*

Proof. By definition, $\mathrm{I}_\mu(N)$ is the ideal generated by minors of size $n-i$ of the matrix $\begin{pmatrix} \boldsymbol{a}_{j_1} & \dots & \boldsymbol{a}_{j_{n-i}} \end{pmatrix}$. Since $\mathrm{I}_\mu(N)$ is a regular ideal, the rank of matrix $\begin{pmatrix} \boldsymbol{a}_{j_1} & \dots & \boldsymbol{a}_{j_{n-i}} \end{pmatrix}$ is $n-i$. So $\mathrm{I}(M') = \mathrm{I}_\mu(N)$ is a regular ideal. For $i = 1, ..., n$, assume that $x_i = e_i + N$ and $y_i = e_i + \langle \boldsymbol{a}_{j_1}, ..., \boldsymbol{a}_{j_{n-i}}\rangle$ where $\{e_1, ..., e_n\}$ is the standard basis for R^n. Thus, A is the matrix presentation of M with respect to $x_1, ..., x_n$ and $(\boldsymbol{a}_{j_1} ... \boldsymbol{a}_{j_{n-i}})$ is the matrix presentation of M' with respect to $y_1, ..., y_n$. Define

$$f : M/\mathrm{T}(M) \longrightarrow M'/\mathrm{T}(M'); f(\sum_{i=1}^{n} b_i x_i + \mathrm{T}(M)) = \sum_{i=1}^{n} b_i y_i + \mathrm{T}(M').$$

Since $\mathrm{I}_\mu(N) = \mathrm{I}(M')$ is a regular ideal, $\mathrm{I}(M)$ contains a regular element. By Lemma 3.2, if $\sum_{i=1}^{n} b_i x_i \in \mathrm{T}(M)$, then

$$\mathrm{rank}((b_1, \dots, b_n)^t | A) = \mathrm{rank}(A).$$

Since $\mathrm{rank}(\boldsymbol{a}_{j_1} \dots \boldsymbol{a}_{j_{n-i}}) = \mathrm{rank}(A)$,

$$\mathrm{rank}((b_1, \dots, b_n)^t \mid \boldsymbol{a}_{j_1} \dots \boldsymbol{a}_{j_{n-i}}) = \mathrm{rank}(A).$$

Again by Lemma 3.2, $\sum_{i=1}^{n} b_i y_i \in \mathrm{T}(M')$. Hence f is well-defined. The same argument as above shows that f is injective. So we are done. $\square$

Below using Lemma 3.3 and a Theorem of J. Ohm, we give a shorter proof for [9, Theorem 3.2].

Theorem 3.4. *Let R be a ring and let N be a submodule of R^n consisting of elements $\boldsymbol{a}_\lambda$ with λ in some index set Λ. Let $M \cong R^n/N$ and let $\mathrm{I}(M)$ be the i-th Fitting ideal of M. The following conditions are equivalent:*

(1) *$M/\mathrm{T}(M)$ is free of rank i;*

(2) *$\mathrm{I}_\mu(N)$ is a principal regular ideal, for some $\mu = \{j_1, ..., j_{n-i}\} \subseteq \Lambda$.*

Proof. (1) $\Rightarrow$ (2) Similar to the proof of [9, Theorem 3.2].

(2) $\Rightarrow$ (1) Let $\mathrm{I}_\mu(N)$ be a regular and principal ideal, for some $\mu = \{j_1, ..., j_{n-i}\} \subseteq \Lambda$. Put $M' = R^n/\langle \mathbf{a}_{j_1}, ..., \mathbf{a}_{j_{n-i}}\rangle$. We have $\mathrm{I}(M') = \mathrm{I}_\mu(N)$ is a principal regular ideal, so [15, Theorem 6.2], implies that $\frac{M'}{\mathrm{T}(M')}$ is free of rank i. By Lemma 3.3, $\frac{M}{\mathrm{T}(M)}$ is free of rank i. $\square$

4 About $M/\,\mathrm{I}(M)M$

In this section we obtain the first nonzero Fitting ideal of $M/\,\mathrm{I}(M)M$.

Lemma 4.1. *Let R be a Noetherian ring and M be an R-module generated by n elements. Let I be an ideal of R. Then for every i,*

$$\mathrm{Fitt}_i(\frac{M}{IM}) = \mathrm{Fitt}_i(M) + I\,\mathrm{Fitt}_{i+1}(M) + \cdots + I^{n-(i+1)}\,\mathrm{Fitt}_{n-1}(M) + I^{n-i}.$$

Proof. Let $I = \langle a_1, \ldots, a_k\rangle$. Let $A = (a_{ij})_{n\times m}$ be a matrix presentation of M and $\bar{A}$ be a matrix presentation of $\frac{M}{IM}$. By [7, Lemma 2.3], we have

$$\bar{A} = \begin{pmatrix} a_{11} & \ldots & a_{1m} & a_1 & \ldots & 0 & a_2 & \ldots & 0 & a_k & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nm} & 0 & \ldots & a_1 & 0 & \ldots & a_2 & 0 & \ldots & a_k \end{pmatrix}$$

