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Characterization of Monoids by Condition (PWP_S) of right acts

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Abstract. In [8] Valdis Laan introduced Condition (PWP). Golchin and Mohammadzadeh in [3] introduced Condition (PWP_E), where Condition (PWP) implies Condition (PWP_E), but the converse is not true in general. In this paper at first we introduce a generalization of Condition (PWP_E), called Condition (PWP_S). Then will give some general properties and a characterization of monoids for which all right acts satisfy this condition. Also, we give a characterization of monoids, by comparing this property of their right acts with some others. Finally, we will give a characterization of monoid S, for which S_S^I, for any non-empty set I and $S_S^{S\times S}$, satisfy Condition (PWP_S).

1 Introduction

For a monoid S, with 1 as its identity, a non-empty set A is called a right S-act, usually denoted by A_S (or simply A), if on which S acts unitarian from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $a(st) = (as)t$ and $a1 = a$, for all $a \in A$ and $s, t \in S$. Let A

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and B be two right S-acts. A mapping $f : A \to B$ is called a homomorphism of right S-acts or just an S-homomorphism if $f(as) = f(a)s$, for $a \in A$ and $s \in S$. The set of all S-homomorphisms from A into B is denoted by $\mathrm{Hom}(A, B)$. Also **Act-**S is the category of right S-acts.

In [8], Condition (PWP) is defined as the principal weak form of Condition (P). In [3], Condition (PWP_E) is defined as the weak form of Condition (PWP).

In this paper at first we introduce a generalization of Condition (PWP_E), called Condition (PWP_S) and will give some general properties, we also show that Condition (PWP_E) implies Condition (PWP_S) but the converse is not true in general. Then, we will give a characterization of monoids S over which all right S-acts satisfy Condition (PWP_S) and also a characterization of monoids S for which this condition of right S-acts has some other properties and vice versa. Finally, we give a characterization of monoid S, for which S_S^I, for any non-empty set I and $S_S^{S\times S}$, satisfy Condition (PWP_S).

We refer the reader to [5, 6], for basic definitions and terminologies relating to semigroups and acts over monoids and to [2, 8, 9], for definitions and results on flatness which are used here.

2 General Properties

In this section we introduce Condition (PWP_S) and give some results on it.

Recall from [3, 6, 8] the following:
The right S-act A satisfies Condition (P), if for all $a, a' \in A$, $s, s' \in S$,

$$as = a's' \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u, a' = a''v \text{ and } us = vs').$$

The right S-act A satisfies Condition (PWP), if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u, a' = a''v \text{ and } us = vs).$$

The right S-act A satisfies Condition (PWP_E), if for all $a, a' \in A$, $s \in S$,

$$\begin{gathered} as = a's \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(\exists e, f \in E(S)) \\ (ae = a''ue, a'f = a''vf, es = s = fs \text{ and } us = vs). \end{gathered}$$

The right S-act A satisfies Condition (PWP_e), if for all $a, a' \in A$, $e \in E(S)$,

$$ae = a'e \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u, a' = a''v \text{ and } ue = ve).$$

We can easily see that the right S-act A satisfies Condition (PWP_E) if and only if $as = a's$, for $a, a' \in A$, $s \in S$, implies that there exist $a'' \in A$, $u, v \in S$ and $e, f \in E(S)$ such that $ae = a''u$, $a'f = a''v$, $es = s = fs$ and $us = vs$.

Definition 2.1. The right S-act A satisfies Condition (PWP_S), if for all $a, a' \in A$, $s \in S$,

$$\begin{gathered} as = a's \Rightarrow (\exists a'' \in A)(\exists u, v, r, r' \in S) \\ (ar = a''u, a'r' = a''v, rs = s = r's \text{ and } us = vs). \end{gathered}$$

Clearly, Condition (PWP_E) implies Condition (PWP_S) but the converse is not true in general, see example 2.4.

Theorem 2.2. *Let S be a monoid and A be a right S-act. Then:*

1. *S_S and Θ_S satisfy Condition (PWP_S).*
2. *Condition (PWP_E) and Condition (PWP_S) are equivalent in idempotent monoids.*
3. *I is a non-empty set and $A = \prod_{i \in I} A_i$, where A_i, $i \in I$, is right S-act. If A satisfies Condition (PWP_S), then A_i satisfies Condition (PWP_S), for every $i \in I$.*
4. *If I is a non-empty set and $A = \coprod_{i \in I} A_i$, where A_i, $i \in I$, is right S-act, then A satisfies Condition (PWP_S) if and only if A_i satisfies Condition (PWP_S), for every $i \in I$.*
5. *Let $\{B_i | i \in I\}$ be a non-empty family of subacts of A. If for any $i_1, i_2 \in I$ there exists $i_0 \in I$ such that $B_{i_1} \cup B_{i_2} \subseteq B_{i_0}$ and B_{i_0} satisfies Condition (PWP_S), then $\bigcup_{i \in I} B_i$ as a subact of A satisfies Condition (PWP_S).*
6. *Let $\{B_i | i \in I\}$ be a non-empty chain of subacts of A. If every B_i, $i \in I$, satisfies Condition (PWP_S), then $\bigcup_{i \in I} B_i$ as a subact of A satisfies Condition (PWP_S).*

(7) *Any retract of an act satisfying Condition* (PWP_S) *satisfies Condition* (PWP_S).

Proof. Parts (1), (2), (3), (4) and (7) are obvious.

(5): Suppose that $as = a's$, for $a, a' \in \bigcup_{i\in I} B_i$ and $s \in S$. Then there exist $i_1, i_2 \in I$ such that $a \in B_{i_1}$ and $a' \in B_{i_2}$, and so, by assumption, there exists $i_0 \in I$ such that $B_{i_1} \cup B_{i_2} \subseteq B_{i_0}$ and B_{i_0} satisfies Condition (PWP_S). Therefore

$$(\exists a'' \in B_{i_0})(\exists u, v, r, r' \in S); ar = a''u, a'r' = a''v,$$
$$rs = s = r's, us = vs.$$

Since $B_{i_0} \subseteq \bigcup_{i\in I} B_i$ thus $a'' \in \bigcup_{i\in I} B_i$, and so $\bigcup_{i\in I} B_i$ satisfies Condition (PWP_S).

(6): is a special case of (5). Let $a \in B_{i_1}$ and $a' \in B_{i_2}$. Without lose of generality, let $B_{i_1} \subseteq B_{i_2}$. Then $a, a' \in B_{i_1} \cup B_{i_2} = B_{i_2}$. By assumption, B_{i_2} satisfies Condition (PWP_S), and so, by (5), $\bigcup_{i\in I} B_i$ satisfies Condition (PWP_S). □

Recall from [6, 8, 10, 11, 15] the following:

An element s of a monoid S is called right cancellable if $ts = t's$, for $t, t' \in S$, implies $t = t'$ and monoid S is called right cancellative if every element s of S is right cancellable. A right S-act A is called torsion free (TF) if $ac = a'c$, for $a, a' \in A$ and right cancellable element $c \in S$, implies $a = a'$.

A right S-act A is called principally weakly flat (PWF) if the functor $A \otimes_S -$, preserves all embeddings of principal left ideals into S. Also, an element s of a monoid S is called left almost regular if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$ and right cancellable elements $c_1, ..., c_m \in S$ such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\cdots \\ s_mc_m &= s_{m-1}r_m \\ s &= s_mrs. \end{aligned}$$

If all elements of S are left almost regular then S is called left almost regular monoid.

A right S-act A is called GP-flat, if $a \otimes s = a' \otimes s$ in $A \otimes_S S$ for $s \in S$ and $a, a' \in A$, implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A \otimes_S S s^n$. It is obvious that every principally weakly flat act is GP-flat and it is proved that every GP-flat act is torsion free, but the converse of both implications are not true in general.

A monoid S is called left PP if every principal left ideal of S is projective, or equivalently for every $s \in S$ there exists an idempotent e of S such that $ker\rho_s = ker\rho_e$. It is left PSF if every principal left ideal of S is strongly flat, as a left S-act. This is equivalent to saying that S is right semi-cancellative, that is, whenever $su = s'u$, for $s, s', u \in S$, there exists $r \in S$ such that $u = ru$ and $sr = s'r$. Obviously every left PP monoid is left PSF.

