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Bounded complexes of objects of finite flat dimensions

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Abstract. Let $(\mathcal{R}, \otimes)$ be a symmetric monoidal closed Grothendieck category which has enough flat objects. It is shown that a given object $\mathcal{G}$ in $\mathcal{R}$ has finite flat dimension if and only if it is quasi-isomorphic to a bounded complex of objects of finite flat dimension. In the case in which $\mathcal{R}$ has enough projective objects, we prove that finite flat dimension in $\mathcal{R}$ implies finite projective dimension if and only if any object in $\mathcal{R}$ that is quasi-isomorphic to a bounded complex of objects of finite flat dimension has finite projective dimension. This leads to a generalization of [4, Proposition 2.3] and [15, Theorem]. Moreover, we present a wide class of n-perfect rings.

1 Introduction

Throughout this article, we will assume that $(\mathcal{R}, \otimes)$ is a symmetric monoidal closed Grothendieck category, i.e. there exists a bifunctor $\mathcal{H}om(\text{-},\text{-}) : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ such that for any $\mathcal{G} \in \mathcal{R}$, we have an adjoint pair $(\text{-}\otimes\mathcal{G}, \mathcal{H}om(\mathcal{G},\text{-}))$ of covariant functors. It is sid that $\mathcal{R}$ has enough flat (respectively, projec-

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tive) objects if every object of $\mathcal{R}$ is a homomorphic image of a flat (respectively, projective). Assume that $\mathcal{R}$ has enough projective objects, then, finite projective dimension implies finite flat dimension, but the converse need not be true. Actually, the converse of this statement leads to the following question.

> **Question**: Assume that $\mathcal{R}$ has enough projective objects and $\mathcal{G} \in \mathcal{R}$ has finite flat dimension. Does $\mathcal{G}$ have finite projective dimension?

The first answer to this question was given by Jensen in the category of left modules over a ring R. He proved in [14, Proposition 6] that if the left Big finitistic projective dimension of R is finite, then any left R-module of finite flat dimension has finite projective dimension. Subsequently, it was shown by Simson in [17] that if R is a ring of cardinality at most $\aleph_n$, $n \geq 0$, then any flat left R-module is of projective dimension at most $n+1$ (see also [18]). Afterwards, Foxby proved in ([3, Corollary 3.4]) that if R is a homomorphic image of a noetherian commutative Gorenstein ring with finite Krull dimension then finite flat dimension implies finite projective dimension. Another answer was provided by Jørgensen who showed that if R is a right-noetherian k-algebra (k a field) with a dualizing complex, then finite flat dimension implies finite projective dimension ([15]). Finally, it was shown in [4] that if R is an n-perfect ring then finite flat dimension implies finite projective dimension. In this work, we will prove that if

$$\mathbf{X} : 0 \longrightarrow \mathcal{X}^0 \longrightarrow \mathcal{X}^1 \longrightarrow \cdots \longrightarrow \mathcal{X}^{n-1} \longrightarrow \mathcal{X}^n \longrightarrow 0$$

is a bounded acyclic complex in $\mathcal{R}$ such that for each $i \neq j$, $\mathcal{X}^i$ has finite flat (respectively, projective) dimension, then, $\mathcal{X}^j$ has finite flat (respectively, projective) dimension. Furthermore, if $\mathcal{R}$ has enough projective objects and any flat objet has finite projective dimension, we will show that any object which is quasi-isomorphic to a bounded complex of objects of finite flat dimension has finite projective dimension. Consequently, we deduce that if $\mathcal{R}$ has enough projective objects and any flat objet has finite projective dimension then any object in $\mathcal{R}$ which is quasi-isomorphic to a bounded complexes of flat objects has finite projective dimension. This leads to a generalization of [4, Proposition 2.3] (see Theorem 3.6).

Before starting, let us fix some definitions and notation that will be used in the sequel. Let $\mathbb{A}$ be a Grothendieck category. The *right orthogonal* of a class $\mathbb{X}$ in $\mathbb{A}$ is defined by $\mathbb{X}^{\perp} := \{\mathbf{B} \in \mathbb{A} \mid \mathrm{Ext}^1_{\mathbb{A}}(\mathbf{X}, \mathbf{B}) = 0, \text{ for all } \mathbf{X} \in \mathbb{X}\}$. The *left* orthogonal is defined dually. The cotorsion pair $(\mathbb{X}, \mathbb{X}^{\perp})$ *has enough injectives* (respectively, *projectives*) if for any $\mathbf{A} \in \mathbb{A}$ there is an exact sequence $0 \to \mathbf{A} \to \mathbf{Y}' \to \mathbf{X}' \to 0$ (respectively, $0 \to \mathbf{Y}' \to \mathbf{X}' \to \mathbf{A} \to 0$), where $\mathbf{X}' \in \mathbb{X}$ and $\mathbf{Y}' \in \mathbb{X}^{\perp}$. The sequence $0 \to \mathbf{A} \to \mathbf{Y}' \to \mathbf{X}' \to 0$ (respectively, $0 \to \mathbf{Y}' \to \mathbf{X}' \to \mathbf{A} \to 0$) is called a *special* $\mathbb{X}^{\perp}$*-preenvelope* (respectively, *special* $\mathbb{X}$*-precover*) of $\mathbf{A}$. Moreover, $(\mathbb{X}, \mathbb{X}^{\perp})$ is *cogenerated* by a set if there is a set Y of objects in $\mathbb{X}$ such that $Y^{\perp} = \mathbb{X}^{\perp}$. Theorem 2.1 clearly shows the role of cogenerating sets in relative homological algebra.

This paper is organized as follows. Section 2 is devoted to the flat cotorsion pair in $\mathcal{R}$. The results are applied in Section 3 where we prove that if

$$\mathbf{X} : 0 \longrightarrow \mathcal{X}^0 \longrightarrow \mathcal{X}^1 \longrightarrow \cdots \longrightarrow \mathcal{X}^{n-1} \longrightarrow \mathcal{X}^n \longrightarrow 0$$

is a bounded acyclic complex in $\mathcal{R}$ such that for any $i \neq j$, $\mathcal{X}^i$ has finite flat (respectively, projective) dimension, then, $\mathcal{X}^j$ has finite flat (respectively, projective) dimension. A characterization of an n-perfect category will be discussed in Section 4. Furthermore, for a given ring R, we show that the ring S_R (constructed by Neeman in [16]) is an n-perfect ring if and only if R is n-perfect.

2 Flat cotorsion pair in symmetric monoidal closed Grothendieck categories

This section is devoted to the flat cotorsion pair in $\mathcal{R}$. Recall that, an object $\mathcal{F}$ in $\mathcal{R}$ is said to be *flat* if $\mathcal{F}\otimes$- preserves short exact sequences in $\mathcal{R}$. Let Flat$\mathcal{R}$ be the class of all flat objects in $\mathcal{R}$ that is closed under pure subobjects and directed colimits. An object $\mathcal{C} \in \mathcal{R}$ is said to be cotorsion if $\mathcal{C} \in \mathrm{Flat}\mathcal{R}^{\perp}$. Let Cot$\mathcal{R}$ be the class of all *cotorsion* objects in $\mathcal{R}$. The cotorsion pair $(\mathrm{Flat}\mathcal{R}, \mathrm{Cot}\mathcal{R})$ is known as flat cotorsion pair. Flat cotorsion pair is a well known pair in relative homological algebra that is first introduced by Enochs and has played an important role in the proof of the flat cover conjecture. In this section, we will prove that the flat cotorsion pair in $\mathcal{R}$ is cogenerated by a set. This ensures the existence of flat covers and cotorsion envelopes in $\mathcal{R}$. In the case in which $\mathcal{R}$ has enough flat objects, the flat cotorsion pair is complete and hereditary, i.e. for any short exact sequence $0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}'' \longrightarrow 0$ in $\mathcal{R}$ starting in the cotorsion object $\mathcal{C}'$, $\mathcal{C}$ is cotorsion if and only if $\mathcal{C}''$ is cotorsion.