Thus,

$$\mathrm{Fitt}_0(\frac{M}{IM}) = \mathrm{Fitt}_0(M) + I\,\mathrm{Fitt}_1(M) + I^2\,\mathrm{Fitt}_2(M) + \cdots + I^{n-1}\,\mathrm{Fitt}_{n-1}(M) + I^n.$$

Similarly for every i, we have

$$\mathrm{Fitt}_i(\frac{M}{IM}) = \mathrm{Fitt}_i(M) + I\,\mathrm{Fitt}_{i+1}(M) + \cdots + I^{n-(i+1)}\,\mathrm{Fitt}_{n-1}(M) + I^{n-i}.$$

$\square$

Theorem 4.2. *Let R be a Noetherian ring and M be a finitely generated regular R-module. Let $\mathrm{I}(M) = \mathrm{Fitt}_i(M)$. Then $\mathrm{I}(\frac{M}{\mathrm{I}(M)M}) = \mathrm{I}(M)^{i+1}$.*

Proof. We have $\mathrm{I}(M) = \mathrm{Fitt}_i(M)$. By Lemma 4.1,

$$\mathrm{Fitt}_0(\frac{M}{\mathrm{I}(M)M}) =$$

$$\mathrm{Fitt}_0(M)+\mathrm{I}(M)\,\mathrm{Fitt}_1(M)+\mathrm{I}(M)^2\,\mathrm{Fitt}_2(M)+\cdots+\mathrm{I}(M)^{n-1}\,\mathrm{Fitt}_{n-1}(M)+\mathrm{I}(M)^n$$

$$= \mathrm{I}(M)^i\,\mathrm{Fitt}_i(M) + \cdots + \mathrm{I}(M)^{n-1}\,\mathrm{Fitt}_{n-1}(M) + \mathrm{I}(M)^n$$

$$= \mathrm{I}(M)^{i+1}+\mathrm{I}(M)^{i+1}\,\mathrm{Fitt}_{i+1}(M)\cdots+\mathrm{I}(M)^{n-1}\,\mathrm{Fitt}_{n-1}(M)+\mathrm{I}(M)^n = \mathrm{I}(M)^{i+1}.$$

Since $\mathrm{I}(M)$ is regular, $\mathrm{Fitt}_0(\frac{M}{\mathrm{I}(M)M}) = \mathrm{I}(M)^{i+1} \neq 0$, So $\mathrm{I}(\frac{M}{\mathrm{I}(M)M}) = \mathrm{I}(M)^{i+1}$. □

Lemma 4.3. *Let (R, Q) be a local ring and M be an R-module generated by n elements with $\mathrm{I}(M) = \mathrm{Fitt}_t(M) \neq R$. Then $\mathrm{Fitt}_i(M) \neq \mathrm{Fitt}_j(M)$, for all $i \neq j$; $i, j = t, ..., n$. In fact $\mathrm{Fitt}_i(M) \subseteq Q\,\mathrm{Fitt}_{i+1}(M)$, $t \leq i \leq n-1$.*

Proof. Let (R, Q) be a local ring and $F \xrightarrow{\varphi} G \xrightarrow{\psi} M \longrightarrow 0$ be a minimal free presentation of M. Assume that $(a_{ij}) \in M_{m\times n}(R)$ be a matrix presentation of φ. Since $F \xrightarrow{\varphi} G \xrightarrow{\psi} M \longrightarrow 0$ is a minimal free presentation of M, so $a_{ij} \in Q$, for all $1 \leq i \leq m$, $1 \leq j \leq n$. By definition of Fitting ideals, we have $\mathrm{Fitt}_i(M) \subseteq Q\,\mathrm{Fitt}_{i+1}(M)$, for $t \leq i \leq n-1$. So if $\mathrm{Fitt}_i(M) = \mathrm{Fitt}_{i+1}(M)$, then $\mathrm{Fitt}_i(M) = Q\,\mathrm{Fitt}_i(M)$ and by Nakayama's Lemma $\mathrm{Fitt}_i(M) = 0$, for $t \leq i \leq n-1$, a contradiction. □

Proposition 4.4. Let (R, Q) be a local ring and M be an R-module generated by a minimal generating set with n elements and $\mathrm{I}(M) = \mathrm{Fitt}_i(M) = Q$. Then $\mathrm{I}(\frac{M}{QM}) = Q^n = (\mathrm{I}(M))^n$.

Proof. By Lemma 4.3, $Q \subseteq Q\,\mathrm{Fitt}_{i+1}(M)$. Since Q is a maximal ideal, $\mathrm{Fitt}_{i+1}(M) = R$. Hence by [3, Proposition 20.6], $i + 1 = n$. So $i = n - 1$. On the other hand we know the cardinals of generating sets of M and $\frac{M}{QM}$ are the same. As M/QM is an R/Q vector space, $\frac{M}{QM} \cong (\frac{R}{Q})^n$. Thus $\mathrm{I}(\frac{M}{QM}) = Q^n = (\mathrm{I}(M))^n$, as Theorem 4.2 says. □

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