An act A_S is called strongly torsion free (STF) if $as = bs$, for any $a, b \in A$ and any $s \in S$, implies $a = b$.

Theorem 2.3. *Let S be a monoid. Then for a right S-act A the following statements hold:*

(1) $(PWP) \Rightarrow (PWP_E) \Rightarrow (PWP_S) \Rightarrow PWF.$

(2) *If S is left PSF, then*

$$(PWP_S) \Leftrightarrow PWF.$$

(3) *If S is left PP, then*

$$(PWP_E) \Leftrightarrow (PWP_S) \Leftrightarrow PWF.$$

(4) *If S is left almost regular, then*

$$(PWP_E) \Leftrightarrow (PWP_S) \Leftrightarrow PWF \Leftrightarrow GP\text{-}flat \Leftrightarrow TF.$$

(5) *If S is right cancellative, then*

$$STF \Leftrightarrow (PWP) \Leftrightarrow (PWP_E) \Leftrightarrow (PWP_S) \Leftrightarrow PWF \Leftrightarrow GP\text{-}flat \Leftrightarrow TF.$$

Proof. (1): Clearly $(PWP) \Rightarrow (PWP_E) \Rightarrow (PWP_S)$. Now let A satisfies Condition (PWP_S) and $as = a's$ for $a, a' \in A$ and $s \in S$ then there exist $a'' \in A$ and $u, v, r, r' \in S$ such that $ar = a''u$, $a'r' = a''v$, $rs = s = r's$ and $us = vs$. So

$$a \otimes s = a \otimes rs = ar \otimes s = a''u \otimes s = a'' \otimes us = a'' \otimes vs = a''v \otimes s = \\ a'r' \otimes s = a' \otimes r's = a' \otimes s$$

in $A_S \otimes Ss$. Thus A is PWF.

(2): Suppose the right S-act A is PWF and let $as = a's$ for $a, a' \in A$ and $s \in S$. By assumption, there exist $n \in \mathbb{N}$ and elements $a_1, a_2, ..., a_n \in A$, $s_1, ..., s_n, t_1, ..., t_n \in S$ such that

$$\begin{array}{cc} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 s = t_1 s \\ a_2 t_2 = a_3 s_3 & s_2 s = t_2 s \\ \cdots & \cdots \\ a_n t_n = a' & s_n s = t_n s. \end{array}$$

Since S is left PSF, $s_1 s = t_1 s$ implies the existence of $r_1 \in S$ such that $r_1 s = s$ and $s_1 r_1 = t_1 r_1$. Also $s_2 s = t_2 s$ implies $s_2 r_1 s = t_2 r_1 s$, and so, there exists $r_2 \in S$ such that $r_2 s = s$, $s_2 r_1 r_2 = t_2 r_1 r_2$ then $r_1 r_2 s = s$, $s_i r_1 r_2 = t_i r_1 r_2$, for $i = 1, 2$.

Continuing this procedure, there exist $r_1, r_2, ..., r_n \in S$ such that $r_1 r_2 ... r_n s = s$, $s_i r_1 r_2 ... r_n = t_i r_1 r_2 ... r_n$, for $1 \leq i \leq n$. Let $r_1 r_2 ... r_n = r$. Thus $rs = s$ and $s_i r = t_i r$, for $1 \leq i \leq n$. So $ar = a_1 s_1 r = a_1 t_1 r = a_2 s_2 r = ... = a_n s_n r = a_n t_n r = a'r$. Let $u = v = r = r'$ and $a'' = a$. So A satisfies Condition (PWP_S).

(3): Suppose S be a left PP monoid. Then by [3, Theorem 2.5], A is principally weakly flat if and only if A satisfies Condition (PWP_E) and by (1),(3) is true.

(4): Suppose S be a left almost regular monoid. Then by [6, IV, Theorem 6.5] every torsion free right S-act is principally weakly flat. Therefore for a right S-act A, torsion freeness and principal weak flatness are equivalent. Also $PWF \Rightarrow GP\text{-}flat \Rightarrow TF$ then for a right S-act A, we will have, $PWF \Leftrightarrow GP\text{-}flat \Leftrightarrow TF$. On the other hand, according to the doual of [6, IV, Proposition 1.3] every left almost regular monoid is left PP. So by (3) for a right S-act A, we will have $(PWP_E) \Leftrightarrow (PWP_S) \Leftrightarrow PWF$. Thus (4) is obtained.

(5): By (1) and definition, we will have $STF \Rightarrow (PWP) \Rightarrow (PWP_E) \Rightarrow (PWP_S) \Rightarrow PWF \Rightarrow GP\text{-}flat \Rightarrow TF$. Since S is a right cancellative monoid for a right S-act A, we will have $STF \Leftrightarrow TF$. Thus (5) is true. $\square$

We recall from [6] that a right ideal K of S satisfies Condition (LU) if for every $k \in K$ there exists $l \in K$ such that $lk = k$.

Example 2.4. Consider the commutative monoid $S = \{x_i^m | i \in \mathbb{R}, m \in \mathbb{N}\} \cup \{1\}$ such that

$$x_i^m x_j^n = \begin{cases} x_j^n & i < j \\ x_i^{m+n} & i = j. \end{cases}$$

Let $K = \{x_i^m | i \in \mathbb{R}, m \in \mathbb{N}\}$. It is evident that K is an ideal of S. Let $x_i^m \in K$ and $j < i$. Then $x_j^m x_i^m = x_i^m$, and so K satisfies Condition (LU). Hence, by [6, III, Proposition 12.19], $A = S \coprod^K S$ is weakly flat and so is principally weakly flat. Since S is left PSF (refer to [12, Example 1.6]), according to Theorem 2.3, A satisfies Condition (PWP_S), Now, we proceed to show that A does not satisfy Condition (PWP_E). Since

$$(1, x)x_i^m = (1, y)x_i^m$$

and $e = 1$ is the only idempotent such that $ex_i^m = x_i^m$, there must exist $a'' \in A$ and $u, u' \in S$ such that $(1, x) = a''u$, $(1, y) = a''u'$ and $ux_i^m = u'x_i^m$. Notice that $(1, x) = a''u$ implies $a'' = (1, x)$ and $u = 1$ but there is no element $u' \in S$ such that $(1, y) = (1, x)u'$.

3 Characterization of monoids by Condition $(\mathbf{PWP_S})$ of right acts

In this section we give a characterization of monoids by Condition (PWP_S) of right S-acts. Also, we give a characterization of monoids, by comparing Condition (PWP_S) of their acts with some others.

Theorem 3.1. *Let S be a left almost regular monoid. Then for a right S-act A, the following statements are equivalent:*

(1) *A satisfies Condition (PWP).*

(2) *A satisfies Conditions (PWP_E) and (PWP_e).*

(3) *A satisfies Conditions (PWP_S) and (PWP_e).*

(4) *A is PWF and satisfies Condition (PWP_e).*

(5) *A is GP-flat and satisfies Condition (PWP_e).*

(6) *A is torsion free and satisfies Condition (PWP_e).*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1): Let $as = a's$, for $a, a' \in A$ and $s \in S$. Since S is left almost regular, there exist elements $r, r_1, r_2, ..., r_m, s_1, s_2, ..., s_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$ such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\cdots \\ s_mc_m &= s_{m-1}r_m \\ s &= s_mrs. \end{aligned}$$

Therefore $as_1c_1 = asr_1 = a'sr_1 = a's_1c_1$, and so, by assumption, $as_1 = a's_1$ becuase A is torsion free. Also

$$as_2c_2 = as_1r_2 = a's_1r_2 = a's_2c_2,$$

which implies $as_2 = a's_2$. Continuing this procedure $as_i = a's_i$, for $1 \leq i \leq m$. On the other hand $s_1c_1 = sr_1 = s_mrsr_1 = s_mrs_1c_1$ which implies $s_1 = s_mrs_1$. Continuing this procedure, $s_i = s_mrs_i$, for $1 \leq i \leq m$. Therefore $s_m = s_mrs_m$, and so, $s_mr, rs_m \in E(S)$. Now $as_m = a's_m$ implies $as_mr = a's_mr$. Since A satisfies Condition (PWP_e), there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$, $us_mr = vs_mr$. Also $s = s_mrs$ implies $us = us_mrs = vs_mrs = vs$, that is, A satisfies Condition (PWP). □

Now, an equivalent condition for a cyclic S-act satisfying Condition (PWP_S) is given.