Recall that a short exact sequence $\mathcal{E}$ in $\mathcal{R}$ is called *pure* if for any $\mathcal{G} \in \mathcal{R}$, $\mathcal{E} \otimes \mathcal{G}$ remains exact. A subobject $\mathcal{G}_0$ of an object $\mathcal{G}$ in $\mathcal{R}$ is called *pure* if the exact sequence $0 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_0 \longrightarrow 0$ is pure. Recall from [13, Theorem 2.3] that, an object $\mathcal{F}$ in $\mathcal{R}$ is flat if and only if any short exact sequence ending in $\mathcal{F}$ is pure. Notice that by [9], this type of flatness and purity are different from the categorical ones. First, we remind the following fundamental Theorem from [1].

Theorem 2.1. (Eklof and Trlifaj). *Let $\mathbb{X}$ be a class of objects in $\mathbb{A}$ such that $(\mathbb{X}, \mathbb{X}^{\perp})$ is cogenerated by a set. Then, $(\mathbb{X}, \mathbb{X}^{\perp})$ has enough injectives. In addition, if any object in $\mathbb{A}$ is a homomorphic image of an object in $\mathbb{X}$ then $(\mathbb{X}, \mathbb{X}^{\perp})$ has enough projectives.*

Proof. See [1, Theorem 10]. □

In the next result, we will see that $(\mathrm{Flat}\mathcal{R}, \mathrm{Cot}\mathcal{R})$ has enough injectives, i.e. any object in $\mathcal{R}$ has a special cotorsion preenvelope. Moreover, if $\mathcal{R}$ has enough flat objects, then, we deduce that $(\mathrm{Flat}\mathcal{R}, \mathrm{Cot}\mathcal{R})$ is a hereditary complete cotorsion pair.

Proposition 2.2. The pair $(\mathrm{Flat}\mathcal{R}, \mathrm{Cot}\mathcal{R})$ has enough injectives. If $\mathcal{R}$ has enough flat objects, then $(\mathrm{Flat}\mathcal{R}, \mathrm{Cot}\mathcal{R})$ has enough projectives.

Proof. Let $\mathcal{G}$ be a non-zero flat object in $\mathcal{R}$. By [2, Theorem 3], there are infinite regular cardinals $\gamma \leq \kappa$ such that any γ-generated subobject of $\mathcal{G}$ is contained in a κ-generated pure subobject of $\mathcal{G}$. So, we can find a non-zero κ-generated pure subobject $\mathcal{G}_0$ of $\mathcal{G}$ such that $\mathcal{G}_0, \mathcal{G}/\mathcal{G}_0$ are flat objects in $\mathcal{R}$. Hence, by transfinite induction we can find an ordinal number λ and a continuous chain $\{\mathcal{G}_\alpha : \alpha < \lambda\}$ of pure subobjects of $\mathcal{G}$, such that $\mathcal{G} = \bigcup_{\alpha<\gamma} \mathcal{G}_\alpha$ (a direct union) and $\mathcal{G}_0$, $\mathcal{G}_{\alpha+1}/\mathcal{G}_\alpha$ are κ-generated flat objects. So, if Y is a representative set of κ-generated flat objects, then $\mathcal{C} \in \mathrm{Cot}\mathcal{R}$ if and only if for any $\mathcal{G} \in Y$, $\mathrm{Ext}^1_{\mathcal{R}}(\mathcal{G}, \mathcal{C}) = 0$. Consequently, $(\mathrm{Flat}\mathcal{R}, \mathrm{Cot}\mathcal{R})$ is cogenerated by Y and so, by Theorem 2.1, for a given object $\mathcal{X}$ in $\mathcal{R}$, there is an exact sequence $0 \longrightarrow \mathcal{X} \xrightarrow{f} \mathcal{C} \xrightarrow{g} \mathcal{F} \longrightarrow 0$ such that $\mathcal{C} \in \mathrm{Cot}\mathcal{R}$ and $\mathcal{F} \in \mathrm{Flat}\mathcal{R}$. Furthermore, if each object is a homomorphic image of a flat object, then for a given $\mathcal{X} \in \mathcal{R}$, we have an exact sequence $0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{X} \longrightarrow 0$ in $\mathcal{R}$ such that $\mathcal{F}'$ is flat. Let $0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{C} \longrightarrow \mathcal{F}' \longrightarrow 0$ be a special cotorsion preenvelope of $\mathcal{H}$. By the following pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{F}'' & = & \mathcal{F}'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

we obtain an exact sequence $0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{F} \longrightarrow \mathcal{X} \longrightarrow 0$ in $\mathcal{R}$ where $\mathcal{F}$ is flat and $\mathcal{C}$ is cotorsion. So, $(\mathrm{Flat}\mathcal{R}, \mathrm{Cot}\mathcal{R})$ has enough projectives. □

The above result shows that for a given object $\mathcal{G}$ in $\mathcal{R}$, the cotorsion dimension of $\mathcal{G}$ (denoted by $\mathrm{cd}\mathcal{G}$) is defined in the usual sense, i.e. $\mathrm{cd}\mathcal{G}$ equals to the length of the minimal cotorsion resolution of $\mathcal{G}$. Unfortunately, we do not know whenever flat covers in $\mathcal{R}$ are epimorphisms. So, we can not talk about minimal flat resolution in the usual way. However, whether this is the case or not, for a given object $\mathcal{X}$ in $\mathcal{R}$, we can construct the complex

$$\cdots \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^{0} \xrightarrow{\epsilon} \mathcal{X} \longrightarrow 0 \tag{2.1}$$

such that $\mathcal{F}^0 \to \mathcal{X}$, $\mathcal{F}^{-1} \to \mathrm{Ker}(\epsilon)$ and $\mathcal{F}^n \to \mathrm{Ker}(\mathcal{F}^{n+1} \to \mathcal{F}^{n+2})$ for $n \geq -2$ are flat covers. In the case in which $\mathcal{R}$ has enough flat objects, (2.1) is called *the minimal flat resolution* of $\mathcal{X}$. The flat dimension of $\mathcal{X}$ (denoted by $\mathrm{fd}\mathcal{X}$) is defined by the length of (2.1). If $\mathcal{R}$ has enough projective objects, the projective dimension of $\mathcal{X}$ is denoted by $\mathrm{pd}\mathcal{X}$. Recall that $\mathcal{R}$ is n-perfect (n is a non-negative integer) if for any $\mathcal{F} \in \mathrm{Flat}\mathcal{R}$, $\mathrm{cd}\mathcal{F} \leq n$ or equivalently $\mathrm{pd}\mathcal{F} \leq n$.

Before we proceed, let us fix some notation. Let $\mathbb{C}(\mathcal{R})$ be the category of all complexes (complexes are written cohomologically) in $\mathcal{R}$ and $\mathbb{C}^{\mathrm{b}}(\mathcal{R})$ its full subcategory consisting of all bounded complexes. A bounded complex

$$\mathbf{G} : 0 \longrightarrow \mathcal{G}^0 \longrightarrow \cdots \longrightarrow \mathcal{G}^{i-1} \xrightarrow{\partial_{\mathbf{G}}^{i-1}} \mathcal{G}^{i} \xrightarrow{\partial_{\mathbf{G}}^{i}} \mathcal{G}^{i+1} \longrightarrow \cdots \longrightarrow \mathcal{G}^{k} \longrightarrow 0$$

is called *acyclic* if for any $n \in \mathbb{Z}$, $\operatorname{Ker} \partial^n_{\mathbf{G}} = \operatorname{Im}\partial^{n-1}_{\mathbf{G}}$ or equivalently, all cohomologies are vanish. Let $\mathbf{X} = (\mathcal{X}^i, \partial^i_{\mathbf{X}})$ be a bounded complex in $\mathcal{R}$. For a given integer $n \in \mathbb{Z}$, $\mathbf{X}[n]$ denotes the complex $\mathbf{X}$ shifted n degrees to the left. Indeed, $\mathbf{X}[n]$ is the following complex

$$\mathbf{X}[n] : 0 \longrightarrow \mathbf{X}[n]^0 \longrightarrow \cdots \longrightarrow \mathbf{X}[n]^i \xrightarrow{\partial^i_{\mathbf{X}[n]}} \mathbf{X}[n]^{i+1} \longrightarrow \cdots \longrightarrow \mathbf{X}[n]^k \longrightarrow 0$$

where $\mathbf{X}[n]^i = \mathcal{X}^{n+i}$ and $\partial^i_{\mathbf{X}[n]} = (-1)^n \partial^{i+n}_{\mathbf{X}}$. A bounded acyclic complex $\mathbf{Y}$ in $\mathcal{R}$ is called *pure acyclic* if for any object $\mathcal{G} \in \mathcal{R}$, $\mathbf{X} \otimes \mathcal{G}$ is acyclic. A morphism $\alpha : \mathbf{A} \to \mathbf{B}$ of bounded complexes is called a *quasi-isomorphism* if the induced morphisms on cohomologies are isomorphisms. Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a morphism of bounded complexes. The *cone* of f, denoted by $\operatorname{Cone}(f)$, is defined by the complex $(\mathbf{X}[1] \oplus \mathbf{Y}, \delta)$ where

$$\delta = \begin{pmatrix} \partial_{\mathbf{X}[1]} & 0 \\ f[1] & \partial_{\mathbf{Y}} \end{pmatrix}.$$

It is known that $f : \mathbf{X} \longrightarrow \mathbf{Y}$ is a *quasi-isomorphism* if and only if $\operatorname{Cone}(f)$ is acyclic.

In the next result, we duplicate the same argument that used in the proof of [11, Proposition 2.3] and deduce the following proposition.

Proposition 2.3. Let $\mathcal{F} \in \mathcal{R}$. Then, we have the following equivalent conditions.

(i) $\operatorname{cd}\mathcal{F} \leq n$.

(ii) There exists an acyclic complex

$$\mathbb{C} : 0 \longrightarrow \mathcal{F} \xrightarrow{\partial^{-1}} \mathcal{C}^0 \xrightarrow{\partial^0} \mathcal{C}^1 \xrightarrow{\partial^1} \cdots \longrightarrow \mathcal{C}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{C}^n \longrightarrow 0$$

such that for any $0 \leq k \leq n$, $\mathcal{C}^k$ is a special cotorsion preenvelope of $\operatorname{Coker}\partial^{k-2}$.

(iii) If

$$\mathbb{C}' : 0 \longrightarrow \mathcal{F} \xrightarrow{\partial^{-1}} \mathcal{C}^0 \xrightarrow{\partial^0} \mathcal{C}^1 \xrightarrow{\partial^1} \cdots \longrightarrow \mathcal{C}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{C}^n \longrightarrow \cdots$$

is an acyclic complex such that for each $0 \leq k$, $\mathcal{C}^k$ is a special cotorsion preenvelope of $\operatorname{Coker}\partial^{k-2}$, then $\operatorname{Im}\partial^{n-1}$ is cotorsion.

Corollary 2.4. *Assume that any flat object in $\mathcal{R}$ has finite cotorsion dimension. Then so has any object in $\mathcal{R}$.*

Proof. Let $\mathcal{X}$ be an object in $\mathcal{R}$. By Proposition 2.2, there exists an exact sequence $0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{C} \longrightarrow \mathcal{F} \longrightarrow 0$ in $\mathcal{R}$ such that $\mathcal{C}$ is cotorsion and $\mathcal{F}$ is flat. By assumption, $\mathcal{F}$ admits a finite minimal cotorsion resolution. So, by Proposition 2.3, we are done. □

Lemma 2.5. *Assume that $0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}' \longrightarrow 0$ is an exact sequence in $\mathcal{R}$ such that $\mathcal{C}$ and $\mathcal{C}'$ are cotorsion. Then,* $\mathrm{cd}\mathcal{X} \leq 1$.

Proof. Let $0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{F} \longrightarrow 0$ be a special cotorsion preenvelope of $\mathcal{X}$. Then, in the following pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{C}_0 & \longrightarrow & \mathcal{P} & \xrightarrow{j} & \mathcal{C}' & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{F} & = & \mathcal{F} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

$\mathcal{P}$ is cotorsion and $\mathcal{P} \cong \mathcal{C} \oplus \mathcal{F}$. It follows that $\mathcal{F}$ is cotorsion and so by Proposition 2.3, $\mathrm{cd}\mathcal{X} \leq 1$. □

Proposition 2.6. Let $0 \longrightarrow \mathcal{X}' \longrightarrow \mathcal{X} \longrightarrow \mathcal{X}'' \longrightarrow 0$ be an exact sequence of flat objects in $\mathcal{R}$. Then, there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns such that $\mathcal{C}', \mathcal{C}, \mathcal{C}''$ (respectively, $\mathcal{F}', \mathcal{F}, \mathcal{F}''$) are cotorsion (respectively, flat) objects in $\mathcal{R}$.

Proof. Let $0 \longrightarrow \mathcal{X}' \xrightarrow{g'} \mathcal{C}' \longrightarrow \mathcal{F}' \longrightarrow 0$ be a special cotorsion preenvelope of $\mathcal{X}'$. In the following pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h} & & \| & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{P} & \xrightarrow{j} & \mathcal{X}'' & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{F}' & = & \mathcal{F}' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

we have $\mathcal{P} = \mathcal{C}' \oplus \mathcal{X}''$ because $\mathcal{C}'$ is cotorsion and $\mathcal{X}''$ is flat.

Let $0 \longrightarrow \mathcal{X}'' \xrightarrow{g''} \mathcal{C}'' \longrightarrow \mathcal{F}'' \longrightarrow 0$ be a special cotorsion preenvelope of $\mathcal{X}''$. Then, we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{P} & \xrightarrow{j} & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle g''} & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C}' \oplus \mathcal{C}'' & \longrightarrow & \mathcal{C}'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}'' & = & \mathcal{F}'' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $t = id_{\mathcal{C}'} \oplus g''$. Therefore, we deduce the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\ 0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C}' \oplus \mathcal{C}'' & \longrightarrow & \mathcal{C}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

such that $g = t \circ h$ and $\mathcal{F}', \mathcal{F}''$ are flat objects. Consequently, $\mathcal{F}$ is flat and hence the proof completes. □

Lemma 2.7. *Assume that $\mathcal{R}$ has enough flat objects and $\mathcal{K} \in \mathcal{R}$ is cotorsion. Then the cohomological functor $\mathrm{Ext}^i_{\mathcal{R}}(-, \mathcal{K})$ vanishes over flat objects for all $i > 0$.*

Proof. It is a direct consequence of [12, Lemma 3.7]. □

Lemma 2.8. *Assume that $\mathcal{R}$ has enough flat objects and*

$$0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}'' \longrightarrow 0$$

is an exact sequence in $\mathcal{R}$ starting in a cotorsion object $\mathcal{C}'$. Then $\mathcal{C}$ is cotorsion if and only if so is $\mathcal{C}''$.