Theorem 3.2. *Let S be a monoid and ρ be a right congruence on S. Then the cyclic right S-act S/ρ satisfies Condition (PWP_S) if and only if $(xt)\rho(yt)$, for $x, y, t \in S$, implies the existence of elements $u, v, r, r' \in S$ such that $ut = vt$, $(xr)\rho u$, $(yr')\rho v$ and $rt = t = r't$.*

Proof. Necessity: Suppose that S/ρ satisfies Condition (PWP_S) and let $(xt)\rho(yt)$, for $x, y, t \in S$. Then $[x]_\rho t = [y]_\rho t$, and so there exist $r, r', w, w_1, w_2 \in S$ such that $rt = t = r't$, $[x]_\rho r = [w]_\rho w_1$, $[y]_\rho r' = [w]_\rho w_2$ and $w_1 t = w_2 t$. If $ww_1 = u$ and $ww_2 = v$ then $[x]_\rho r = [1]_\rho u$ and $[y]_\rho r' = [1]_\rho v$ and so, $(xr)\rho u$, $(yr')\rho v$ and $ut = vt$.

Sufficiency: Let $[x]_\rho t = [y]_\rho t$, for $x, y, t \in S$. Thus $(xt)\rho(yt)$ and so, by assumption, there exist $u, v, r, r' \in S$ such that $ut = vt$, $(xr)\rho u$, $(yr')\rho v$ and $rt = t = r't$. Hence $[x]_\rho r = [1]_\rho u$ and $[y]_\rho r' = [1]_\rho v$. So S/ρ satisfies Condition (PWP_S). □

Corollary 3.3. *Let $z \in S$. Then the principal right ideal zS satisfies Condition (PWP_S) if and only if $zxt = zyt$, for $x, y, t \in S$, implies the existence of elements $u, v, r, r' \in S$ such that $ut = vt$, $zxr = zu$, $zyr' = zv$ and $rt = t = r't$.*

Proof. Since $zS \cong S/ker\lambda_z$, the result follows from Theorem 3.2 if we put $\rho = ker\lambda_z$. □

Theorem 3.4. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfy Condition (PWP_S).*

(2) *All finitely generated right S-acts satisfy Condition (PWP_S).*

(3) *All cyclic right S-acts satisfy Condition (PWP_S).*

(4) *All monocyclic right S-acts satisfy Condition (PWP_S).*

(5) *All monocyclic right S-acts of the form $S/\rho(s, s^2)$, $s \in S$, satisfy Condition (PWP_S).*

(6) *All right Rees factor acts of S satisfy Condition (PWP_S).*

(7) *All right Rees factor acts of S of the form S/sS, $s \in S$, satisfy Condition (PWP_S).*

(8) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (3) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(5) $\Rightarrow$ (8): All monocyclic right S-acts of the form $S/\rho(s, s^2)$, $s \in S$, are principally weakly flat, by Theorem 2.3(1). Thus by [6, IV, Theorem 6.6], S is regular.

(7) $\Rightarrow$ (8): All right Rees factor acts of the form S/sS, $s \in S$, are principally weakly flat, by Theorem 2.3(1). Thus by [6, IV, Theorem 6.6], S is regular.

(8) $\Rightarrow$ (1): Since S is regular then it is left PP. So by [6, IV, Theorem 6.6] and by Theorem 2.3(3), all right S-acts satisfy Condition (PWP_S). $\square$

Recall from [10], that a right S-act A satisfies Condition (EP), if for all $a \in A$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A)(\exists u, v \in S)(a = a'u = a'v \text{ and } us = vs').$$

It satisfies Condition $(E'P)$, if for all $a \in A$, $s, s', z \in S$,

$$(as = as', sz = s'z) \Rightarrow (\exists a' \in A)(\exists u, v \in S)(a = a'u = a'v \text{ and } us = vs').$$

Recall from [3], that a right S-act A satisfies Condition (E), if for all $a \in A$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \text{ and } us = us').$$

It satisfies Condition (E'), if for all $a \in A$, $s, s', z \in S$,

$$(as = as', sz = s'z) \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \text{ and } us = us').$$

Example 3.5. Condition (E) does not imply Condition (PWP_S) in general, because if $S = (\mathbb{N}, \cdot)$ then $A_{\mathbb{N}} = \mathbb{N} \coprod^{2\mathbb{N}} \mathbb{N} = (1, x)\mathbb{N} \cup (1, y)\mathbb{N}$. Clearly, $(1, x)\mathbb{N} \cong \mathbb{N}_{\mathbb{N}} \cong (1, y)\mathbb{N}$. Since, $\mathbb{N}_{\mathbb{N}}$ satisfies Condition (E) then subacts $(1, x)\mathbb{N}$ and $(1, y)\mathbb{N}$ satisfy Condition (E). So $A_{\mathbb{N}} = (1, x)\mathbb{N} \cup (1, y)\mathbb{N}$ satisfy Condition (E). But $A_{\mathbb{N}}$ does not satisfy Condition (PWP_S), because on the other hand $(1, x)2 = (1, y)2$ implies that there exist $\alpha \in A_{\mathbb{N}}$ and $r, r', u, v \in S$ such that $(1, x)r = \alpha u$, $(1, y)r' = \alpha v$, $r.2 = 2 = r'.2$ and $u2 = v2$. Since $r.2 = 2 = r'.2$ implies $r = r' = 1$ so $(1, x) = \alpha u$ and $(1, y) = \alpha v$. Now, $(1, x) = \alpha u$ implies that there exists $l \in \mathbb{N} \backslash 2\mathbb{N}$ such that $\alpha = (l, x)$. So $(1, y) = \alpha v = (l, x)v$, which is contradiction. So $A_{\mathbb{N}}$ does not satisfy Condition (PWP_S).

Theorem 3.6. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfy Condition (PWP_S).*

(2) *All right S-acts satisfying Condition $(E'P)$ satisfy Condition (PWP_S).*

(3) *All right S-acts satisfying Condition (E') satisfy Condition (PWP_S).*

(4) *All right* S*-acts satisfying Condition* (EP) *satisfy Condition* (PWP_S).

(5) *All right* S*-acts satisfying Condition* (E) *satisfy Condition* (PWP_S).

(6) S *is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6): Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$, and so, s is regular. Now let $sS \neq S$. Then

$$A = S \coprod^{sS} S = \{(l,x) | l \in S \setminus sS\} \mathbin{\dot\cup} sS \mathbin{\dot\cup} \{(t,y) | t \in S \setminus sS\}$$

is a right S-act and

$$B = \{(l,x) | l \in S \setminus sS\} \mathbin{\dot\cup} sS \cong S_S \cong \{(t,y) | t \in S \setminus sS\} \mathbin{\dot\cup} sS = C.$$

where B and C are subacts of A. Also, $A = B \bigcup C$ is generated by two elements $(1,x)$ and $(1,y)$. Since S satisfies Condition (E), B and C satisfy Condition (E), and so, A satisfies Condition (E). Hence, by assumption, A satisfies Condition (PWP_S). Since $(1,x)s = (1,y)s$, then there exist $a \in A$ and $u, v, r, r' \in S$ such that $(1,x)r = au$, $(1,y)r' = av$, $rs = s = r's$ and $us = vs$. Now $(1,x)r = au$ and $(1,y)r' = av$ imply either $r \in sS$ or $r' \in sS$. If $r \in sS$ there exists $s' \in S$ such that $r = ss'$, and so, $s = rs = ss's$. Thus s is regular. If $r' \in sS$, then Similarly s is regular. Therefore S is regular.