Proof. Let $\mathcal{K}$ be a cotorsion object. By Lemma 2.7, the cohomology functors $\mathrm{Ext}^i_{\mathcal{R}}(-, \mathcal{K})$ vanish over flat objects, for all $i > 0$. So, for a given flat object $\mathcal{F}$, we have the following exact sequence

$$\mathrm{Ext}^i_{\mathcal{R}}(\mathcal{F}, \mathcal{C}') \longrightarrow \mathrm{Ext}^i_{\mathcal{R}}(\mathcal{F}, \mathcal{C}) \longrightarrow \mathrm{Ext}^i_{\mathcal{R}}(\mathcal{F}, \mathcal{C}'') \longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{R}}(\mathcal{F}, \mathcal{C}').$$

It follows that, $\mathcal{C}$ is cotorsion if and only so is $\mathcal{C}''$. □

Proposition 2.9. Assume that $\mathcal{R}$ has enough flat objects and

$$0 \longrightarrow \mathcal{X}' \longrightarrow \mathcal{X} \longrightarrow \mathcal{X}'' \longrightarrow 0$$

is an exact sequence in $\mathcal{R}$. Then, there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns such that $\mathcal{C}', \mathcal{C}, \mathcal{C}''$ (respectively, $\mathcal{F}', \mathcal{F}, \mathcal{F}''$) are cotorsion (respectively, flat) objects in $\mathcal{R}$.

Proof. Let $0 \longrightarrow \mathcal{X}' \longrightarrow \mathcal{C}' \longrightarrow \mathcal{F}' \longrightarrow 0$ be a special cotorsion preenvelope of $\mathcal{X}'$. Consider the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{P} & \xrightarrow{j} & \mathcal{X}'' & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{F}' & = & \mathcal{F}' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and let $0 \longrightarrow \mathcal{P} \xrightarrow{i} \mathcal{C} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be a special cotorsion preenvelope of $\mathcal{P}$. By the pushout of i and j, we obtain the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{P} & \xrightarrow{j} & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}'' & = & \mathcal{F}'' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where, by Lemma 2.8, $\mathcal{C}''$ is cotorsion. So, we deduce the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

such that $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{F}''$ are flat objects. □

In the next, by the same argument that used in the proof of [10, Lemma 2.1], we deduce the following result.

Lemma 2.10. *Assume that $\mathcal{R}$ has enough flat objects and*

$$0 \longrightarrow \mathcal{X}' \longrightarrow \mathcal{X} \longrightarrow \mathcal{X}'' \longrightarrow 0$$

is an exact sequence in $\mathcal{R}$. Then, there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns such that $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ (respectively, $\mathcal{C}', \mathcal{C}, \mathcal{C}''$) are flat (respectively, cotorsion) objects in $\mathcal{R}$.

3 Finite flat and cotorsion (respectively, projective) dimension

In this section we will prove some general results on flat, projective and cotorsion dimensions. In particular, we will show that if

$$\mathbf{X} : 0 \longrightarrow \mathcal{X}^0 \xrightarrow{\partial^0} \mathcal{X}^1 \xrightarrow{\partial^1} \mathcal{X}^2 \xrightarrow{\partial^2} \cdots \longrightarrow \mathcal{X}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{X}^n \longrightarrow 0$$

is a bounded acyclic complex in $\mathcal{R}$ such that for each $i \neq j$, $\mathcal{X}^i$ has finite flat (respectively, projective, cotorsion) dimension, then the flat (respectively, projective, cotorsion) dimension of $\mathcal{X}^j$ is also finite.

Theorem 3.1. *Let*

$$\mathbf{X} : 0 \longrightarrow \mathcal{X}^0 \xrightarrow{\partial^0} \mathcal{X}^1 \xrightarrow{\partial^1} \mathcal{X}^2 \xrightarrow{\partial^2} \cdots \longrightarrow \mathcal{X}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{X}^n \longrightarrow 0$$

be a bounded acyclic complex of flat objects in $\mathcal{R}$ such that for each $i \neq j$, $\mathrm{cd}\mathcal{X}^i < +\infty$. Then, $\mathrm{cd}\mathcal{X}^j < +\infty$.

Proof. By Proposition 2.6 and Proposition 2.3, there exists a positive integer k and the following commutative diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}^0 & \longrightarrow & \mathcal{X}^1 & \longrightarrow & \mathcal{X}^2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{X}^{n-1} & \longrightarrow & \mathcal{X}^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}_0^0 & \longrightarrow & \mathcal{C}_0^1 & \longrightarrow & \mathcal{C}_0^2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{C}_0^{n-1} & \longrightarrow & \mathcal{C}_0^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \cdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}_{k-1}^0 & \longrightarrow & \mathcal{C}_{k-1}^1 & \longrightarrow & \mathcal{C}_{k-1}^2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{C}_{k-1}^{n-1} & \longrightarrow & \mathcal{C}_{k-1}^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_k^0 & \xrightarrow{\partial_k^0} & \mathcal{F}_k^1 & \xrightarrow{\partial_k^1} & \mathcal{F}_k^2 & \xrightarrow{\partial_k^2} & \cdots & \longrightarrow & \mathcal{F}_k^{n-1} & \xrightarrow{\partial_k^{n-1}} & \mathcal{F}_k^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & & & &
\end{array} \tag{3.1}$$

with exact rows and pure exact columns such that for any $0 \leq j \leq k-1$ and $0 \leq i \leq n$, $\mathcal{C}_j^i$ is a cotorsion flat object and $\mathcal{F}_k^i$ is flat. Consider the following three cases.

1: Let $j = n$. By Proposition 2.3, $\mathcal{F}_k^0, ..., \mathcal{F}_k^{n-1}$ are cotorsion objects in $\mathcal{R}$. Then, the bottom row of (3.1) splits and hence $\mathcal{F}_k^n$ is cotorsion. So by Proposition 2.3, we have $\mathrm{cd}\mathcal{X}^n \leq k$.

2: Let $j = 0$. By Proposition 2.3, $\mathcal{F}_k^1, \mathcal{F}_k^2, ..., \mathcal{F}_k^{n-1}, \mathcal{F}_k^n$ are cotorsion objects and hence, the pure exact sequence

$$0 \longrightarrow \mathcal{X}^0 \longrightarrow \mathcal{C}_0^0 \longrightarrow \cdots \longrightarrow \mathcal{C}_{k-1}^0 \longrightarrow \mathcal{F}_k^1 \longrightarrow \mathcal{F}_k^2 \longrightarrow \cdots \longrightarrow \mathcal{F}_k^{n-1} \longrightarrow \mathcal{F}_k \longrightarrow 0 \tag{3.2}$$

implies that $\mathrm{cd}\mathcal{X}^0 \leq n+k$.