(6) $\Rightarrow$ (1): It is obvious, by Theorem 3.4. □

If $s \in S$ such that $sS \neq S$, then, by [6, III, Proposition 12.19], the right S-act $S \coprod^{sS} S$ is principally weakly flat if and only if sS satisfies Condition (LU) and this is equivalent to saying that s is regular. On the other hand if S is regular, then S is left PP, and so, by Theorem 2.3(3) for every right S-act, Condition (PWP_S) is equivalent to principally weakly flat. Hence Theorem 3.6 is true, if we substitute Condition (PWP_S) by principally weakly flat. Moreover, for finitely generated right S-acts and for right S-acts generated by at most (exactly) two elements Theorem 3.6 is also true.

Theorem 3.7. *A right* S*-act* A *satisfies Condition* (PWP_S) *if and only if for* $a, a' \in A$, $s \in S$ *and homomorphism* $f : {}_SS \to {}_SS$, $af(s) = a'f(s)$ *implies that there exist* $a'' \in A$, $u, v, r, r' \in S$ *such that* $f(u) = f(v)$, $f(r) = f(1) = f(r')$ *and* $a \otimes sr = a'' \otimes u$, $a' \otimes sr' = a'' \otimes v$ *in* $A \otimes_S S$.

Proof. Necessity: Let $af(s) = a'f(s)$, for homomorphism $f : {}_SS \to {}_SS$, $a, a' \in A$ and $s \in S$. Then, $asf(1) = a'sf(1)$, and so, there exist $a'' \in A$, $u, v, r, r' \in S$ such that $asr = a''u$, $a'sr' = a''v$, $rf(1) = f(1) = r'f(1)$ and $uf(1) = vf(1)$. Thus $f(r) = f(1) = f(r')$ and $f(u) = f(v)$. Now, by [6, II, Proposition 5.13], $asr = a''u$ and $a'sr' = a''v$ imply $a \otimes sr = a'' \otimes u$ and $a' \otimes sr' = a'' \otimes v$ in $A \otimes_S S$, as required.

Sufficiency: Let $as = a's$, for $a, a' \in A$ and $s \in S$. Define

$$\begin{aligned} f = \rho_s : {}_SS &\to {}_SS \\ x &\mapsto xs. \end{aligned}$$

It is obvious that f is a homomorphism and $af(1) = a'f(1)$. Thus, by assumption, there exist $a'' \in A$, $u, v, r, r' \in S$ such that $a \otimes r = a'' \otimes u$, $a' \otimes r' = a'' \otimes v$ in $A \otimes_S S$, $f(u) = f(v)$ and $f(r) = f(1) = f(r')$. By [6, II, Proposition 5.13], $a \otimes r = a'' \otimes u$ and $a' \otimes r' = a'' \otimes v$ imply $ar = a''u$ and $a'r' = a''v$ respectively. Also, $f(u) = f(v)$ implies $us = vs$ and $f(r) = f(1) = f(r')$ implies $rs = s = r's$, by definition f. Hence A satisfies Condition (PWP_S), as required. $\square$

By putting $r = r' = 1$ in the above theorem, we have the following corollary.

Corollary 3.8. *A right S-act A satisfies Condition (PWP) if and only if for $a, a' \in A$, $s \in S$ and homomorphism $f : {}_SS \to {}_SS$, $af(s) = a'f(s)$, implies that there exist $a'' \in A$ and $u, v \in S$ such that $f(u) = f(v)$ and $a \otimes s = a'' \otimes u$, $a' \otimes s = a'' \otimes v$ in $A \otimes_S S$.*

Theorem 3.9. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfy Condition (PWP_S).*

(2) *All generators in right S-acts satisfy Condition (PWP_S).*

(3) *All finitely generated generators in right S-acts satisfy Condition (PWP_S).*

(4) *All generators generated by at most three elements in right S-acts satisfy Condition (PWP_S).*

(5) *$S \times A_S$ satisfies Condition (PWP_S), for every generator right S-act A.*

(6) *$S \times A_S$ satisfies Condition (PWP_S), for every finitely generated generator right S-act A.*

(7) *$S \times A_S$ satisfies Condition (PWP_S), for every generator right S-act A generated by at most three elements.*

(8) *$S \times A_S$ satisfies Condition (PWP_S), for every right S-act A.*

(9) *$S \times A_S$ satisfies Condition (PWP_S), for every finitely generated right S-act A.*

(10) *$S \times A_S$ satisfies Condition (PWP_S), for every right S-act A generated by at most two elements.*

(11) *A right S-act A satisfies Condition (PWP_S), if $Hom(A_S, S_S) \neq \emptyset$.*

(12) *A finitely generated right S-act A satisfies Condition (PWP_S), if $Hom(A_S, S_S) \neq \emptyset$.*

(13) *A right S-act A generated by at most two elements satisfies Condition (PWP_S), if $Hom(A_S, S_S) \neq \emptyset$.*

(14) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6) $\Rightarrow$ (7), (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (11) $\Rightarrow$ (12) $\Rightarrow$ (13), (1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (11) are obvious.

(1) $\Leftrightarrow$ (14): It is obvious, by Theorem 3.4.

(2) $\Rightarrow$ (8): It is obvious that the mapping $\pi : S \times A_S \to S_S$, where $\pi(s, a) = s$, for $a \in A$ and $s \in S$, is an epimorphism in Act-S, and so, by [6, II, Theorem 3.16], $S \times A_S$ is a generator. Thus by assumption, $S \times A_S$ satisfies Condition (PWP_S).

(10) $\Rightarrow$ (1): Suppose $as = a's$ for $a, a' \in A$ and $s \in S$. If $B_S = aS \bigcup a'S$. It is obvious that B_S is a subact of A_S and B_S is generated by at most two elements. Then by assumption, right S-act $S \times B_S$ satisfies Condition (PWP_S). Since, $as = a's$ implies $(1, a)s = (1, a')s$, there exist $(w, b) \in S \times B_S$, $u, v, r, r' \in S$ such that $(1, a)r = (w, b)u$, $(1, a')r' = (w, b)v$, $us = vs$ and $rs = s = r's$. Then $ar = bu$ and $a'r' = bv$ and so A_S satisfies Condition (PWP_S).

(13) $\Rightarrow$ (2): Suppose A be a generator right S-act, and $as = a's$ for $a, a' \in A$ and $s \in S$. Let $B_S = aS \bigcup a'S$. It is obvious that B_S is a subact of A_S generated by at most two elements. Since A_S is a generator, there exists an epimorphism $\pi : A_S \to S_S$. Obviously $\pi^* = \pi|_{B_S} : B_S \to S_S$ is a homomorphism, then $Hom(B_S, S_S) \neq \emptyset$. Thus, by assumption, B_S satisfies Condition (PWP_S). Now $as = a's$ in B_S implies that there exist

$a'' \in B_S \subseteq A_S$, $u, v, r, r' \in S$ such that $ar = a''u$, $a'r' = a''v$, $us = vs$ and $rs = s = r's$. Hence, A_S satisfies Condition (PWP_S), as required.

(7) $\Rightarrow$ (2): Suppose A be a generator right S-act and $as = a's$ for $a, a' \in A$ and $s \in S$. Since A_S is a generator there exists an epimorphism $\pi : A_S \to S_S$. Let $\pi(z) = 1$. If $B_S = aS \bigcup a'S \bigcup zS$. It is obvious that B_S is a subact of A_S which is generated by at most three elements. Obviously the mapping $\pi^* = \pi|_{B_S} : B_S \to S_S$ is an epimorphism, and so B_S is a generator. Therefore by assumption, right S-act $S \times B_S$, satisfies Condition (PWP_S). Now $as = a's$ in B_S implies $(1, a)s = (1, a')s$ in $S \times B_S$ and so, there exist $(w, a'') \in S \times B_S$, $u, v, r, r' \in S$ such that $(1, a)r = (w, a'')u$, $(1, a')r' = (w, a'')v$, $us = vs$ and $rs = s = r's$. Thus $ar = a''u$ and $a'r' = a''v$. Hence, A_S satisfies Condition (PWP_S), as required.

(4) $\Rightarrow$ (2): Suppose A be a generator right S-act and $as = a's$ for $a, a' \in A$ and $s \in S$. Since A_S is a generator there exists an epimorphism $\pi : A_S \to S_S$. Let $\pi(z) = 1$. If $B_S = aS \bigcup a'S \bigcup zS$, then B_S is a subact of A_S generated by at most three elements. It is obvious that the mapping $\pi^* = \pi|_{B_S} : B_S \to S_S$ is an epimorphism, then B_S is a generator, and so, by assumption, B_S satisfies Condition (PWP_S). Now $as = a's$ in B_S implies that there exist $a'' \in B_S \subseteq A_S$, $u, v, r, r' \in S$ such that $ar = a''u$, $a'r' = a''v$, $us = vs$ and $rs = s = r's$. Hence, A_S satisfies Condition (PWP_S), as required. $\square$

A right S-act A is called $\mathfrak{R}$-torsion free if for any $a, b \in A$ and $c \in S$, c right cancellable, $ac = bc$ and $a \ \mathfrak{R} \ b$ ($\mathfrak{R}$ is the Green's equivalence) imply that $a = b$.

Theorem 3.10. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfy Condition (PWP_S).*

(2) *All $\mathfrak{R}$-torsion free right S-acts satisfy Condition (PWP_S).*

(3) *All $\mathfrak{R}$-torsion free finitely generated right S-acts satisfy Condition (PWP_S).*

(4) *All $\mathfrak{R}$-torsion free right S-acts generated by at most two elements satisfy Condition (PWP_S).*

(5) *All $\mathfrak{R}$-torsion free right S-acts generated by exactly two elements satisfy Condition (PWP_S).*

(6) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6): Every right S-act satisfies Condition (EP) is $\mathfrak{R}$-torsion free, by [14, Proposition 1.2]. Therefore by assumption, all right S-acts satisfying Condition (EP) generated by exactly two elements satisfies Condition (PWP_S). Since Theorem 3.6, for right S-acts generated by exactly two elements is true, S is a regular monoid.

(6) $\Rightarrow$ (1): It is obvious, by Theorem 3.4. $\square$

Notation: C_l (C_r) is the set of all left (right) cancellable elements of S. It is clear that C_l (C_r) is not empty. because $1 \in C_l(C_r)$.

We recall from [6] that a right S-act A is (strongly) faithful if for $s, t \in S$ the equality $as = at$ for (some) all $a \in A$ implies that $s = t$. It is straightforward that every strongly faithful right S-act is faithful, but the converse is not true in general.

By [1, Lemma 2.10], there exists at least one strongly faithful cyclic right (left) S-act if and only if S_S ($_SS$) is a strongly faithful right (left) S-act, which it is equivalent to saying that S is a left (right) cancellative monoid.

Lemma 3.11. *For any monoid S the following statements are equivalent:*

(1) *There exists at least one strongly faithful right(left) S-act.*

(2) *There exists at least one strongly faithful finitely generated right(left) S-act.*

(3) *There exists at least one strongly faithful cyclic right(left) S-act.*

(4) *There exists at least one strongly faithful monocyclic right(left) S-act.*

(5) *For every $s \in S$, $sS(Ss)$ is a strongly faithful right(left) S-act.*

(6) *There exists $s \in S$ such that $sS(Ss)$ is a strongly faithful right(left) S-act.*

(7) *$S_S(_SS)$ is a strongly faithful right(left) S-act.*

(8) *For every $s \in S$, $sS \subseteq C_l(Ss \subseteq C_r)$.*

(9) *There exists $s \in S$ such that $sS \subseteq C_l(Ss \subseteq C_r)$.*

(10) *S is a left(right) cancellative monoid, that is, $S = C_l(S = C_r)$.*

Proof. By [1, Lemma 2.10], it is suffices to show that statements (4) and (7) are equivalent.

(7) $\Rightarrow$ (4): Since $S/\rho(s,s) = S/\Delta_S \cong S_S$; $(s \in S)$, the result is obvious.

(4) $\Rightarrow$ (7): Suppose there exists at least one strongly faithful monocyclic right(left) S-act, then there exists at least one strongly faithful right(left) S-act. Let A be a strongly faithful right(left) S-act, and let $ls = lt(sl = tl)$, for $l, t, s \in S$. Then for every $a \in A$, $als = alt(sla = tla)$. Since A is strongly faithful, the last equality implies that $s = t$. Hence S is a left(right) cancellative monoid and so the result follows. $\square$

Theorem 3.12. *For any monoid S the following statements are equivalent:*

(1) *All strongly faithful right S-acts satisfy Condition (PWP_S).*

(2) *All finitely generated strongly faithful right S-acts satisfy Condition (PWP_S).*

(3) *All strongly faithful right S-acts generated by at most two elements satisfy Condition (PWP_S).*

(4) *All strongly faithful right S-acts generated by exactly two elements satisfy Condition (PWP_S).*

(5) *Either S is not left cancellative or S is regular.*

(6) *Either S is not left cancellative or S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5): If S is not left cancellative, then we are done proved. Otherwise, if $sS = S$, for $s \in S$, then s is regular. Now let $sS \neq S$. Then

$$A = S \coprod^{sS} S = \big\{(l,x) | l \in S \setminus sS\big\} \,\dot{\cup}\, sS \,\dot{\cup}\, \big\{(t,y) | t \in S \setminus sS\big\}$$

is a right S-act and

$$B = \big\{(l,x) | l \in S \setminus sS\big\} \,\dot{\cup}\, sS \cong S_S \cong \big\{(t,y) | t \in S \setminus sS\big\} \,\dot{\cup}\, sS = C.$$

Since S is left cancellative, it is strongly faithful, by Lemma 3.11. Therefore B and C are strongly faithful as subacts of A. Thus A is strongly faithful, and so, by assumption, it satisfies Condition (PWP_S). Now by the proof (5) $\Rightarrow$ (6) of Theorem 3.6, S is regular.

$(5) \Rightarrow (6)$: If S is left cancellative, then it is regular. Thus for every $s \in S$, there exists $x \in S$ such that $sxs = s$, which implies $xs = 1$. Hence $Ss = S$, for every $s \in S$ and so S is group.

$(6) \Rightarrow (1)$: If S is not left cancellative, by Lemma 3.11, we obtain the result. Otherwise, S is regular because it is group, and so, by Theorem 3.4, the result is proved. □

Using a similar argument as in the proof of above theorem and that S_S is always a faithful right S-act, we have the following theorem.

Theorem 3.13. *For any monoid* S *the following statements are equivalent:*

(1) *All right* S*-acts satisfy Condition* (PWP_S).

(2) *All faithful right* S*-acts satisfy Condition* (PWP_S).

(3) *All finitely generated faithful right* S*-acts satisfy Condition* (PWP_S).

(4) *All faithful right* S*-acts generated by at most two elements satisfy Condition* (PWP_S).

(5) *All faithful right* S*-acts generated by exactly two elements satisfy Condition* (PWP_S).

(6) S *is regular.*

For fixed elements $u, v \in S$, a binary relation $P_{u,v}$ on S can be defined as follows:

$$(x, y) \in P_{u,v} \Leftrightarrow ux = vy(x, y \in S).$$

For $s, t \in S$, let $\mu_{s,t} = ker\lambda_s \vee ker\lambda_t$ and for any right ideal I of S, let ρ_I denote the right Rees congruence on S, i.e., for $x, y \in S$,

$$(x, y) \in \rho_I \Leftrightarrow (x = y \vee x, y \in I).$$

For $x, y \in S$

$$L(x, y) = \{(a, b) \in S \times S | ax = by\}$$

is either empty or a subact of ${}_S(S \times S)$. Similarly, we define

$$R(x, y) = \{(a, b) \in S \times S | xa = yb\}.$$

Therefore $P_{u,v} = R(u,v)$, for every $u, v \in S$.