3: Let $0 < j < n$. By Proposition 2.3, $\mathrm{Ker}\,\partial_k^j$ in (3.1) is cotosion flat and $\mathrm{Im}\partial_k^j$ is a flat object of finite cotrsion dimension. So, $\mathcal{F}_k^j = \mathrm{Ker}\,\partial_k^j \oplus \mathrm{Im}\partial_k^j$ has finite cotorsion dimension. Then, the pure exact sequence

$$0 \longrightarrow \mathcal{X}^j \longrightarrow \mathcal{C}_0^j \longrightarrow \mathcal{C}_1^j \longrightarrow \cdots \longrightarrow \mathcal{F}_k^j \longrightarrow 0 \tag{3.3}$$

shows that $\mathrm{cd}\mathcal{X}^j \leq n+k$. $\square$

Theorem 3.2. *Assume that $\mathcal{R}$ has enough flat objects and $\mathcal{G}$ is an object in $\mathcal{R}$. Then, the following conditions are equivalent*

(i) $\mathrm{cd}\mathcal{G} < +\infty$.

(ii) *$\mathcal{G}$ is quasi-isomorphic to a bounded complex of cotorsion objects.*

(iii) *$\mathcal{G}$ is quasi-isomorphic to a bounded complex $\mathbf{X}$ such that any component of $\mathbf{X}$ has finite cotorsion dimension.*

Proof. (i)$\rightarrow$(ii) and (ii)$\rightarrow$(iii) are clear.

(iii)$\rightarrow$(i). Assume that $\mathcal{G}$ is quasi-isomorphic to a bounded complex $\mathbf{X} = (\mathcal{X}^i, \partial^i_{\mathbf{X}})_{i=0}^{i=n}$ such that any of its component has finite cotorsion dimension. Then, we obtain a bounded acyclic complex

$$0 \longrightarrow \mathcal{Y}^0 \longrightarrow \mathcal{Y}^1 \longrightarrow \mathcal{Y}^2 \longrightarrow \cdots \longrightarrow \mathcal{Y}^{r-1} \longrightarrow \mathcal{Y}^r \longrightarrow 0$$

such that for some $0 \leq j \leq r$, $\mathcal{Y}^j = \mathcal{G} \oplus \mathcal{X}^s$ $(0 \leq s \leq n)$, and for each $i \neq j$, $\mathcal{Y}^i$ has finite cotorsion dimension. By Lemma 2.9 and Proposition 2.3, we deduce a positive integer k and the following commutative diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{Y}^0 & \longrightarrow & \mathcal{Y}^1 & \longrightarrow & \mathcal{X}^2 & \longrightarrow \cdots \longrightarrow & \mathcal{Y}^{r-1} & \longrightarrow & \mathcal{Y}^r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^0_0 & \longrightarrow & \mathcal{C}^1_0 & \longrightarrow & \mathcal{C}^2_0 & \longrightarrow \cdots \longrightarrow & \mathcal{C}^{r-1}_0 & \longrightarrow & \mathcal{C}^r_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & \cdots & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^0_{k-1} & \longrightarrow & \mathcal{C}^1_{k-1} & \longrightarrow & \mathcal{C}^2_{k-1} & \longrightarrow \cdots \longrightarrow & \mathcal{C}^{r-1}_{k-1} & \longrightarrow & \mathcal{C}^r_{k-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^0_k & \xrightarrow{\partial^0_k} & \mathcal{F}^1_k & \xrightarrow{\partial^1_k} & \mathcal{F}^2_k & \xrightarrow{\partial^2_k} \cdots \longrightarrow & \mathcal{F}^{r-1}_k & \xrightarrow{\partial^{r-1}_k} & \mathcal{F}^r_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array} \tag{3.4}$$

with exact rows and pure exact columns such that for any $0 \leq j \leq k-1$ and $0 \leq i \leq r$, $\mathcal{C}^i_j$ is a cotorsion object and $\mathcal{F}^i_k$ is flat. By Proposition 2.3

and a similar method used in Theorem 3.1, we have $\mathrm{cd}\mathcal{Y}^j < +\infty$. We know that direct sum of any pair of minimal cotorsion resolutions of $\mathcal{G}$ and $\mathcal{X}^s$ is a minimal cotorsion resolution of $\mathcal{Y}^j = \mathcal{G} \oplus \mathcal{X}^s$. Therefore, by Proposition 2.3, any minimal cotorsion resolution of $\mathcal{G}$ stops and hence $\mathrm{cd}\mathcal{G}^j < +\infty$. □

Lemma 3.3. *Assume that $\mathcal{R}$ has enough flat objects and $\mathcal{G}$ is an object in $\mathcal{R}$. Then, the following conditions are equivalent.*

(i) $\mathrm{fd}\mathcal{G} \leq n$.

(ii) *There exists an acyclic complex*

$$\mathbb{C} : 0 \longrightarrow \mathcal{F}^n \xrightarrow{\partial^n} \mathcal{F}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{F}^{n-2} \longrightarrow \cdots \longrightarrow \mathcal{F}^1 \xrightarrow{\partial^1} \mathcal{F}^0 \xrightarrow{\partial^0} \mathcal{G} \longrightarrow 0$$

such that for any $0 \leq k \leq n$, $\mathcal{F}^k$ is a special flat precover of $\operatorname{Ker} \partial^{k-1}$.

(iii) *If*

$$\mathbb{C}' : \cdots \longrightarrow \mathcal{F}^n \xrightarrow{\lambda^n} \mathcal{F}^{n-1} \xrightarrow{\lambda^{n-1}} \mathcal{F}^{n-2} \longrightarrow \cdots \longrightarrow \mathcal{F}^1 \xrightarrow{\partial^1} \mathcal{F}^0 \xrightarrow{\partial^0} \mathcal{G} \longrightarrow 0$$

is an acyclic complex such that for each $0 \leq k$, $\mathcal{F}^k$ is a special flat precover of $\operatorname{Ker} \lambda^{k-1}$, *then* $\operatorname{Ker} \lambda^{n-1}$ *is flat.*

Proof. (i) ⇒ (ii) Clear.

(ii) ⇒ (i) Let

$$\mathbf{G} : \cdots \longrightarrow \mathcal{K}^n \xrightarrow{\lambda^n} \mathcal{K}^{n-1} \xrightarrow{\lambda^{n-1}} \mathcal{K}^{n-2} \xrightarrow{\lambda^{n-2}} \cdots \longrightarrow \mathcal{K}^1 \xrightarrow{\lambda^1} \mathcal{K}^0 \xrightarrow{\lambda^0} \mathcal{G} \longrightarrow 0$$

be the minimal flat resolution of $\mathcal{G}$, i.e for each $1 \leq m$, $\mathcal{K}^m$ is the flat cover of $\operatorname{Ker} \lambda^{m-1}$. Then, we have the following commutative diagrams

$$\begin{array}{ccccccccccccccccccc}
\mathbb{C}: \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}^n & \xrightarrow{\partial^n} & \mathcal{F}^{n-1} & \xrightarrow{\partial^{n-1}} & \mathcal{F}^{n-2} & \xrightarrow{\partial^{n-2}} \cdots \longrightarrow & \mathcal{F}^1 & \xrightarrow{\partial^1} & \mathcal{F}^0 & \xrightarrow{\partial^0} & \mathcal{G} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathbf{G}: \cdots & \longrightarrow & \mathcal{K}^{n+2} & \xrightarrow{\lambda^{n+2}} & \mathcal{K}^{n+1} & \xrightarrow{\lambda^{n+1}} & \mathcal{K}^n & \xrightarrow{\lambda^n} & \mathcal{K}^{n-1} & \xrightarrow{\lambda^{n-1}} & \mathcal{K}^{n-2} & \xrightarrow{\lambda^{n-2}} \cdots \longrightarrow & \mathcal{K}^1 & \xrightarrow{\lambda^1} & \mathcal{K}^0 & \xrightarrow{\lambda^0} & \mathcal{G} & \longrightarrow & 0.
\end{array}$$

and

$$\begin{array}{ccccccccccccccccccc}
\mathbb{C}: \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}^n & \xrightarrow{\partial^n} & \mathcal{F}^{n-1} & \xrightarrow{\partial^{n-1}} & \mathcal{F}^{n-2} & \xrightarrow{\partial^{n-2}} \cdots \longrightarrow & \mathcal{F}^1 & \xrightarrow{\partial^1} & \mathcal{F}^0 & \xrightarrow{\partial^0} & \mathcal{G} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
\mathbf{G}: \cdots & \longrightarrow & \mathcal{K}^{n+2} & \xrightarrow{\lambda^{n+2}} & \mathcal{K}^{n+1} & \xrightarrow{\lambda^{n+1}} & \mathcal{K}^n & \xrightarrow{\lambda^n} & \mathcal{K}^{n-1} & \xrightarrow{\lambda^{n-1}} & \mathcal{K}^{n-2} & \xrightarrow{\lambda^{n-2}} \cdots \longrightarrow & \mathcal{K}^1 & \xrightarrow{\lambda^1} & \mathcal{K}^0 & \xrightarrow{\lambda^0} & \mathcal{G} & \longrightarrow & 0.
\end{array}$$

such that for any $n \geq 0$ and any $\mathcal{H}$,

$$\mathrm{H}^n(\mathcal{H} \otimes \mathbf{C}) \cong \mathrm{H}^n(\mathcal{H} \otimes \mathbf{G}).$$

is an isomorphism. This shows that

$$0 \longrightarrow \mathrm{Ker}\,\lambda^{n+1} \longrightarrow \mathcal{K}^{n+1} \longrightarrow \mathrm{Im}\lambda^{n+1} \longrightarrow 0$$

is a pure exact sequence and hence $\mathrm{Im}\lambda^{n+1} = \mathrm{Ker}\,\lambda^n$ is flat. By the same method, we deduce that $\mathrm{Ker}\,\lambda^n$ is a pure subobject of $\mathcal{K}^n$. Consequently, $\mathrm{Im}\lambda^n = \mathrm{Ker}\,\lambda^{n-1}$ is flat and so, $\mathrm{fd}\mathcal{G} \leq n$.

(ii) $\Rightarrow$ (iii) Similar to the proof of (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (i) Trivial. □

Theorem 3.4. *Assume that $\mathcal{R}$ has enough flat objects and*

$$\mathbf{X} : 0 \longrightarrow \mathcal{X}^0 \xrightarrow{\partial^0} \mathcal{X}^1 \xrightarrow{\partial^1} \mathcal{X}^2 \xrightarrow{\partial^2} \cdots \longrightarrow \mathcal{X}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{X}^n \longrightarrow 0$$

is a bounded acyclic complex in $\mathcal{R}$ such that for each $i \neq j$, $\mathrm{fd}\mathcal{X}^i < +\infty$. Then, $\mathrm{fd}\mathcal{X}^j < +\infty$.

Proof. By Lemma 2.10 and Lemma 3.3, there exists a positive integer k and the following commutative diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{C}^0 & \xrightarrow{\partial^0} & \mathcal{C}^1 & \xrightarrow{\partial^1} & \mathcal{C}^2 & \xrightarrow{\partial^2} & \cdots & \longrightarrow & \mathcal{C}^{n-1} & \xrightarrow{\partial^{n-1}} & \mathcal{C}^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow t \\
0 & \longrightarrow & \mathcal{F}_0^0 & \longrightarrow & \mathcal{F}_0^1 & \longrightarrow & \mathcal{F}_0^2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_0^{n-1} & \longrightarrow & \mathcal{F}_0^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_{k-1}^0 & \longrightarrow & \mathcal{F}_{k-1}^1 & \longrightarrow & \mathcal{F}_{k-1}^2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{k-1}^{n-1} & \longrightarrow & \mathcal{F}_{k-1}^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{X}^0 & \longrightarrow & \mathcal{X}^1 & \longrightarrow & \mathcal{X}^2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{X}^{n-1} & \longrightarrow & \mathcal{X}^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array} \tag{3.5}$$

with exact rows and columns such that for all $0 \leq j \leq k-1$ and $0 \leq i \leq n$, $\mathcal{F}_j^i$ is flat and $\mathcal{C}^0, ..., \mathcal{C}^{n-1}, \mathcal{C}^n$ are cotorsion objects. Consider the following three cases.

1: Let $j = n$. By Lemma 3.3, the following exact sequence

$$0 \longrightarrow \mathcal{C}^0 \xrightarrow{\partial^0} \cdots \longrightarrow \mathcal{C}^{n-1} \xrightarrow{t\partial^{n-1}} \mathcal{F}_0^n \longrightarrow \cdots \longrightarrow \mathcal{F}_{k-1}^n \longrightarrow \mathcal{X}^n \longrightarrow 0 \tag{3.6}$$

implies that $\mathrm{fd}\mathcal{X}^n \leq k + n$.

2: Let $j = 0$. By Lemma 3.3 and [13, Theorem 2.3], the first row of (3.5) is pure and so $\mathcal{C}^0$ is flat. Therefore, $\mathrm{fd}\mathcal{X}^0 \leq n$.

3: Let $0 < j < n$. By Lemma 3.3 and [13, Theorem 2.3], $\mathrm{Ker}\,\partial^j$ is a cotorsion object of finite flat dimension and $\mathrm{Im}\partial^j$ is a flat object. Then, $\mathcal{F}^j = \mathrm{Ker}\,\partial^j \oplus \mathrm{Im}\partial^j$ has finite flat dimension. □

Now, by the same method used in the proof of Theorem 3.4, we obtain the following result.

Theorem 3.5. *Assume that $\mathcal{R}$ has enough projective objects and*

$$\mathbf{X} : 0 \longrightarrow \mathcal{X}^0 \xrightarrow{\partial^0} \mathcal{X}^1 \xrightarrow{\partial^1} \mathcal{X}^2 \xrightarrow{\partial^2} \cdots \longrightarrow \mathcal{X}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{X}^n \longrightarrow 0$$

is a bounded acyclic complex in $\mathcal{R}$ such that for each $i \neq j$, $\mathrm{pd}\mathcal{X}^i < +\infty$. Then, $\mathrm{pd}\mathcal{X}^j < +\infty$.

The next result is a generalization of [4, Proposition 2.3] and the main Theorem of [15].

Theorem 3.6. *Assume that $\mathcal{R}$ has enough projective objects. Then the following conditions are equivalent.*

(i) *$\mathcal{R}$ is n-perfect for some positive integer n.*

(ii) *Finite flat dimension implies finite non-negative dimension.*

(iii) *If*

$$\mathbf{X} : 0 \longrightarrow \mathcal{X}^0 \xrightarrow{\partial^0} \mathcal{X}^1 \xrightarrow{\partial^1} \mathcal{X}^2 \xrightarrow{\partial^2} \cdots \longrightarrow \mathcal{X}^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{X}^n \longrightarrow 0$$

is a bounded acyclic complex in $\mathcal{R}$ such that for each $i \neq j$, $\mathrm{fd}\mathcal{X}^i < +\infty$. Then, $\mathrm{pd}\mathcal{X}^j < +\infty$.