Recall from [6] that a right S-act is called cofree if it is isomorphic to the act $X^S = \{f | f$ is a mapping from S to $X\}$, where fs is defined by $fs(t) = f(st)$, for $f \in X^S$ and $s, t \in S$.

An S-act Q_S is called injective (Inj), if for any monomorphism $\iota : A_S \to B_S$ and any homomorphism $f : A_S \to Q_S$ there exists a homomorphism $\overline{f} : B_S \to Q_S$ such that $f = \overline{f}\iota$. It is called (fg-) weakly injective ((fg-)WI), if it is injective relative to all embeddings of (finitely generated) right ideals into S.

Theorem 3.14. *For any monoid S the following statements are equivalent:*

(1) *All fg-weakly injective right S-acts satisfy Condition (PWP_S).*

(1) *All weakly injective right S-acts satisfy Condition (PWP_S).*

(2) *All injective right S-acts satisfy Condition (PWP_S).*

(3) *All cofree right S-acts satisfy Condition (PWP_S).*

(4) $(\forall s \in S)\ (\exists u, v, r, r' \in S)(rs = s = r's \wedge us = vs)$ *and the following conditions hold:*

(i) $P_{u,v} \subseteq P_{r,s} \circ ker\lambda_s \circ P_{s,r'}$

(ii) $ker\lambda_u \subseteq ker\lambda_r$

(iii) $ker\lambda_v \subseteq ker\lambda_{r'}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious, because $cofree \Rightarrow Inj \Rightarrow WI \Rightarrow fg - WI$.

(4) $\Rightarrow$ (5): Let $s \in S$ and S_1, S_2 be two distinct sets, where $|S_1| = |S_2| = |S|$ and $\alpha : S \to S_1$, $\beta : S \to S_2$ are bijections. Put $X = (S/ker\lambda_s)\ \dot\cup\ S_1\ \dot\cup\ S_2$, and define the mappings $f, g : S \to X$ as follows:

$$f(x) = \begin{cases} [y]_{ker\lambda_s} & if\ there\ exists\ y \in S;\ x = sy \\ \alpha(x) & if\ x \in S \setminus sS \end{cases}$$

$$g(x) = \begin{cases} [y]_{ker\lambda_s} & if\ there\ exists\ y \in S;\ x = sy \\ \\ \beta(x) & if\ x \in S \setminus sS \end{cases}.$$

If there exist $y_1, y_2 \in S$, such that $sy_1 = sy_2$, then $(y_1, y_2) \in ker\lambda_s$, and so, $[y_1]_{ker\lambda_s} = [y_2]_{ker\lambda_s}$, that is, $f(sy_1) = f(sy_2)$. So f is well-defined. Similarly, g is well-defined. According to our definition of f and g, we clearly have $fs = gs$. By assumption, the cofree right S-act $X^S = \{h : S \to X\}$ satisfies Condition (PWP_S), and so, there exist mapping $h : S \to X$, $u, v, r, r' \in S$ such that $fr = hu$, $gr' = hv$, $rs = s = r's$ and $us = vs$. Let $(l_1, l_2) \in P_{u,v}$, for $l_1, l_2 \in S$, then

$$f(rl_1) = (fr)l_1 = (hu)l_1 = h(ul_1) = h(vl_2) = (hv)l_2 = (gr')l_2 = g(r'l_2).$$

Thus there exist $y_1, y_2 \in S$ such that $rl_1 = sy_1$ and $r'l_2 = sy_2$, and so $f(rl_1) = [y_1]_{ker\lambda_s}$ and $g(r'l_2) = [y_2]_{ker\lambda_s}$, which imply $sy_1 = sy_2$. Also

$$\left.\begin{array}{r} rl_1 = sy_1 \Rightarrow (l_1, y_1) \in P_{r,s} \\ sy_1 = sy_2 \Rightarrow (y_1, y_2) \in ker\lambda_s \\ sy_2 = r'l_2 \Rightarrow (y_2, l_2) \in P_{s,r'} \end{array}\right. \Rightarrow (l_1, l_2) \in P_{r,s} \circ ker\lambda_s \circ P_{s,r'}$$

that is, $P_{u,v} \subseteq P_{r,s} \circ ker\lambda_s \circ P_{s,r'}$, and so (i) is proved.
Now let $(t_1, t_2) \in ker\lambda_u$, for $t_1, t_2 \in S$. Then $ut_1 = ut_2$ and so

$$f(rt_1) = (fr)t_1 = (hu)t_1 = h(ut_1) = h(ut_2) = (hu)t_2 = (fr)t_2 = f(rt_2).$$

From definition f, we consider two cases as follows:

Case 1. $rt_1, rt_2 \in S \setminus sS$, then $\alpha(rt_1) = \alpha(rt_2)$, which implies $(t_1, t_2) \in ker\lambda_r$.

Case 2. $rt_1, rt_2 \in sS$ then there exist $y_1, y_2 \in S$ such that $rt_1 = sy_1$ and $rt_2 = sy_2$. Therefore $f(rt_1) = f(rt_2)$ implies $rt_1 = sy_1 = sy_2 = rt_2$, that is $(t_1, t_2) \in ker\lambda_r$. Similarly, (iii) is proved.

(5) $\Rightarrow$ (1): Suppose that A is a fg-weakly injective right S-act and $as = a's$, for $a, a' \in A$ and $s \in S$. By assumption, there exist $u, v, r, r' \in S$ such that $rs = s = r's$, $us = vs$ and conditions (i), (ii), and (iii) hold. Define

$$\varphi : uS \cup vS \to A$$
$$x \mapsto \begin{cases} arp & \exists p \in S : \ x = up \\ & \\ a'r'q & \exists q \in S : \ x = vq \end{cases}.$$

First we show that φ is well-defined. If there exist $p, q \in S$ such that $up = vq$, then $(p, q) \in P_{u,v}$. By (i), there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{r,s}$, $(y_1, y_2) \in ker\lambda_s$ and $(y_2, q) \in P_{s,r'}$. Thus $rp = sy_1$, $sy_1 = sy_2$ and $sy_2 = r'q$. Therefore $arp = asy_1 = a'sy_1 = a'sy_2 = a'r'q$. If there exist $p_1, p_2 \in S$ such that $up_1 = up_2$ then $(p_1, p_2) \in ker\lambda_u$, and so by (ii), $rp_1 = rp_2$, which implies $arp_1 = arp_2$. If there exist $q_1, q_2 \in S$ such that $vq_1 = vq_2$, by (iii), similar to the pervious case, $a'r'q_1 = a'r'q_2$. Thus, φ is well-defined, and obviously it is a homomorphism. Since, by assumption, A is fg-weakly injective, there exists a homomorphism $\psi : S \to A$ such that $\psi|_{uS\cup vS} = \varphi$. Let $a'' = \psi(1)$. Then

$$\begin{cases} ar = \varphi(u) = \psi(u) = \psi(1)u = a''u \\ \\ a'r' = \varphi(v) = \psi(v) = \psi(1)v = a''v \end{cases},$$

that is, A satisfies Condition (PWP_S). □

In the following, we give a classification of monoids when Condition (PWP_S) of their acts implies some other flatness properties.

Theorem 3.15. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (PWP_S) are generator.*

(2) *All finitely generated right S-acts satisfying Condition (PWP_S) are generator.*

(3) *All cyclic right S-acts satisfying Condition (PWP_S) are generator.*

(4) *All right Rees factor acts of S satisfying Condition (PWP_S) are generator.*

(5) *$S = \{1\}$.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$: $\Theta_S \cong S/S_S$ satisfies Condition (PWP_S), by Theorem 2.2, and so, by assumption, $\Theta_S \cong S/S_S$ is generator. Hence there exists an epimorphism $\pi : \Theta_S \to S_S$, which implies $S = \{1\}$.

$(5) \Rightarrow (1)$: Since $S = \{1\}$, all right S-acts are generator. □

Recall from [2, 4], the following:

A right S-act A is called principally weakly kernel flat $(PWKF)$ if the mapping φ is bijective for every pullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$ and it is translation kernel flat (TKF) if the mapping φ is bijective for every pullback diagram $P(S, S, f, f, S)$.

A right S-act A satisfies Condition (P'), if for all $a, a' \in A$, $s, s', z \in S$,

$$(as = a's', sz = s'z) \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u, a' = a''v \text{ and } us = vs').$$

Theorem 3.16. *For any monoid S the following statements are equivalent:*

(1) *S is left PSF and every right S-act satisfying Condition (PWP_S) is $PWKF$.*

(2) *S is left PSF and every right S-act satisfying Condition (PWP_S) is TKF.*

(3) *S is left PSF and every right S-act satisfying Condition (PWP_S) satisfy Condition (PWP).*

(4) *S is left PSF and every right S-act satisfying Condition (PWP_S) satisfy Condition (P').*

(5) *S is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are obvious, since $PWKF \Rightarrow TKF \Rightarrow (PWP)$ and $(P') \Rightarrow (PWP)$.

(3) $\Rightarrow$ (5): Suppose S is not right cancellative. If $I = S \setminus C_r$ then, by [7, Lemma 3.12], I is a proper right ideal of monoid S. Let

$$A = S \coprod^{I} S = \{(l, x) | l \in S \setminus I\} \,\dot{\cup}\, I \,\dot{\cup}\, \{(t, y) | t \in S \setminus I\}.$$

and

$$B = \{(l, x) | l \in S \setminus I\} \,\dot{\cup}\, I \cong S_S \cong \{(t, y) | t \in S \setminus I\} \,\dot{\cup}\, I = C,$$

So $A = B \cup C$ is generated by two elements $(1, x)$ and $(1, y)$. Since S is left PSF, by [7, Lemma 3.12], I satisfies Condition (LU). Thus, by [6, III, Proposition 12.19], A is PWF. Also, by Theorem 2.3(2), A satisfies Condition (PWP_S), and so, by assumption A satisfies Condition (PWP). Therefore $(1, x)i = (1, y)i$ for $i \in I$, implies that there exist $a'' \in A$ and

$u, v \in S$ such that $(1, x) = a''u$, $(1, y) = a''v$ and $ui = vi$. But $(1, x) = a''u$ and $(1, y) = a''v$ imply that there exist $w_1, w_2 \in S \setminus I$ such that $a'' = (w_1, x) = (w_2, y)$, which is a contradiction.

(5) $\Rightarrow$ (4): Since S is right cancellative, by [4, Theorem 2.2], all torsion free right S-acts satisfy Condition (P'), but by Theorem 2.3(1), $(PWP_S) \Rightarrow PWF \Rightarrow TF$. Thus all right S-acts satisfying Condition (PWP_S) satisfy Condition (P'). Also, every right cancellative monoid is left PSF.

(5) $\Rightarrow$ (1): Since S is right cancellative, by [7, Lemma 3.13] and Theorem 2.3(5), Condition (PWP_S) and $PWKF$ are equivalent. Also, every right cancellative monoid is left PSF. □

It is clear that the above theorem is also true for finitely generated right S-acts and right S-acts generated by at most (exactly) two elements.

We recall from [7] that a right S-act A satisfies Condition (PWP_{ssc}) if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists r \in S)(ar = a'r \text{ and } rs = s).$$

Corollary 3.17. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (PWP_S) are $PWKF$ and satisfy Condition (PWP_{ssc}).*

(2) *All right S-acts satisfying Condition (PWP_S) are TKF and satisfy Condition (PWP_{ssc}).*

(3) *All right S-acts satisfying Condition (PWP_S) satisfy Conditions (PWP) and (PWP_{ssc}).*

(4) *All right S-acts satisfying Condition (PWP_S) satisfy Conditions (P') and (PWP_{ssc}).*

(5) *S is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (5): S_S satisfies Condition (PWP_S), by Theorem 2.2(1). Thus by assumption, S_S satisfies Condition (PWP_{ssc}), and so, S is left PSF. Also, all right S-acts satisfying Condition (PWP_S) satisfy Condition (PWP). Thus S is right cancellative, by Theorem 3.16.

(5) $\Rightarrow$ (4): All right S-acts satisfying Condition (PWP_S) satisfy Condition (P'), by Theorem 3.16. Now let A satisfies Condition (PWP_S) and

$as = a's$ for $a, a' \in A$ and $s \in S$. Since A satisfies Condition (PWP_S), there exist $a'' \in A$, $u, v, r, r' \in S$ such that $ar = a''u$, $a'r' = a''v$, $rs = s = r's$ and $us = vs$. Since S is right cancellative, $r = r' = 1$ and $u = v$. Hence $a = a'$, and so, A satisfies Condition (PWP_{ssc}).

(5) $\Rightarrow$ (1): All right S-acts satisfying Condition (PWP_S) are principally weakly kernel flat, by Theorem 3.16. Also by the proof of (5) $\Rightarrow$ (4), all right S-acts satisfying Condition (PWP_S) satisfy Condition (PWP_{ssc}). $\square$

By the proof of Theorem 3.16, we conclude that the above corollary is true for finitely generated right S-acts and also right S-acts generated by at most (exactly) two elements.

Theorem 3.18. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (PWP_S) are (strongly) faithful.*

(2) *All finitely generated right S-acts satisfying Condition (PWP_S) are (strongly) faithful.*

(3) *All cyclic right S-acts satisfying Condition (PWP_S) are (strongly) faithful.*

(4) *All right Rees factor acts of S satisfying Condition (PWP_S) are (strongly) faithful.*

(5) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) are obvious.

(4) $\Rightarrow$ (5): $\Theta_S \cong S/S_S$ satisfies Condition (PWP_S), by Theorem 2.2(1). Thus, by assumption, it is (strongly) faithful, and so $S = \{1\}$. $\square$

We recall from [6] that a right S- act A is called divisible if $Ac = A$, for any left cancellable element $c \in S$.

Theorem 3.19. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are divisible.*

(2) *All right S-acts satisfying Condition (PWP_S) are divisible.*

(3) *All finitely generated right S-acts satisfying Condition (PWP_S) are divisible.*

(4) *All cyclic right S-acts satisfying Condition (PWP_S) are divisible.*

(5) *All monocyclic right S-acts satisfying Condition (PWP_S), are divisible.*

(6) $Sc = S$, *for every* $c \in C_l$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6): The monocyclic right S-act $S_S \cong S/\Delta_S = S/\rho_{(x,x)}$, $x \in S$, satisfies Condition (PWP_S), and so, it is divisible, that is $Sc = S$, for every $c \in C_l$.

(6) $\Rightarrow$ (1): It is obvious, by [6, III, Proposition 2.2]. □

Theorem 3.20. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (PWP_S) satisfy Condition (PWP_{ssc}).*

(2) *All finitely generated right S-acts satisfying Condition (PWP_S) satisfy Condition (PWP_{ssc}).*

(3) *All cyclic right S-acts satisfying Condition (PWP_S) satisfy Condition (PWP_{ssc}).*

(4) *All monocyclic right S-acts satisfying Condition (PWP_S) satisfy Condition (PWP_{ssc}).*

(5) *S is left PSF.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5): The monocyclic right S-act $S_S \cong S/\Delta_S = S/\rho_{(x,x)}$, $x \in S$, satisfies Condition (PWP_S), and so, it satisfies Condition (PWP_{ssc}). Thus S is left PSF, by [7, Theorem 2.2].

(5) $\Rightarrow$ (1): Suppose A satisfies Condition (PWP_S). Thus A is PWF, by Theorem 2.3(1), and so, A satisfies Condition (PWP_{ssc}), by [7, Theorem 2.8]. □

4 Characterization of monoid S by Condition $(\mathbf{PWP_S})$ of $\mathbf{S_S^I}$ and $\mathbf{S_S^{S\times S}}$

In this section, we give equivalent conditions that S_S^I, for any nonempty set I and $S_S^{S\times S}$, satisfy Condition (PWP_S).

Recall that for any nonempty set I, S_S^I is the product of a family of S in **Act-**S.