(iv) *Any object in $\mathcal{R}$, which is quasi-isomorphic to a bounded complex of objects of finite flat dimension, has finite projective dimension.*

Proof. The equivalence (i)$\leftrightarrow$(ii) deduced by the similar method used in the proof of [4, Proposition 2.3]. The implication (i)$\rightarrow$(iii) is a direct consequence of Theorem 3.5. For the implication (i)$\rightarrow$(iii), we proceed as follows. Assume that $\mathcal{G} \in \mathcal{R}$ is quasi-isomorphic to a bounded complex $\mathbf{X} = (\mathcal{X}^i, \partial^i_{\mathbf{X}})_{i=0}^{i=n}$ such that any of its component has finite flat dimension. So, there exists a bounded acyclic complex

$$0 \longrightarrow \mathcal{Y}^0 \longrightarrow \mathcal{Y}^1 \longrightarrow \mathcal{Y}^2 \longrightarrow \cdots \longrightarrow \mathcal{Y}^{r-1} \longrightarrow \mathcal{Y}^r \longrightarrow 0$$

such that for some $0 \leq j \leq r$, $\mathcal{Y}^j = \mathcal{G} \oplus \mathcal{X}^s$, and for each $i \neq j$, $\mathcal{Y}^i$ has finite flat dimension. By Lemma 2.10 and Lemma 3.3, we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{C}^0 & \xrightarrow{\partial^0} & \mathcal{C}^1 & \xrightarrow{\partial^1} & \mathcal{C}^2 & \xrightarrow{\partial^2} & \cdots & \longrightarrow & \mathcal{C}^{n-1} & \xrightarrow{\partial^{r-1}} & \mathcal{C}^r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle t} \\
0 & \longrightarrow & \mathcal{F}^0_0 & \longrightarrow & \mathcal{F}^1_0 & \longrightarrow & \mathcal{F}^2_0 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}^{r-1}_0 & \longrightarrow & \mathcal{F}^r_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}^0_{k-1} & \longrightarrow & \mathcal{F}^1_{k-1} & \longrightarrow & \mathcal{F}^2_{k-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}^{r-1}_{k-1} & \longrightarrow & \mathcal{F}^r_{k-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{Y}^0 & \longrightarrow & \mathcal{Y}^1 & \longrightarrow & \mathcal{Y}^2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{Y}^{r-1} & \longrightarrow & \mathcal{Y}^r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array} \tag{3.7}$$

with exact rows and columns such that for all $0 \leq j \leq k-1$ and $0 \leq i \leq r$, $\mathcal{F}^i_j$ is flat and $\mathcal{C}^0, ..., \mathcal{C}^{r-1}, \mathcal{C}^r$ are cotorsion objects. By the same method used in the proof of Theorem 3.4, we deduce that $\text{fd}\mathcal{Y}^j < +\infty$. Consequently, by Lemma 3.3, $\text{fd}\mathcal{G} < +\infty$ and hence by Theorem 3.5 we are done.

The implication (iv)$\rightarrow$(i) is straightforward. □

4 n-perfect categories

In this section, we assume that $(\mathcal{A}, \otimes)$ is a locally noetherian symmetric monoidal closed Grothendieck category which has enough flat objects. It is known that, projective objects are rare in some situations (see [10, §3]) and so we can not talk about the projective dimension. This issue was resolved in [9, 11] by using the cotorsion dimension rather than the projective dimension and was characterized non-affine schemes which over them any flat object has finite cotorsion dimension (see [9, 11] for more details). So, the study around the objects of finite cotorsion dimension is interesting for us. In the following, we give some examples of Grothendieck categories which has enough flat objects and do not have non-zero projective objects.

Example 4.1. Let $(X, \mathcal{O}_X)$ be a locally noetherian scheme (see [7] for more details on sheaves and schemes).

(i) The category $(\mathfrak{Mod}X, \text{-} \otimes_{\mathcal{O}_X} \text{-})$ of all sheaves of $\mathcal{O}_X$-modules is locally noetherian ([8, Theorem II.7.8]) and has enough flat objects, see [8, Proposition II.1.2] for more details.

(ii) The category $(\mathfrak{Qcoh}X, \text{-} \otimes_{\mathcal{O}_X} \text{-})$ of all quasi-coherent sheaves of $\mathcal{O}_X$-modules is locally noetherian. Furthermore, by [19], $\mathfrak{Qcoh}X$ has a flat generator if and only if X is quasi-compact and semi-separated.

(iii) Let R be a commutative ring and $X = \mathbb{P}^n_R$ be the projective n-space over R. Then, by [6], $\mathfrak{Qcoh}X$ dose not have non-zero projective object. But, X is quasi-compact and semi-separated so $\mathfrak{Qco}X$ has a flat generator.

(iv) Let k be a commutative ring with identity. It was shown in [5] that if $X = \mathbb{P}^1_k$ is the projective line over k, then $\mathfrak{Qco}X$ has a flat generator.

The next result concerns a characterization of objects of finite cotorsion dimension.

Proposition 4.2. Let $\mathcal{G}$ be an object in $\mathcal{A}$. Then, we have the following equivalent conditions.

(i) $\mathrm{cd}\mathcal{G} \leq n$.

(ii) For any flat object $\mathcal{F}$, $\mathrm{Ext}^{n+1}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) = 0$.

(iii) For any flat object $\mathcal{F}$ and any $i \geq 1$, $\mathrm{Ext}_{\mathcal{A}}^{n+i}(\mathcal{F}, \mathcal{G}) = 0$.

Proof. The proof is a standard homological algebra fare. □

In the next proposition, we will give a characterization of n-perfect categories.

Proposition 4.3. The following conditions are equivalent.

(i) $\mathcal{A}$ is n-perfect, for some non-negative integer n.

(ii) Finite flat dimension in $\mathcal{A}$ implies finite cotorsion dimension.

Proof. (i) ⇒ (ii). Let $\mathcal{G}$ be an object in $\mathcal{A}$ of finite flat dimension. Then, $\mathcal{G}$ is quasi-isomorphic to a bounded complex of flat objects. So, by Theorem 3.1, $\mathcal{G}$ has finite cotorsion dimension.

(ii) ⇒ (i). Assume by contradiction that for any integer n, $\mathcal{A}$ is not n-perfect. Then, for any n, we have a flat object $\mathcal{G}_n$ of cotorsion dimension n. For such n, consider the following minimal injective resolution

$$0 \longrightarrow \mathcal{G}_n \xrightarrow{\partial_n} \mathcal{I}_n^0 \xrightarrow{\partial_n^0} \mathcal{I}_n^1 \xrightarrow{\partial_n^1} \cdots \longrightarrow \mathcal{I}_n^m \xrightarrow{\partial_n^m} \cdots .$$

of $\mathcal{G}_n$. Then,

$$0 \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathcal{G}_n \xrightarrow{\oplus\partial_n} \bigoplus_{n\in\mathbb{Z}} \mathcal{I}_n^0 \xrightarrow{\oplus\partial_n^0} \bigoplus_{n\in\mathbb{Z}} \mathcal{I}_n^1 \longrightarrow \cdots \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathcal{I}_n^m \longrightarrow \cdots .$$

is an injective resolution of $\bigoplus_{n\in\mathbb{Z}} \mathcal{G}_n$ (direct sum of injectives is injective in a locally noetherian Grothendieck category). But, $\bigoplus_{n\in\mathbb{Z}} \mathcal{G}_n$ is flat and hence it has finite cotorsion dimension. Proposition 4.2 implies that there is a non-negative integer k_0 where $\mathrm{cd}(\bigoplus_{n\in\mathbb{Z}})\mathcal{G}_n \leq k_0$. So, $\mathrm{Ker}(\oplus\partial_n^{k_0}) \cong \oplus(\mathrm{Ker}\,\partial_n^{k_0})$ and $\mathrm{Ker}\,\partial_n^{k_0}$ are cotorsion for any n and so for any n $\mathrm{cd}\mathcal{G}_n \leq k_0$. But this is a contradiction. It follows that, $\mathcal{A}$ is n-perfect for some non-negative integer n. □

Finally, in the rest of this paper, we give some examples of Grothendieck categories that our results are hold for them.