Recall from [1, 10, 13] following:

The right S-act $S \times S$ equipped with the right S-action $(s,t)u = (su, tu)$, for $s, t, u \in S$ is called the diagonal act of S and will be denoted by $D(S)$ or S_S^2.

A monoid S is called left PCP, if all principal left ideals of S satisfy Condition (P) or equivalently $sz = tz$, for $s, t, z \in S$, implies that there exist $u, v \in S$ such that $z = uz = vz$ and $su = tv$. Note that in some papers, such as [13], left PCP is denoted by left $P(P)$.

A monoid S is called weakly left $P(P)$ if $as = bs$, $xb = yb$, for $s, x, y, a, b \in S$, imply the existence of $r \in S$, such that $xar = yar$ and $rs = s$. It is obvious that left $PP \Rightarrow$ left $PSF \Rightarrow$ left PCP and by [13, Proposition 2.2] it is proved that every left PCP monoid is weakly left $P(P)$ monoid.

Theorem 4.1. *For any monoid S the following statements are equivalent:*

(1) *S is left PSF.*

(2) *S is left PCP and S_S^n satisfies Condition (PWP_S) for every $n \in \mathbb{N}$.*

(3) *S is weakly left $P(P)$ and S_S^n satisfies Condition (PWP_S) for every $n \in \mathbb{N}$.*

(4) *S is left PCP and $D(S)$ satisfies Condition (PWP_S).*

(5) *S is weakly left $P(P)$ and $D(S)$ satisfies Condition (PWP_S).*

Proof. Implications $(2) \Rightarrow (3)$ and $(4) \Rightarrow (5)$ follow from [13, Proposition 2.2].

Implications $(2) \Rightarrow (4)$ and $(3) \Rightarrow (5)$ are obvious.

$(1) \Rightarrow (2)$: Every left PSF monoid is left PCP. Also, S_S^n is principally weakly flat for every $n \in \mathbb{N}$, by [10, Corollary 2.16]. Since S is left PSF, S_S^n satisfies Condition (PWP_S) for every $n \in \mathbb{N}$, by Theorem 2.3(2).

$(5) \Rightarrow (1)$: S is weakly left $P(P)$ and $D(S)$ is principally weakly flat, by Theorem 2.3(1). Hence S is left PSF, by [13, Theorem 2.5]. $\square$

Theorem 4.2. *For any commutative monoid S the following statements are equivalent:*

(1) *S is left PSF.*

(2) S_S^n *satisfies Condition* (PWP_S), *for every* $n \in \mathbb{N}$.

(3) $D(S)$ *satisfies Condition* (PWP_S).

Proof. Implication (1) $\Rightarrow$ (2) is obvious, by Theorem 4.1.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (1): $D(S)$ is principally weakly flat, by Theorem 2.3(1), and so, S is left PSF, by [13, Proposition 3.2]. □

Theorem 4.3. *For any monoid* S *the following statements are equivalent:*

(1) S *is left* PP.

(2) S *is left* PSF *and* $[1]_{ker\rho_s}$ *as a submonoid of* S, $s \in S$, *contains a right zero.*

(3) S *is left* PCP *and* $[1]_{ker\rho_s}$ *as a submonoid of* S, $s \in S$, *contains a right zero.*

(4) S_S^I *satisfies Condition* (PWP_{ssc}), *for any nonempty set* I.

(5) $S_S^{S\times S}$ *satisfies Condition* (PWP_{ssc}).

(6) S *is left* PSF *and* S_S^I *satisfies Condition* (PWP_S), *for any nonempty set* I.

(7) S *is left* PCP *and* S_S^I *satisfies Condition* (PWP_S), *for any nonempty set* I.

(8) S *is weakly left* $P(P)$ *and* S_S^I *satisfies Condition* (PWP_S), *for any nonempty set* I.

(9) S *is left* PSF *and* $S_S^{S\times S}$ *satisfies Condition* (PWP_S).

(10) S *is left* PCP *and* $S_S^{S\times S}$ *satisfies Condition* (PWP_S).

(11) S *is weakly left* $P(P)$ *and* $S_S^{S\times S}$ *satisfies Condition* (PWP_S).

Proof. Implications (1) $\Leftrightarrow$ (6), (1) $\Leftrightarrow$ (7) and (1) $\Leftrightarrow$ (8) are obvious, by [13, Corollary 2.6] and Theorem 2.3.

Implications (2) $\Rightarrow$ (3), (9) $\Rightarrow$ (10), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (9) are obvious.

(6) $\Rightarrow$ (4): It is obvious, by Theorem 2.3(2) and [7, Theorem 2.8].

(5) $\Rightarrow$ (6): By [7, Theorem 2.8], $S_S^{S\times S}$ is PWF, then S_S^I is PWF, for any nonempty set I, by [12, Proposition 2.2]. Now let $xs = ys$, for $x, y, s \in S$. Put $S \times S = I$, take $i_0 \in I$ and define

$$x_i = \begin{cases} x & i = i_0 \\ 1 & i \neq i_0 \end{cases} \qquad\qquad y_i = \begin{cases} y & i = i_0 \\ 1 & i \neq i_0 \end{cases}$$

for every $i \in I$. Then $(x_i)_I s = (y_i)_I s$, and so, by assumption, there exists $r \in S$ such that $(x_i)_I r = (y_i)_I r$ and $rs = s$. Thus $xr = x_{i_0} r = y_{i_0} r = yr$, that is, S is left PSF, and so, S is left PP, by [13, Corollary 2.6]. Hence (PWP_S) and PWF are equivalent, by Theorem 2.3(2).

(1) $\Rightarrow$ (2): Clearly left PP implies left PSF and $[1]_{ker\rho_s}$, $s \in S$, is a submonoid of S. Since S is left PP, there exists $e \in E(S)$ such that $ker\rho_s = ker\rho_e$, and so, $(1, e) \in ker\rho_e = ker\rho_s$, which implies $e \in [1]_{ker\rho_s}$. Let $t \in [1]_{ker\rho_s}$. Then $(1, t) \in ker\rho_s = ker\rho_e$ implies $te = e$, that is, e is a right zero element in submonoid $[1]_{ker\rho_s}$.

(3) $\Rightarrow$ (1): By assumption, for $s \in S$, there exists $e \in [1]_{ker\rho_s}$ such that $te = e$, for any $t \in [1]_{ker\rho_s}$. Now $(l_1, l_2) \in ker\rho_e$ implies $l_1 e = l_2 e$. Also $e \in [1]_{ker\rho_s}$ implies $s = es$, and so $l_1 s = l_1 es = l_2 es = l_2 s$, that is, $ker\rho_e \subseteq ker\rho_s$. Take $(x, y) \in ker\rho_s$ which implies $xs = ys$. Since S is left PCP, there exist $u, v \in S$ such that $xu = yv$ and $s = us = vs$ so $(1, u), (1, v) \in ker\rho_s$, and so, $u, v \in [1]_{ker\rho_s}$. Therefore $ue = e = ve$ which implies $xue = xe$, $yve = ye$, that is, $ker\rho_s \subseteq ker\rho_e$. Hence S is left PP, as required.

(10) $\Rightarrow$ (11): It is obvious, by [13, Proposition 2.2].

(11) $\Rightarrow$ (1): By Theorem 2.3(1), $S_S^{S\times S}$ is PWF. So S_S^I is PWF, by [12, Proposition 2.2]. Since by assumption, S is weakly left $P(P)$, S is left PP, by [13, Corollary 2.6]. $\square$

Now investigate the previous theorem for commutative monoid S.

Theorem 4.4. *For any commutative monoid S the following statements are equivalent:*

(1) *S is left PP.*

(2) *S_S^I satisfies Condition (PWP_S), for every nonempty set I.*

(3) *$S_S^{S\times S}$ satisfies Condition (PWP_S).*

Proof. Implication (1) $\Rightarrow$ (2) is obvious, by Theorem 4.3.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (1): By Theorem 2.3(1), $S_S^{S\times S}$ is PWF, and so, S_S^I is PWF, by [12, Propositions 2.2]. Hence S is left PP, by [13, Proposition 3.2]. $\square$

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