4.1 The category of sheaves Let $(X, \mathcal{O}_X)$ be an arbitrary scheme, $\mathfrak{Qco}X$ be the category of all quasi-coherent $\mathcal{O}_X$-modules and $\mathfrak{Mod}X$ be the category of all sheaves of $\mathcal{O}_X$-modules (for more details on sheaves and schemes see [7]). It is known that $\mathfrak{Qco}X$ is a full subcategory of $\mathfrak{Mod}X$. If $\mathcal{R}$ is one of $\mathfrak{Qco}X$ or $\mathfrak{Mod}X$, Theorems 3.1 and Theorem 3.4 hold true (see [4, 10, 11] for more details).

4.2 The category of R-modules Assume that R is an associative ring with $1 \neq 0$ and $\mathbf{C}(R)$ is the category of complexes of left R-modules. In this subsection, we will present a method to construct new n-perfect rings (for more details and examples see [14], [17, 18], [4], [10]). For this purpose, we recall some notations from [16, Remark 2.4]. Let S_R be the R-algebra of the quiver

$$\cdots \longrightarrow \bullet \overset{\delta^{n-1}}{\longrightarrow} \bullet \overset{\delta^{n}}{\longrightarrow} \bullet \overset{\delta^{n+1}}{\longrightarrow} \bullet \overset{\delta^{n+2}}{\longrightarrow} \cdots$$

with the relation $\delta^{n+1}\delta^{n} = 0$ and S_R-Mod is the category of all left S_R-modules. Every representation of this quiver is in fact a complex of R-modules. By [16], we have a fully faithful functor

$$\mathrm{inc} : \mathbf{C}(R) \longrightarrow S_R\text{-Mod}$$

where for any complex $\mathbf{X}$, $\mathrm{inc}(\mathbf{X}) = \bigoplus_{n\in\mathbb{Z}} \mathbf{X}^n$. This functor has a right adjoint

$$C : S_R\text{-Mod} \longrightarrow \mathbf{C}(R).$$

In addition, C is a colimit preserving exact functor. By [16, Proposition 2.8], for any flat (projective) S_R-module F, $C(F)$ is a flat (projective) complex of R-modules and for any flat (projective) complex $\mathbf{F}$ of R-modules $\mathrm{inc}(\mathbf{F})$ is a flat (projective) S_R-module.

Theorem 4.2.1. *The following conditions are equivalent.*

(i) *Finite flat dimension in R-Mod implies finite projective dimension.*

(ii) *Finite flat dimension in S_R-Mod implies finite projective dimension.*

Proof. (i) $\Rightarrow$ (ii). Let F be a flat S_R-module and

$$\mathcal{E} : \cdots \longrightarrow P^n \overset{\partial^n_{\mathcal{E}}}{\longrightarrow} P^{n-1} \overset{\partial^{n-1}_{\mathcal{E}}}{\longrightarrow} P^{n-2} \longrightarrow \cdots \longrightarrow P^0 \overset{\partial^0_{\mathcal{E}}}{\longrightarrow} F \longrightarrow 0$$

be a projective resolution of F. Then, $\mathrm{inc}(\mathcal{E})$ gives a projective resolution of $\mathrm{inc}F$ where $\mathrm{Ker}\,\partial^{n-1}_{\mathrm{inc}(\mathcal{E})}$ is projective by assumption. Therefore, $\mathrm{Ker}\,\partial^{n-1}_{\mathcal{E}} = C(\mathrm{Ker}\,\partial^{n-1}_{\mathrm{inc}(\mathcal{E})})$ is projective and so we are done.

(ii) $\Rightarrow$ (i). Assume that F is a flat R-module. By assumption, $\mathrm{pd}(\mathrm{inc}(F)) \leq n$. Let

$$\mathcal{E}: 0 \longrightarrow P^n \overset{\partial^n}{\longrightarrow} P^{n-1} \overset{\partial^{n-1}}{\longrightarrow} P^{n-2} \longrightarrow \cdots \longrightarrow P^0 \overset{\partial^0}{\longrightarrow} \mathrm{inc}(F) \longrightarrow 0$$

be the minimal projective resolution of $\mathrm{inc}(F)$. Therefore, $\mathrm{C}(\mathcal{E})$ gives a projective resolution of F and hence $\mathrm{pd}F \leq n$. Consequently, we are done by [4, Proposition 2.3]. □

Corollary 4.2.2. *The following conditions are equivalent.*

(i) *R is n-perfect.*

(ii) *S_R is n-perfect.*

Assume that k is a field, R (S) is a right-noetherian (left-noetherian) k-algebra.

Definition 4.2.3. Let ${}_S\mathcal{D}_R$ be a bounded complex in the derived category $\mathcal{D}(S \otimes_k R^{\mathrm{op}})$ with finitely generated cohomology both over S and R^{op}. It is called a dualizing complex if the following conditions hold.

(i) The injective dimensions $\mathrm{id}_S\mathcal{D}$ and $\mathrm{id}_{R^{\mathrm{op}}}\mathcal{D}$ are finite.

(ii) Both morphisms $R \longrightarrow \mathrm{RHom}_S(\mathcal{D},\mathcal{D})$ in $\mathcal{D}(R^e)$, $R \longrightarrow \mathrm{RHom}_{R^{\mathrm{op}}}(\mathcal{D},\mathcal{D})$ in $\mathcal{D}(S^e)$ are isomorphisms.

If $\mathcal{D}^{\mathrm{fd}}(R)$ is the subcategory of the derived category $\mathcal{D}(R)$ consisting of complexes which are isomorphic to a bounded complex of flat R-modules, and $\mathcal{D}^{\mathrm{id}}(S)$ is the subcategory of the derived category $\mathbf{D}(S)$ consisting of complexes which are isomorphic to a bounded complex of injective S-modules. Then, there is an equivalence

$$\mathbf{D}^{\mathrm{fd}}(R) \underset{\mathrm{RHom}_S(\mathbf{D},-)}{\overset{\mathbf{D}^L\otimes_R -}{\rightleftarrows}} \mathbf{D}^{\mathrm{id}}(S) \tag{4.1}$$

of triangulated categories. Now, we use 4.1 and deduce the main result of [15]. It was shown in [4] that an associative ring is n-perfect if and only if finite flat dimension implies finite projective dimension if and only if finite flat dimension implies finite cotorsion dimension.

Corollary 4.2.4. *Let M be a left R-module. Then* $\mathrm{fd}M < \infty$ *implies* $\mathrm{pd}M < \infty$.

Proof. By 4.1, any flat R-module F is quasi-isomorphic to a bounded complex of cotorsion R-modules. Therefore, by Corollary 3.2, F has finite cotorsion dimension and so, by Corollary 3.6, we are done. □

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